PART 2: THEORETICAL SURVEY

CHAPTER 2

RELEVANT THEORY OF ELASTICITY

This chapter discusses the theory of elasticity required for fracture mechanics and the boundary value problem in continuum mechanics. The finite element method is a numerical method for the solution of a boundary value problem, but the finite element formulation for boundary value problems in continuum mechanics is also based on the theory of elasticity. Enough details are given to logically follow the arguments and evaluate the assumptions, but some mathematical proofs, which can be followed from the references, are omitted.

The theoretical development is only followed for linear elasticity, which forms the basis for linear elastic fracture mechanics (LEFM) and linear elastic finite element analysis. It is explained later that this is applicable to most large structures, especially those operating at relatively low temperature, and in particular to the case study of the LP turbine.

The basis of the theory of elasticity can be followed through the definition of the boundary value problem in continuum mechanics. The boundary value problem defines all the governing equations to ensure single valued solutions. A number of simplifications are made to linearize some of the relations leading to restrictions in its application and inherent errors.

While general 3D continuum problems are mostly solved by numerical methods, analytical solutions to plane problems are readily achievable. The general theory is simplified for plane problems and used to find solutions to stress field problems with singularities as is found around crack tips.

Knowledge of matrix algebra and tensor component notation [T2], including the summation convention, is assumed in this chapter (see conventions on page iv).

2.1. THE BOUNDARY VALUE PROBLEM IN CONTINUUM MECHANICS

The continuum problem in solid mechanics is described as follows. Assume a structural body with an outer surface C (see figure 2.1). The body has surface traction, t^*, on C_t and displacement constraints, u^*, on C_u. There is also an arbitrary temperature distribution through the body. The general problem is to find the stress field, s , the strain field, e , and the displacement field, **u**, for all points in the body.

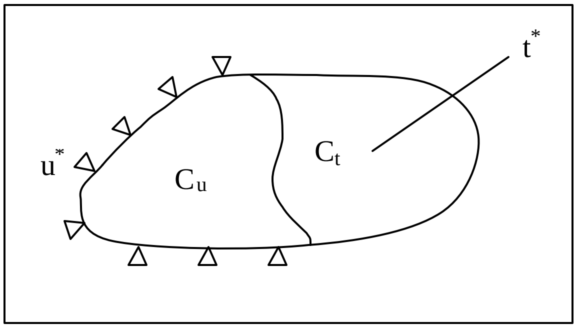

Figure 2.1: Structural body

The solution is naturally subjected to some conditions. From practical considerations the following conditions apply [T3,T6]:

- *Equilibrium*: The sum of all forces acting on the body, including acceleration, equals zero.
- *Compatibility*: The strain-displacement solution is compatible with a body that is still continuous after straining has taken place.
- *Constitutive*: The constitutive law describes the stress-strain behaviour of the material.
- *Boundary conditions*: The prescribed boundary conditions must be satisfied at all times. The stress solution must satisfy the traction on C_t and the displacement solution must satisfy the constraints on C_u.

The compatibility equations and constitutive law are investigated first because the results are used in the derivation of expressions used in the equilibrium relations.

2.1.1. Compatibility

Compatibility is achieved through the strain-displacement relationship. Examining the mathematical behaviour of a line element when it goes through deformation develops compatibility equations.

2.1.1.1 Stretch of a Line Element [T4]

Consider element P_oQ_o of a body, in its reference configuration, with length δL aligned in the direction of unit vector **A** (see figure 2.2). If P_o has co-ordinates X_R, then Q_o has co-ordinates $X_R + A_R\delta L$. Suppose the body goes through a deformation such that P_oQ_o moves to PQ and has a new length δl directed along a unit vector, **a**. If P has co-ordinates, x_i, then Q has co-ordinates $x_i + a_i\delta l$. Since P was initially at P_o, a Lagrange description of the displaced co-ordinates as a function of the reference co-ordinates is:

$$x_i = x_i(X_R) \tag{2.1.1}$$

and similarly,

$$x_i + a_i\delta l = x_i(X_R + A_R\delta L) \tag{2.1.2}$$

assuming static conditions such that time dependency can be omitted. A Lagrange formulation describes changes in the body as a function of its initial or reference co-ordinates, opposed to an Euler description that is a function of the strained co-ordinates.

A Taylor series expansion around $A_R\delta L$ gives:

$$x_i + a_i\delta l = x_i(X_R) + A_S\delta L\frac{\partial x_i(X_R)}{\partial X_S} + O\{(\delta L)^2\} \tag{2.1.3}$$

In the limit as $\delta L \to 0$ expression 2.1.3 becomes:

$$a_i\frac{\delta l}{\delta L} = A_S\frac{\partial x_i(X_R)}{\partial X_S} \tag{2.1.4}$$

The ratio of the final and initial lengths ($\delta l/\delta L$) of an infinitesimal material line element as described is called the stretch, denoted by λ.

Squaring equation 2.1.4 and summing on the index, i, give,

$$(\lambda a_i)(\lambda a_i) = \left(A_S \frac{\partial x_i}{\partial X_S} \right)\left(A_T \frac{\partial x_i}{\partial X_T} \right)$$

$$\therefore \qquad \lambda^2 = A_S A_T \frac{\partial x_i}{\partial X_S} \frac{\partial x_i}{\partial X_T} \qquad (2.1.5)$$

with $a_i a_i = 1$, because **a** is a unit vector.

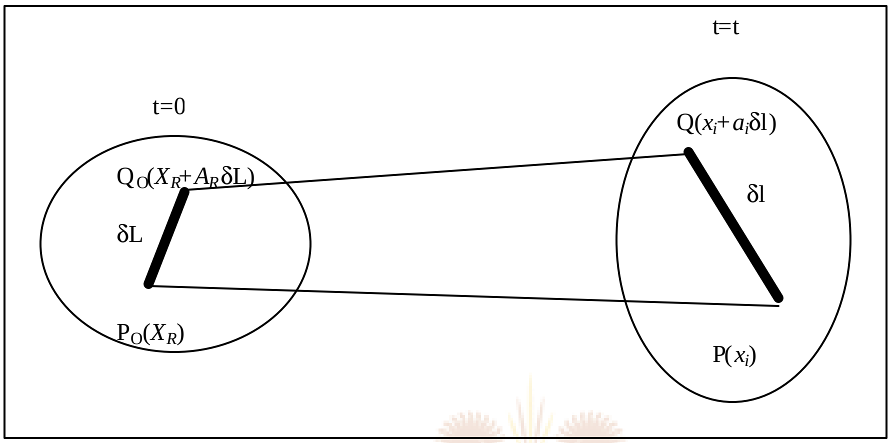

Figure 2.2 : Stretch of a line element

The 9 quantities $\partial x_i/\partial X_R$ are called deformation gradients denoted:

$$F_{iR} = \frac{\partial x_i}{\partial X_R} \qquad (2.1.6)$$

F_{iR} are components of a second order tensor, **F**, because **F** conforms to the tensor transformation law [T4]. $\mathbf{F}^T$ and $\mathbf{F}^{-1}$ would also be second order tensors provided the inverse exists i.e. the determinant is not equal to zero.

The components of the displacement vector **u** are given by $u_i = x_i - X_i$. The displacement gradients are given by:

$$\frac{\partial u_i}{\partial X_R} = \frac{\partial (x_i - X_i)}{\partial X_R} = \frac{\partial x_i}{\partial X_R} - \delta_{iR} = F_{iR} - \delta_{iR} \qquad (2.1.7)$$

where δ_{iR} is the Kronecker delta. The displacement gradients are therefore components of the tensor **F**-**I** called the displacement gradient tensor. Although **F** is important in the analysis of deformation it is not a suitable measure of strain because it varies in rigid body motion. In rigid body motion for the general case, **F** is a function of time i.e. **F** = **Q**(t).

2.1.1.2 Deformation and Strain Tensors [T2,T4,T5,T6]

The results of section 2.1.1.1 are further developed to derive suitable measures for strain as follows.

Define the tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F}$

$$\therefore C_{RS} = F_{iR}F_{iS} = \frac{\partial x_i}{\partial X_R}\frac{\partial x_i}{\partial X_S} \qquad (2.1.8)$$

From equation 2.1.8, it is evident that $C_{RS} = C_{SR}$ so that **C** is symmetric. Equation 2.1.5 can now be written as:

$$\lambda^2 = C_{RS}A_RA_S = \mathbf{A}^T\mathbf{C}\mathbf{A} \qquad (2.1.9)$$

in matrix notation.

For a rigid body motion $\mathbf{C} = \mathbf{Q}^T\mathbf{Q} = \mathbf{I}$ giving a constant value [T2]. **C** is called the Cauchy-Green deformation tensor. **C**, although not uniquely defined, forms the basis of a suitable measure for strain. Any tensor function of **C**, such as $\mathbf{C}^2$ would also be suitable.

The Lagrange strain tensor, **L** , is defined as:

$$\mathbf{L} = \tfrac{1}{2}(\mathbf{C} - \mathbf{I}) \qquad (2.1.10)$$

From equation 2.1.7, it follows that

$$F_{iR} = \frac{\partial x_i}{\partial X_R} = \frac{\partial u_i}{\partial X_R} + \delta_{iR} \qquad (2.1.11)$$

Substitution of equation 2.1.11 in 2.1.10 gives:

$$L_{RS} = \frac{1}{2}\left\{\left(\frac{\partial u_i}{\partial X_R} + \delta_{iR}\right)\left(\frac{\partial u_i}{\partial X_S} + \delta_{iS}\right) - \delta_{RS}\right\} \qquad (2.1.12)$$

$$= \frac{1}{2}\left(\frac{\partial u_R}{\partial X_S} + \frac{\partial u_S}{\partial X_R} + \frac{\partial u_i}{\partial X_R}\frac{\partial u_i}{\partial X_S}\right) \qquad (2.1.13)$$

This would give, for example,

$$L_{11} = \frac{\partial u_1}{\partial X_1} + \frac{1}{2}\left\{\left(\frac{\partial u_1}{\partial X_1}\right)^2 + \left(\frac{\partial u_2}{\partial X_1}\right)^2 + \left(\frac{\partial u_3}{\partial X_1}\right)^2\right\}$$

Similarly, an Euler description can be followed to give:

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_R}{\partial x_i}\frac{\partial u_S}{\partial x_j} \right) \tag{2.1.14}$$

2.1.1.3 Infinitesimal Strain [T2,T4,T5,T6]

In typical engineering applications, only small changes of shape are experienced. Expressions 2.1.13 and 2.1.14 are simplified considerably if all the components in these expressions are assumed to be small compared to 1 i.e.

$$\left| \frac{\partial u_i}{\partial X_R} \right| << 1 \text{ and } \left| \frac{\partial u_i}{\partial x_j} \right| << 1$$

such that squares and products can be neglected. The validity of this is evaluated by considering a strain in the range of 0.002 as is commonly encountered at the yield limit for steel. A square of this strain would be $4x10^{-6}$; far smaller than unity. Omission of the squares would result in an error in the order of 0.2%.
Since $u_i = x_i - X_i$, the derivative with respect to x_i gives:

$$\frac{\partial u_i}{\partial x_j} = \left(\delta_{ij} - \frac{\partial X_i}{\partial x_j} \right) = \mathbf{I} - \mathbf{F}^{-1} \tag{2.1.15}$$

A binomial expansion [T8] of 2.1.15 gives:

$$\begin{aligned} \mathbf{I} - \mathbf{F}^{-1} &= \mathbf{I} - \{\mathbf{I} + (\mathbf{F} - \mathbf{I})\}^{-1} \\ &= \mathbf{I} - \{\mathbf{I} - (\mathbf{F} - \mathbf{I}) + (\mathbf{F} - \mathbf{I})^2 - (\mathbf{F} - \mathbf{I})^3 \; ..... \} \\ \therefore \frac{\partial u_i}{\partial x_j} &= (\mathbf{F} - \mathbf{I}) - (\mathbf{F} - \mathbf{I})^2 + (\mathbf{F} - \mathbf{I})^3 \; ..... \end{aligned} \tag{2.1.16}$$

Equation 2.1.7 gives **F** - **I** as $\frac{\partial u_i}{\partial X_R} = F_{iR} - \delta_{ij}$ and it follows that:

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial u_i}{\partial X_j} - \frac{\partial u_i}{\partial X_R}\frac{\partial u_R}{\partial X_j} + \frac{\partial u_i}{\partial X_R}\frac{\partial u_R}{\partial X_S}\frac{\partial u_S}{\partial X_j} - ..... \tag{2.1.17}$$

If squares and products are neglected the result is:

$$\frac{\partial u_i}{\partial x_j} \cong \frac{\partial u_i}{\partial X_j} \tag{2.1.18}$$

From equations 2.1.13 and 2.1.14 it follows that:

$$L_{ij} = E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad (2.1.19)$$

The infinitesimal strain tensor is defined from expressions 2.1.16 and 2.1.19 as:

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) = \frac{1}{2}\left(\mathbf{F} + \mathbf{F}^{\mathbf{T}}\right) - \mathbf{I} \qquad (2.1.20)$$

E cannot be an exact measure of strain because it does not remain constant for rigid-body rotations. In static analysis, the component must have a fixture preventing rigid body motion. Because the components of the tensor contain only second order terms for small rotations, **E** would be a suitable measure of strain for such cases.

The infinitesimal strain tensor (2.1.20) consists of 9 equations (components), but since **E** is symmetric in i and j, there are only 6 independent partial differential equations for three unknown displacements, u_i. The equations are:

$$\left.\begin{aligned}
\frac{\partial u_1}{\partial X_1} &= e_{11} \\
\frac{\partial u_2}{\partial X_2} &= e_{22} \\
\frac{\partial u_3}{\partial X_3} &= e_{33} \\
\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} &= 2e_{12} \\
\frac{\partial u_1}{\partial X_3} + \frac{\partial u_3}{\partial X_1} &= 2e_{13} \\
\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2} &= 2e_{23}
\end{aligned}\right\} \qquad (2.1.21)$$

The infinitesimal strain relations produce some mathematical difficulty when a plausible strain solution is sought. The reason is that the three unknown displacements are over-defined by six independent partial differential equations. The compatibility equations define conditions that must be satisfied by the strain distribution in order for equations 2.1.19 and 2.1.20 to be integrable. A generalised derivation of compatibility equations produces, in tensor component notation:

$$e_{ij,kl} + e_{kl,ij} = e_{ik,jl} + e_{jl,ik} \qquad (2.1.22)$$

Equation 2.1.22 consists of 3^4 = 81 equations, but some of these are identically satisfied and some are repetitions due to symmetry in ij and kl so that only 6 essential equations exist.

The compatibility equations can also be derived from equation 2.1.21. Differentiating the fourth equation in 2.1.21 with respect to X_1 and X_2 produces:

$$2\frac{\partial^2 e_{12}}{\partial X_1 \partial X_2} = \frac{\partial^3 u_1}{\partial X_2^2 \partial X_1} + \frac{\partial^3 u_2}{\partial X_1^2 \partial X_2} \tag{2.1.23}$$

Similar operations on the first two equations in 2.1.21 produce:

$$\frac{\partial^2 e_{11}}{\partial X_2^2} + \frac{\partial^2 e_{22}}{\partial X_1^2} = \frac{\partial^3 u_1}{\partial X_2^2 \partial X_1} + \frac{\partial^3 u_2}{\partial X_1^2 \partial X_2} \tag{2.1.24}$$

Combining 2.1.23 and 2.1.24 gives

$$\frac{\partial^2 e_{11}}{\partial X_2^2} + \frac{\partial^2 e_{22}}{\partial X_1^2} = 2\frac{\partial^2 e_{12}}{\partial X_1 \partial X_2} \tag{2.1.25}$$

Two similar equations are found by starting with the other two shear components. A further three equations are derived as follows. From 2.1.21, differentiate e_{12} with respect to X_1 and X_3 and e_{13} with respect to X_1 and X_2 and add. The result is

$$\frac{\partial^2 e_{11}}{\partial X_2 \partial X_3} = \frac{\partial^2 e_{23}}{\partial X_1 \partial X_1} + \frac{\partial^2 e_{31}}{\partial X_1 \partial X_2} + \frac{\partial^2 e_{12}}{\partial X_1 \partial X_3} \tag{2.1.26}$$

Two similar equations are found by starting with combinations of e_{23} and e_{12} as well as e_{23} and e_{13}. Defining functions R_i and U_i, the six compatibility equations are:

$$\left.\begin{aligned} R_3 &= \frac{\partial^2 e_{11}}{\partial X_2^2} + \frac{\partial^2 e_{22}}{\partial X_1^2} - 2\frac{\partial^2 e_{12}}{\partial X_1 \partial X_2} = 0 \\ R_1 &= \frac{\partial^2 e_{22}}{\partial X_3^2} + \frac{\partial^2 e_{33}}{\partial X_2^2} - 2\frac{\partial^2 e_{23}}{\partial X_2 \partial X_3} = 0 \\ R_2 &= \frac{\partial^2 e_{33}}{\partial X_1^2} + \frac{\partial^2 e_{11}}{\partial X_3^2} - 2\frac{\partial^2 e_{31}}{\partial X_3 \partial X_1} = 0 \end{aligned}\right\} \tag{2.1.27.a}$$

$$\left.\begin{aligned}
U_1 &= -\frac{\partial^2 e_{11}}{\partial X_2 \partial X_3} + \frac{\partial}{\partial X_1}\left(-\frac{\partial e_{23}}{\partial X_1} + \frac{\partial e_{31}}{\partial X_2} + \frac{\partial e_{12}}{\partial X_3} \right) = 0 \\
U_2 &= -\frac{\partial^2 e_{22}}{\partial X_3 \partial X_1} + \frac{\partial}{\partial X_1}\left(\frac{\partial e_{23}}{\partial X_1} - \frac{\partial e_{31}}{\partial X_2} + \frac{\partial e_{12}}{\partial X_3} \right) = 0 \\
U_3 &= -\frac{\partial^2 e_{33}}{\partial X_1 \partial X_2} + \frac{\partial}{\partial X_1}\left(\frac{\partial e_{23}}{\partial X_1} + \frac{\partial e_{31}}{\partial X_2} - \frac{\partial e_{12}}{\partial X_3} \right) = 0
\end{aligned}\right\} \qquad (2.1.27.b)$$

The equations in 2.1.27 are not independent since they satisfy the following 3 identities.

$$\left.\begin{aligned}
\frac{\partial R_1}{\partial X_1} + \frac{\partial U_3}{\partial X_2} + \frac{\partial U_2}{\partial X_3} = 0 \\
\frac{\partial U_3}{\partial X_1} + \frac{\partial R_2}{\partial X_2} + \frac{\partial U_1}{\partial X_3} = 0 \\
\frac{\partial U_2}{\partial X_1} + \frac{\partial U_1}{\partial X_2} + \frac{\partial R_3}{\partial X_3} = 0
\end{aligned}\right\} \qquad (2.1.28)$$

The displacement gradient tensor **F - I** (expression 2.1.7) can be decomposed into symmetric and asymmetric parts as follows:

$$\begin{aligned}
\mathbf{F} - \mathbf{I} &= \frac{u_i}{X_R} = \frac{1}{2}\left(\frac{u_i}{X_R} + \frac{u_R}{X_i} \right) + \frac{1}{2}\left(\frac{u_i}{X_R} - \frac{u_R}{X_i} \right) \\
&= \left\{ \frac{1}{2}\left(\mathbf{F} + \mathbf{F}^{T}\right) - \mathbf{I} \right\} + \left\{ \frac{1}{2}\left(\mathbf{F} - \mathbf{F}^{T}\right) \right\} \\
&= \mathbf{E} + \mathrm{O} \qquad (2.1.29)
\end{aligned}$$

The symmetric part, **E,** is the infinitesimal strain tensor, while the asymmetric part, W**,** is the infinitesimal rotation tensor. The infinitesimal strain tensor plays an important role in the definition of the constitutive laws, which is defined next.

2.1.2. Constitutive Law For Linear Elasticity [T2,T4,T5]

The constitutive law defines the material stress strain behaviour. Various non-linear laws can be developed for cases where plasticity is encountered or the material is visco-elastic. Only linear elastic models will be considered as they form the basis of LEFM. The materials will further be considered as isotropic.

A linear elastic solid is one for which the energy density, in the reference configuration, has the following properties:

- The internal energy density, w or $\rho_o e$ (per unit volume), is a function of infinitesimal strain only, or may be well approximated by, and is a quadratic function of strain components.

  $$w = \tfrac{1}{2}(C_{ijkl} E_{ij} E_{kl}) \qquad (2.1.30)$$

  C is a 4th order tensor of material constants.

- Heat flux is negligible and energy is conserved in the deformation. In this case, the equation for the conservation of energy reduces to:

  $$\rho \frac{De}{Dt} = \frac{\rho}{\rho_o} \frac{Dw}{Dt} = \sigma_{ij} D_{ij} \qquad (2.1.31)$$

  with D_{ij} components of the rate of deformation tensor.

The internal energy density is equal to the strain energy for the case under consideration. Certain symmetries can be assumed for C_{ijkl} due to **C** being an isotropic fourth order tensor and because E_{ij} is symmetric i.e.

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij} \qquad (2.1.32)$$

The symmetries of equation 2.1.24 will be further explored at a later stage.

An isotropic tensor is one whose components remain unchanged by any orthogonal transformation of the co-ordinate axis. From conditions 2.1.30 and 2.1.31 and since $\rho/\rho_o \approx 1$ for small deformations,

$$\sigma_{ij} D_{ij} = \frac{Dw}{Dt} = \frac{\partial w}{\partial E_{ij}} \frac{DE_{ij}}{Dt} = \frac{\partial w}{\partial E_{ij}} D_{ij} \qquad (2.1.33)$$

Identity 2.1.33 must hold for all values of D_{ij} and therefore

$$\sigma_{ij} = \frac{\partial w}{\partial E_{ij}} \qquad (2.1.34)$$

Substituting equation 2.1.30 in 2.1.34 gives

$$\begin{aligned}\sigma_{ij} &= \frac{1}{2} C_{pqrs} \frac{\partial (E_{pq} E_{rs})}{\partial E_{ij}} \\ &= \frac{1}{2} C_{pqrs} \left(\delta_{ip} \delta_{jq} E_{rs} + \delta_{ir} \delta_{js} E_{pq} \right) \\ &= \frac{1}{2} \left(C_{ijrs} E_{rs} + C_{pqij} E_{pq} \right) \\ &= C_{ijrs} E_{rs} \qquad (2.1.35)\end{aligned}$$

Equation 2.1.35 is the constitutive equation for a linear-elastic solid with the stress components as linear functions of the infinitesimal strain components.

Substituting equation 2.1.35 in 2.1.30 gives the strain energy function as:

$$w = \frac{1}{2} \sigma_{ij} E_{ij} \qquad (2.1.36)$$

The most general fourth order isotropic tensor has rectangular Cartesian components of the form:

$$C_{ijrs} = \lambda \delta_{ij} \delta_{rs} + \mu (\delta_{ir} \delta_{js} + \delta_{is} \delta_{jr}) + \nu (\delta_{ir} \delta_{js} - \delta_{is} \delta_{jr}) \qquad (2.1.37)$$

where λ, μ and ν have the same values for all possible rotations of the co-ordinate system. Further simplifications can be made when C_{ijrs} are coefficients in a linear homogeneous equation such as 2.1.35 where at least σ or $\mathbf{E}$ is symmetric.

If $E_{rs} = E_{sr}$ there is no loss in generality in choosing the coefficients C_{ijrs} as symmetric in indices r and s. Similarly, if $\sigma_{ij} = \sigma_{ji}$ there is no loss in generality in choosing the coefficients C_{ijrs} as symmetric in indices i and j. Evaluating equation 2.1.37 it is seen that if C_{ijrs} is required to be symmetric in either ij or rs, the constant, ν, must be equal to zero.

Equation 2.1.37 reduces to:

$$C_{ijrs} = \lambda \delta_{ij} \delta_{rs} + \mu (\delta_{ir} \delta_{js} + \delta_{is} \delta_{jr}) \qquad (2.1.38)$$

Substituting equation 2.1.38 in 2.1.35 gives,

$$\begin{aligned}\sigma_{ij} &= [\lambda \delta_{ij} \delta_{rs} + \mu (\delta_{ir} \delta_{js} + \delta_{is} \delta_{jr})] E_{rs} \\ &= \lambda \delta_{ij} E_{rr} + 2\mu E_{ij} \qquad (2.1.39)\end{aligned}$$

The two Lamé constants λ and μ are related to the more commonly known material constants as follows:

$$\mu = G = \frac{E}{2(1+?)} \qquad (2.1.40.a)$$

$$? = \frac{?E}{(1+?)(1-2?)} \qquad (2.1.40.b)$$

Equation 2.1.35 represents only six independent equations and is simplified by introducing the following enumeration for elements of the tensors:

$$\begin{array}{lll} s_1 = s_{11}, & s_2 = s_{22}, & s_3 = s_{33}, \\ s_4 = s_{23}, & s_5 = s_{31}, & s_6 = s_{12}, \\ e_1 = E_{11}, & e_2 = E_{22}, & e_3 = E_{33}, \\ e_4 = E_{23}, & e_5 = E_{31}, & e_6 = E_{12} \end{array}$$

Equation 2.1.35 becomes

$$s_i = C_{ij} e_j \qquad (2.1.41)$$

or in matrix form:

$$\begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \end{bmatrix} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{bmatrix} \qquad (2.1.42)$$

Substituting equation 2.1.40 in 2.1.42 gives

$$\begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \end{bmatrix} = D \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5-\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5-\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5-\nu \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{bmatrix} \qquad (2.1.43)$$

with, $D = \frac{E}{(1+?)(1-2?)}$

The inverse of equation 2.1.43 is

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+\nu \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} \tag{2.1.44}$$

2.1.3. Equilibrium [T2,T3,T4,T5,T6,T7]

The equilibrium equation in solid mechanics contains partial derivatives of the stress tensor and has the form in, index notation:

$$\sigma_{ij,j} + \rho b_i = 0 \tag{2.1.45.a}$$
$$\sigma_{ij} = \sigma_{ji} \tag{2.1.45.b}$$

The principal of virtual displacement can be used to derive an alternative formulation of equilibrium that is more convenient for the formulation of numerical approximations like finite element analysis.

From equation 2.1.45.a, equilibrium is specified by:

$$\frac{\partial \sigma_{ji}}{\partial x_i} + \rho b_j = 0 \tag{2.1.46}$$

With reference to the continuum mechanics problem in figure 2.1, a virtual displacement field, $\delta \mathbf{u}$, is introduced which is differentiable and satisfies the essential boundary conditions i.e. the virtual displacement is compatible to the prescribed displacement on C_u.

Multiplying 2.1.46 by δu_j and integrating over the volume give:

$$\int_B \left(\frac{\partial \sigma_{ij}}{\partial x_i} \delta u_j + \rho b_j \delta u_j \right) dV = 0 \tag{2.1.47}$$

with *B* indicating integration over the volume (body). Applying the chain rule gives:

$$\int_B \left(\frac{\partial (\sigma_{ij} \delta u_j)}{\partial x_i} - \frac{\partial \delta u_j}{\partial x_i} \sigma_{ij} + \rho b_j \delta u_j \right) dV = 0 \tag{2.1.48}$$

From equation 2.1.29 the second term in equation 2.1.48 is expanded as follows:

$$\frac{\partial \delta u_j}{\partial x_i} = \frac{1}{2}\left(\frac{\partial \delta u_j}{\partial x_i} + \frac{\partial \delta u_i}{\partial x_j}\right) + \frac{1}{2}\left(\frac{\partial \delta u_j}{\partial x_i} - \frac{\partial \delta u_i}{\partial x_j}\right) = \delta E_{ij} + \delta \Omega_{ij} \qquad (2.1.49)$$

Also:

$$\sigma_{ij}\frac{\partial \delta u_j}{\partial x_i} = \sigma_{ij}\left(\delta E_{ij} + \delta \Omega_{ij}\right) = \sigma_{ij}\delta E_{ij} \qquad (2.1.50)$$

due to the stress tensor being symmetrical and the infinitesimal rotation tensor being skew symmetric.

Substituting equation 2.1.50 in 2.1.48 yields:

$$\int_B \left(\frac{\partial(\sigma_{ij}\delta u_j)}{\partial x_i} - \sigma_{ij}\delta E_{ij} + \rho b_j \delta u_j\right) dV = 0 \qquad (2.1.51)$$

Therefore

$$\begin{aligned}\int_B \sigma_{ij}\delta E_{ij}\, dV &= \int_B \left(\frac{\partial(\sigma_{ij}\delta u_j)}{\partial x_i} + \rho b_j \delta u_j\right) dV \\ &= \int_B \rho b_j \delta u_j\, dV + \int_C \sigma_{ij}\delta u_j n_i\, dS \\ &= \int_B \rho b_j \delta u_j\, dV + \int_{Ct} t_j^{(n)*}\delta u_j\, dS + \int_{Cu} t_j^{(n)}\delta u_j^{*}\, dS \end{aligned} \qquad (2.1.52)$$

using the divergence theorem [T8]. $\mathbf{t}^{(n)}$ is the traction on the surface of the body, C_t is the surface area where traction is prescribed and C_u is the surface area where displacement is prescribed. The symbol, *, denotes prescribed or known quantities.

If δu_j^* is chosen such that it satisfies the homogeneous essential boundary conditions i.e. $\delta u_j^* = 0$, the last term of equation 4.52 is omitted to give:

$$\int_B \sigma_{ij}\delta E_{ij}\, dV = \int_B \rho b_j \delta u_j\, dV + \int_{Ct} t_j^{(n)*}\delta u_j\, dS \qquad (2.1.53)$$

The left hand side of equation 2.1.53 represents the internal virtual work or strain energy (δW_{int}) while the right hand side is the external (δW_{ext}) work of the applied forces displacing through virtual displacements. Equation 2.1.53 gives:

$$\delta W_{int} = -\delta W_{ext} \qquad (2.1.54)$$

The total potential energy Π is given by,

$$\delta W_{int} + \delta W_{ext} = \delta\Pi = 0 \qquad (2.1.55)$$

For equilibrium, the total potential energy must be stationary for variations of admissible displacements as indicated in equation 2.1.55. It should be noted that, even with small virtual displacements, no compromise was made in the equilibrium equation, as there are no restrictions to real displacements.

The same steps are used to derived the variational or weak form of the equilibrium equation, expressed as

$$\int_B \sigma_{ij} E_{ij} dV = \int_B \rho b_j u_j dV + \int_{Ct} t_j^{(n)*} u_j dS \qquad (2.1.56)$$

with u_j as a weighing or trial function.

2.1.4. The Continuum Problem In Structural Mechanics

The continuum problem in structural mechanics (neglecting heat flux and assuming static conditions) has 15 unknown variables namely:

- 3 displacements
- 6 strains
- 6 stresses

In this section, 15 equations were derived for solving the unknowns. They are:

- 6 compatibility equations (2.1.27)
- 6 constitutive equations (2.1.43)
- 3 equilibrium equations (2.1.45)

This fully defines the boundary value problem in continuum mechanic s.

2.2. PLANE STRAIN LINEAR ELASTICITY

Analytical solutions to crack tip stress fields in three dimensions are complex. Most of the initial work in LEFM was developed for plane strain conditions. The governing equations, derived in section 2.1, can be simplified for the description of plane strain conditions.

In plane strain, the strains in the direction of the third co-ordinate system i.e. ε_3, ε_4 and ε_5 are equal to zero (see section 2.1.2 for notation). The use of E_{ij} as components of the infinitesimal strain tensor will be deviated from in this section. E_{ij} will be replaced by the more commonly used symbol, ε, for strain to prevent confusion with the use of E as the modulus of elasticity.

For plane strain:

$$\left.\begin{array}{ll}\varepsilon_{11} \neq 0, & \sigma_{11} \neq 0 \\ \varepsilon_{22} \neq 0, & \sigma_{22} \neq 0 \\ \varepsilon_{33} = 0, & \sigma_{33} \neq 0 \\ \varepsilon_{12} \neq 0, & \sigma_{12} \neq 0 \\ \varepsilon_{13} = 0, & \sigma_{13} = 0 \\ \varepsilon_{23} = 0, & \sigma_{23} = 0 \end{array}\right\} \qquad (2.2.1)$$

2.2.1. Compatibility

The infinitesimal strain tensor was derived in section 2.1.1 as (equation 2.1.20):

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad (2.2.2)$$

Equation 2.2.2 is expanded for the two dimensional case with the indices running over 1,2. This will give infinitesimal strain, in matrix form, as:

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} = 2\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x_1} & 0 \\ 0 & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad (2.2.3)$$

Five of the compatibility equations (2.1.27) are automatically satisfied for plane strain conditions. The only equation applicable to plane strain is:

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} = 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} \tag{2.2.4}$$

2.2.2. Constitutive Law For Plane Strain

The constitutive equations for linear elasticity are given by equations 2.1.43 and 2.1.44 for the general three dimensional case. The constitutive equations for plane strain are derived by applying the conditions in equation 2.2.1 to equations 2.1.43 and 2.1.44 to give:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \frac{\mathrm{E}}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 0.5-\nu \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{bmatrix} \tag{2.2.5}$$

or inversely,

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{bmatrix} = \frac{(1+\nu)}{\mathrm{E}} \begin{bmatrix} 1-\nu & -\nu & 0 \\ -\nu & 1-\nu & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} \tag{2.2.6}$$

The stress in the x_3 direction is also not equal to zero and is given by:

$$\sigma_{33} = \nu(\sigma_{11} + \sigma_{22}) \tag{2.2.7}$$

2.2.3. Equilibrium

The equilibrium equation (2.1.45) is expanded to give two separate expressions for the indices running over 1 and 2 for a two dimensional case. In the absence of body forces the expansions are:

$$\left.\begin{aligned} \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = 0 \\ \frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = 0 \end{aligned}\right\} \tag{2.2.8}$$

2.2.4. Formulation Of The Plane Strain Boundary Value Problem

In the analysis of plane strain there are nine unknowns namely:

- 2 displacements
- 3 strains
- 3 stresses

Equations for solving the variables are provided as follows:

- 3 strain-displacement relations (2.2.3)
- 1 compatibility equation (2.2.4)
- 3 constitutive equations (2.2.6)
- 2 equilibrium equations (2.2.8)

The only approximation used at this stage is implied by equation 2.1.18 which means small strain is considered to the extend that squares and products of displacement derivatives can be neglected. This implies that the difference between material co-ordinates, **X**, and spatial co-ordinates, **x**, can be neglected.

The compatibility equation are stated in terms of stress by substituting equation 2.2.6 into 2.2.4. This gives:

$$\frac{\partial^2}{\partial x_2^2}\left(\frac{1-\nu^2}{E}\sigma_{11}-\frac{\nu(1+\nu)}{E}\sigma_{22}\right)+\frac{\partial^2}{\partial x_1^2}\left(-\frac{\nu(1+\nu)}{E}\sigma_{11}+\frac{1-\nu^2}{E}\sigma_{22}\right)= 2\frac{\partial^2}{\partial x_1 \partial x_2}\left(\frac{(1+\nu)}{E}\sigma_{12}\right) \qquad (2.2.9)$$

From the equilibrium equation (2.2.8), differentiating the first equation with respect to x_1 and the second with respect to x_2 and adding give:

$$-\frac{\partial^2 \sigma_{11}}{\partial x_1^2}-\frac{\partial^2 \sigma_{22}}{\partial x_2^2}=2\frac{\partial^2 \sigma_{12}}{\partial x_1 \partial x_2} \qquad (2.2.10)$$

Substituting equation 2.2.10 in 2.2.9 to eliminate the shear terms yields:

$$\frac{\partial^2}{\partial x_2^2}\left(\frac{1-\nu^2}{E}\sigma_{11}-\frac{\nu(1+\nu)}{E}\sigma_{22}\right)+\frac{\partial^2}{\partial x_1^2}\left(-\frac{\nu(1+\nu)}{E}\sigma_{11}+\frac{1-\nu^2}{E}\sigma_{22}\right)= \left(-\frac{(1+\nu)}{E}\frac{\partial^2\sigma_{11}}{\partial x_1^2}-\frac{(1+\nu)}{E}\frac{\partial^2\sigma_{22}}{\partial x_2^2}\right) \qquad (2.2.11)$$

Dividing equation 2.2.11 by (1+ν)/E and collecting results in

$$(1-\nu)\frac{\partial^2\sigma_{11}}{\partial x_1^2}+(1-\nu)\frac{\partial^2\sigma_{11}}{\partial x_2^2}+(1-\nu)\frac{\partial^2\sigma_{22}}{\partial x_1^2}+(1-\nu)\frac{\partial^2\sigma_{22}}{\partial x_2^2}=0 \qquad (2.2.12)$$

or, $$\nabla^2(\sigma_{11}+\sigma_{22})=0 \qquad (2.2.13)$$

with ∇ as the two dimensional Laplace operator. Equation 2.2.13 is the Laplace equation and defines the 2D boundary value problem in elastostatics together with the boundary conditions. Compatibility, constitutive and equilibrium equations were combined in the derivation of equation 2.2.13.

2.3. ANALYTICAL SOLUTIONS [T8,T9]

Although it is laborious to obtain analytical solutions to the boundary value problem, they can provide invaluable insight into the mechanical behaviour and governing parametric relations for some structural members; including fracture mechanics.

2.3.1. Stress Functions [T2,T5,T6,T10,T11]

Equation 2.2.13 is exactly satisfied if the stresses are related to a scalar function, $f(x_1,x_2)$, called an Airy stress function, as follows:

$$\left.\begin{aligned}\sigma_{11}&=\frac{\partial^2 f}{\partial x_2^2}\\ \sigma_{22}&=\frac{\partial^2 f}{\partial x_1^2}\\ \sigma_{12}&=-\frac{\partial^2 f}{\partial x_1\partial x_2}\end{aligned}\right\} \qquad (2.3.1)$$

In addition to the conditions of equation 2.3.1, $f(x_1,x_2)$ must also satisfy the boundary conditions. Substitution of equation 2.3.1 into 2.2.13 results in the bi-harmonic equation given by 2.3.2.

$$\nabla^4 f=0 \qquad (2.3.2)$$

or in rectangular co-ordinates:

$$\frac{\partial^4 f}{\partial x_1^4}+2\frac{\partial^4 f}{\partial x_1^2\partial x_2^2}+\frac{\partial^4 f}{\partial x_2^4}=0 \qquad (2.3.3)$$

Any function ϕ that satisfies the bi-harmonic equation and the boundary conditions is a solution to the plane strain boundary value problem. It is difficult to find suitable stress functions given the complexity of these conditions.

Various solution routines are available of which the complex function methods (potential theory) are the most elegant for analytical solutions. The Airy stress function is especially well suited if all boundary conditions are specified as traction.

2.3.2. Potential Theory

Potential theory is the theory of solutions to the Laplace equation expressed as:

$$\nabla^2 f = 0 \qquad (2.3.4)$$

Solutions to equation 2.3.4 that have continuous second partial derivatives are called harmonic functions. The nature of harmonic functions can be best studied through the properties of analytic functions of a complex variable.

2.3.2.1 Analytical Functions In A Complex Variable

A function of a complex variable is defined as a function, f(z), which assigns to each complex number, z, a unique complex function value, w with:

$$z = x_1 + ix_2 \qquad (2.3.5)$$

with i as the square root of -1, f(z) can be written as:

$$w = f(z) = u(x_1,x_2) + iv(x_1,x_2) \qquad (2.3.6)$$

with functions u and v the real and imaginary parts of f(z).

The function, f(z), is said to be analytic in a domain, S, if it is defined and differentiable at all points in the domain. All the familiar rules of real calculus hold in complex calculus. The corresponding proofs of the rules are also literally the same for complex calculus.

If f(z) is differentiable, it implies that the following limit exists (similar to real calculus):

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \qquad (2.3.7)$$

Expression 2.3.7 is further developed to form two limits depending on whether z is approached from the x_1 or x_2 direction as the limit is sought. These are:

$$f'(z) = Lim_{\Delta x_1 \to 0} \frac{u(x_1 + \Delta x_1, x_2) - u(x_1, x_2)}{\Delta x_1} + i\, Lim_{\Delta x_1 \to 0} \frac{v(x_1 + \Delta x_1, x_2) - v(x_1, x_2)}{\Delta x_1}$$

$$f'(z) = Lim_{\Delta x_2 \to 0} \frac{u(x_1, x_2 + \Delta x_2) - u(x_1, x_2)}{i\Delta x_2} + i\, Lim_{\Delta x_2 \to 0} \frac{v(x_1, x_2 + \Delta x_2) - v(x_1, x_2)}{i\Delta x_2}$$

reducing to:

$$\left.\begin{aligned} f'(z) &= \frac{\partial u}{\partial x_1} + i\frac{\partial v}{\partial x_1} \\ f'(z) &= -i\frac{\partial u}{\partial x_2} + \frac{\partial v}{\partial x_2} \end{aligned}\right\} \qquad (2.3.8)$$

Equating the real and imaginary parts of equations 2.3.8 results in the expressions

$$\left.\begin{aligned} \frac{\partial u}{\partial x_1} &= \frac{\partial v}{\partial x_2} \\ \frac{\partial u}{\partial x_2} &= -\frac{\partial v}{\partial x_1} \end{aligned}\right\} \qquad (2.3.9)$$

better known as the Cauchy-Riemann equations.

A geometrical interpretation of f(z) is that f(z) or w is the complex potential associated with real equipotential lines, u = constant, intersecting the lines, v = constant, at right angles.

Three theorems are important with regard to the analyticity of solutions. They are briefly presented for completeness of the argument.

Theorem 1: If f(z) = $u(x_1,x_2)$ + i$v(x_1,x_2)$ is analytic in a domain, D, then f(z) must satisfy the Cauchy Riemann equations for all points in D.

Theorem 2: If two real valued functions $u(x_1,x_2)$ and $v(x_1,x_2)$ have continuous first partial derivatives that satisfy the Cauchy-Riemann equations (2.3.9) in a domain, D, then the function, f(z) = $u(x_1,x_2)$ + i$v(x_1,x_2)$, is analytic in the domain.

Theorem 3: The real and imaginary parts of a complex function, f(z) = $u(x_1,x_2)$ + i$v(x_1,x_2)$, that is analytic in a domain, D, are solutions to the Laplace equation (2.3.4) and have continuous second partial derivatives in D.

A solution to the Laplace equation having continuous second partial derivatives, is called a harmonic function. The real and imaginary parts of an analytic function, $f(z) = u(x_1,x_2) + iv(x_1,x_2)$, are therefore harmonic functions such that:

$$\left.\begin{aligned}\nabla^2 u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = 0 \\ \nabla^2 v = \frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_2^2} = 0\end{aligned}\right\} \qquad (2.3.10)$$

2.3.2.2 General Solution To The Bi-Harmonic Equation [T5]

In section 2.3.1 the boundary value problem for plane elasticity was reduced to the bi-harmonic equation by means of Airy stress functions. The bi-harmonic equation is written as:

$$\nabla^4 f = \nabla^2\left(\nabla^2 f\right) = 0 \qquad (2.3.11)$$

The second order derivatives of ϕ are the components of the stress tensor as expressed by equation 2.3.1. The solution of 2.3.11 is equivalent to the solution of two equations namely,

$$\nabla^2 f = \mathrm{P} \qquad \text{and,} \qquad (2.3.12)$$

$$\nabla^2 \mathrm{P} = 0 \qquad (2.3.13)$$

If functions f_1 and f_2 satisfies equation 2.3.12 for given P, it follows that:

$$\nabla^2 (f_1 - f_2) = \nabla^2 (\mathrm{W}) = \mathrm{P} - \mathrm{P} = 0 \qquad (2.3.14)$$

Solutions to 2.3.12 have the form $(f_1 + \mathrm{W})$ where f_1 is particular and W is harmonic due to $\mathrm{W} = f_1 - f_2$. Solutions to 2.3.13 are also harmonic functions. Equation 2.3.11 is written as:

$$\nabla^2\left[\nabla^2\left(f_1 + \mathrm{W}\right)\right] = 0 \qquad (2.3.15)$$

Real harmonic functions, u and v, are chosen such that:

$$\frac{\partial u}{\partial x_1} = \mathrm{P} = \frac{\partial v}{\partial x_2} \qquad (2.3.16)$$

Examining the function, $x_1 u + x_2 v$, and substituting 2.3.10 it is found that,

$$\nabla^2\left(x_1 u + x_2 v\right) = x_1 \nabla^2 u + 2\frac{\partial u}{\partial x_1} + x_2 \nabla^2 v + 2\frac{\partial v}{\partial x_2} = 4\mathrm{P} \qquad (2.3.17)$$

The desired particular solution, ϕ_1, can therefore be expressed as

$$\phi_1 = \frac{1}{4}\left(x_1 u + x_2 v\right), \tag{2.3.18}$$

provided u and v can be found as specified in expression 2.3.16. From equations 2.3.11 and 2.3.15 it follows that:

$$\phi = \phi_1 + \Omega = \frac{1}{4}\left(x_1 u + x_2 v\right) + \Omega \tag{2.3.19}$$

Expression 2.3.19 is simplified by defining two analytical functions, φ and χ such that:

$$\left.\begin{array}{l} \phi_1 = \mathrm{Re}\left[\bar{z}\varphi(z)\right] \text{with } \varphi(z) = \frac{1}{4}(u + iv) \\ \Omega = \mathrm{Re}\left[\chi(z)\right] \text{with } \chi(z) = \theta_1 + i\theta_2 \end{array}\right\} \tag{2.3.20}$$

with χ analytic and θ_1 harmonic due to Ω being harmonic. $\bar{z}$ denotes the complex conjugate of z such that $\bar{z} = x_1 - ix_2$. The general solution, ϕ, of equation 2.3.11 can therefore be written as

$$\phi = \mathrm{Re}\left[\bar{z}\varphi(z) + \chi(z)\right] \tag{2.3.21}$$

or alternatively

$$2\phi = \bar{z}\varphi(z) + z\overline{\varphi(z)} + \chi(z) + \overline{\chi(z)} \tag{2.3.22}$$

The bars indicate complex conjugates.

2.3.2.2 Stress And Displacement From Complex Potentials [T5,T10]

Partial differentiation of ϕ with respect to z is given by

$$\frac{\partial \phi}{\partial x_i} = \frac{\partial \phi}{\partial z}\frac{\partial z}{\partial x_i} + \frac{\partial \phi}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial x_i} \tag{2.3.23}$$

Applying equation 2.3.23 to 2.3.22 and substituting in 2.3.1 gives [T5]:

$$\left.\begin{array}{l} \sigma_{11} + \sigma_{22} = 4\mathrm{Re}\left[\varphi'(z)\right] \\ \sigma_{11} - \sigma_{22} + 2i\sigma_{12} = 2\left[\bar{z}\varphi''(z) + \chi''(z)\right] \end{array}\right\} \tag{2.3.24}$$

Separation of the real and imaginary parts of equation 2.3.24 enables the calculation of (s_{11} - s_{22}) and s_{12} separately. That leaves two equations with two unknowns for the calculation of s_{11} and s_{22}; a relatively straightforward procedure.

Denoting the displacement in the x_1 direction as u_1 and in the x_2 direction as u_2 expressions for u_1 and u_2 are derived by integration of the stress-strain relationships (equation 2.2.5).

This gives, for plane strain,

$$2\mu(u_1 + iu_2) = (3 - 4\nu)j(z) - z\overline{j'(z)} - \overline{c'(z)} \qquad (2.3.25)$$

Individual values of u_1 and u_2 can be obtained by equating the real and imaginary parts of equation 2.3.24.

Various solutions to plates with elliptical holes and cracks were obtained by people like Westergaard [P2], Inglis, Williams, Dugdale [T10] etc., using the formulae discussed. Methods based on polynomial stress functions i.e. Williams, are particularly well suited for more complex geometries where the constants can be derived through boundary collocation methods. The Westergaard solution will be used, in this dissertation, to develop expressions for stresses in the vicinity of a crack.

2.4 WESTERGAARD SOLUTION [P2,P3,P4,T10]

2.4.1 General Symmetric Problems

Consider a general plane problem in the x_1, x_2 co-ordinate system. If external loads are placed symmetrically with respect to the x_1 axis, the shear stress must vanish at x_2=0. This means the imaginary part of the second equation in 2.3.24 must vanish i.e.

$$\mathrm{Im}\left\{2\left[\bar{z}j''(z) + c''(z)\right]\right\} = 0 \qquad (2.4.1)$$

Some arbitrariness in the selection of functions, j and χ, are suggested in the discussions of section 2.3.2.2. The restrictions are that the functions must provide a valid solution to the bi-harmonic equation and the boundary conditions must be met. Investigations into the uniqueness of solutions prove that valid solution functions are related so that equation 2.4.1 is satisfied at x_2 = 0 by the expression [T5,P3]:

$$c''(z) + zj''(z) + A = 0 \qquad (2.4.2)$$

with A a real constant.

Substituting 2.4.2 in 2.3.24 gives

$$\begin{aligned}\sigma_{11} - \sigma_{22} &= 2\,\mathrm{Re}\left[\bar{z}\varphi''(z) - z\varphi''(z) - A\right] \\ &= 2\,\mathrm{Re}\left[-2ix_2\varphi''(z) - A\right] \\ &= -4x_2\,\mathrm{Im}\left[\varphi''(z)\right] + A \end{aligned} \qquad (2.4.3)$$

absorbing constant terms into *A*. Solving equation 2.4.3 and the first equation in 2.3.24 simultaneously and absorbing constants into *A* gives

$$\left.\begin{aligned}\sigma_{11} &= \frac{\partial^2\phi}{\partial x_2^2} = \mathrm{Re}\left[\varphi'(z)\right] - x_2\,\mathrm{Im}\left[\varphi''(z)\right] + A \\ \sigma_{22} &= \frac{\partial^2\phi}{\partial x_1^2} = \mathrm{Re}\left[\varphi'(z)\right] + x_2\,\mathrm{Im}\left[\varphi''(z)\right] - A \\ \sigma_{12} &= -\frac{\partial^2\phi}{\partial x_1 \partial x_2} = -x_2\,\mathrm{Re}\left[\varphi''(z)\right]\end{aligned}\right\} \qquad (2.4.4)$$

with $A = 0$ for the case of uniform tension at infinity. This means only a single harmonic function, φ, must be found to solve this particular group of problems i.e. symmetric around $x_2 = 0$.

Consider a harmonic function $\varphi(z)$ with first and second derivatives with respect to *z* denoted by $\varphi'(z)$ and $\varphi''(z)$ as well as the first integral with respect to *z* denoted by $\bar{\varphi}(z)$. Consider the stress function defined by

$$\phi = \mathrm{Re}\left[\bar{\varphi}(z)\right] + x_2\,\mathrm{Im}\left[\varphi(z)\right] \qquad (2.4.5)$$

The second derivatives of ϕ gives the stresses defined in 2.4.4 proving that an equation in the form of 2.4.5 is the correct solution for the condition of symmetry earlier defined.

Equation 2.4.5 can be substituted in the constitutive law for plane strain, equation 2.2.5, and integrated in accordance with the compatibility equation, equation 2.2.3, to give the displacements in terms of the stress function. The displacements are given by

$$\left.\begin{aligned}2\mu u_1 &= (1-2\nu)\mathrm{Re}\left[\varphi(z)\right] - x_2\,\mathrm{Im}\left[\varphi'(z)\right] \\ 2\mu u_2 &= 2(1-\nu)\mathrm{Im}\left[\varphi(z)\right] - x_2\,\mathrm{Re}\left[\varphi'(z)\right]\end{aligned}\right\} \qquad (2.4.6)$$

with μ as defined in 2.1.40.a.

The main condition implied from symmetry and hence equation 2.4.4 is when $x_2 = 0$, then $\sigma_{12} = 0$ and $\sigma_{11} = \sigma_{22}$. The displacements are then

$$\left.\begin{aligned} u_1 &= \frac{(1-2\nu)(1+\nu)}{E}\mathrm{Re}[\varphi(z)] \\ u_2 &= \frac{2(1-\nu^2)}{E}\mathrm{Im}[\varphi(z)] \end{aligned}\right\} \qquad (2.4.7)$$

2.4.2 Centre Cracked Panel

Consider the centre cracked panel in biaxial tension ($\sigma_{11} = \sigma_{22} = \sigma$) depicted in figure 2.3. The objective is to determine the stress intensity for an uni-axial stress perpendicular to the crack front. The model can be modified at a later stage by superposition, with $\sigma_{11} = -\sigma$ assuming this condition will not generate any stress concentration effects in the x_2 direction.

Any stress function with the properties outlined in section 2.3.2 is a solution to the bi-harmonic equation. If the equation also satisfies the boundary conditions it is the solution to the linear elasticity problem at hand.

Examination of the boundary conditions in figure 2.3 reveals that

- $\sigma_{11} = \sigma_{22} = \sigma$ at infinity
- $\sigma_{22} = 0$ when $x_2 = 0$ and $-a < x_1 < a$
- $\sigma_{22} > \sigma$ when $x_2 = 0$ and $-a > x_1 > a$ due to the stress concentration effect

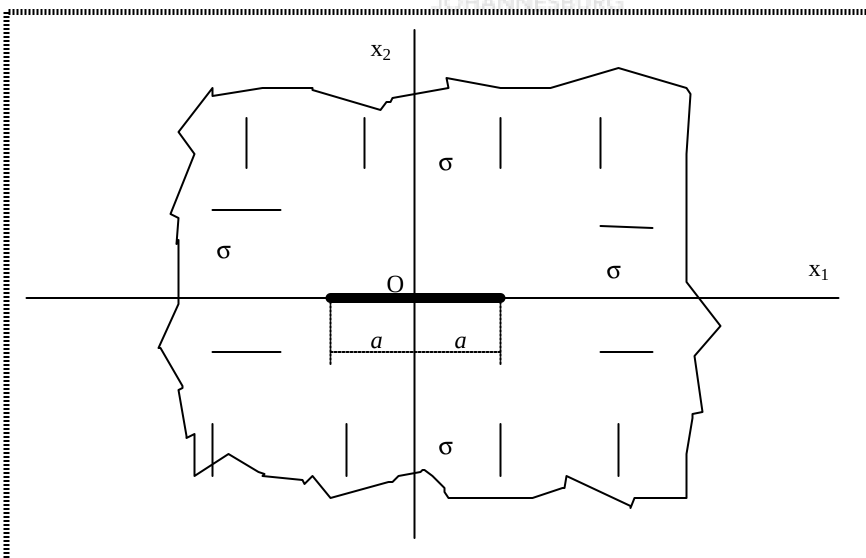

Figure 2.3 : Centre cracked panel with bi-axial tension at infinity

When $x_2 = 0$, a suitable function for σ_{22} is derived as

$$\sigma_{22} = \frac{\sigma}{\sqrt{1 - \frac{a^2}{x_1^2}}} \tag{2.4.8}$$

Noting that $\sigma_{22} = \mathrm{Re}\left[\varphi(z)\right]$ for $x_2 = 0$ (from equation 2.4.4), equation 2.4.8 is satisfied by:

$$j\,'(z) = \frac{s}{\sqrt{1 - \frac{a^2}{z^2}}} \tag{2.4.9}$$

The constant, *A*, in equation 2.4.4 is identically equal to zero for this particular problem [P4]. Westergaard investigated equation 2.4.9 as a solution to the problem in figure 2.3. Equation 2.4.9 was verified to be compatible to all the required conditions. The main interest in fracture mechanics lies in obtaining a solution for σ_{22} when $x_2 = 0$. The origin of the co-ordinate axis is moved to the crack tip by the substitution, $r = x_1 - a$, and deriving an expression in terms of the distance, r, from the crack tip.

$$\sigma_{22} = \frac{\sigma}{\sqrt{1 - \frac{a^2}{(r+a)^2}}} = \frac{\sigma(r+a)}{\sqrt{2ra}}\left(1 + \frac{r}{2a}\right)^{-\frac{1}{2}} \tag{2.4.10}$$

A Maclauren series expansion of $\left(1 + \frac{r}{2a}\right)^{-\frac{1}{2}}$ gives

$$\left(1 + \frac{r}{2a}\right)^{-\frac{1}{2}} = 1 - \frac{r}{4a} + \frac{3r^2}{32a^2} - \frac{5r^3}{128a^3} + \ \ldots - \ldots + \ldots \tag{2.4.11}$$

Substituting 2.4.11 in 2.4.10 gives

$$\sigma_{22} = \frac{\sigma(r+a)}{\sqrt{2ra}}\left(1 - \frac{r}{4a} + \frac{3r^2}{32a^2} - \frac{5r^3}{128a^3} + \ \ldots - \ldots + \ldots\right) \tag{2.4.12}$$

An approximation of 2.4.12 is made under the condition that $r/a<<1$ to evaluate the stress field close to the crack tip. Omitting terms containing $(r/a)^n$, equation 2.4.12 is written as

$$\sigma_{22} = \frac{\sigma(r+a)}{\sqrt{2ra}} = \frac{\sigma a\left(\frac{r}{a}+1\right)}{\sqrt{2ra}} = \frac{\sigma a}{\sqrt{2ra}} = \sigma\sqrt{\frac{a}{2r}} \qquad (2.4.13)$$

The angular dependency of stress around the crack tip is obtained by writing $\varphi(z)$ in equation 2.4.9 as a function of η where $\eta = z - a$, similar to moving the origin of the axis for calculation of σ_{22}, and writing η in polar co-ordinates as $re^{i\theta}$. In polar co-ordinates, $x_2 = r\sin(\theta)$ with θ as the angle with respect to the x_1-axis.

From equation 2.4.4, σ_{22} is expressed as:

$$\begin{aligned}
\sigma_{22} &= \mathrm{Re}[\varphi'(z)] + x_2\,\mathrm{Im}[\varphi''(z)] - A \\
&= \mathrm{Re}\left[\sigma\sqrt{\frac{a}{2re^{i\theta}}}\right] + r\sin\theta\,\mathrm{Im}\left[\frac{\sigma}{2}\sqrt{\frac{a}{2\left(re^{i\theta}\right)^3}}\right] + \sum_n O\left[\left(\frac{r}{a}\right)^n\right] \\
&= \sigma\sqrt{\frac{a}{2r}}\,\mathrm{Re}\left[e^{-\frac{\theta}{2}i}\right] + \sigma\sqrt{\frac{a}{2r}}\,\mathrm{Im}\left[e^{-\frac{3\theta}{2}i}\right]\frac{\sin\theta}{2} + \sum_n O\left[\left(\frac{r}{a}\right)^n\right] \\
&= \sigma\sqrt{\frac{a}{2r}}\cos\left(-\frac{\theta}{2}\right) + \sigma\sqrt{\frac{a}{2r}}\frac{\sin\theta}{2}\sin\left(\frac{-3\theta}{2}\right) + \sum_n O\left[\left(\frac{r}{a}\right)^n\right] \qquad (2.4.14) \\
&= \sigma\sqrt{\frac{a}{2r}}\cos\left(\frac{\theta}{2}\right) - \sigma\sqrt{\frac{a}{2r}}\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right) + \sum_n O\left[\left(\frac{r}{a}\right)^n\right] \\
&= \sigma\sqrt{\frac{a}{2r}}\cos\left(\frac{\theta}{2}\right)\left[1+\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right] + \sum_n O\left[\left(\frac{r}{a}\right)^n\right]
\end{aligned}$$

O is a function containing the higher order terms. Expressions for the other stress components are derived in a similar way to give the stress solution as:

$$\left.\begin{aligned}
\sigma_{11} &= \sigma\sqrt{\frac{a}{2r}}\cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}+\ldots\right) \\
\sigma_{22} &= \sigma\sqrt{\frac{a}{2r}}\cos\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2}+\ldots\right) \\
\sigma_{12} &= \sigma\sqrt{\frac{a}{2r}}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2}+\ldots
\end{aligned}\right\} \qquad (2.4.15)$$

Equation 2.4.15 tends to the same form as equation 2.4.13 as x_2 tends to zero. Integrating 2.4.9 and substituting in 2.4.7 gives

$$u_2 = \frac{2s\left(1-n^2\right)}{E}\sqrt{\left(a^2 - x_1^2\right)} \qquad (2.4.16)$$

A number of papers were published in which stress solutions to other crack models were investigated, including 3D models. Sneddon [P5] compared the solution of a crack in a 2D solid to a circular crack in a 3D solid and found that they only differ by a constant factor of $2/\pi$. The singularity of $r^{-1/2}$, as found in equation 2.4.15, was also valid for the 3D case. The $r^{-1/2}$ singularity is generally accepted to hold true for all linear elastic stress solutions around crack tips.

2.5 CONCLUSION

This chapter was compiled from a number of textbook and paper references with the aim to develop a thorough understanding of the theory of elasticity relating to fracture mechanics and finite element analysis. Although all the calculations are not shown, most derivations were calculated in detail by the author to check for approximations and assumptions. Where such approximations and assumptions exist, it is shown in the text.

Many of the theoretically derived expressions in this section will be applied in the next chapter, on fracture mechanics. Chapter 3 is a continuation of the theoretical discussion of this chapter, but focuses on linear elastic fracture mechanics.