CHAPTER 4

FINITE ELEMENT FORMULATION [T3,T6,T7,T9]

Analytical solutions to the general boundary value problem in structural mechanics are often difficult to obtain. To this end, a number of numerical solution routines were developed from the beginning of the 1900's of which the finite element method (FEM) has proven to be the most versatile and is the widest used.

The finite element method is a numerical mathematical method for the solution of boundary value problems. The development of FEM in structural mechanics is primarily for standard problems i.e. discontinuities needs some special treatment.

In this chapter, the FEM implementation of discontinuous models, as discussed in the previous chapters, is investigated in some detail. The aim of the chapter is to ensure a thorough understanding of the theory and FEM calculation of stress intensity.

4.1. THE FINITE ELEMENT METHOD [T3,T6,T7,T9]

Finite element analysis is a numerical solution routine for solving differential equations. The finite element theory is based on the principle of minimum potential energy, which states:

The kinimatically admissible displacement fields corresponding to equilibrium extrimises the potential energy of conservative systems. If this extremum is a minimum, the equilibrium state is stable.

or $\Pi = U + WP$

where Π = Potential energy.
U = strain energy and
WP = work potential.

4.1.1 General Formulation

The strain energy, U, for linear elasticity is given in equation 2.1.36 as

$$U = \frac{1}{2}\int_V s^T e dV \qquad (4.1.1)$$

The work potential or work done by external forces is given by

$$WP = -\int_V \mathbf{u}^T \mathbf{b} dV - \int_S \mathbf{u}^T \mathbf{T} dS - \sum_i \mathbf{u}_i^T \mathbf{P}_i \qquad (4.1.2)$$

where **u** is the displacement, **b** is the body forces, **T** is the traction on surface, S, and **P** is point loads. Substitution of equations 4.1.1 and 4.1.2 in the principal of minimum potential energy gives :

$$? = \frac{1}{2}\int_V s^{?} e dV - \int_V \mathbf{u}^{?} \mathbf{b} dV - \int_S \mathbf{u}^{?} \mathbf{T} dS - \sum_i \mathbf{u}_i^{?} \mathbf{P}_i \qquad (4.1.3)$$

4.1.2 Discretised Presentation Of A Continuum

An elastic continuum can be represented by separating the continuum along imaginary boundaries into a number of finite elements. The elements are assumed to be interconnected at a number of discrete points called nodes. The displacements at these nodes are the basic unknowns in the problem. A set of functions, called shape functions, uniquely defines the state of displacement and hence strain and stress in each finite element.

The displacement vector is defined by

$$\mathbf{u}^h = \mathbf{N}^k\mathbf{q}^k \qquad k = 1,2,3 \ldots n \qquad (4.1.4)$$

where $\mathbf{u}$ is the displacement vector, $\mathbf{N}^k$ are shape functions and $\mathbf{q}^k$ are constants associated with a displacement degree of freedom for each node. The displacement in an element is therefore defined as an accumulation of the product of shape functions, **N**, and associated constants, **q**.

Each degree of freedom has a shape function with the property that it has a value of one at the node and zero at all other nodes. This means the displacement for a particular degree of freedom is equal to its associated constant, q.

Equation 4.1.4 can be substituted into equation 4.1.3 together with the compatibility and constitutive laws. The constitutive law and compatibility equations are given by equations 2.1.41 and 2.1.20 and are written as:

$$\text{Constitutive} \quad s = \mathbf{C}e \qquad (4.1.5)$$

$$\text{Compatibility} \quad e = \mathbf{Lu} = \mathbf{LNq} = \mathbf{Bq} \qquad (4.1.6)$$

where $\mathbf{L}$ is a differential operator. Substitution of 4.1.5 in 4.1.3 gives

$$? = \frac{1}{2}\int_V e^{\mathbf{T}}\mathbf{C}e \, dV - \int_V \mathbf{u}^{\mathbf{T}}\mathbf{b} \, dV - \int_S \mathbf{u}^{\mathbf{T}}\mathbf{T} \, dS - \sum_i \mathbf{u}_i^{\mathbf{T}}\mathbf{P}_i \qquad (4.1.7)$$

Substitution of 4.1.6 in 4.1.7 gives

$$? = \frac{1}{2}\mathbf{q}^{\mathbf{T}}\int_V \mathbf{B}^{\mathbf{T}}\mathbf{CBq} \, dV - \mathbf{q}^{\mathbf{T}}\int_V \mathbf{N}^{\mathbf{T}}\mathbf{b} \, dV - \mathbf{q}^{\mathbf{T}}\int_S \mathbf{N}^{\mathbf{T}}\mathbf{T} \, dS - \mathbf{q}^{\mathbf{T}}\sum_i \mathbf{N}_i^{\mathbf{T}}\mathbf{P}_i \qquad (4.1.8)$$

The minimum potential energy is obtained by setting $\frac{\partial ?}{\partial q_i} = 0$ giving

$$\int_V \mathbf{B}^{\mathbf{T}}\mathbf{CB}dV\mathbf{q} = \int_V \mathbf{N}^{\mathbf{T}}\mathbf{b}dV + \int_S \mathbf{N}^{\mathbf{T}}\mathbf{T}dS + \sum_i \mathbf{N}_i^{\mathbf{T}}\mathbf{P}_i \qquad (4.1.9)$$

from equation 4.1.8. The boundary value problem in linear elasticity is fully defined by equation 4.1.9 provided the displacements, defined in equation 4.1.4, are kinematically permissible, satisfies the essential boundary conditions and have continuous first order derivatives.

The right hand side of equation 4.1.9 represents the applied loads while the left hand side represents the stiffness matrix, **K**, times the nodal displacements, **q**. The finite element method is implemented by evaluating expression 4.1.9 for each element and assembling the elements into a global matrix system in line with standard stiffness principles.

The formulae for plane strain are obtained by replacing **C** and **L** in equations 4.1.5 and 4.1.6 with equations 2.2.6 and 2.2.3.

4.1.3 Crack Tip Elements [T6]

One of the simplest methods of modelling crack tips involves the modification of isoparametric elements in the region of the crack tip. The purpose of the modification is to impose the $r^{-1/2}$ singularity, as indicated by equation 2.4.15, in the stress and strain fields. The method of modification is only applicable to isoparametric elements with polynomial shape functions of quadratic or higher degree. Practical use is largely confined to quadratic elements as higher order elements impose computational restrictions for large models.

The method of modification will be demonstrated with the aid of a quadratic line element. The element (figure 4.1) is mapped into an isoparametric co-ordinate, ξ.

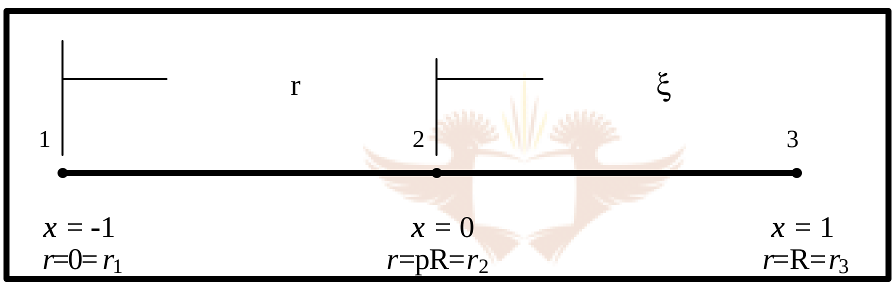


Figure 4.1 : Quadratic isoparametric element

The element has the following polynomial shape functions:

$$N_1 = -0.5x + 0.5x^2 \qquad (4.1.10.a)$$
$$N_2 = 1 - x^2 \qquad (4.1.10.b)$$
$$N_3 = 0.5x + 0.5x^2 \qquad (4.1.10.c)$$

The shape function, N_i, of node i has a value of 1 at node i and zero at all other nodes. The shape functions are used together with the co-ordinate values to map the element into the new co-ordinate system (in the same way as the displacements are expressed in terms of the shape functions and constants, q_i, by expression 4.1.4) as follows.

$$r = N_i r_i \qquad i = 1,2,3$$

The symbol r, represents the physical distance from node 1. With node 3 at R, node 2 would be at R/2 for an undistorted element.

If the distance to node 2 is assigned as pR, with p undefined at this stage, r is expressed as:

$$
\begin{aligned}
r &= N_1 r_1 + N_2 r_2 + N_3 r_3 \\
&= (-0.5x + 0.5x^2)*0 + (1 - x^2)*\mathrm{pR} + (0.5x + 0.5x^2)*\mathrm{R} \\
&= \mathrm{pR} - \mathrm{p}x^2\mathrm{R} + 0.5x\mathrm{R} + 0.5x^2\mathrm{R} \\
&= \mathrm{R}[\mathrm{p} + 0.5x + x^2(0.5-\mathrm{p})] \qquad (4.1.11)
\end{aligned}
$$

Differentiating with respect to ξ,

$$\frac{\mathrm{d}r}{\mathrm{d}x} = \mathrm{R}\left[\frac{1}{2} + (1-2\mathrm{p})x\right] \qquad (4.1.12)$$

Denoting u as the displacement in the r direction, the direct strain in the r direction is given by

$$\frac{du}{dr} = \frac{du}{dx}\frac{dx}{dr} = \frac{du}{dx}\frac{1}{\mathrm{R}}\left[\frac{1}{2} + (1-2\mathrm{p})x\right]^{-1} \qquad (4.1.13)$$

and is singular when

$$0.5 + (1-2\mathrm{p})x = 0 \qquad (4.1.14)$$

Note that this condition is never satisfied when p=0.5. The strain field is always non-singular for an undistorted element with the middle node in the centre. If a singularity is required for r=0 where x =-1, p must be chosen such that 4.1.14 is satisfied. For x =-1:

$$
\begin{aligned}
&0.5 - (1 - 2\mathrm{p}) = 0 \\
&\therefore\ \mathrm{p} = 0.25 \qquad (4.1.15)
\end{aligned}
$$

This means the mid-node must be moved to a point at a quarter of the element length from the node at which the singularity is desired. The same strategy applies to solid elements in general i.e. quadrilaterals, triangles, bricks, etc.

4.1.4 Singularity Of A Quarter Node Element

The order of the singularity is evaluated by substituting equation 4.1.15 in 4.1.11 and solving for x to give

$$x = -1 + \sqrt{\frac{4r}{\mathrm{R}}} \tag{4.1.16}$$

The displacement is given by equation 4.1.4. Substitution of the shape functions into equation 4.1.4 and collection of terms gives the displacement as a function of x in the general form:

$$u = b_1 + b_2 x + b_3 x^2 \tag{4.1.17}$$

Equation 4.1.16 in 4.1.17 gives

$$\begin{aligned} u &= b_1 + b_2\left(-1 + 2\sqrt{\frac{r}{\mathrm{R}}}\right) + b_3\left(-1 + 2\sqrt{\frac{r}{\mathrm{R}}}\right)^2 \\ &= \left(b_1 - b_2 + b_3\right) + 2\left(b_2 - 2b_3\right)\sqrt{\frac{r}{\mathrm{R}}} + 2b_3\frac{r}{\mathrm{R}} \end{aligned}$$

The strain is obtained from the derivative of u with respect to r as

$$\frac{du}{dr} = \frac{\left(b_2 - 2b_3\right)}{\sqrt{r\mathrm{R}}} + \frac{2b_3}{\mathrm{R}} \tag{4.1.18}$$

which displays the correct order of singularity in $r^{-½}$. The stiffness matrix for the line element is derived using the formulae of section 4.1.2. Equation 4.1.4 gives

$$u = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \tag{4.1.19}$$

where q_i represents the nodal displacements. For the constitutive relation of equation 4.1.5, **C** is simply replaced by E. The compatibility operator, **L**, is given by

$$\mathrm{L} = \frac{d}{dr} = \frac{d}{dx}\frac{dx}{dr} \tag{4.1.20}$$

The strain is given by e $= \mathbf{LNq} = \mathbf{Bq}$. Equations 4.1.20 and 4.1.10 in 4.1.6 give

$$\mathbf{B} = \begin{bmatrix} \frac{dN_1}{dx} & \frac{dN_2}{dx} & \frac{dN_3}{dx} \end{bmatrix} \frac{dx}{dr} \tag{4.1.21}$$

The stiffness integral of equation 4.1.9 is given by

$$\int_v \mathbf{B}^{\mathrm{T}}\mathbf{C}\mathbf{B}dV = \mathrm{EA}\int_{-1}^{1}\mathbf{B}^{\mathrm{T}}\mathbf{B}\frac{\partial r}{\partial x}dx \tag{4.1.22}$$

assuming a bar element with an cross sectional area of A. The components of **B** are given by

$$\left.\begin{aligned} \frac{dN_1}{dx} &= x-\frac{1}{2} \\ \frac{dN_2}{dx} &= -2x \\ \frac{dN_3}{dx} &= x+\frac{1}{2} \end{aligned}\right\} \tag{4.1.23}$$

The Jacobian is given in equation 4.1.12 as

$$\left.\begin{aligned} \frac{dr}{dx} &= \frac{\mathrm{R}}{2}(1+x) \quad \text{for quarter node element} \\ \frac{dr}{dx} &= \frac{\mathrm{R}}{2} \qquad\qquad \text{for undistorted element} \end{aligned}\right\} \tag{4.1.24}$$

Substituting 4.1.21, 4.1.23 and 4.1.24 in 4.1.22 gives

$$\mathbf{K} = \frac{2\mathrm{AE}}{\mathrm{R}}\int_{-1}^{1}\begin{bmatrix} \frac{(x-0.5)^2}{(x+1)} & \frac{-2(x-0.5)x}{(x+1)} & \frac{(x^2-0.25)}{(x+1)} \\ \frac{-2(x-0.5)x}{(x+1)} & \frac{4x^2}{(x+1)} & \frac{-2(x+0.5)x}{(x+1)} \\ \frac{(x^2-0.25)}{(x+1)} & \frac{-2(x+0.5)x}{(x+1)} & \frac{(x+0.5)^2}{(x+1)} \end{bmatrix}dx \tag{4.1.25}$$

for quarter node elements and

$$\mathbf{K} = \frac{2\mathrm{AE}}{\mathrm{R}}\int_{-1}^{1}\begin{bmatrix} (x-0.5)^2 & -2(x-0.5)x & (x^2-0.25) \\ -2(x-0.5)x & 4x^2 & -2(x+0.5)x \\ (x^2-0.25) & -2(x+0.5)x & (x+0.5)^2 \end{bmatrix}dx \tag{4.1.26}$$

for undistorted elements. The singularity of the quarter node element does not allow an analytical solution of equation 4.1.25. This is a requirement for the singularity in the strain or stress field, which is the first derivative of the displacement.

The strain is given by **Bq** and is expressed as

$$\varepsilon = \frac{1}{\mathrm{R}}\begin{bmatrix} \frac{2(x-0.5)}{(x+1)} & \frac{-4x}{(x+1)} & \frac{2(x+0.5)}{(x+1)} \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \tag{4.1.27}$$

for the quarter node element and

$$e = \frac{1}{\mathrm{R}}\begin{bmatrix} 2(x-0.5) & -4x & 2(x+0.5) \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \tag{4.1.28}$$

for an undistorted element. The stress in both cases is simply given by Ee in accordance with equation 4.1.5.

4.1.5 Numerical Integration

The integration required in equation 4.1.9 is complex for the general case. The Gausian quadrature approach for evaluating integrals of polynomials is ideally suited for computer applications and is used in FEM code for this purpose. The scheme is explained by equation 4.1.29.

$$I = \int_{-1}^{1} f(x)dx = w_1 f(x_1) + w_2 f(x_2) + \cdots + w_n f(x_n) \tag{4.1.29}$$

where $w_1, w_2, \ldots, w_n$ are weights and $x_1, x_2, \ldots, x_n$ are sampling points or Gauss points. The object is to select n Gauss points and n weights such that equation 4.1.29 gives an exact answer for a polynomial of as high as possible order.

Consider a cubic function $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$. Applying equation 4.1.29 with a 2-point integration gives

$$I = \int_{-1}^{1} f(x)dx \approx w_1 f(x_1) + w_2 f(x_2) \tag{4.1.30}$$

The error can be calculated by

$$Error = \int_{-1}^{1} \left(a_0 + a_1x + a_2x^2 + a_3x^3\right)dx - \left[w_1 f(x_1) + w_2 f(x_2)\right] \tag{4.1.31}$$

Evaluation of 4.1.31 with an error equal to zero requires the following conditions to hold true:

$$\left.\begin{aligned} w_1 + w_2 &= 2 \\ w_1 x_1 + w_2 x_2 &= 0 \\ w_1 x_1^2 + w_2 x_2^2 &= 0 \\ w_1 x_1^3 + w_2 x_2^3 &= 0 \end{aligned}\right\} \qquad (4.1.32)$$

The equations in 4.1.32 have the unique solution:

$$w_1 = w_2 = 1 \quad ; -x_1 = x_2, = 1/\sqrt{3} = 0.5773502691...... \qquad (4.1.33)$$

In general, n point Gauss quadrature would provide exact solutions to polynomials of order (2n - 1) or less. Two-dimensional integration is a simple expansion of equation 4.1.29. Two-point integration in two dimensions is given by:

$$I \approx \sum_{i=1}^{2}\sum_{j=1}^{2} w_i w_j f\left(\xi_i, \eta_j\right) \qquad (4.1.34)$$

Numerical evaluation of equations 4.1.25 and 4.1.26 by 2 point Gauss quadrature, with A = 1 and replacing R with L, gives

$$\mathbf{K} = \frac{\mathrm{E}}{\mathrm{L}}\begin{bmatrix} 5.5 & -6.0 & 0.5 \\ -6.0 & 8.0 & -2.0 \\ 0.5 & -2.0 & 1.5 \end{bmatrix} \qquad (4.1.35)$$

for the quarter node element and

$$\mathbf{K} = \frac{\mathrm{E}}{3\mathrm{L}}\begin{bmatrix} 7.0 & -8.0 & 1.0 \\ -8.0 & 16.0 & -8.0 \\ 1.0 & -8.0 & 7.0 \end{bmatrix} \qquad (4.1.36)$$

for an undistorted element. Two point Gauss quadrature would give exact solutions for the integration of the 2nd order polynomials of equation 4.1.28. The accuracy of integration of equation 4.1.27 is difficult to judge as the integral does not theoretically exist. The two point Gauss quadrature gives the value of the integral between the boundaries of approximately -0.9 to 1. This means the stiffness contribution of the first 5 % of the quarter node element is neglected.

4.1.6 Displacements In The Vicinity Of A Quarter Node Element Singularity

The finite element implementation consists of a quarter node element, with the singularity simulating a crack tip, surrounded by regular second order elements. The field variables in structural finite element analysis are displacements. Stresses and strains are calculated from first derivatives of the displacements in accordance with the compatibility and constitutive laws.

Substitution of equation 4.1.10 in 4.1.19 with $q_1 = 0$ (the displacement at the singular node is used as reference) gives the displacement in terms of x.

$$u_{FE} = \left(1 - \xi^2\right)q_2 + \frac{\xi}{2}\left(1 + \xi\right)q_3 \tag{4.1.36}$$

The subscript, FE, indicates finite element results. The isoparametric co-ordinate x, is written in terms of r, where r is the radius from the singular node, by substituting equation 4.1.16. The general solution is:

$$u_{FE} = C_0 + C_1\sqrt{r} + C_2 r \tag{4.1.37}$$

where C_i are constants. The displacement of an undistorted element is given by the general solution:

$$u_{FE} = D_0 + D_1 r + D_2 r^2 \tag{4.1.38}$$

where D_i are constants. Equations 4.1.36 and 4.1.37 can be compared to the analytical solution of equation 2.4.16, written in terms of r with $r = x_1 - a$ as substituted in equation 2.4.10,

$$u_{AN} = A\sqrt{2ar - r^2} \tag{4.1.39}$$

with A as a constant. In the finite element method, equation 4.1.39 is modelled by piecewise functions consisting of equation 4.1.37, at the crack tip, followed by segments of equation 4.1.38. Equations 4.1.37 and 4.1.39 are of the same order in r.

4.1.7 Multi-Dimensional Quarter Node Elements

It was demonstrated in section 4.1.6 that, for a quarter node line element, the displacement function is of order $r^{½}$ with the resulting stress function, by differentiation, of order $r^{-½}$. The order of the singularity will now be investigated for a plane triangular iso-parametric element with 6 nodes, as displayed in figure 4.2. As a result of the iso-parametric mapping, generality is not lost due to the x_1, x_2 co-ordinate values in figure 4.2. The element is mapped from the x_1, x_2 co-ordinate system to the ξ_1, ξ_2, ξ_3 area co-ordinate system.

The element has the following shape functions:

$$\left.\begin{aligned} N_1 &= \xi_1(2\xi_1 - 1) \\ N_2 &= \xi_2(2\xi_2 - 1) \\ N_3 &= \xi_3(2\xi_3 - 1) \\ N_4 &= 4\xi_1\xi_2 \\ N_5 &= 4\xi_2\xi_3 \\ N_6 &= 4\xi_1\xi_3 \end{aligned}\right\} \qquad (4.1.40)$$

with u_i defined by equation 4.1.4 as

$$\mathbf{u}^h = \mathbf{N}^k\mathbf{q}^k \qquad k = 1,2,3 \ldots n \qquad (4.1.4)$$

or

$$\left.\begin{aligned} u_1 &= \sum_{i=1}^{6} N_i q_{1i} \\ u_2 &= \sum_{i=1}^{6} N_i q_{2i} \end{aligned}\right\} \qquad (4.1.41)$$

and x_i defined as

$$\left.\begin{aligned} x_1 &= \sum_{i=1}^{6} N_i x_{1i} \\ x_2 &= \sum_{i=1}^{6} N_i x_{2i} \end{aligned}\right\} \qquad (4.1.42)$$

through iso-parametric mapping.

The intent is to describe the function order of the displacements through an arbitrary line that passes through the crack tip (node 1 in figure 4.2). The first step is to relate the area co-ordinates, ξ's, to the cartesian co-ordinates, x's, along the line $x_2 = mx_1$, with m the constant slope of the line. The relationship, $\xi_2 = n\xi_3$, applies to the same line.

$$x_2 = mx_1 \qquad (4.1.43)$$

$$\xi_2 = n\xi_3 \qquad (4.1.44)$$

Substitution of the x_i values, from figure 4.2, into equation 4.1.42 gives the mapping functions for x_1 and x_2.

$$\left.\begin{aligned} x_1 &= N_2 + \frac{N_4}{4} + \frac{N_5}{2} \\ x_2 &= N_3 + \frac{N_5}{2} + \frac{N_6}{4} \end{aligned}\right\} \qquad (4.1.45)$$

Substitution of 4.1.40 in 4.1.45 yields

$$x_1 = 2\xi_2^2 - \xi_2 + \xi_1\xi_2 + 2\xi_2\xi_3 \qquad (4.1.46)$$

$$x_2 = 2\xi_3^2 - \xi_3 + 2\xi_2\xi_3 + \xi_1\xi_3 \qquad (4.1.47)$$

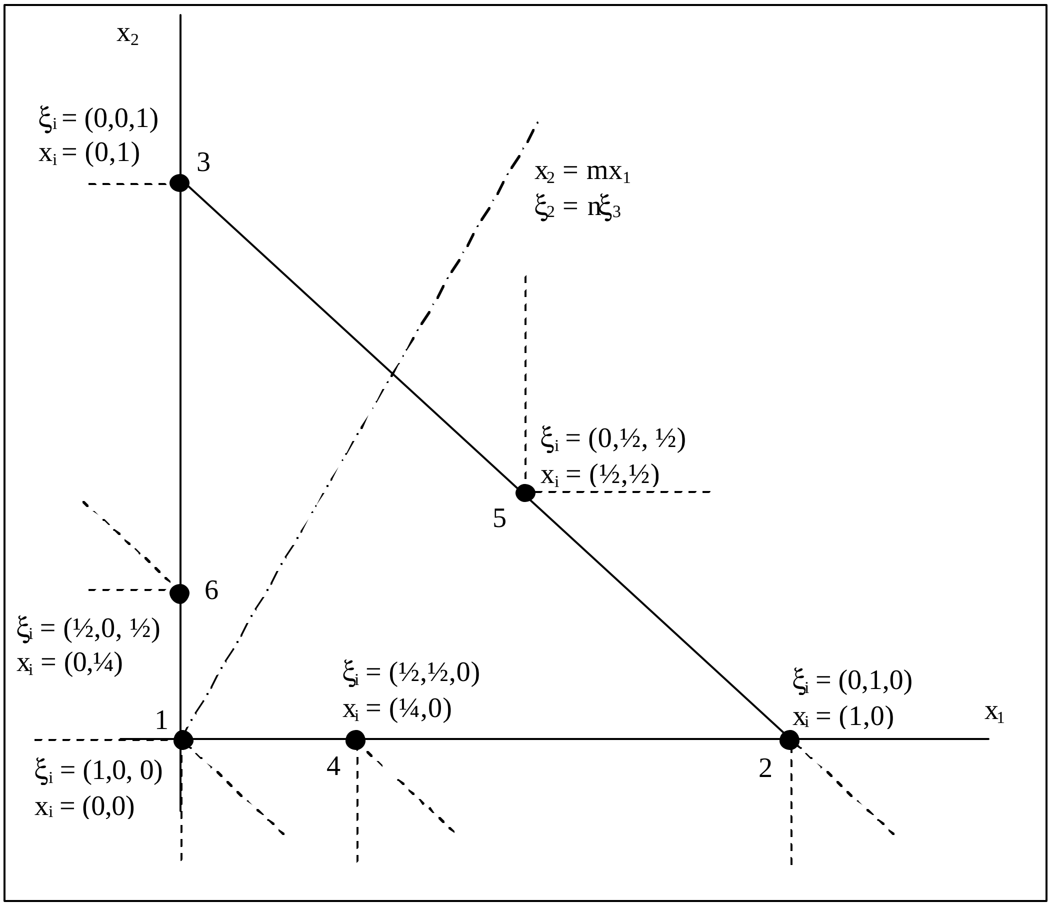


Figure 4.2: Crack tip element

Subtracting equation 4.1.46-4.1.47 and substituting equations 4.1.43 and 4.1.44 gives ξ_2 and ξ_3 as a function of x_1 or x_2. The relationship is

$$\left.\begin{aligned} \xi_2 &= C_0\sqrt{x_1} = C_1\sqrt{x_2} \\ \xi_3 &= D_0\sqrt{x_1} = D_1\sqrt{x_2} \end{aligned}\right\} \qquad (4.1.48)$$

with C_i and D_i constants. Area co-ordinates are always defined such that their sum total is equal to one i.e.

$$\xi_1 + \xi_2 + \xi_3 = 1 \tag{4.1.49}$$

A general equation for displacement, u, is derived by expansion of equation 4.1.41 to give:

$$\begin{aligned} u = {} & \beta_1\xi_1^2 + \beta_2\xi_2^2 + \beta_3\xi_3^2 + \beta_4\xi_1\xi_2 + \beta_5\xi_1\xi_3 + \\ & \beta_6\xi_2\xi_3 + \beta_7\xi_1 + \beta_8\xi_2 + \beta_9\xi_3 \end{aligned} \tag{4.1.50}$$

Equations 4.1.48 and 4.1.49 in 4.1.50 give the expression for u as a function of x_1, in a general form,

$$u = E_0 + E_1 x_1 + E_2\sqrt{x_1} \tag{4.1.51}$$

with E_i constants. The distance from node one, along the line $x_2 = m x_1$, is denoted r such that

$$\begin{aligned} r &= \sqrt{x_1^2 + x_2^2} \\ &= \sqrt{x_1^2 + m^2 x_1^2} \\ &= x_1\sqrt{1 + m^2} \\ &= \text{Constant}.x_1 \end{aligned} \tag{4.1.52}$$

The x_1 in equation 4.1.51 can, therefore, be replaced by r without any loss in generality.

$$u = E_0 + E_1 r + E_2\sqrt{r} \tag{4.1.53}$$

Strain (and stress) is obtained by the differentiation of the displacements such that an $r^{-½}$ term appears in the stress solution, as required by crack tip stress field theory. The derivation of equation 4.1.53 proves that the 6 node triangular element in figure 4.2, has the correct function order for displacements along all lines passing through the crack tip node. It was proven, along similar lines, that this is not the case for a 8 node quadrilateral element where the $r^{½}$ term in the displacement fields only features along lines that coincide with the element edges passing through the crack tip node.

The 6 node triangular element can be expanded to a 3-D wedge or tetrahedron element (see figure 4.3) with the same function order of displacement and strain fields as the plane element. Displacements in the wedge element will vary in accordance with equation 4.1.53 through all lines passing perpendicularly through the crack front, while a quadratic function is maintained for lines that run in parallel with the crack front.

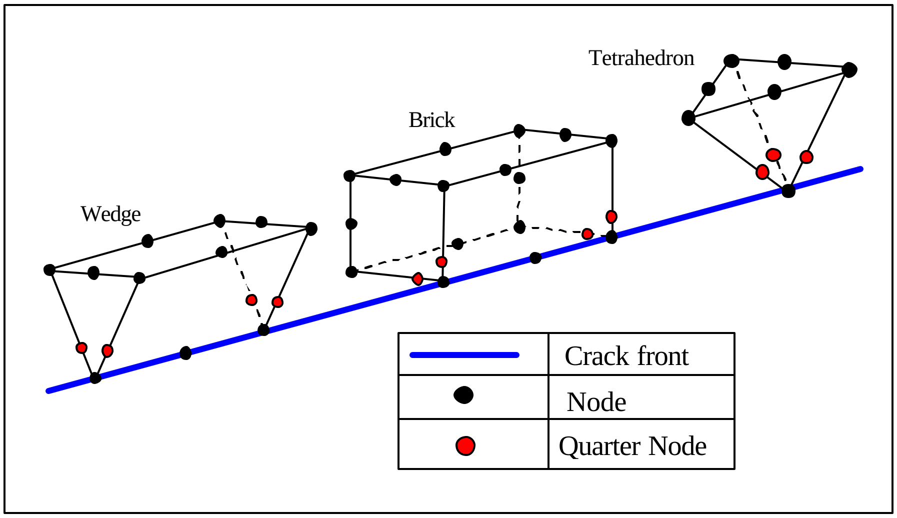


Figure 4.3: 3-D Quarter Node Elements

4.1.8 Finite Element Convergence

There are some basic conditions that finite element modelling must comply with to ensure convergence. Most of these conditions are derived from simple logic and includes

- straining of an element must not be caused by rigid body motion,
- the displacement functions must be compatible with a constant strain condition and
- there must be displacement continuity at the element interfaces.

Convergence is naturally also influenced by factors specific to the problem under consideration and the numerical accuracy of the computational system used. These two factors are described as a discretization error and a round off error.

The discretization error has to do with the capability of the shape functions to model the real gradients. If the shape function is linear, fine discretization is required to model a steep gradient while a single cubic shape function can model a cubic real displacement exactly. Smaller elements are therefore required in areas of steep gradients (displacement-strain-stress gradients).

Round off errors occur due to the finite significant numbers used in digital computation. Finite element analysis entails the solution of a set of linear equations of which the coefficients consists of a square matrix, the stiffness matrix. The stiffness matrix becomes ill-conditioned if the ratio of the smallest to the largest coefficient on the main diagonal is too large. Errors are accumulated in numerical addition or subtraction of two numbers with too little or no overlap in significant digits. The result in such a case is always equal to the larger number and the computation has no effect at all.

4.2 CONVERGENCE BEHAVIOUR

Section 4.1.8 discusses a few basic pre-requisites for finite element convergence. In this section, the accuracy of the stress intensity calculation will be investigated in comparison to some known solutions.

4.2.1 Convergence Of Calculation Method

The primary field variable in finite element analysis is displacement meaning that the solution provides a displacement for each node. The displacement of the free crack surface, u, is used in accordance with equation 3.2.5 for the calculation of a stress intensity.

$$K_I = \lim_{r \to 0}\left(\frac{-u}{4}\frac{E}{(1-\nu^2)}\sqrt{\frac{2\pi}{r}}\right) = \frac{-E\sqrt{2\pi}}{4(1-\nu^2)}\lim_{r \to 0}\left(\frac{u}{\sqrt{r}}\right) \qquad (3.2.5)$$

An exact analytical solution for u is obtained from equation 2.4.16 by substituting r = a – x_1, noting that equation 2.4.16 was derived for $\theta = \pi$ or $x_2 = 0$ i.e. free surface of the crack. The substitution gives

$$u = \frac{2\sigma(1-\nu^2)}{E}\sqrt{(2ar - r^2)} \qquad (4.2.1)$$

The physical interpretation of r is, in this case, the distance from the crack tip along the free surface of the crack face.

Finite element displacements would be available for the quarter node and the node furthest away from the crack front. The u/sqrt(r) results for these two nodes would have to be extrapolated to zero to find the limit in equation 3.2.5.

The displacements through the entire volume of the element can be interpolated using the interpolation or shape functions of equation 4.1.40 in combination with equation 4.1.41. The convergence behaviour of the stress intensity solution, equation 3.2.5, is evaluated under the assumption that the finite element solution on the nodes corresponds exactly to the analytical solution of equation 4.2.1.

The evaluation considers solutions for a range of crack tip element size to crack length ratios, c/a, of 0.05 to 1. The following values were used in the evaluation:

E = 200 000 N/mm^2
v = 0.3
a = 1 mm
σ = 1 MPa

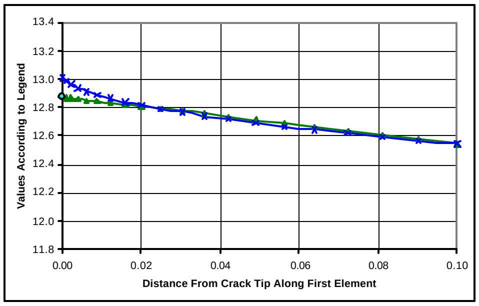


Figure 4.4.1: c/a = 0.1

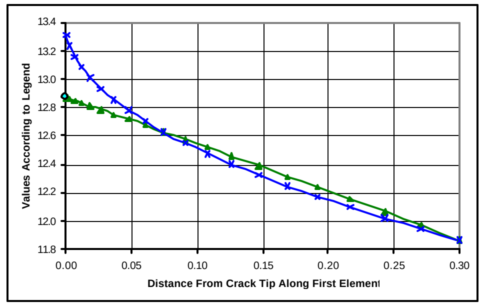


Figure 4.4.2: c/a = 0.3

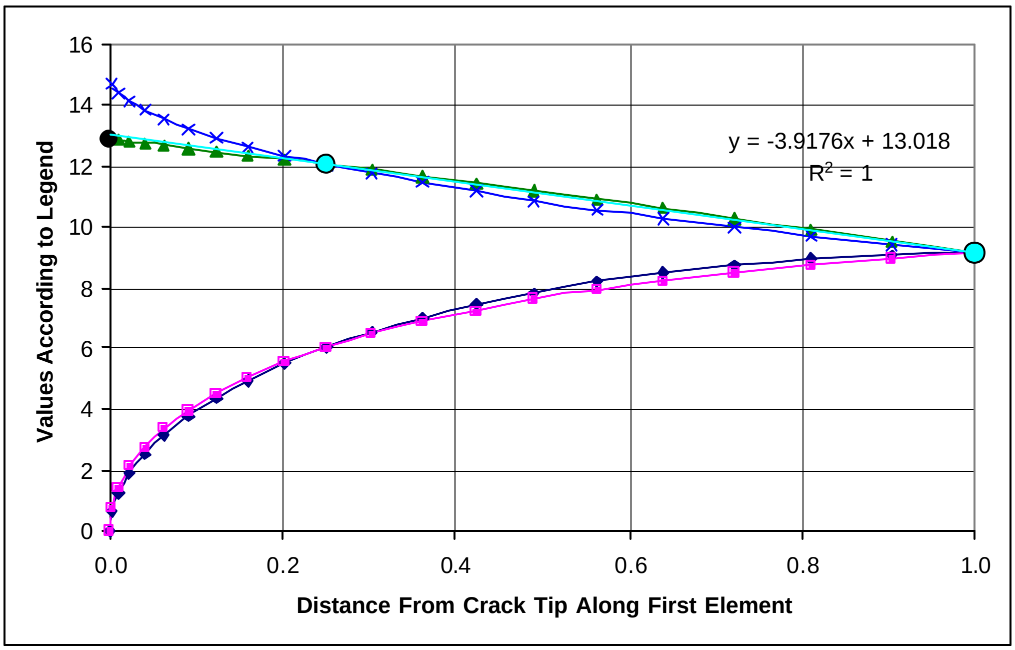
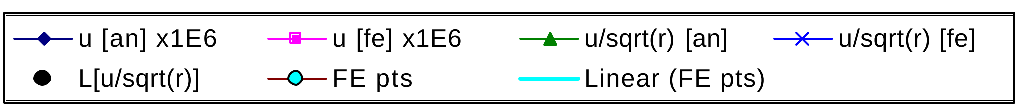


Figure 4.4.3: c/a = 1



Legend to figure 4.4

Figures 4.4.1 to 4.4.3 give the finite element, fe, and analytical, an, results of u/sqrt(r) for three cases of crack tip element length to crack length ratios, c/a, namely 0.1, 0.3 and 1 respectively. Figure 9.3 includes the displacement curves and a linear fit of u/sqrt(r) through the finite element nodes. The analytical solution as well as the interpolated finite element solution is displayed for displacements as well as for the limit of equation 3.2.5.

It is evident that the zero intersection of a linear extrapolation through the nodes gives the most accurate result compared to the analytical solution. The interpolated finite element limit deviates significantly from the analytical solution for higher c/a ratios.

One must bear in mind that this discussion assumes an exact solution to the finite element nodes. The high stress gradients around the crack tip demand a relatively high mesh density for good convergence. A parametric convergence study was performed by the author, but is not included as part of this dissertation.

In conclusion, the zero offset limit of a linear fit through the finite element nodes provides an accurate answer in accordance with equation 3.2.5 provided the displacement results are accurate at the nodes.

4.3 CONCLUSIONS

It was demonstrated that the theoretical approximation and assumptions in the stress intensity approach accumulate little error i.e. small strain, higher order terms in crack tip stresses etc. Finite element analysis of the boundary value problem in structural mechanics violates none of the governing laws in the extreme i.e. infinitely small discretization. The only limitation is in computing resources. An optimum approach would therefore have a balance between the element size and the order of the shape functions that would give sufficiently converged solutions.

The ability of equations 4.1.37 and 4.1.38 to follow the curve of equation 4.1.39 was evaluated by the author in a convergence study that is not presented in this dissertation. The methods derived in the convergence study was verified by calculating stress intensity factors for a number of known and referenced solutions including the centre cracked panel [T14], embedded elliptical crack [T11] and rotating disk with quarter-elliptical corner crack in the bore [P6]. The verification studies proved that the methods derived in the convergence studies can guarantee an optimum solution.