UNCERTAINTY ANALYSIS OF HEAT EXCHANGERS

By

LIORA CHANA COBLENTZ

Submitted in partial fulfilment of the requirement for the degree

MAGISTER INGENERIAE

in

MECHANICAL ENGINEERING

in the

FACULTY OF ENGINEERING

at the

RAND AFRIKAANS UNIVERSITY

STUDY LEADER:

PROF. J.P. MEYER

Table of Contents

List of Equations

List of Figures

List of Tables

Preface

G-d does not play dice with the world
Einstein

I would like to pay tribute to G-d for creating this world and all that He enables us to make of it.

I would like to acknowledge the continual support and advice of Prof. J.P. Meyer.

I would like to thank my family, whose encouragement and support made this research possible.

This research project was sponsored by the Mechanical Engineering department of the Rand Afrikaans University and by the National Research Foundation.

Abstract

Experiments are being conducted with regard to heat exchange systems. However, there are errors and uncertainties attached to each system. Journals, which publish articles concerning heat transfer experiments, require an estimate of this uncertainty. These uncertainties must be calculated in order to determine how valid a set of results is. The uncertainty describes to what level one may rely on a set of experimental results and conclusions.

The uncertainty was calculated by the formulation of an uncertainty equation with the use of various statistical methods. Adjustments or modifications had to be made to the present uncertainty equations in order to calculate the uncertainty in heat transfer systems.

Uncertainty based on a general uncertainty equation by Schultz and Cole (1979) enabled the derivation of the equations to calculate the necessary uncertainty factor for heat transfer systems. Implementation of the equations in various experimental set-ups was achieved.

The uncertainty equations yielded results that seemed consistent with the subjective view of the experimenter.

Therefore, the equations were considered valid.

Nomenclature

A	heat transfer surface area
B_x	bias uncertainty
$\lvert C \rvert$	determinant of a $(n \times n)$ matrix
C_i	constant for inside heat transfer correlation
C_o	constant for outside heat transfer correlation
C_p	specific heat
D	diameter of the pipe
d_i	deviation of a measurement
d_i	diameter of the annulus
$f(v)$	frequency distribution function
$f(x)$	height of the distribution at x
F	net inflow of the species in to the control volume in a transport transformation equation
h	convection heat transfer coefficient
k	convergence factor
k	thermal conductivity
L	length of the tube
$\dot{m}$	mass flow rate
N	total number of observations
n	number of variables whose errors of measurement affect the uncertainty
Nu	Nusselt number
$P(a,b)$	probability of observing a value in the range of a to b
P_i	power of Reynolds number for heat transfer coefficient correlation

Pr	Prandtl number
P_x	precision uncertainty
Q_w	heat transfer of the water
q	heat flux
R	desired variable in a general equation
R	net rate of production of the species due to chemical or radioactive processes in the transport transformation equation
Re	Reynolds number
s	standard deviation
S	the net rate of injection of the species into the control volume in the transport transformation equation
T_{ri}	inside R22 temperature
T_{ro}	outside R22 temperature
T_{sat}	saturation temperature
T_{wi}	inside water temperature
T_{wo}	outside water temperature
t	quantity used for the confidence interval for a student distribution
t_s	saturation temperature
t_w	wall temperature
U	overall heat transfer coefficient
U_R	estimate of the uncertainty in the calculated value of the desired variable due to the independent uncertainty in the primary measurements of n numbers of variables affecting the results
U_{vi}	independent uncertainties
u_c^2	estimated variance
u_i	standard uncertainty
V_i	different variables

v_1, v_2	variables
w	approximation of the uncertainty interval
X	central value of the distribution
$\overline{x}$	mean of a variable
x_i	measured value
Y	input variable
y	output variable
$\frac{dc}{dt}$	rate of change of a species concentration in a control volume in a transport transformation equation
Δ_A	uncertainty associated with the measurement of A
ΔT_{LMTD}	logarithmic mean temperature difference
s_A	standard deviation on A for a statistical number of observations
s_{R1}	standard deviation in the calculated result as a consequence of independent and correlated errors
$s_{x1,x2}$	covariance interval
r	density
r_{AB}	correlation coefficient for the pairs of uncertainties in the measurement of A and B
$f(x)$	normal probability density function

Introduction

In the area of heat transfer applications and systems, there is much developmental work being done. There are basically three ways in which the development is achieved, namely numerical methods, analytical methods and experimental methods. However, even with the two former methods, experimental procedures must be performed in order to test the hypotheses. Therefore, much experimentation in the field of heat transfer is performed.

Due to the very nature of experimentation and its associated techniques, errors are present in each experimental system. Due to these errors and additional factors such as age, sensitivity and accuracy, there are uncertainties connected to every system. Therefore the uncertainties need to be calculated.

The calculation of the uncertainty of the system has a number of advantages. Firstly, it provides a measure of validity to the experimental results. Secondly, it describes to what degree one may rely on the results of the experiment.

In some journals, for example the Journal of Heat Transfer, an estimate of the uncertainty of the experimental system must be given for each experiment. There are references to determine the uncertainty from various statistical uncertainty equations, for example the equation by Kline and McC lintock (1953) or Schultz and Cole (1979). However, these equations are general in format and with regard to application. No work has been done on developing uncertainty equations with respect to a final equation format in heat exchange mechanisms.

Therefore the purpose of this study is to develop specific equations that can be used in an uncertainty analysis for heat exchange mechanisms.

The heat exchange mechanisms that were examined involved tube-in-tube heat exchangers. The experiments for the tube-in-tube heat exchangers were conducted by Dirker (1999) and Coetzee (2000).

Definitions

Accuracy: This is the difference between the measured value (or a set of observations) and the true value. The higher the accuracy the lower the error.

Calibration and validation data: Sets of data used to calibrate respectively valid 0 of a model.

Cross-validation: A procedure used for calibrating and validating a model with a limited number of representative data sets. It consists of repeated subdivision of all the data into calibration and validation data, followed by corresponding calibration and validation. The average of the observed prediction errors over the subdivisions provides an estimate of the prediction error in an entirely new situation. There are several variants of cross-validation. In the most popular one, called leave-one-out validation, each independent data set gets the role of validation data exactly once, on which occasion the complementary set gets the role of the calibration set.

Data: This is the recorded value of the variables. In some cases, the results will be the same as the data.

Elicitation: A formal procedure to translate expert knowledge regarding input uncertainty into probability distributions.

Evaluation: The broadest term used to describe the action of judging the accuracy of a model. Evaluation includes checking internal consistency and units used in computer programming, comparison of model output with an independent data set of real world observations, uncertainty analysis and judgement of utility. The term “test” sometimes has the same meaning.

Factorial Sensitivity Analysis: Analysis where inputs are varied according to a factorial design. In the most common factorial design, called two-level design, each input has two levels, low and high. A full two-level factorial design for n inputs requires 2n model runs. If this number is prohibitive, it is possible to apply a fractional factorial design, in which only a fraction of the input combinations is realised.

Input: All parameters, initial values, tabulated function and driving variables in the model. In some analyses, tabulated functions may have to be parameterised.

Linearity: This is a measure of the linearity of the output with respect to the measured variable (departure from a straight line in percent full scale).

Local Sensitivity Analysis: An analysis of responses to very small variations around some setting of the input (nominal values). Logical sensitivity analysis is the effect to establish by a theoretical study of the model, or by inspection of the results of sensitivity or uncertainty analysis, whether the model is sensitive at all for changes in the input.

Observation: An observation is the single measurement or reading.

One-at-a-time Sensitivity Analysis: An analysis of responses to variation of one input at a time, where all the other inputs are kept at nominal values.

Precision: This is the difference between the values of multiple observations of the same quantity, or the repeatability of a measurement.

Propagation of Uncertainty: This is the way in which uncertainties in the variables affect the uncertainty in the results.

Resolution: This is the smallest increment of change in measured variable that can be detected. Resolution is often the same order as precision but can be smaller.

Result: The result is obtained by making corrections to or calculations with the recorded values of the variables.

Sensitivity Analysis: It is the study of model properties through changes in variables and the analysis of the effect on model outputs.

Sensitivity: This is the change in the transducer's output per unit change in measured quantity. Typically an instrument with higher sensitivity will also have better precision and higher accuracy.

Sources of Uncertainty: Uncertainty exists at the level of inputs and outputs of the model. Uncertainty at the level of the model formulation also exists. Here it is assumed that the model is deterministic, and the uncertainties are introduced via the inputs. Input uncertainty is caused by the natural variation as well as by the imperfection of the data. Although the causes of data may differ, their effect is the same; namely uncertainty about the model outputs. It is the modeller's choice whether to incorporate natural variation in the model; the choice depends entirely on the spatial or temporal scale at which the model is used. The input uncertainty of different parameters may contain correlations caused by biological or physical mechanisms. Correlation may also be caused by the nature of the data analysed to estimate parameters.

Transport-transformation models: This refers to a wide range of biological, environmental and engineering problems.

Uncertainty Analysis: It is the study of output uncertainty as a function of a careful inventory of the different sources of uncertainty present in the model.

Uncertainty: A possible value the error might have. For a single observation, the error, which is the difference between the observed and true values, is a fixed number. But the uncertainty values vary depending on the particular circumstances of the observation.

Uncertainty: In this context, this refers to the imperfect knowledge regarding aspects of model. Uncertainty regarding model variables is usually specified by a probability distribution or by a sample of measured values or by a set of possible values. Here, the probabilistic concept of uncertainty is used as well as variances as a measure of uncertainty.

Validation: The term is used in a sense of establishing the usefulness and relevance of a model for a predefined purpose. It is a recurrent activity in a phase of a model development. Models always have a limited range of validity, and it is necessary to specify clearly what it is. In the case of predicative models, a major part of the validation consists of an assessment of prediction accuracy.

Variability: This is the heterogeneity between individual members of a population of some type and is typically characterised through a frequency distribution.

Variable: A basic quantity observed directly in the laboratory. For example 20°C is an observation or the indicated value, and temperature is the variable. A discrete variable has discrete values like heads or tails on a coin. A continuous variable has a continuous range of values. Pressure, temperature, velocity, length etc. are continuous variables. Continuous variables are generally used.

Verification: This term designates the inspection of the internal consistency of the model and its software implementation. Some important elements are: analysis of dimensions and units, on-line checks on mass conservation, detection of violation of natural ranges of parameters and variables. It also comprises the inspection of qualitative behaviour of the model and its implementation, for example, a check whether the response of one model output to changing values of one parameter conforms to theoretical insights.

Literary Research

Background to Statistics

Areas of Statistical Activity

According to Hodges (1987), there are three broad areas of statistical activity, namely discovery/imposition of structure, assessment of variation conditional structure and execution of the techniques selected. The above-mentioned three areas are discussed in the following paragraphs.

Firstly, structure, once discovered and imposed on the system, has several elements associated with it.

The value used in the area of structure is generally captured in a loss or utility function. Several actions are usually available in pursuit of a payoff to this function.

In scientific activity, this action is actually a choice from among possible assertions and possible future observations. The actions that result of the assertions and observations can be large or small. Selections from these observations and assertions are better made if they are based on facts and beliefs about the nature of the relevant part of the situation. The facts and beliefs often take on the form of a model of some process central to the problem and some more-or-less specific expression about features of the model. After this, the use of facts and beliefs is informed in some way by the data, which can be either continuous or discrete.

The process that turns actual events into data can introduce systematic effects that must be accounted for, or modelled. In some cases, it may not be possible to use data arising from the process of interest - it might be necessary to use a proxy or analogous data source. The error introduced into an analysis by a substitute data source can be considered in an analogous fashion.

The second area of statistical activity is the assessment of variation conditional structure, which comprises the most known area of statistical work. This area provides the answers to the following question: Which of the possible structures (usually values of parameters or models) is the more or less plausible? This question can be considered under the rubric of estimation. Also, conditional on some structure, the question can therefore be decided about future observational features of the modelled process.

The third area is the execution of techniques. This occurs in conjunction with the other two areas, but is nevertheless distinct and therefore must be considered separately. It comprises of a couple of steps.

The first step is that the data must be processed. This processing includes the extraction of items from databases, the conversion of raw numbers or characters into usable quantities, aggregation, counting and computation of estimates, and descriptive or diagnostic quantities. Secondly, in executing model fitting and prediction techniques, analytical or numerical approximations must be considered and employed.

Each of the above three areas has an associated type of uncertainty. For a sizeable analysis to be complete all three types of uncertainty must be assessed and their effects on the product of the analysis weighed.

Theory of Probability

Another way of considering uncertainty is to consider the theory of probability.

The mathematical theory of probability is also called the theory of the games of chance. This is because probability is connected to a general framework for evaluating evidence and weighing arguments.

There are two distinct strategies of determining the probability, as explained by Shafer (1987). The strategies are known as the frequentist approach (direct probability strategy) and the Bayesian approach (conditional probability strategy). Both strategies are discussed below.

The direct probability strategy relies on the direct application of the idea that in life, as in the game of chance, what happens most is most likely to happen in a particular case under consideration.

The conditional probability philosophy limited the domain of numerical probability. This body of theory is called the Bayesian approach, since it uses Bayes' theorem.

However, this Bayesian approach has its shortcomings. The most important shortcoming is its inability to explain how the quality of a probability analysis depends on the availability and quality of relevant evidence.

The above strategy observes that the game of chance unfolds step by step, with the probabilities for each of the possible outcomes changing at each step. Therefore the accumulation of evidence should change probabilities in a similar step to step manner.

Therefore, this recognises the relevance of evidence but claims objectivity by limiting the numerical probability judgement to cases where the evidence is of an ideal form. The Bayesian philosophy recognises the subjectivity of all probability judgement but ignores quality of evidence and claims that it is normative to force all probability judgement into one mould.

Neither of the above two strategies is inherently more objective or subjective than the other. The direct probability strategy is more likely to result in probability judgements based on actual frequency counts. But the objectivity of the frequencies must be coupled with a subjective judgement of their relevance. Therefore, there are more often hunches and impressions of the frequencies than counts. Nevertheless, the direct probability strategy has come to be associated with objectivity, while the conditional probability approach is associated with terms of rationality.

Types of Errors

Errors can be classified in different groups. However, authorities differ on the classification and grouping of the errors. Webster, Morten (1997) and Kline and McClintock (1953), as detailed below, are some authorities which differ on their classification and grouping of errors.

According to Webster, there are three types of errors, namely stupid error or blunders, systematic or bias errors, and random or precision errors. Each of these three errors is discussed in the following paragraphs.

Blunders result from mistakes in procedure, operator bias, and unreliable equipment. These problems produce irretrievable errors and inaccuracies. There is no way to account for the inaccuracies. The only solution to this type of error and inaccuracy is not to make such mistakes.

Secondly, the systematic or bias errors result from incorrect calibration of the equipment (zero-offset error, scale error), or errors due to specific but perhaps uncontrolled effects (changes in viscosity with temperature for example). These errors cannot be studied theoretically but can be determined by comparison with theory (and then correction) or by alternate measurements (better calibration). This type of error can be minimised by re-calibration or by comparison with theory. If the error is small, the average measured value will be accurate.

Finally, random errors result from the accumulation of many small, normally independent effects, for example disturbances to measuring system (i.e. temperature changes, vibration, voltage fluctuations), variations in the actual quantity being measured .The latter is not strictly an error but can be treated in the same way. These random errors arise due to the very nature of physics and the real world. It can be seen that unrecognised systematic errors, which depend on varying external effects, may appear to be random errors. The concepts of probability and statistics are used to study random errors. Random errors are related to repeatability or precision.

According to Morten (1997) there are three sources of error, namely prior knowledge, calibration process and design process. All three are discussed in the next paragraphs.

Prior knowledge can lead to uncertainty due to neutral variability, or unobserved input disturbances as well as measurement and sampling errors of observed input and output can also arise. Three specific errors occur as a result of prior knowledge, namely aggregation error, numerical error and structural error.

The second source of error is the calibration process. The calibration process can lead to various errors due to the start up, the T-S input data file and the model itself. The resulting errors would be discretization error, input environmental datafile error, model structure and state-parameter error and parameter optimisation error.

The final source of errors is the design process. The design process can cause a number of errors to occur as a result of uncertainty of parameters and user output-

interpretation error. The resulting errors would be parameter propagation error and error analysis.

According to Kline and McClintock (1953), there are three different types of errors, namely accidental errors, fixed errors and mistakes. All of these are discussed in the coming paragraphs.

The first type is accidental errors, which are those varying errors that cause repeated reading to differ without any apparent reason. Accidental errors arise from instrument friction and time lag, personal errors etc. Accidental errors are studied by taking repeated observations of the value of a variable and considering the sequence. Such a sequence may fall into patterns.

The second type of errors is the fixed errors. Fixed errors are those errors that cause repeated reading to be in error by the same amount without any apparent reason. If a reason were known, presumably a suitable correction would be made to eliminate the error. Examples of causes of fixed errors are a burr on a lip of a Pitot tube or a lever arm of erroneous length.

The examination of fixed errors shows that the classification of errors as accidental or fixed is a relative matter. An examination of this type of error shows that which errors are called fixed and which are called accidental depends on the scope of the experiment. If one observer, using one thermocouple and one potentiometer, were to make readings of temperature at one point, then nearly all the errors would be considered fixed, even if the reading would be taken many times. However, if several observers, using several different types of temperature measuring devices, measured the temperature repeatedly at a given point, almost all the errors would be considered accidental. If all the observers use the same type of apparatus, a fixed error may remain.

The third type of errors is mistakes. Mistakes are those errors that arise from completely erroneous readings of watches, scales etc. A careful observer will discard these errors if they are very large.

No matter how the errors are classified, they all contribute to the general and total uncertainty of the system.

Types of Uncertainties

There is a number of ways to classify uncertainties due to different methods as shown by Robinson from the Sandio National Laboratories (1999), Hodges (1987), Schultz and Cole (1979) and Isukapalli (1991) and Naidoo (2001). Each of different classifications is discussed below.

According to Robinson, Sandio National Laboratories (1999) there are two types of uncertainties, namely probabilistic uncertainty and possibilistic uncertainty. Both are discussed below.

Probabilistic uncertainties make use of random variables to describe the various sources of uncertainty, which are determined with the use of reliability methods. These uncertainties apply when the system is of small to medium complexity, which means 100 to 150 variables or less as opposed to the probabilistic approach.

In the probabilistic approach, uncertainties are characterised by the probabilities associated with events. The probability of an event can be interpreted in terms of frequency of the occurrence of the event. When a large number of samples or experiments are considered, the probability of an event is defined as the ratio of the number of times the event occurs to the total number of samples or experiments. Therefore it is the most widely used method for characterising uncertainty in physical systems, especially when estimates of the probability distributions of uncertain parameters are available. This approach describes uncertainty arising from variable conditions and risk considerations. The uncertainties associated with model inputs are described by probability distributions, while the objective is to estimate the output probability distributions.

The process comprises of two stages:

1. Probability of encoding inputs: This process involves the determination of the probabilistic distribution of the input parameters, and the incorporation of random variations due to both natural variability and errors. This is accomplished by using either statistical estimation techniques or expert judgements. Statistical estimation techniques involve estimating probability distributions from available data, or by collecting a large number of representative samples. For example, a uniform distribution is selected only if a range of possible values for an input is available, but there is no information about which values is more likely to occur. A normal distribution is selected to describe unbiased measurement errors.

2. Propagation of uncertainty through models: Each calculated output value of the mode to change in inputs, is characterised by a probability density function (*pdf*) that will depend on the *pdf*s of the inputs

Possibilistic uncertainties make use of fuzzy set theory or possibility theory. This is used in large complex systems or in systems that do not lend themselves to statistical characterisation. It is applicable when an exact system model is impractical or impossible to obtain.

According to Hodges (1987), there are three types of uncertainty that correspond to the three previously mentioned areas of statistical activity, which are discussed below.

Firstly, structural uncertainty corresponds to the discovery or imposition of structure. Structural uncertainty is concerned about the accuracy of the model as a surrogate for the actual process of interest and the transparency of the system that turns raw events into data.

The elements of structure are the payoff, choices, facts and belief arranged into a model format and the processes that turns raw events into data. These elements always contain errors. It is possible to use a probabilistic scheme to discount the usable elements, explicitly incorporating the uncertainty about the actual payoff.

Eventhough action will be taken concerning the elements, it is possible to be uncertain about which actions are or will be available. Many analyses have among their policy choices different levels of lateral capability to decide on the course of action.

The second type of uncertainty concerns the accuracy of one's model as a substitute of the process of interest, since a model that is satisfactory now may be deficient later. It is therefore the consideration of uncertainty about the past and the future, conditional on a model. This type of uncertainty is called risk. Risk has two aspects corresponding to the forward and backward looking elements of the assessment of variation, conditional on structure.

Finally, there is uncertainty type concerning the data sources themselves. This concerns uncertainties about inaccuracies introduced by repeated manipulation of raw data items, by numerical instability or numerical approximations. This form of uncertainty is known as technical uncertainty.

Technical uncertainty is due to the fact that there is often a difficulty in reconciling the sometimes-gross discrepancies between the description of events provided by two different data systems. It is not clear how to allow for biases that may be contained in the data systems.

Schultz and Cole (1979), classify experimental uncertainties as independent, non-independent or correlated. The independent and non-independent classifications represent the two limiting situations. Completely random errors are independent; systematic or consistent errors are non-independent. Uncertainties that are neither truly independent nor non-independent are called correlated.

According to Isukapalli (1991), there are three types of uncertainty, namely model uncertainty, parametric or data uncertainty and reducible and irreducible uncertainty. Each is discussed further in the chapter.

There are two types of model uncertainty, namely model formulation and model application. Examples of each type of model uncertainty are shown in the table below.

Uncertainty in model formulation (structural uncertainty)	**Uncertainty in model application (data / parametric uncertainty)**
Simplifications in conceptual formula	Constitutive parameter selection
Simplifications in mathematical formulation	Design / structural parameter selection
Ergodic-type hypotheses Idealisations in constitutive relations	Input data development / selection • Source information
Independent hypotheses	• Meteorology (air quality models)
Spatial averaging	• Hydrology (water quality models)
Temporal averaging	• Initial and boundary conditions
Process de-coupling	Operational model evaluation
Lumping of parameters	• Uncertainty in model estimates
Discretization	• Uncertainty in observations
Numerical algorithm / operator splitting	• Non-existence of observations
Approximations in computer coding	Response interpretation
	• Hypotheses

Table 1:Uncertainty in model formulation and model application

The second type of uncertainty is due to parameters or data. Uncertainties in model parameters arise from a number of sources. Even though many parameters are measurable up to any desired precision, the are often significant uncertainties associated with their estimates. One source of the uncertainties arises from measurement errors. These errors involve random errors in analytic devices, systematic biases that occur due to imprecise calibration or inaccuracies in the assumptions used to infer the actual quality of interest from the observed reading of a proxy variable. Other possible sources in estimates of parameters include misclassification, estimation of parameters through a small sample, and estimation of parameters through non-representative samples. Also, uncertainties in model application arise from the uncertainties associated with measurement data used for the model evaluation.

The third type of uncertainties comprises of reducible and irreducible factors. The reducible and irreducible factors lead to uncertainty associated with model formulation and application. Natural uncertainty is inherent or irreducible, while data and model uncertainty contain both reducible and irreducible components.

The irreducible uncertainty in models and data is a result of the presence of natural uncertainty, as opposed to reducible uncertainty.

Reducible uncertainty can be lowered by better inventorying methods, by improved instrumentation, improvements in model formulation etc. However, there is a distinction between reducible and irreducible model and data uncertainties, since it is not always feasible to eliminate the presence of an error (reducible uncertainty) in measurement or modelling beyond a certain level. Also, what is perceived as irreducible natural uncertainty, may be statistically classified and mechanistically modelled.

Irreducible modelling reflects the current model formulation and may actually change when improved theories become available.

According to Naidoo (2001) there are two types of uncertainties in industry, namely lack of accuracy and repeatability, where accuracy is the device reporting the correct result and repeatability is the device reporting the same result for repeated experimentation.

All the above types of uncertainty arise from a number of various sources during experimentation.

Sources of Uncertainty

There are four sources of uncertainty according to Robinson, Sandio National Laboratories (1999), namely from the external system parameters, the internal system parameters, the system model itself and the experimenter's observation of the system. All these are discussed below.

The external system parameters include a variety of possible sources of uncertainty, for example temperature, radiation, road surfaces etc. These parameters are common causes of uncertainty.

The internal system parameters comprise of the materials of the system chosen for their specific strength, wear, and corrosion characteristics. These parameters vary in magnitude of uncertainty.

The system model is an abstract representation of the physical world. Even a finite element model with many elements provides only a very simple representation of the behaviour of the physical system, which results in many uncertainties.

The observational uncertainty is caused by observation of the behaviour of the system. Uncertainties are caused by human perception and sensor error compounded by data fusion.

According to Isukapalli (1991), there are five different sources of model uncertainties, namely model structure, model detail, extrapolation, model resolution and model boundaries. They are discussed separately in the next few paragraphs.

Model structure can result in uncertainty. This uncertainty can be the result of alternative sets of scientific or technical assumptions for developing a model. In this instance, if the results from different models have similar conclusions, then one can be confident about the decision with regard to uncertainty.

Model detail contributes to the uncertainty. Models are often simplified for purposes of tractability. These purposes include assumptions to convert a complex non-linear model to a simpler linear model in a parameter space of interest. Uncertainty in the predictions of simplified models can be characterised by the comparison of the predictions to more detailed models.

Extrapolation causes uncertainties by the fact that models that are validated for one portion of input space may be completely inappropriate for making predictions in other regions of the parameter space.

Model resolution affects the uncertainty. In the application of numerical models, the selection of the spatial and/or temporal grid involves uncertainty. There is a trade-off between the computation time and the precision accuracy. However, there is also a trade-off between resolution and the validity of the governing equations of the model. Often a coarse grid resolution introduces approximations and uncertainties into the model results. However, a finer grid resolution does not necessarily result in more accurate predictions. This type of uncertainty is dealt with through the appropriate

selection of the model domain parameter values or by comparing results based on different grid sizes.

Finally, model boundaries cause uncertainties to be present in the system. A model can have limited boundaries in terms of time, space etc. The selection of a model boundary can be a type of simplification. Within the boundary, the model may be an accurate representation, but other phenomena being overlooked may play a role in the model.

All the above sources of uncertainty must be considered for the complete uncertainty factor to be calculated.

The Reason for Introducing Uncertainty

Many industries are concerned with uncertainty. This means that industries concerned with measurements identify uncertainties, minimise them or adapt the process.

The industrial sectors promote this factor since a low uncertainty value has many implications. Low uncertainty means tighter process control, higher product quality, lower rework production and improved safety. These all contribute to higher profitability for the industry. Therefore the measurement taken must have an associated uncertainty value.

Most measurement techniques do not measure the property required, but rather infer it from other properties that are easier to measure. Examples of this is include traditional flow measurements in process industry where the traditional flow measurements are based on differential pressure measurements across a constriction in the pipe. Similar are pressure measurements that are based on changes in capacitance or displacement and temperature based on voltage. The relationships between the properties are known, but there are other factors that influence the relationship that result in approximations. An example of this is that flow measurement from an orifice plate is also dependent upon the properties of the fluid and the shape of the orifice. In addition, as the material of the orifice degrades, the more uncertainty is introduced into the measurement. Sometimes it is possible to compensate for these factors, but mostly they just introduce more factors that need to be accounted for. Therefore according to Naidoo (2001), one should identify the factor with the greatest influence and compensate for those only.

Guidelines to General Uncertainty Format

There is a number of guidelines that should enable complete uncertainty analyses. By following the following steps, an uncertainty analysis can be completed.

Firstly, consider one reading that was taken during the experiment.

- About how large were most of the readings?
- What uncertainty is there in reading the instrument?
- The uncertainty can be calculated using percent uncertainty, which is the uncertainty divided by the approximate reading size and multiplied by 100%.
- Repeat the above process for all other measurements made in the experiment.

Consider the calculations done with the measurements.

- Could any uncertainty be associated with the calculations?
- How were the uncertainties that were calculated altered in the calculations?
- Follow through the calculation to see whether any uncertainties will be magnified or reduced.

Was there anything else in the procedure that would have introduced uncertainty?

Finally, the question is whether the percent difference between the data and the calculated values is less than the estimate of the total error (i.e. the "margin of error").

There is a number of guidelines that can be followed in order to obtained an uncertainty analysis, according to the CAMSE project.

- All parameters should be accessible for uncertainty- and sensitivity analysis. The source code of a model should not contain unexplained numerical values.
- Perform sensitivity analysis for verification of the model and its implications. Repeated running of the software over a broad range of circumstances already constitutes a non-trivial test. Then check whether the qualitative behaviour of the responses conforms to theoretical expectations.
- A logical sensitivity analysis can help to detect inputs for which an output is entirely insensitive. These sleeping inputs may be ignored in subsequent analyses.
- Apply factorial sensitivity analysis if you are interested in the interaction between inputs. This is important when the response to an input depends on the settings of other inputs.
- Use one-at-a-time sensitivity analysis to detect irregularities (discontinuities) in the response that may preclude specific model – calibration techniques.
- For research papers on models and validation studies, it is recommended to use an uncertainty analysis.
- The establishment of input uncertainty constitutes the most elaborate and critical stage of uncertainty analysis Literature and experiments constitute the most natural source of information. E xpert knowledge is another source.
- Data providing information about input uncertainty often pertain to separate sub-models. Information about correlation in uncertainty inputs is useful, since this information can reduce output uncertainty greatly.

- If it is possible, perform uncertainty analysis for all variables simultaneously. However, in the case of large models, the analysis may have to be performed separately.
- A good starting point is simple random sampling from the input uncertainty distribution. However, for efficiency, it is preferable to start with Latin hypercube sampling.
- When comparing alternative scenarios, calculate the relevant contrasts with the same values of the input sample. This provides the most efficient estimates of the scenario effects.
- Uncertainty analysis may be used as, and is regarded as, partial validation. The validation is only partial because structural uncertainty in the model is not described as "input" uncertainty.
- Large uncertainty contributions of individual inputs or group of inputs to model output, indicate that it is worthwhile to know more about these inputs. Therefore, this indicates that uncertainty analysis provides information to support decisions on research priority.
- By the same token, uncertainty analysis provides support in the selection of calibration parameters.
- Compare estimated model uncertainty with the current empirical uncertainty. Differences may be due to: structural errors in the model and errors in the presumed input uncertainty distribution, such as absence of uncertain inputs, absence of correlation between inputs, erroneous specification of distributions etc.

According to Isukapalli (1991), the purpose of a quantitative uncertainty analysis is to use currently available information in order to quantify the degree of confidence in the existing data and models. The purpose is not to reduce the uncertainty, since reduction in uncertainty can only come from gathering additional information and filling gaps.

He adds that there are four stages involved in the uncertainty analysis of a model. They are: the estimation of uncertainties in model inputs and parameter (characterisation of input uncertainties); estimation of the uncertainty in model outputs resulting from uncertainty in model inputs and model parameters (uncertainty propagation); characterisation of uncertainties associated with different model structures and model formulations (characterisation of model uncertainty) and characterisation of the uncertainties in model predictions resulting from uncertainties in the evaluation data.

Uncertainties in Multiple Sample Experiments

In multiple sample experiments, there is much data with which to work. This data allows a more accurate estimate of the readings to be obtained. Using this data, the appropriate uncertainty equations can easily be used. It therefore should be easier to calculate uncertainties in multiple sample experiments.

However, there are some problems associated with multiple sample systems. According to Kline and McClintock (1953), there are 4 factors that tend to lessen the effect of repetition in experimentation. These factors, discussed below, show that multiple sample experiments are not necessarily uncertainty free.

The first factor was discovered by Pearson (1902). He showed that the observation of scales by a single observer did not give consistent results, even though all extraneous variables appeared to have been removed. He showed that even a sample of 20 to 30 readings might have a significantly different mean value than the true mean established by 500 or more samples.

Pearson (1902) also showed that observations of scales by different observers are not necessarily independent due to some unexplained causes as well as to number bias which is the tendency to read consistently high or low ate certain points.

Tuemmler noted a difference between the results of various laboratories using equipment of the same design to perform the same tests.

The last factor is the most significant since it gives rise to the largest errors. Instruments of different designs will not give the same results. Therefore if a single instrument is used for a set of observations, some inherent error in the instrument will be sampled only once, no matter how many times each reading is repeated.

Describing Uncertainties in Single Sample Experiments

In the area of heat transfer, most experiments are not of the multiple sample variety, but are rather single sample experiments.

In single sample experiments, it is more difficult to gain accurate results, since there is not much data. Therefore calculating the uncertainty associated with single sample experiments becomes complicated.

In single-sample experiments it is inevitable that the statements of reliability will be partly based on estimates. This is true since by definition statistics cannot be applied to all of the errors. Often these estimates will be no better that ±50% of the uncertainty; but it could be satisfactory if the uncertainty is of the order of a few percent or less of the original data.

In order to treat uncertainties completely in a given experiment, three questions must be answered:

1. What is a rational way for estimating and describing the uncertainties in the variables?
2. What is a proper method for calculating the propagation of these uncertainties into the results?
3. What must be presented in a report to give a reasonably complete but concise picture of the reliability of the experiments?

In answering the above questions, it is found that in order to derive a rational method for describing the uncertainties in the variables, it is necessary to define the nature of the uncertainties based on three acceptable areas of errors, namely accidental errors, fixed errors and mistakes. Each of these is discussed below.

For accidental errors, if the sequence is predictable, even if there is a variation from one reading to another, it is called a homogenous sequence. The homogenous sequence causes the experiment to become a controlled one. If a very large number of data is obtained from this controlled experiment, then a frequency-distribution function can be used to describe them. The defining characteristic of the frequency-distribution function, $f(v)$, is those values lying between v and $v + dv$ is $f(v)dv$.

The frequency distribution for accidental errors is assumed to be normal or Gaussian. However, this is not true is all cases as shown below.

In a case of a small number of observations, it is not possible to describe the frequency distribution function exactly. However, it is still possible to make some precise statement about the characteristics of the distribution function, using the statistical concept of a confidence limit as well as using the errors present in the system.

Since no measure of the scatter of the errors can be obtained from a single-sample experiment the experimenter must rely on his past experience and judgement to describe the experiment, not statistical science. Statistical uncertainty analyses are properly based on calculations with measured numbers. Therefore a statistical

description of the errors would be misleading. The best he can do is make a statement concerning what would happen if the experiment were repeated an infinite number of times. Therefore statements concerning single-sample experiments must be based on what he thinks these numbers will be. Therefore the term "uncertainty distribution" will be used for the statement of uncertainty.

The uncertainty distribution is therefore defined as the distribution of errors which the experimenter believes would be found if the variables were sampled a great many times.

The uncertainty distribution caused by accidental errors would be measured by repeated readings of the variable in question. Therefore the existing frequency distribution for accidental errors can be used to determine what the shape of uncertainty distributions for accidental errors should be.

The few available frequency distributions indicate that the uncertainty distribution caused by accidental errors should have a similar shape to the Gaussian distribution. Therefore its general characteristics are similar to Gaussian distributions, namely that small errors are more likely than large errors, plus and minus errors are equally likely and no finite maximum error exists.

The nature of the uncertainties based on fixed errors is discussed next. Fixed errors can be estimated by theoretical means, and the uncertainty in these calculations can be considered as having an uncertainty distribution. Therefore fixed errors have an uncertainty distribution which can be visualised in terms of calculations and the use of instruments and observers. Therefore, the uncertainty distribution developed in connection with the uncertainties resulting from accidental errors can be applied directly to the uncertainties that result from fixed errors. The shape of the uncertainty distribution due to fixed errors is believed to be similar to a normal distribution.

Finally, the nature of mistakes is that, if they are large, a careful observer will discard them. Therefore, small errors are more likely than large errors, positive and negative errors are equally likely and no finite maximum error can be stated. Although the errors occur in discrete steps, they have an uncertainty distribution that can be visualised in terms of the use of many observers and scale intervals.

The entire error is a given reading due to all the above causes has no distribution function. However, the entire uncertainty or lack of knowledge about the value of a reading can be described completely in terms of an uncertainty distribution, since each component can be described this way. This method would therefore be too cumbersome to use.

Therefore, a different methods must be used. A satisfactory notation for the uncertainty of a variable must include both a statement of the best estimate of the true value as well as a statement about the magnitude of the error in the estimate.

The best estimate of the true value is given by the mean of the readings.

In frequency distributions, the standard deviations are often used as the mean. However, there are problems in using this method in single-sample experiments.

Firstly, for nearly normal distributions, single sample experiments describe an interval such that the odds are about 2:1 that the error in a particular reading will lie in that interval. However, the experimenter wants his odds to be at least 20:1. Secondly, the respective numbers must be estimated in single-sample experiments.

The range is sometimes used as the statement of magnitude about the error under the name "maximum error" or "maximum uncertainty".

Another method of notation for describing a distribution that is more suited for description of uncertainty distributions is to specify an interval based on certain odds.

Let the result R be a function of *n* independent variables, $v_1, v_2, \ldots, v_n$.

For small variations in the variables, this expression can be expressed in the linear form as follows.

$$dR = \frac{dR}{dv} dv_1 + \frac{dR}{dv_2} dv_2 + \ldots + \frac{dR}{dv_n}$$

Equation 1: Small variations in a result

The uncertainties in the variable v_i are represented completely by an uncertainty distribution but can adequately be described by uncertainty intervals w_i based on certain odds. Therefore we must examine how to find the uncertainty interval for the result w_R based on essentially the same odds as the interval for each of the variables. The appropriate statistical theorems are shown in the appendix.

Examples were used to compare the accuracy of the linear and second-power equations, as shown in the appendix, for predicting the appropriate interval in the result in the case of different distributions based on the above equations. Three different distributions were chosen, namely the normal, the sinusoidal (corresponding to 1 wavelength of a sine curve) and the triangular (corresponding to an isosceles triangle).

Odds of 9:1, 19:1 and 99:1 were chosen as being of interest to the engineering applications. These odds were all assumed to have a zero mean and unit standard deviation since outside the specified regions, the frequency-distribution functions are zero.

For the first experiment, the result was chosen to be proportional to the sum of 2 variables as shown below.

$$R = (v_1 + v_2)/\sqrt{2}$$

Equation 2: Result is the sum of two variables

The factor $1/\sqrt{2}$ in the above equation is introduced to give the result a standard deviation of unity. The frequency-distribution function of this result can be found by calculating the following integral:

$$f(R) = \int f\left(\sqrt{2}R - v_1\right) f(v_1) \sqrt{2} dv_1$$

Equation 3: Frequency distribution function of the sum of two variables

Calculation of the distribution function of the sum of more than 2 variables becomes complex. For the case of an infinite number of variables, the distribution is normal and comparison becomes easier. The result of the distribution will have a unit standard deviation if R is taken as:

$$R = \lim_{n \to \infty} (v_1 + v_2 + \ldots + v_n) / \sqrt{n}$$

Equation 4: Result of sum of more than two variables

According to the above statistical theory, the second-power equation gives odds nearest to the desired odds in all 3 cases. This equation predicts the uncertainty to within ±10% of the correct value while the linear equation predicts uncertainty intervals varying from the correct value by as much as 40%.

The difference between intervals given by the linear and second-power equations increases as the square root of the number of variables, if each variable has about and equals effect on the result.

In the second example, the result was taken to be proportional to the sum of an infinite number of variables. The same method as above was used and it could be seen that the second-power equation is superior. The odds given by this equation for the uncertainty interval are still reasonable, while the linear equation gives infinite odds. The error in the interval introduced by use of the second-power equation increases to ±15%, while the error due to the linear equation becomes infinite. Therefore it can be seen that the second-power equation should be used for the calculation of the uncertainty interval for the result.

Problems Associated with Uncertainties

According to Tavener (2001), there are a number of concerns about uncertainties. Tractability and the near obsession for uncertainties can halt the improvement of apparatus. Examples were given to illustrate the point.

A triple point cell is as good as its purity, its isotopic composition and its lack of dissolved air. A calibration certificate can be purchased for the cell that the cell's temperature is 0.01° ± 100 micro Kelvin. The cell received such a certificate for calibration after comparison to other cells. However the National Laboratory does not know the isotopic composition of its cells and in only some designs of cells can any unwanted cells be assessed. This demonstrates that the certificate of calibration does not actually give the true error or uncertainty.

It is therefore concluded that the national standards are not necessarily the highest quality primary standards. Also, it can be concluded that inter-comparison uncertainties should be reduced, as the result would better qualify national standards. Finally it can be concluded that the national standards and the primary standards can all agree within very small temperatures, suggesting that uncertainties of realisation can also be reduced.

Computational Uncertainty Analysis

Computer Model Uncertainty

The differences between the physical world and the predictions of mathematical models are not only due to uncertainty in what has been modelled but also due to incomplete descriptions of very complex physical procedures.

According to Koley and Hofer (1996), results from application of computer model are subject to uncertainty due to model and parameter uncertainty. Model uncertainty arises in areas such as phenomenology, model formulation and numerical algorithms are partially due to internal simplifications. An uncertainty analysis can provide a quantitative statement about the combined influence of potentially important uncertainties on the results. This analysis provides quantitative sensitivity statements that rank the uncertainties with respect to their contribution to model output uncertainty. This analysis approach is based on well-established concepts and tools from probability, calculus and statistical theory. Once the potentially important uncertainties are identified and the current state of knowledge of each is quantitatively expressed by subjective probability distributions, very little subjectivity is involved.

Before the uncertainty can be identified, it is customary to perform an error analysis of both measurements and computations. The inherent uncertainty of the various methods and models used in complicated computer codes is often ignored in applications. However, by combining statistical methods with parallel computational capabilities allows for uncertainty analysis for the application of complicated engineering software.

There are two main questions that must be asked with regard to this type of uncertainty analysis. How do these uncertainties influence the most important results of the code application? Which are the major contributors to the uncertainty of the results? The following paragraphs provide a way to answer these questions.

The analytical approach requires identification of potentially important contributors to uncertainty of the code results and the quantification of the respective states of knowledge by subjective probability distributions. This type of distribution shows how well an uncertain parameter of the code application is known in the face of all the available evidence.

This analytical input requires expert judgement to varying degrees. It is the only instance where subjectivity enters the analysis.

Subjective probability distributions for the code results follow in a logistically consistent manner from the state of knowledge quantified at the level of models, parameters and application-specific input data. They are the consequence of the propagation of this state of knowledge quantification through the code and will generally remain unknown except for extremely simple codes. The aim of the analysis is to obtain approximations to these distributions and to derive quantitative uncertainty statements for them.

In order to perform a computational uncertainty analysis, a random sample is drawn up. An element of this sample is called a parameter vector and is composed of one value for each of the uncertain quantities. The computational code is run with each parameter vector in the sample. The alternative sets of output values obtained also constitute a random sample, but this is drawn according to the unknown subjective probability distributions of the respective code results. From this sample, by applying statistical concepts and methods quantitative uncertainty statements are derived.

For quantitative uncertainty measures, a two-sided statistical tolerance limit or confidence statement for each of the code results of interest is chosen. The confidence statement quantifies the possible influence of the fact that only a limited number of code runs have been performed. The minimum number of uncertain quantities needed for these limits is independent of the number of uncertain quantities taken into account, and rather depends on the two percentages as shown above.

Simply choosing combinations of parameter values at will and performing the respective code runs does not permit one to make quantitative tolerance or confidence statements about the combined influence of the identified uncertainties. Without the use of statistics, statements could not be made even after many code runs.

The results from the code runs, together with the respective parameter vector of the sample, are used to derive global sensitivity measures and statements. These sensitivity measures account for the sensitivity of the code result with respect to the uncertain parameter and the parameter uncertainty. Therefore they are suited for rank uncertainty according to their contribution to code output uncertainty. Standardised rank regression coefficients are chosen to aid in calculating the rank uncertainty. A standardised regression coefficient shows how many standard deviations the code result will change if the uncertain parameter is changed by one deviation. These indicate the direction of the contribution and allow for the computation of correlation ratios. The correlation ratio is the square root of the quotient of the variance of the conditional mean value of the code result divided by the total variance of the code result due to all the uncertainties.

It can be seen therefore that the analytical results are only as good as the input to the analysis. Information uncertainty and sensitivity statements are obtained by the identification of the potentially important uncertainties, in the quantification of the respective states of knowledge and in the selection of measures for uncertainty and sensitivity.

Computationally Efficient Methods for Uncertainty Propagation

Mechanistic modelling of physical systems is often complicated by the presence of uncertainties. The implications of these uncertainties are particularly important in the assessment of several potential regulatory options. Even though significant effort may be needed to incorporate uncertainties into the modelling process, an uncertainty analysis results in providing useful information that can aid in decision making.

A systematic uncertainty analysis provides insight into the level of confidence in model estimates, and aids in assessing how various possible model estimates should be weighed. Further, a systematic uncertainty analysis can lead to the identification of the key sources of uncertainty (such as data gaps) which merit further research, as well as the sources of uncertainty that are not important with respect to a given response.

The purpose of quantitative uncertainty analysis is to use currently available information in order for quantifying the degree of confidence in the existing data and models. The purpose is not to reduce the uncertainty. Reduction in uncertainty can only come from gathering additional information and filling the data gaps. Even though the applicability of a model is limited by the model assumptions and the uncertainties in the evaluation data, understanding the judgements associated with the modelling process is more valuable than side-stepping the uncertainty analysis. In fact, it is precisely for problems where data is limited and where simplifying assumptions have been used that a quantitative uncertainty analysis can provide an illuminating role, to help identify how robust the conclusions about model results are, and to help target data gathering efforts.

The following stages are involved in the uncertainty analysis of a model:
(1) Estimation of uncertainties in model inputs and parameter (characterisation of input uncertainties)
(2) Estimation of the uncertainty in model outputs resulting from the uncertainty in model inputs and model parameters (uncertainty propagation)
(3) Characterisation of uncertainties associated with different model structures and model formulations (characterisation of model uncertainty)
(4) Characterisation of the uncertainties in model predictions resulting from uncertainties in the evaluation data

Uncertainty Analysis Techniques

Uncertainty Analysis

Mechanistic modelling of uncertainties in physical systems is complicated by the presence of uncertainties.

A systematic uncertainty analysis provides insight into the level of confidence in model estimates, and can aid in assessing the weighing of various possible model estimates. The uncertainty analysis can also lead to the identification of the key sources of uncertainty which need further investigation, as well as the sources of uncertainty that are not so important with respect to a given response.

The method presented by Isukapalli (1991), as well as the various methods given by Sikanta Mishra are further detailed below.

Isukapalli (1991) made use of transport transformation function in order to calculate uncertainty. The term "transport" refers to the movement of material through the surroundings as a result of the associated flows and diffusive processes. The term "transformation" refers to the change of a species (physical, chemical or biological).

The transport transformation models are similar to each other in structure and are often described by ordinary or partial differential equations. These models are based on the continuity equation and on a mass balance in a control volume.

Many transport-transformation models can be represented in simplified form by the following equation:

$$\frac{dc}{dt} = F + R + S$$

Equation 5: Transport transformation equation

Uncertainty occurs in transport-transformation models due to a number of factors. The randomness inherent in natural systems, errors in the estimates of transport properties of a species in a medium, and inaccurate estimates of transformation properties, such as reaction rates, contribute to the overall uncertainty in these models.

The main limitation in performing comprehensive uncertainty analyses of transport-transformation models is the associated cost and effort. The computer resources needed for uncertainty propagation using conventional methods can sometimes be expensive. Also the incorporation of uncertainty associated with structural and formulation aspects of the models requires further effort. Finally, the data need for characterising input uncertainties are substantial.

Due to randomness or unavoidable unpredictability, the data present in environmental and biological systems are inherently scholastic. Some quantities are random in principle, while other quantities that are precisely measurable are modelled as random quantities.

Some quantities vary over time, over space or across individuals in a population. This is called variability. It is possible to interpret variability as uncertainty under certain conditions, since both can be expressed in terms of frequency distributions.

However, there is a difference between the implications of the uncertainty and variability in decision making. The knowledge of the frequency distribution for variability can lead to the identification of significant sub-populations that require a more focussed study. The knowledge of the uncertainty can lead to the determination of areas where additional research or alternative techniques are needed in order to reduce the uncertainty.

Conventional methods for sensitivity analysis and uncertainty propagation can be classified broadly into 4 categories namely sensitivity testing, analytical methods, sampling based methods and computer algebra based methods. All of the methods are discussed below.

Sensitivity testing involves studying model response for a set of changes in model formulation and for selected model parameter combinations.

In this approach, the model is run for a set of sample points for the parameter of concern or with straightforward changes in the model structure. This approach is often used to evaluate the robustness of the model by testing whether the model response changes significantly in relation to changes in model parameters and structural formulation of the model. This approach can be used in conjunction with transport-transformation modelling.

The primary advantage of this model is that it accommodates both qualitative and quantitative information regarding variation in the model. But there is a disadvantage to this method. Using this approach, the detailed information about the uncertainties is difficult to obtain. Also, the sensitivity information obtained depends greatly on the choice of the sample points, especially when only a small number of simulations can be performed.

Analytical methods involve either the differentiation of model equations and subsequent solution of a set of auxiliary sensitivity equations, or the reformation of original model using stochastic algebraic or differential equations. There are four analytical methods used for sensitivity or uncertainty analysis, namely differential analysis, Green's function, spectral based stochastic finite element method and coupled and decoupled direct methods. These are discussed in the next couple of paragraphs.

Differential analysis methods include the Neumann expansion by Adomain (1980) and Tatang (1992), and the perturbation method by Adomain (1980) and Tatang (1995). The Neumann expansion method involves finding the inverse of the model operator through the expansion of model equations and therefore has limitations on the type of model it can address. The perturbation method involves the expansion of model outputs as a series in terms of small random perturbations in model parameters and the subsequent solution of the series coefficients. The main limitation of these methods is the requirement that the perturbation terms be small. Also, these methods

are difficult to apply in conjunction with the modelling of complex non-linear systems since the model equations are often mathematically intractable.

In Green's function method, the sensitivity equations of a model are obtained by differentiating the model equations. Constructing an auxiliary set of Green's functions then solves the sensitivity equations. This method minimises the number of differential equations that are solved for sensitivity and replaces them with integrals that can be easily evaluated as shown by Doughtery and Rabitz (1979) and by Doughtery, Hwang and Rabitz (1979).

Spectral based stochastic finite element method relies on the use of representing stochastic processes in terms of series expansion like the Karhunen-Loeve expansion by Ghanema and Spanos (1991) and Papoulis (1991). For finite element method problems, this approach results in a set of linear matrix equations with deterministic matrices multiplied by random vectors. These matrix equations are solved either by the use of operator expansions or by using the Garlerkin's method as set out by Villadsen and Michelsen (1978).

Coupled / decoupled direct method involves the differentiation of model equations and the subsequent solution of the sensitivity equations. These sensitivity equations are then solved along with the original model equations by using the coupled direct method as illustrated by Tomovi'c and Vukobratovi'c (1972) or they are solved separately as shown by Dunker (1984) or by Prokopakis (1993). The decoupled method is advantageous in terms of its computational efficiency as well as the stability of the solution.

Sampling based methods involve running the original model for a set of input / parameter combinations (sample points) and estimating the sensitivity or the uncertainty using the model outputs at these points. These methods do not require access to model equations or even the model code. These methods involve the running a set of model at a set of sample points and establishing a relationship between inputs and outputs using the model results at the sample points. There are four sampling based sensitivity or uncertainty methods, namely Monte Carlo and Latin Hypercube-Sampling methods, Fourier Amplitude Sensitivity Test (FAST), reliability-based methods and response surface methods. All are discussed below.

Monte Carlo and Latin Hypercube-sampling methods are based on the same principle. Monte Carlo methods are the most widely used means for uncertainty analysis, with applications in aeronautical engineering as illustrated by Barrett (1996). These methods involve random sampling from the distribution of inputs and successive model runs until a statistically significant distribution of outputs is obtained. They are used in the solution of problems that can be modelled by a sequence of a set of random steps that eventually converge to a desired solution. Since these methods require a large number of samples or model runs, their applicability is limited to simple models.

The Latin Hypercube-sampling, as shown by Iman and Conover (1980), Loh (1996) and Stein (1987), is a variant of the Monte Carlo method. The range of probable values for each uncertain parameter is divided into ordered segments of equal probability. Therefore the whole parameter space, consisting of all the uncertain

parameters, is partitioned into cells having equal probability, and they are sampled in an efficient manner so that each parameter is sampled once from each of its possible segments. The advantage of this approach is that the random samples are generated from all the ranges of possible values, thereby giving insight into the extremes of the probability distributions of the outputs.

Fourier amplitude sensitivity test (FAST) is based on Fourier transformation of uncertain model parameters into a frequency domain, thereby reducing the multidimensional model into a single dimensional model.

For a model with *m* parameters, $k_1, k_2, \ldots, k_m$, and *n* outputs $u_1, u_2, \ldots, u_m$, such that $u_i = f_i(t; k_1, k_2, \ldots, k_m); i = 1,2,\ldots,n$, the FAST method involves the transformation of the parameters into a frequency domain spanned by a scalar *s* as shown below.

$$k_j = G_j(\sin \nu_j s), i = 1,2,\ldots,m$$

Equation 6: Transformation of parameter into FAST

The outputs are then approximated as shown.

$$\overline{u_i}(t) = \frac{1}{2p}\int_{-p}^{p} u_i(t; k_1(s), k_2(s), \ldots, k_m(s))ds$$

Equation 7: Uncertainty as calculated by the FAST method

$$s_i^2(t) = \frac{1}{2p}\int_{-p}^{p} u_i^2(t; k_1(s), k_2(s), \ldots, k_m(s))ds - \overline{u_i^2}$$

Equation 8: Standard deviation as calculated by the FAST method

The integrals are evaluated by repeatedly sampling the parameter space of *s*, which corresponds to the sampling in the multidimensional model parameter space.

Reliability based methods comprise of first and second order methods. First and second order reliability methods are approximation methods that estimate the probability of an event under consideration. These methods are useful in uncertainty analysis of models with a single failure criterion. For a model with random parameters $X = (X_1, X_2, \ldots, X_n)$ and a failure criterion $g(X_1, X_2, \ldots, X_n) < 0$, the objective of he reliability based approach is to estimate the probability of failure.

These methods use analytical schemes to approximate the probability integral, through a series of simple step, as illustrated by Bjerager (1990).

- Mapping the basic random variables X and the failure function g(X) into a vector of standardised and uncorrelated normal variates U, as X(U) and G(U) respectively.
- Approximating the function G(U) by a tangent or a paraboloid at a failure point u* closest to the origin.
- Calculating the probability of failure as a simple function of u*.

Response surface methods consists of screening the subset of important model input parameters, making multiple runs of the computer model using specific values and pairing of these input parameters and fitting a general polynomial model to the model data by using the method of least squares. This fitted response surface is then used as a replacement or proxy for the computer model and all inference related to the sensitivity or uncertainty analysis for the original model are derived from this fitted model. This response is also called the secondary model technique. Box and Draper (1987) and Box et al (1978) describe the adaptive nature of these methods and the methodology used in identifying the sample points.

Computer based algebra methods are based on direct manipulation of the computer code of the model and are also called automatic differentiation. These methods involve the direct manipulation of the computer code and the estimation of the sensitivity and uncertainty of model outputs with respect to model inputs. These methods do not need information about the model structure or the model equations, but use mechanical patter-matching algorithms to generate a derivative code based on the model code. One of these methods is the automatic differentiation, which is sometimes called automated differentiation.

Automated differentiation involves direct manipulation of a model code to generate a corresponding derivative calculating code. Given the source code and the information about what the dependent and independent variables of interest are, the automatic differentiation tools can generate derivative information without further user intervention.

According to Sikanta Mishra, there are four types of uncertainty analysis techniques.

The first method is the First-order Second Moment Method. The mean and variance of model outcome are computed from the mean and variance of uncertain inputs, their correlation matrix and sensitivity coefficients via a first-order Taylor expansion approach.

The second method is the Point Estimation Method. The mean and variance of model output are estimated from the mean and variance of uncertain inputs via a weighted average of functional evaluations carries out with parameter sets sampled using an eigen transformation of the input correlation matrix.

The third method is the First-Order Reliability Method. The probability of exceeding a given performance "target" is estimated from the mean, variance and correlation matrix of uncertain inputs using a first-order approximation to the multifold probability integral.

The final method is the Monte-Carlo Simulation method. Multiple model calculations are performed with parameter values drawn randomly from specified probability distribution to estimate the range of possible model outcomes and the likelihood of each outcome. This method is discussed in more detail in the following chapter.

Monte Carlo Simulation Method

The Monte Carlo Simulation method as discussed in the previous chapter, is a numerical technique that estimates the solutions of problems with multidimensional integrals. As a scholastic technique, the method can be used in a number of applications. Typical usage of the Monte Carlo method involves the prediction of manufacturing variations and tolerance analyses.

In essence, the basic procedure of Monte Carl integration estimates the integral of a function over a multidimensional volume by substituting information over the discrete set of functional values at samples inputs. A major feature of this method is its dimensional independence.

The probability that the calculated integral using the Monte Carlo method is within plus or minus some distance of the true value can readily be determined from the sample size N, irrespective of the number of parameters.

Sampling consists of a statistical exploration of the parameter space using computer generated pseudo-random numbers. Random numbers are essentially uniform deviated - independent random variables, distributed uniformly over the interval [0,1]. By evaluating prescribed functions of random numbers, random variations from different distribution can be generated.

The mean and standard deviation are estimated using Monte Carol integration by numerical summations as follows.

$$m = \frac{1}{n}\sum_{i=1}^{n} x_i$$

Equation 9: Mean according to the Monte Carlo simulation method

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - m)^2$$

Equation 10: Standard deviation according to the Monte Carlo simulation method

The single variable definitions above can be generalised to the case of multiple variables by introducing the concept of joint probability density functions (jpdf).

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(x_1, x_2, \ldots, x_n)dx_1 dx_2 \ldots dx_n = 1$$

Equation 11: Joint probability density function

However, in the presence of more than one random variable, one has to take into account the relationship between the different distributions. A measure of this correlation is their covariance integral (s_{x1x2}) as shown below.

$$s_{x1x2} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x_1 - m_{x1})(x_2 - m_{x2})f(x_1, x_2)dx_1 dx_2$$

Equation 12: Covariance interval

s (x_1,x_2) is the joint probability density function of x_1 and x_2. This may be estimated by the following equation.

$$s_{x1,x2} = \frac{1}{n}\sum_{i=1}^{n}(x_{1i} - m_{x1})(x_{1i} - m_{x2})$$

Normalising the covariance integral results in the correlation coefficient of two distributions *p* such that $-1 \le r \le 1$ gives the following equation.

$$r_{x1,x2} = \frac{s_{x1,x2}}{s_{x1}s_{x2}}$$

Equation 13: Covariance integral

Using the above definitions, a normal probability density function is characterised by its mean μ and standard deviation σ:

$$f(x) = \frac{1}{s\sqrt{2p}}\exp\left[-\frac{(x-m)^2}{2s^2}\right]$$

Equation 14: Normal probability density function

The multidimensional normal joint probability density function is expressed as follows.

$$f(X) = \frac{1}{\sqrt{2|C|(2p)^2}}\exp\left[-\frac{1}{2}(X-m)^T C^{-1}(X-m)\right]$$

Equation 15: Multidimensional normal joint probability density function

$|C|$ is the determinant of the $(n \times n)$ covariance matrix C.

$$C = \begin{bmatrix} s_{11}s_{12}\cdots s_{1n} \\ s_{21}s_{22}\cdots s_{2n} \\ \vdots \\ s_{n1}s_{n2}\cdots s_{nn} \end{bmatrix}$$

Equation 16: Determinant of the (*n* x *n*) covariance matrix

Each one of the diagonal terms of this matrix s_{ii} is the variance of the i^{th} variant, and each off-diagonal term s_{ij} is the covariance between the i^{th} and j^{th} variants.

For normal distributions, it can be shown that given two uniform deviates random numbers U_1 and U_2, two independent normal deviates with zero mean and a standard deviation of one can be generated according to the system of equations:

$$Z_1 = \sqrt{-2\ln(U_1)}\cos(2pU_2)$$

Equation 17: First of the normal deviates

$$Z_2 = \sqrt{-2\ln(U_1)}\sin(2pU_2)$$

Equation 18: Second of the normal deviate s

The transformation $X = m + sZ$ is then used to calculate the values of the variants when chosen from the normal probability density function with mean μ and standard deviation σ.

For the multiple normal distribution, the covariance matrix C is a positive, definite and symmetric matrix that can be decomposed into the following matrix.

$$C = LL^T$$

Equation 19: Covariance matrix for multiple normal distributions

$$L = \begin{bmatrix} l_{11}0\cdots 0 \\ l_{21}l_{22}\cdots 0 \\ \vdots \\ l_{n1}l_{n2}\cdots l_{nn} \end{bmatrix}$$

Equation 20: Lower triangular matrix for multiple normal distributions

A random vector $X = (x_1, x_2, \ldots, x_n)$ with a multiple normal joint probability density function can be generated from the following matrix equation.

$$X = LZ + m$$

Equation 21: Matrix equation for a random variable

Z is a normal vector of independent zero mean and one standard deviation normal deviates, and μ is the vector of the means.

The decomposition can be performed using the square-root method, which is based on a set of recursive formulae, for the computation of the elements of L.

$$l_{ij} \leftarrow \frac{s_{ij} - \sum_{k=1}^{j-1} l_{ik} l_{jk}}{\left(s_{jj} - \sum_{k=1}^{j-1} l_{jk}^2 \right)^2}$$

Equation 22: Square root method for the computation of the elements of L

Calculating Uncertainty: Method 1

The experimental uncertainty is defined as the absolute value of the maximum expected deviation from the reported experimental value. Schultz and Cole (1979) suggested the following method of calculating a value for the uncertainty.

$$U_R = \left[\sum_{i=1}^{n} \left(\frac{\partial R}{\partial Vi} U_{Vi} \right)^2 \right]^{\frac{1}{2}}$$

Equation 23: Schultz and Cole (1979) UncertaintyEquation

According to Schultz and Cole (1979), there is a procedure for combining independent and correlated uncertainties. For an equation written in terms of measured quantities A and B, it is therefore possible to calculate the uncertainty using the following equation.

$$s_{R_1} = \left[\left(\frac{\partial R}{\partial A} s_A \right)^2 + \left(\frac{\partial R}{\partial B} s_B \right)^2 + 2 r_{AB} \frac{\partial R}{\partial A} \frac{\partial R}{\partial B} s_A s_B \right]^{\frac{1}{2}}$$

Equation 24: Combined correlation and independent uncertianties

However, if the uncertainties in A and B are completely non-correlated, then $r_{AB} = 0$ and the latter equation can be reduced in order to obtain the following equation.

$$s_{R_1} = \left[\left(\frac{\partial R}{\partial A} s_A \right)^2 + \left(\frac{\partial R}{\partial B} s_B \right)^2 \right]^{\frac{1}{2}}$$

Equation 25: Non-correlated uncertainty

However, if the deviations are completely correlated, then $r_{AB} = \pm 1$ and the latter equation can be reduced to the following.

$$s_{R_1} = \frac{\partial R}{\partial A} s_A \pm \frac{\partial R}{\partial B} s_B$$

Equation 26: Completely correlated uncertainty

For $\rho_{AB} = +1$, the maximum uncertainty in the result is obtained due to a lack of cancellation of any uncertainties in A and B.

Non-independent uncertainties are of a different nature than completely correlated uncertainties because the former are of a systematic nature and cannot be determined through the examination of the deviations as required in the latter case. Since, for non-independent uncertainties, it is not possible to cancel the uncertainties in the

experimentally measured quantities; an expression analogous to the latter equation is appropriate for their combined uncertainty.

$$\Delta_R = \frac{\partial R}{\partial A}\Delta_A + \frac{\partial R}{\partial B}\Delta_B$$

Equation 27: Non-independent uncertainties

The final result may have all the uncertainties (namely correlated, independent and non-independent) associated with its determination. Since the one equation combines correlated and independent uncertainties, is independent of the equation which expresses the non-independent uncertainty, they should be combined in accordance to the general equation. . Therefore the total uncertainty in R is represented as follows.

$$s_R = \left[\left(\frac{\partial R}{\partial A}\right)^2\left(s_A^2 + \Delta_B^2\right) + \left(\frac{\partial R}{\partial B}\right)^2\left(s_B^2 + \Delta_B^2\right) + 2r_{AB}\frac{\partial R}{\partial A}\frac{\partial R}{\partial B}s_A s_B\right]^{\frac{1}{2}}$$

Equation 28: Total uncertainty

In single sample experiments, when only single values of A and B are determined, it is not possible to approach the analysis of experimental uncertainties on a truly statistical basis.

However, with various modifications, the form of the equation of the total uncertainty can be used. Since there is no meaning for standard deviations in single sample experiments, Kline and McClintock (1953) suggest that they be replaced by the uncertainty U, as shown below.

$$U_R = \left[\left(\frac{\partial R}{\partial A}\right)^2\left(U_A^2 + \Delta_A^2\right) + \left(\frac{\partial R}{\partial B}\right)^2\left(U_B^2 + \Delta_B^2\right) + 2r_{AB}\frac{\partial R}{\partial A}\frac{\partial R}{\partial B}U_A U_B\right]^{\frac{1}{2}}$$

Equation 29: Uncertainty in single sample experiments

Calibration procedures are used to minimise the non-independent uncertainties Δ_A and Δ_B that are present in calculating the uncertainty U. It is suggested that purely independent errors are more common than correlated errors, so r_{AB} is assumed to be zero. This results in the following equation.

$$U_R = \left[\left(\frac{\partial R}{\partial A}U_A\right)^2 + \left(\frac{\partial R}{\partial B}U_B\right)^2\right]^{\frac{1}{2}}$$

Equation 30: Uncertainty in single sample experiments using only independent uncertainties

Calculating Uncertainty: Method 2

A different method in calculating uncertainty was established by Webster. Webster analysed the random error analysis of a single variable. In general, the true value of the variable, *x*, is not known. Therefore the average of the measured values must be used.

$$\bar{x} = \frac{1}{N_i}\sum_{i=1}^{N} x_i$$

Equation 31: True value of the variable

The mean or average value is the sum of the values divided by the total number of observations. The mean is the value that is reported as the "measured value" or the result. The measured values are scattered about the mean value. The amount by which a single measurement differs from the mean is called the deviation of the measurement, calculated as shown below.

$$d_i = x_i - \bar{x}$$

Equation 32: Deviation of the measurement

In order to determine the random error in the result, the spread of the data is analysed. Statistical analysis of error generally assumes a Gaussian or normal distribution of data. The easiest manner in analysing spreads is to plot a histogram. This is a plot of frequency of occurrenc e of values in a range $D(x)$ centred at x, plotted versus x.

For a Gaussian distribution, $P(a, b)$ is the probability of observing a value of x in the range of a to b, as shown below.

$$P(a,b) = \int_a^b f(x)dx$$

Equation 33: Probability equation

$$P(a,b) = \int_a^b \frac{1}{s\sqrt{2p}} \exp\left(\frac{-(x-X)^2}{2s^2}\right)dx$$

Equation 34: Probability equation

$$\int_{-\infty}^{\infty} f(x) = 1$$

Equation 35: Height of the distribution

The Gaussian distribution describes the population of possible errors as a measurement where many independent sources of error contribute to the total random

error. However, data do not always abide by this distribution. The limitation on the Gaussian distribution is that the sources of error must be unrelated and random.

For a true normal or Gaussian distribution, the data are centred about the mean value *X*. The value of *X* is determined by the following equation.

$$X = \int_{-\infty}^{\infty} xf(x)dx$$

Equation 36: Gaussian distribution

For a finite number of readings, it can be shown by the method of least squares fitting that the best estimate of X is the average.

The variance of the distribution is the second moment about X.

$$V = \int_{-\infty}^{\infty} (x - X)^2 f(x)dx = s^2$$

Equation 37: Variance of the distribution

If applied to this expression as the least square fit method, the standard deviation of the sample would be:

$$s = \left[\frac{1}{n} \sum \left(x_i - \bar{x} \right)^2 \right]^{\frac{1}{2}}$$

Equation 38: Standard deviation of the distribution

The best estimate of the standard deviation from the data is:

$$s^2 = \frac{n}{n-1} s^2 = \frac{1}{n-1} \sum \left(x_i - \bar{x} \right)^2$$

Equation 39: Best estimate of standard deviation

The latter value is slightly larger than the mean square of the data. This value (s) is called the sample standard deviation and is used if the number of samples (n) is less than thirty.

The former quantity s is called the standard deviation of the distribution. The standard deviation quantifies the typical deviation of the measurements from the mean value. The smaller the standard deviation, the more precise the data.

The standard deviation, in a Gaussian distribution, is associated with various levels of confidence.

The above expression of standard deviation gives an expression for the precision. This is not an estimate of the mean as an approximation to the true mean. This is only a single estimate of the true value.

In order to calculate the uncertainty completely, an estimate of the true value as well as a confidence interval must be given. There is a number of ways to determine the confidence interval based on a small sample size, a large sample size or a single sample experiment. The following paragraphs discuss the confidence intervals that are based on small and large samples respectively.

For a large confidence interval, meaning that a test is repeated many times, then a set of the values of the mean values is obtained. This set of mean values will have a Gaussian distribution, regardless of the type of distribution of the data (Central Limit Theorem). The standard deviation of the mean is give by the following equation.

$$s_m = \frac{s}{\sqrt{n}}$$

Equation 40: Standard deviation of the mean

Therefore for a sample of n measurements, the best estimate of the true value X is $X = \bar{x} \pm s/\sqrt{n}$ at the 68% confidence interval.

When the sample size is less than thirty, and the data follows a Gaussian distribution, the confidence intervals for the sample mean are calculated from a student distribution of a quantity t.

$$t = \frac{x - \bar{x}}{s_s/\sqrt{n}}$$

Equation 41: Student distribution

In this case, with a confidence interval of c%, the true mean lies in the interval, where $a = 1 - c$ and $n = N - 1$, the following confidence interval results.

$$\bar{x} - t_{\frac{a}{2y}} \frac{s_s}{\sqrt{n}} < m < \bar{x} + t_{\frac{a}{2y}} \frac{s_s}{\sqrt{n}}$$

Equation 42: Confidence interval of student distribution

This interval is narrower than the dispersion of the data, because of the focus on the precision error of the estimate.

Single sample experiments are experiments for which n=1. I such a case, statistical analysis is not possible and either the manufacturer's stated accuracy of best judgement is used. Experience, intuition and understanding are depended upon. If the manufacturer's accuracy is not stated, then ± ½ of the smallest division is used.

Calculating Uncertainty: Method 3

According to Webster, there is another method of calculating the uncertainty of a system. The total uncertainty is a measurement of x can be calculated after the precision and bias uncertainties have been estimated by the following expression:

$$U_x = \sqrt{B_x^2 + P_x^2}$$

Equation 43: Total uncertainty

B_x is the bias uncertainty (normally given by the manufacture as the accuracy) and P_x is the precision uncertainty calculated as shown above. Both of these uncertainties must be calculated at the same confidence level.

The uncertainty of the measurement result y arises from the uncertainty $u(x_i)$ or u_i of the input estimate x_i that enter the output estimate equation. In general, components of uncertainty, namely type A and type B may be categorised according to the method used to evaluate them. Both methods are discussed in the following paragraphs.

Type A evaluation is a method of evaluation of uncertainty by the statistical analysis of series of observations.

The standard uncertainty of type A is an uncertainty component obtained by type A evaluation, represented by a statistically estimated deviation s_i, equal to the positive square root of the statistically estimated variance s_i^2, and the associated number of degrees of freedom, v_i. For such a component of type A, the standard uncertainty is $u_i = s_i$.

A type A evaluation of standard uncertainty may be based on any valid statistical method for treating data. There are different statistical ways of achieving this. The different ways include calculating the standard mean of a series of independent observations; using least squares method to fit a curve to data in order to estimate the parameters of the curve and their standard deviations; and carrying out an analysis of variance in order to identify and quantify the random effects in certain types of measurements.

For the type A method, consider an input quantity X_i whose value is estimated from n independent observations $X_{i,k}$ of X_i obtained under the same conditions of measurement. In this case, the input estimate x_i is usually the sample mean, as shown below.

$$x_i = \overline{X_i} = \frac{1}{n}\sum_{k=1}^{n} X_{i,k}$$

Equation 44: Sample mean

The standard uncertainty $u(x_i)$, shown below, to be associated with x_i is the estimated standard deviation of the mean.

$$u(x_i) = s(\overline{X}) = \left(\frac{1}{n(n-1)} \sum_{k=1}^{n} \left(X_{i,k} - \overline{X}_i \right)^2 \right)^{\frac{1}{2}}$$

Equation 45: Estimated standard deviation of the mean

Type B evaluation is a method of evaluation of uncertainty by means other than the statistical analysis of series of observations.

The standard uncertainty of type B is an uncertainty component obtained by a type B evaluation, represented by a quantity u_j, which may be considered an approximation to the corresponding standard deviation. The standard deviation is equal to the positive square root of u_i^2, which may be considered an approximation to the variance and which is obtained from an assumed probability distribution based on all the available information. Since the quantity u_i^2 is treated like a variance and u_j like a standard deviation, for such a component the standard uncertainty is simply u_j.

Type B evaluation of standard uncertainty is based on scientific judgement using all the relevant information available. This information includes: previous measured data, experience with, or general knowledge of, the behaviour and property of relevant materials and instruments, manufacturer's specifications, data provided in calibration and other reports and uncertainties assigned to reference data taken from handbooks.

The combined standard uncertainty of the measurement result y, designated by $u_c(y)$ and taken to represent the estimated standard deviation of the result, is the positive square root of the estimated variance $u_c^2(y)$ obtained from the following equation.

$$u_c^2(y) = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial x_i} \right)^2 u^2(x_i) + 2 \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} u(x_i, x_j)$$

Equation 46: Estimated variance

The above equation of estimated variance is based on a first-order Taylor series approximation of the measurement equation is referred to the law of propagation of uncertainty. The equation contains the partial derivatives of f with respect to the X_i evaluated at $X_i = x_i$ where $u(x_i)$ is the standard uncertainty associated with the input estimate x_i and $u(x_i, x_j)$ is the estimated covariance associated with x_i and x_j.

The estimated variance equation often reduces to a simple form in practical cases. If the input estimate x_i of the input quantities X_i can be assumed to be uncorrelated, then the second term vanishes. Also, if the input estimates are uncorrelated and the measurement equation is in one of the following forms, then the equation simplifies further.

The measurement equation is a sum of the quantities X_i multiplied by constants a_i, or is a product of quantities X_i, raised to powers $a, b, \ldots, p$, multiplied by a constant A.

The combined standard uncertainties for the measurement equation multiplied by consonants as well as the measurement equation multiplied by a product of quantities are given below.

$$u_c^2(y) = a_1^2 u^2(x_1) + a_2^2 u^2(x_2) + \ldots + a_N^2 u^2(x_N)$$

Equation 47: Combined standard uncertainty for a measurement equation multiplied by constants

$$u_{c,r}^2(y) = a^2 u_r^2(x_1) + b^2 u_r^2(x_2) + \ldots + p^2 u_r^2(x_N)$$

Equation 48: Combined standard uncertainty for a measurement equation multiplied by a product of quantities

For the measurement equation, if the probability distribution characterised by the measurement result y and its combined standard uncertainty $u_c(y)$ is approximately normal, and $u_c(y)$ is a reliable estimate of the standard deviation of y, then the interval $y - u_c(y)$ to $y + u_c(y)$ is expected to encompass approximately 68% of the distribution of values that could reasonably be attributed to the value of the quantity Y of which y is an estimate. This implies that it is believed with an approximate level of confidence of 68% that Y is greater than or equal to $y - u_c(y)$ and is less than or equal to $y + u_c(y)$, which is commonly written as $Y = y \pm u_c(y)$.

Although the combined standard uncertainty is used to express the uncertainty of many measurement results, what is often required is a measure of uncertainty that defines an interval about the measurement y within which the value of the measurand Y can be confidently asserted to lie. The measurement of uncertainty intended to meet this requirement is termed expanded uncertainty, U, and is obtained by multiplying $u_c(y)$ by a coverage factor, k. Therefore $U = k u_c(y)$ and is confidently believed that Y is greater than or equal to y-U, and is less than or equal to y+U, which is commonly written as $Y = y \pm U$.

Generally, the value of the coverage factor k is chosen on the basis of the desired level of confidence to be associated with the interval defined by $U = k u_c$. Typically, k is in the range 2 to 3. When the normal distribution applies, and u_c is a reliable estimate of the standard deviation of y, k = 2 and $U = 2u_c$ defines an interval having a level of confidence of approximately 95% and when k = 3, $U = 3u_c$ defines an interval having a level of confidence greater than 99%.

If it can be assumed that the possible estimated values of the standard are approximately normally distributed with approximate standard deviation u_c, the unknown value of the standard is believed to lie in the interval $m_s \pm u_c$ with a level of confidence of approximately 68%.

If it can be assumed that the possible estimated values of the standard are approximately normally distributed with approximate standard deviation u_c, the unknown values of the standard is believed to lie in the interval defined by U with a level of confidence of approximately 95%.

Calculating Uncertainties Using Other Methods

According to Klir (1994), there is a number of theories from which uncertainties can be calculated. These include the classical set theory, the probability theory, the fuzzy set theory, the fuzzy measure theory, the rough set theory and interval mathematics. Each of these theories will be discussed below.

In the **classical set theory**, uncertainty is expressed by sets of mutually exclusive alternatives in situations where one alternative is desired. This includes diagnostic, predictive and retrodictive uncertainties. Therefore the uncertainty arises from the non-specificity inherent in each set. Large sets result in less specific predictions, and retrodictions, than smaller sets. Full specificity is obtained when only one alternative is possible.

In the **probability theory**, uncertainty is expressed in terms of a measure on subsets of a universal set of alternatives or events. The uncertainty measure is a function that assigns a number between 0 and 1 to each subset of the universal set. This number, the probability of the subset, expresses the likelihood that the desired unique alternative is in this subset. Here, the uncertainty arises from the conflict among the likelihood claims associated with the subsets of the universal set, each consisting of exactly one alternative. Since these alternatives are mutually exclusive, nonzero probabilities assigned to two or more events conflict with one another, since only one of them can be the desired one.

Fuzzy set theory is similar to classical set theory in that it expresses non-specificity. Also, they express vagueness. Vagueness is different from non-specificity in the sense that vagueness comes from imprecision of definitions. In fuzzy sets, the membership is not of affirmation or denial, but rather a matter of degree.

Fuzzy theory is a method that facilitates the uncertainty analysis where uncertainty arises due to vagueness or "fuzziness" rather than due to only randomness. This is based on Boolean logic that has been extended to handle the concept of partial truth - truth values between "completely true" and "completely false". It was introduced as a means to model the uncertainty of natural language. Fuzzy theory uses the process of "fuzzification" as a methodology to generalise any specific theory from crisp (discrete) to a continuous (fuzzy) form.

Classical set theory has a "crisp" definition as to whether an element is a member of a set or not. However, certain attributes of systems cannot be defined in one set or another. In such a case, uncertainty arises out of vagueness involved in the definition of that attribute. Classical set theory allows for either one or the other value. Fuzzy theory, on the other hand, allows for a gradual degree of membership.

In the classical set theory, the truth-value of a statement can be given by the membership function $m_A(x)$ as follows:

$$m_A(x) = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

Equation 49: Truth value as defined by classical theory

Fuzzy set theory allows for a continuous value of m_A between 0 and 1 as follows:

$$m_A(x) = \begin{Bmatrix} 1 \\ 0 \\ p; 0 < p < 1 \end{Bmatrix}$$

Equation 50: Truth-value as defined by the fuzzy set theory

The fuzzy measure theory consists of special classes of measures, each of which is classified by a special property. Some measures used are plausibility and belief measures. In fuzzy set theory, the condition for membership into a set are vague, whereas in the fuzzy measure theory the conditions are precise, but the information about an element is not sufficient to determine whether it satisfies the conditions.

The **rough set theory** is an imprecise representation of a crisp set in terms of two subsets, a lower approximation and an upper approximation. Also, the approximations themselves could be imprecise or fuzzy.

Interval mathematics is used to address data that arise due to imprecise measurement and due to the existence of several alternative methods, techniques or theories that are used to estimate model parameters. In many cases, it may not be possible to obtain the probabilities of different values of imprecision in data - in some cases only errors can be obtained. This is especially true in a case of conflicting theories for the estimation of model parameters, in the sense that probabilities cannot be assigned to the validity of one theory over another. Therefore interval mathematics can be used for the uncertainty estimation.

The objective of interval mathematics is to estimate the bounds on various model outputs based on the bounds of model inputs and parameters. In the interval mathematics approach, uncertain parameters are assumed to be known but bounded and each of them has upper and lower limits without a probability structure. Every uncertain parameter is described by an interval.

If a parameter x_i of a model is know to be between $\overline{x_i} - e_i$ and $\overline{x_i} + e_i$, the interval representation of x_i us given by $\left[\overline{x_i} - e_i, \overline{x_i} + e_i\right]$. Therefore the model estimates would also belong to another interval.

If two variables a and b are given by $[a_1, a_u]$ and $[b_i, b_u]$, where $a_1 \le a_u$ and $b_1 \le b_u$, simple operations are given by the following:

$$a \pm b = [a_1 \pm b_1, a_u \pm b_u]$$

Equation 51: Addition and subtraction of interval mathematics

$$a \cdot b = [\min(a_1 b_1, a_1 b_u, a_u b_1, a_u b_u) \max(a_1 b_1, a_1 b_u, a_u b_1, a_u b_u)]$$

Equation 52: Multiplication of interval mathemetics

$$a / b = [a_1, a_u] \cdot \left[\frac{1}{b_u}, \frac{1}{b_1} \right]; 0 \notin [b_1, b_u]$$

Equation 53: Division of interval mathematics

The primary advantage of interval mathematics is that it can address problems of uncertainty analysis that cannot be studied through probabilistic analysis. It is useful for cases in which the probability distributions of the inputs are not known. However, this method does not provide adequate information about the nature of the output uncertainty, since all the uncertainties are forced into one arithmetic interval. When the probability structure of the inputs in known, the application of interval analysis would ignore this information and therefore would not be recommended.

ISO Regarding Measurement Uncertainty

ISO Requirements Regarding Uncertainty Measurement

The new ISO 17025 standard imposes requirements regarding the uncertainty statements for measurement results in testing laboratories. According to ILAC, the compliance to the ISO 17025 standard should be completed by the end of 2002. There are specific requirements to the standard, as discussed below.

According to the ISO 17025, it is important that the single measurement is not only considered, but also the overall result of a test. In this case uncertainty of measurement embraces all components of a test. Some components of unc ertainty are obtained by interpreting the statistical spread of results of a series of measurements. Other components of uncertainty have to be derived from complementary methods such as sampling plans or experience.

According to Müller (2001), there is a number of factors that contribute to the uncertainty of measurement. These factors may contribute to the coverall uncertainty of a measurement. These factors include the following:

- definition of the measurand
- sampling
- transportation, storage and handling of samples
- preparation of samples
- environmental and measurement conditions
- the personnel carrying out the tests
- variations in test procedure
- the measuring instruments
- calibration standards or reference materials
- software and methods associated with the measurement
- uncertainty arising from correction of the measurement results for systematic effects

The ISO 17025 requires that a calibration or testing laboratory performing its own calibration should have and apply a procedure to estimate the uncertainty of measurement for all calibrations and all types of calibrations (ISO 17025 5.4.6.1).

In addition, the testing laboratories should have to apply procedures for estimating uncertainties of measurement, except where test methods preclude such rigorous calculations. The laboratory should at least attempt to identify all the components of the uncertainty and make a reasonable estimation and should ensure that the statement does not give an exaggerated impression of accuracy (ISO 17025 5.4.6.2).

According to Bremser (2001) accuracy and expenditure of measurement uncertainty estimation depend on a series of factors. These factors are the type of measurement performed, the type of laboratory, the analytical requirements, the economic impact of the results and the multiplication factor or degree of proliferation.

The consideration of the above factors ensures whether a detailed measurement uncertainty estimation should be carried out and to what extent. The decision can result in various actions. These actions vary from rough estimation of the uncertainty to scientific and statistic investigation into the issue.

Guidance on Implementation

According to Müller (2001), the implementation of the concept of uncertainty of measurement must be in line with the implementation of the standard. The following fundamental points must be noted.

- The statement of the uncertainty of measurement should contain sufficient information for comparative purposes.
- The ISO 17025 forms the basic document, but specific interpretation may be needed.
- Only uncertainty in quantitative testing is considered.
- The basic requirement should be either an estimation of the overall uncertainty, or identification of the major components followed by an attempt to estimate their size and the size of the combined uncertainty.
- The basis for the estimation of uncertainty of measurement is to use existing knowledge including existing experimental data.
- When a standardised test method is used, which contains guidance to the uncertainty evaluation, testing laboratories should only follow the uncertainty evaluation procedure as given in the standard.
- If a standard gives a typical uncertainty of measurement for test results, laboratories are allowed to quote this figure if full compliance with the test method is shown.
- If a standard implicitly includes the uncertainty of measurement in the test results, there is no need for further action. Testing laboratories are expected to only quote the applicable figure or perform the applicable procedure for uncertainty estimation.
- The required depth of the uncertainty estimations may be different in different technical fields. The factors that need to be considered are common sense, influence of the uncertainty of measurement on the result, appropriateness and classification of the degree of rigour in the determination of the uncertainty of measurement.
- In certain cases it is only necessary to report the reproducibility.
- When the estimation of the uncertainty of measurement is limited, the report should state this clearly.
- There should be no development of new guides where usable guides already exist.

Standard Procedures According to the ISO Guides

In principle, the uncertainty estimation is simple. There are basically four steps involved in the process of uncertainty estimation. All four steps will be discussed in turn.

The first step is to specify the measurand. A clear statement of what is being measured, including the relationship between the measurand and the parameters upon which it depends, should be noted. If it is possible, the corrections for the known systematic effects should be given. This information should be given in the relevant standard operating procedure.

The second step is to identify the uncertainty sources. All the possible uncertainty sources should be listed. This includes sources that contribute to the uncertainty of the parameters from the first step as well as other necessary sources.

The third step is to quantify the uncertainty components. The size of the uncertainty component associated with each potential source of uncertainty identified should be measured or estimated. It is possible to determine a single contribution to uncertainty associated with a number of separate sources. It should also be noted if the available data accounts sufficiently for all sources of uncertainty.

The final step is the calculation of the combined uncertainty. The information obtained from the previous step consists of a number of quantified contributions to the overall uncertainty that either is associated with the individual sources or with the combined effects of several sources. These contributions have to be expressed as standard deviations and combined according to the appropriate rules, which are either empirical or approximated. This results in the combined standard uncertainty. The appropriate coverage factor should be applied to give an expanded uncertainty. This step involves the development of the model describing how the values of the measurand, auxiliar and influential quantities are interrelated, or how the measurand depends on the input and other influential quantities.

Uncertainty Calculations

Uncertainties in a Basic Heat Transfer System

The experimental uncertainty of heat transfer coefficient, based on Schultz and Cole (1979), is given below.

$$U_h = \left[\sum_{i=1}^{n} \left(\frac{\partial h}{\partial Vi} U_{Vi} \right)^2 \right]^{\frac{1}{2}}$$

Equation 54: Uncertainty for the heat transfer coefficient

The experimental value of heat transfer coefficient is calculated by:

$$h = \frac{q}{(t_s - t_{wi})}$$

Equation 55: Heat transfer coefficient

Applying the former equation to the latter results in the following uncertainty equation.

$$U_h = \left\{ \left(\frac{U_q}{(t_s - t_{wi})} \right)^2 \left(\frac{-q \cdot U_{ts}}{(t_s - t_{wi})^2} \right)^2 \left(\frac{q \cdot U_{twi}}{(t_s - t_{wi})^2} \right)^2 \right\}^{\frac{1}{2}}$$

Equation 56: Uncertainty of the heat transfer coefficient in terms of heat flux, saturation temperature and average tube inside temperature

Therefore, in order to obtain the total uncertainty for the heat transfer coefficient *h*, the formula of on Schultz and Cole (1979) can be used to obtain the associated uncertainties and the substituting these values in the above formula.

$$U_q = \left[\left(\frac{1}{A} \left[(C_{pw} \cdot \Delta T_w \cdot U_{mw})^2 + \left(m_w \cdot C_{pw} \left[2 (U_{Tw})^2 \right]^{\frac{1}{2}} \right)^2 \right]^{\frac{1}{2}} \right)^2 + \left(\frac{-Q}{A} \left[(pDU_L)^2 + (pLU_D)^2 \right]^{\frac{1}{2}} \right)^2 \right]^{\frac{1}{2}}$$

Equation 57: Uncertainty for the heat flux

The saturation temperature, t_s, is calculated according to the pressure of the condensing fluid. Therefore the uncertainty of the saturation temperature is given below.

$$U_{ts} = \left[\left(\frac{\partial t_s}{\partial P}\right) U_P\right]$$

Equation 58: Uncertainty of the saturation temperature

The value of $\frac{\partial t_s}{\partial P}$ is calculated from the refrigeration tables for the range of the external parameter.

$$U_{twi} = \left[\left(\left[4\left(\frac{U_{two}}{4}\right)^2\right]^{\frac{1}{2}}\right)^2 + \left\{\left[\frac{D}{2k_w}\ln\left(\frac{D_o}{D}\right)U_q\right]^2 + \left[\left(\frac{q}{2k_w}\ln\left(\frac{D_o}{D}\right) - \frac{q}{2k_w}\right)U_D\right]^2 + \left[\frac{q}{2k_w}\frac{1}{D_o}\right]^2 + \left[\frac{q \cdot D}{2k_w^2}\ln\left(\frac{D_o}{D}\right)U_{kw}\right]^2\right\}^{\frac{1}{2}}\right)^2\right]^{\frac{1}{2}}$$

Equation 59: Uncertainty of the average inside tube temperature

Uncertainties in Boiling Nucleation

The distribution of temperature of a liquid adjacent to a heated surface strongly influences the phenomena of bubble nucleation.

Experiments conducted by Schultz, Kasturirangan and Cole (1975) involved a study of the nucleation characteristics of single cavities under uniform superheat conditions.

The emergent superheat (superheat required for a vapour bubble to emerge or cap from a surface cavity) is given by the following equation.

$$T - T_{sat}^{L} = 2s \left. T_{sat}^{L} \right/ r^{V} h_{fg} t_{m} \equiv \Delta T_{\infty}$$

Equation 60: Emergent superheat in nucleation

This equation is obtained by combing the Laplace-Kelvin equation with the Clausius-Clapeyron equation. For uniform superheat conditions, T need not be identified since $T^{L} = T^{V} = T_{w}$.

An equation can be developed for the uncertainty in the experimentally determined ribbon temperature, which includes the uncertainties in the oscilloscope voltage traces, and the resistance temperature-calibration equations.

Case Study 1: Uncertainty of a Tube-in-Tube Heat Exchanger

Introduction to the Case Study

A heat transfer experiment was conducted by Dirker (1999). He proposed the improvement of the performance of a heat transfer system or augmentation, by increasing the heat transfer coefficients.

It is possible to improve the heat transfer by the use of two methods, namely passive and active methods. Passive methods require no direct application of power and include treated surfaces and extended surfaces. Active methods, on the other hand, require external power and include surface vibration and mechanical aids. Both methods may be employed simultaneously to achieve compound augmentation.

The effectiveness of an augmentation is dependent upon the mode of heat transfer, the type of heat exchanger as well as the pressure drop across the given device.

An efficiency improvement in commercial and residential air conditioning due to heat transfer augmentation would save on the consumption of oil and would reduce Carbon Dioxide emissions. Therefore the heat transfer augmentation is of considerable importance in various applications.

An example of heat transfer augmentation is heat pumps. There are two different types of heat exchangers that are used in conjunction with air source hot water heat pumps. One is an air heated fin-and-tube evaporator with cold refrigerant flowing on the inside of the tubes and warm air flowing over the tubes in a cross flow arrangement. The other type of heat exchangers is a water-cooled condenser, either of the tube-in-tube or fluted tube variety. Both are discussed briefly below.

Tube-in-tube heat exchangers are easy and inexpensive to manufacture, but not very effective. This inefficiency can be seen when R22 hot water heat pumps are operated close to their maximum safe condensing temperature of 55° C. The result is that it is difficult to achieve a hot water outlet temperature of 60° C and higher for the tube-in-tube heat exchangers.

Fluted tubes are difficult and expensive to manufacture, but are very effective. This efficiency can be seen when R22 hot water heat pumps are operated close to their maximum safe condensing temperature of 55° C. It is possible to achieve a hot water outlet temperature of 60° C and higher relatively easily. This is attributed to the higher heat transfer coefficient in the annulus of the fluted tubes as opposed to that of ordinary tubes in heat exchangers.

Therefore it is necessary to determine an easy to manufacture and inexpensive alternative for heat transfer augmentation in tube-in-tube heat exchangers.

Aim of the Case Study

The establishment of swirl in flowing liquids is one of the options available in heat transfer augmentation. This swirl can be obtained by inserting wires that are spiralled around the inside tube into the annulus. By variations of the wire in terms of thickness, number and angle, the intensity of the swirl can be altered. The effect of two wires spiralled symmetrically in a tube-in-tube heat exchanger is considered. The heat exchanger is of the counter flow variety with the refrigerant cooling in the annulus, as shown in the figure below.

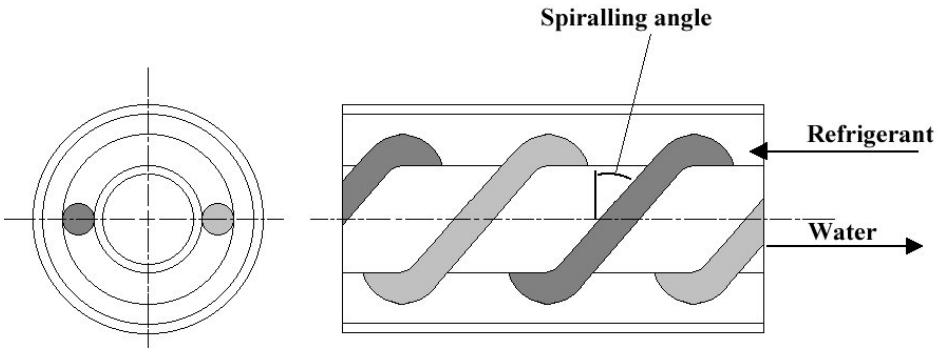

Figure 1: Schematic representation of the heat exchanger with the spiralled wire

The heat transfer characteristics are determined for a hot water heat pump, where the heating of the water occurs in above mentioned tube-in-tube heat exchanger. The heat transfer and the pressure drop characteristics of the refrigerant R22 (Monochlorodifluoromethane $CHClF_2$) during condensation are investigated.

Experimental Method of the Case Study

The tube-in-tube heat exchangers consist of two concentric soft drawn copper refrigerant tubes. The inner tubes had outer diameters of 9.53 mm and inner diameters of 8.11 mm, while the outer tubes had outer diameters of 15.9 mm and 14.3 mm. There were two copper wires of 1.7 mm diameter spiralled round the inner tube of the heat exchanger. Five heat exchangers, three with variations in the angles of the two spiralling wires in the annulus, namely 50°, 60° and 70°, one heat exchanger with only one spiralled wire at 60° and an ordinary tube-in-tube heat exchanger, were used in the investigation. The heat exchangers were 12.5 m in length and spiralled to a diameter of 400 mm.

The heat exchangers used in the experiment operated in a counter flow manner, with water flowing in the inner tube and the refrigerant R22 in the annulus in the experimental set-up shown below.

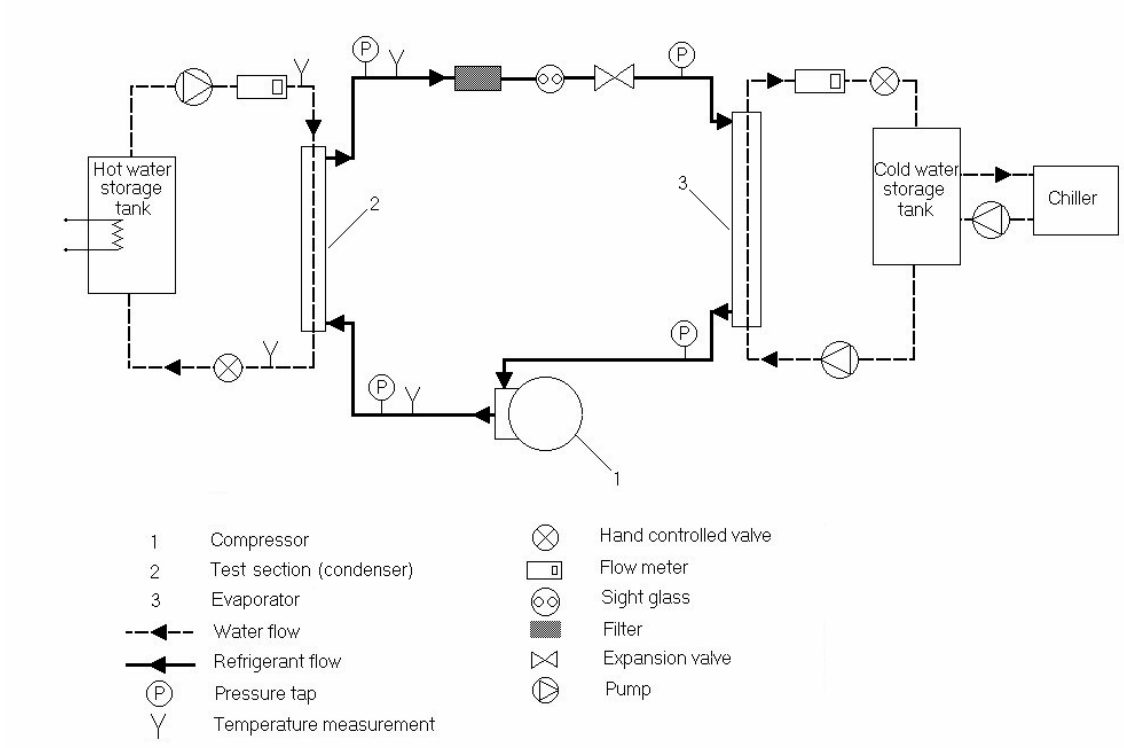

Figure 2: Schematic representation of the experimental set-up

The set-up for the experiment was a closed system that used R22 in the vapour compression cycle together with a hot water and cold water cycle. The vapour compression system consisted of a compressor, water-cooled condenser, filter, sight-glass, expansion valve and water-heated evaporator. Pressure gauges were situated before and after the evaporator. Pt 100s were installed on the tube surface of both the inlet and the outlet of the water and R22 at the condenser by the use of aluminium tape. All the external surfaces of the heat exchanger were covered with insulating

material. The water from the water storage tank was connected to a chiller, which was pumped through the evaporator of the heat exchanger.

Measurement taken from the heat exchanger yielded values from which the results could be processed.

The total heat transfer over the entire length of the condenser is determined from both the water side and the refrigerant side to yield two equations. The first equation is used for determining the heat transfer, while the second is used to estimate the refrigerant flow rate.

The average inside heat transfer coefficient of water is expressed in terms of the Reynolds number, the Prandtl number, water temperature and inside tube wall temperature by means of the Sieder and Tate correlation. This equation is valid for flow that is characterised by large property variations. C_i and P_i are correlation constants in this equation, which are dependent only on the water side heat exchange surface geometry and water flow conditions. By fixing the flow conditions of water, these constants are determined for the inside of a tube-in-tube heat exchanger. The heat exchangers under consideration were constructed in such a manner that they have the same inside tube geometry. Therefore it is possible to apply the same inside tube correlations constants to all heat exchangers.

These correlation constants of the heat exchangers are determined by the use of the Modified Wilson Plot technique. Due to the experiments with the heat exchangers operated in a counter flow arrangement with water both inside and outside the annulus, data was obtained from which the Wilson Plot functions are calculated. The inside correlation constant is calculated from the graph of linear regression of the Wilson Plots.

The correlation for the annulus side could not be found similarly since there is a difference in the annulus geometry of the heat exchangers. However, it is possible to find the relationship between the inside and outside convection heat transfer coefficients by using the overall heat transfer coefficient. The overall heat transfer coefficient is obtained from the data processed.

The results of the tube-in-tube heat exchange experiment seemed to show that there was a drastic improvement in the heat transfer.

Investigation of the Uncertainty of the Case Study

The uncertainty equations for the above tube-in-tube heat exchange experiment are based on the general equation for uncertainty by Schultz and Cole (1953).

$$U_R = \left[\sum_{i=1}^{n} \left(\frac{\partial R}{\partial Vi} U_{Vi} \right)^2 \right]^{\frac{1}{2}}$$

Equation 61: Equation for general uncertainty by Schultz and Cole

The respective equations that were used to calculate the uncertainty equation for the overall heat transfer coefficient are shown in appendix 4. The final uncertainty equation for the overall heat transfer coefficient was derived from the above formula, using the appropriate heat transfer equations and is shown below.

$$U_{U_i} = \left\{ \left(\frac{1}{A\Delta T_{LMTD}} U_{Q_w} \right)^2 + \left(\frac{Q_w}{\Delta T_{LMTD} A^2} U_A \right)^2 + \left(\frac{Q_w}{A(\Delta T_{LMTD})^2} U_{\Delta T_{LMTD}} \right)^2 \right\}^{\frac{1}{2}}$$

Equation 62: Uncertainty equation for the overall heat transfer coefficient

Using the respective uncertainty values and associative values of the errors, the uncertainty of the overall heat transfer coefficient was calculated as 5.5%, as shown in appendix 4. Since this uncertainty is in the range of 2% and 6%, the range for statistically acceptable values, the results of the case study for the overall heat transfer coefficients can be accepted as relatively valid.

The uncertainty equation for the Wilson Plots was derived by use of the general uncertainty equation of Schultz and Cole. Using the respective uncertainties and associative errors, the uncertainty of the correlation constants based on the Wilson Plots was calculated and found to be small enough to be discarded as negligible to the overall uncertainty value. Since the uncertainty of the correlation constants is extremely small, the results of the case study for the correlation constants based on the Wilson Plots can be accepted as valid.

The respective heat transfer equations that were used to derive the uncertainty equation for the convection heat transfer coefficient are shown in appendix 4. The final uncertainty equation for the convection heat transfer coefficient is shown below.

$$U_{hi} = \left\{ \left(\frac{k_i}{d_i} U_{Nui} \right)^2 + \left(\frac{Nu_i}{d_i} U_{ki} \right)^2 + \left(-\frac{Nu_i k_i}{d_i^2} U_{di} \right)^2 \right\}^{\frac{1}{2}}$$

Equation 63: Equation for the convection heat transfer coefficient

However, the calculations for the uncertainty of the convection heat transfer coefficient resulted in an uncertainty factor that was high, namely 50%. This shows

that there is too much uncertainty surrounding the experiment to deem the results valid. It can therefore be assumed that the results of this experiment cannot be relied upon.

Case Study 2: Uncertainty of a Tube-in-Tube Heat Exchanger

Introduction to the Case Study

A heat transfer experiment was conducted by Coetzee. The experiment considered the involvement of enhancement techniques in a heat exchanger.

Enhancement techniques are classified as either passive or active. Active techniques require an external power input. Examples include mechanical aids, surface vibration, fluid vibration etc. Passive techniques require no direct application of external power. Examples are treated surfaces, rough surfaces etc.

One of the common passive techniques is the use of different types of insertions. They have relatively low costs associated with them, they are easy to install and to remove.

Numerous investigations have been conducted studying the effect of different passive techniques on the augmentation of heat transfer coefficients inside horizontal tubes. However, there have been few experiments on heat transfer enhancement in the annulus of a tube-in-tube heat exchanger.

Coetzee therefore proposed the development of an experimental set-up that is used to determine the enhancement of the heat transfer and pressure characteristics of the refrigerant condensing in the inner tube of a tube-in-tube heat exchanger with water flowing in the annulus. This enabled the heat transfer correlations for water flowing inner the inner tube and the annulus to be derived.

Aim of the Case Study

The aim of this case study is to develop an experimental set-up that is used to determine the heat transfer and pressure drop characteristics of several refrigerants condensing in the inner tube of a tube-in-tube heat exchanger. These properties aid in the development of a heat transfer correlation for the single-phase heat transfer that occurs in the annulus.

Experimental Method of the Case Study

A test facility was constructed specifically to measure the in-tube condensation of pure refrigerants and refrigeration mixtures. The test facility comprised of a vapour compression refrigeration and/or heat pump system. The specifications for the system are given in the appendix.

In the analysis of tube-in-tube heat exchangers, the heat transfer coefficients for the inner tube and the annulus are both not known. In order to calculate the heat transfer coefficients, a process such as the Wilson plot technique, is used. This process is time and energy consuming.

It would be beneficial to know the heat transfer coefficient for the water side of a tube-in-tube heat exchanger. The condensing test section was used to determine this heat transfer coefficient. The test section was used as a water-to-water tube-in-tube heat exchanger. Hot water flowed through the inner tube while cold water flowed through the annulus in a counter flow manner. RTDs were used to measure the temperature. Measurements were taken over the full range of water flows necessary for future experimentation. In each experiment the temperatures of the hot water inlet and outlet and the cold water inlet and outlet were measured. The hot and cold water mass flow rates were recorded. A program was written to calculate the mean temperature differences as well as the specific heat capacity for each flow. The energy balance was calculated from the heat rejected from the hot water and the heat gained by the cold water. This ensures the Wilson plot techniques is used only once and the heat transfer coefficient for the water side as well as the Nusselt number correlation is known.

The Wilson Plots, using equations from Seider and Tate (1996) enable the correlations for the inner tube and annulus to be derived. The complete mathematical equations as well as the modified equations can be found in the appendix.

Once the Wilson plot technique had been completed for each heat exchanger, the heat transfer rate can be predicted using the calculated inner and annulus heat transfer coefficients. The heat transfer rate was compared with the measured heat transfer rate. The heat transfer correlations could then be calculated and the results are shown in the appendix.

From the developed correlation it is possible to predict the temperatures of the water inlet and outlet as well as the water mass flow rate quite accurately.

Investigation of the Uncertainty of the Case Study

The uncertainty equations for the above tube-in-tube heat exchange experiment were derived, based on the general equation for uncertainty by Schultz and Cole (1953), shown below.

$$U_R = \left[\sum_{i=1}^{n} \left(\frac{\partial R}{\partial Vi} U_{Vi} \right)^2 \right]^{\frac{1}{2}}$$

Equation 64: General uncertainty equation by Schultz and Cole (1953)

The respective values and equations for the uncertainty are shown in appendix 5. The uncertainty of the overall heat transfer coefficient, the uncertainty of the correlation constants based on the Wilson Plots and the uncertainty of the convection heat transfer coefficient were calculated.

The uncertainty for the overall heat transfer coefficient was derived from the above formula from the appropriate heat transfer equations. The uncertainty equation for the overall heat transfer coefficient is given by the equation below.

$$U_U = \sqrt{\left(\frac{1}{A\Delta T_{LMTD}} U_Q \right)^2 + \left(\frac{Q}{A^2 \Delta T_{LMTD}} U_A \right)^2 + \left(\frac{Q}{A\Delta T_{LMTD}^2} U_{\Delta T_{LMTD}} \right)^2}$$

Using the respective uncertainties and associative errors, the uncertainty of the overall heat transfer coefficient was calculated as 59%. Since this uncertainty is definitely above the accepted range of between 2% and 6%, the results of the case study for the overall heat transfer coefficients cannot be accepted as relatively valid without further investigation. The uncertainty is too high for the results to be accepted without further experimentation.

Using the respective uncertainties and associative errors, the uncertainty of the Reynolds numbers for the inner tube and annulus side were calculated as 0.0004% and 0.001% respectively. Since this uncertainty is extremely small, the results of the case study for the Reynolds numbers can be accepted as completely valid.

Use was made of the general uncertainty equation by Schultz and Cole (1953) in deriving the equation for the uncertainty of the convection heat transfer coefficient for the inner tube of a tube-in-tube heat exchanger, as shown below.

$$U_{h_i} = \left\{ \begin{array}{l} \left(\frac{k_i}{d_i} \mathrm{Re}_i^{0.8} \mathrm{Pr}_i^{\frac{1}{3}} \left(\frac{m}{m_w} \right)_i^{0.14} U_{C_i} \right)^2 + \left(C_i \frac{1}{d_i} \mathrm{Re}_i^{0.8} \mathrm{Pr}_i^{\frac{1}{3}} \left(\frac{m}{m_w} \right)_i^{0.14} U_{k_i} \right)^2 \\ + \left(C_i \frac{k_i}{d_i^2} \mathrm{Re}_i^{0.8} \mathrm{Pr}_i^{\frac{1}{3}} \left(\frac{m}{m_w} \right)_i^{0.14} U_{d_i} \right)^2 + \left(0.8 C_i \frac{k_i}{d_i} \mathrm{Re}_i^{-0.2} \mathrm{Pr}_i^{\frac{1}{3}} \left(\frac{m}{m_w} \right)_i^{0.14} U_{\mathrm{Re}_i} \right)^2 \\ + \left(\frac{1}{3} C_i \frac{k_i}{d_i} \mathrm{Re}_i^{0.8} \mathrm{Pr}_i^{-\frac{2}{3}} \left(\frac{m}{m_w} \right)_i^{0.14} U_{\mathrm{Pr}_i} \right)^2 + \left(0.14 C_i \frac{k_i}{d_i} \mathrm{Re}_i^{0.8} \mathrm{Pr}_i^{\frac{1}{3}} \left(\frac{m^{-0.86}}{m_w^{0.14}} \right)_i U_{m_i} \right)^2 \\ + \left(-0.14 C_i \frac{k_i}{d_i} \mathrm{Re}_i^{0.8} \mathrm{Pr}_i^{\frac{1}{3}} \left(\frac{m^{0.14}}{m_w^{1.14}} \right)_i U_{m_w} \right)^2 \end{array} \right\}^{\frac{1}{2}}$$

Equation 65: Equation of the convection heat transfer coefficient

The calculations for the uncertainty of the convection heat transfer coefficient resulted in an uncertainty factor that was slightly high, namely 59%. Since this uncertainty falls well outside the 2% to 5% interval, it cannot be accepted as a valid set of results.

Use was made of the general uncertainty equation by Schultz and Cole (1953) in deriving the equation for the uncertainty of the convection heat transfer coefficient for the annulus of a tube -in-tube heat exchanger, as shown below.

$$U_{h_o} = \left\{ \left(-\left(\frac{1}{U_a} - \frac{1}{h_i}\frac{A_a}{A_i} - \frac{A_a \ln\left(\frac{D_o}{D_i}\right)}{2pk_{Cu}L} \right)^{-2} \right)^2 \left(\begin{array}{l} \left(\frac{1}{U_a^2}U_{U_a}\right)^2 + \left(-\frac{1}{h_i^2}\frac{A_a}{A_i}U_{h_i}\right)^2 \\ + \left(-\frac{1}{h_i}\frac{1}{A_i}U_{A_a}\right)^2 + \left(-\frac{1}{h_i}\frac{A_a}{A_i^2}U_{A_i}\right)^2 \\ + \left(-\frac{\ln\left(\frac{D_o}{D_i}\right)}{2pk_{Cu}L}U_{A_a}\right)^2 + \left(-\frac{A_a\left(\frac{D_i}{D_o}\right)\left(\frac{1}{D_i}\right)}{2pk_{Cu}L}U_{D_o}\right)^2 \\ + \left(-\frac{A_a\left(\frac{D_i}{D_o}\right)\left(\frac{D_o}{D_i^2}\right)}{2pk_{Cu}L}U_{D_i}\right)^2 + \left(\frac{A_a \ln\left(\frac{D_o}{D_i}\right)}{2pk_{Cu}^2L}U_{k_{Cu}}\right)^2 \\ + \left(\frac{A_a \ln\left(\frac{D_o}{D_i}\right)}{2pk_{Cu}L^2}U_L\right)^2 \end{array} \right)^{\frac{1}{2}} \right\}$$

Equation 66: Equation of the uncertainty for the convection heat transfer coefficient

The calculations for the uncertainty of the convection heat transfer coefficient resulted in an uncertainty factor that was slightly high, namely 5.4%. Since this uncertainty falls inside the 2% to 5% interval, the results are accepted as valid and one can rely on such results.

Conclusion

Statistical uncertainty in all areas but with specific regard to heat transfer was considered and researched, including uncertainties based on objective and subjective uncertainty.

The literature research of statistical uncertainty was completed and the respective uncertainty equations were considered and thereby analysed with regard to their correct application in heat transfer systems.

Various studies have been considered for which the associated uncertainties have been determined. It has been found that the general formula by Cole (1979) yields equations from which it is possible to determine an uncertainty value. The results that were obtained seemed to correlate with the subjective uncertainty given by the experimenter.

References

Books

Adomian G, (1980), Stochastic system analysis, *Applied stochastic processes, Academic press, New york*, pp. 1-17

Box GEP and Draper NR, (1987), Empirical model building and response surfaces, *John wiley & sons, New york*

Box GEP, Hunter WG and Hunter SJ, (1978), Statistics for experimenters: and introduction to design, data analysis and model building, *John wiley & sons, New york*

Ghanem RG and Spanos PD, (1991), Springer-verlag, New york, *Stochastic finite elements: a spectral approach*

Papoulis A, (1991), Direct incorporation of uncertainty in chemical and environmental engineering systems *McGraw-Hill, New York*

Tomovi'c R and Vukobratovi'c M, (1972), General sensitivity theory, *Elsevier, New york*

Villadsen J and Michelsen ML, (1978), Solution of differential equation models by polynomial approximation, *Prentice-hall, englewood cliffs, New jersey*

Internet Articles

http://members.tripod.com/sxmishra/pra.htm

http://physics.nist.gov/cuu/Uncertainty/basic.html

http://physics.nist.gov/cuu/Uncertainty/basic.html

http://webster.me.ume.edu/courses/me3701/classnotes/error/error.html

http://webster.me.umn.edu/course/me3701/classnotes/error/error.html

http://www.addlinks.es/product/mathsoft/mathcad/apps/asee.html

http://www.bib.wau.nl/casme/modguide.html

http://www.eos.uoguelph.ca/webfiles/james/Wkbk02/sld001.htm

http://www.gum.dk/e-wb-princip/gw_principle.html

http://www.ite.tuwien.ac.at/diss/khalil/diss/node25.html

Sandio National Laboratories

Articles

Agrawal KN, Kumar A, Behabadi MAA and Varma HK, (1998), Heat transfer augmentation by coiled wire inserts during forced convection condensation of R22 inside horizontal tubes, *International journal of multiphase flow*, Vol. 24, pp. 635-650

Bremser W, (2001), Measurement uncertainty in testing exemplified: an application to gas metering, *Test and measurement 2001 workshop / conference*, September 9 – 12 2001, South Africa

Doughtery EP and Rabitz H, (1979), A computational algorithm for the Green's function method of sensitivity analysis in chemical kinetics, *International journal of chemical kinetics*, Vol. 11, pp. 1237-1249

Doughtery EP, Hwang JT and Rabitz H, (1979), Further developments and applications of the green's function method of sensitivity analysis in chemical kinetics, *International journal of chemical kinetics*, Vol. 71 (4), pp. 1794-1808

Dunker AM, (1984), The decoupled direct method for calculating sensitivity coefficients in chemical kinetics, *Journal of chemical physics*, Vol. 51(5), pp. 2385-2393

Hodges James S, (1987), Uncertainty, policy analysis and statistics, *Statistical science*, Vol. 2, no. 3, pp. 259-291

Iman RL and Conover WJ, (1980), Small sample sensitivity analysis techniques for computational models with an application to risk assessment, *Communications in statistics part a - theory and methods*, Vol. 17, pp. 1749-1842

Kline SJ and McClintock FA, (1953), Describing uncertainties in single-sample experiments, M*echanical engineering*, Vol. 75, pp. 3-8

Koley NI and Hofer E, (1996), Uncertainty and sensitivity analysis of a post-experiment simulation of non-explosive melt-water interaction, *Experimental thermal and fluid science*, Vol. 13, pp. 98-116

Loh WL, (1996), On Latin hypercube sampling, *Annals of statistics*, Vol. 24(5), pp. 2058-2080

Mosteller F and Youtz C, (1990), Quantifying probabilistic expressions, *Statistical Science*; Vol. 5; pp. 1-34

Müller N, (2001), Introducing the concept of uncertainty of measurement in testing in association with the application of the standard ISO/IEC 17025, *Test and measurement 2001 workshop / conference*, September 9 – 12 2001, South Africa

Naidoo K, (2001), Identifying and dealing with measurement uncertainty in the process industry, *Test and measurement 2001 workshop / conference*, September 9 – 12 2001, South Africa

Pearson K, (1902), On the mathematical theory of errors in judgement with special reference to the personal equation, *Philosophical transactions of the royal society of London*, series A, Vol. 198, pp. 235-299

Prokopakis GJ, (1993), The decoupled direct method for calculating sensitivity coefficients in chemical kinetics, *Applied mathematical modelling*, Vol. 17(9), pp. 499-503

Schultz RR and Cole R, (1979), Uncertainty analysis in boiling nucleation, *AIChE Symposium Sr.*, Vol. 75, no. 189, pp. 32-38

Schultz RR, Kasturirangan S and Cole R, (1975), Experimental studies of incipient vapour nucleation, *The Canadian journal of chemical engineering*, Vol. 35, pp. 408-413

Shafer G, (1987), Probability judgement in artificial intelligence and expert systems, *Statistical science*, Vol. 2, pp. 1-44

Stein M, (1987), Large sample properties of simulations using latin hypercube sampling, *Technometrics*, Vol. 29(2), pp. 143-151

Tavener JP, (2001), Uncertainties: a bar to progress, *Test and measurement 2001 workshop / conference*, September 9 – 12 2001, South Africa

Theses

Barrett TS, (1996), Probabilistic design for rotordynamic simulations using receptance-based re-analysis, *PhD thesis, Texas A&M University*

Coetzee S, (2000), The development of an experimental set-up to investigate heat transfer enhancement in tube-in-tube heat exchangers, *MEng thesis, Rand Afrikaans University*

Dirker J, (1999), Heat transfer augmentation during condensation of r-22 with two spiralled wires in the annulus of a coiled tube-in-tube heat exchanger for hot water heat pumps, *BSc thesis, Rand Afrikaans University*

Tatang MA, (1992), Combined stochastic and deterministic approach in solving stochastic differential equations, *Master's thesis Carnegie Mellon university*

Tatang MA, (1995), Direct incorporation of uncertainty in chemical and environmental engineering systems, *PhD thesis Massachusetts Institute of Technology*

Appendix 1: Statistical Theorems

There are a number of statistical theorems that apply to the calculation of uncertainty. They are discussed below.

Theorem 1

If R is a linear function of n independent variables, and if the maximum deviation of the Ith variable from its mean is $(\pm\delta v_I)_{max}$, then the maximum deviation from its own mean value is given as:

$$dR = \left|\frac{dR}{dv_1}dv_{1\max}\right| + \left|\frac{dR}{dv_2}dv_{2\max}\right| + \ldots + \left|\frac{dR}{dv_n}dv_{n\max}\right|$$

Equation 67: Maximum deviation of a result

The above equation of maximum deviation may be used as an approximation for calculating the uncertainty interval in the result by simply substituting w_I for v_I as shown below.

$$dR = \left|\frac{dR}{dw_1}dw_{1\max}\right| + \left|\frac{dR}{dw_2}dw_{2\max}\right| + \ldots + \left|\frac{dR}{dw_n}dw_{n\max}\right|$$

Equation 68: Linear equation of uncertainty

The above equation is referred to as the linear equation. If it is used, the odds on the uncertainty interval in the result will be much higher than the odds used in the variables. This is because the errors in each variable can have a range of values and it is very unlikely that all of them will have the most adverse values at the same time.

Theorem 2

If R is a linear function of n independent variables, each of which is distributed with a standard deviation σ_I, then the standard deviation of R is given by:

$$s_R = \left[\left(\frac{\partial R}{\partial v_1}\right)^2 s_1^2 + \left(\frac{\partial R}{\partial v_2}\right)^2 s_2^2 + \ldots + \left(\frac{\partial R}{\partial v_n}\right)^2 s_n^2\right]$$

Equation 69: Standard deviation of a result

However, it can be shown that the best measure of uncertainty is neither the maximum value (theorem 1) nor the standard deviation (theorem 2), but rather to use an interval based on certain odds. Theorem 3 applies to the latter case. Theorem 3 is used where the variables are distributed normally and the resulting distribution is also normal.

Theorem 3

If R is a linear function of n independent variables, each of which is normally distributed, then the relation between the interval for the variables w_i, and the interval for the result w_R, which gives the same odds for each of the variables and for the result is:

$$w_R = \left[\left(\frac{\partial R}{\partial v_1} w_1 \right)^2 + \left(\frac{\partial R}{\partial v_2} w_2 \right)^2 + \ldots + \left(\frac{\partial R}{\partial v_n} w_n \right)^2 \right]$$

Equation 70: Approximation of the uncertainty interval

The above equation may be used as an approximation for calculating the uncertainty interval in the result. This equation is referred to as the second-power equation.

Appendix 2: Statistical Terminology

- Measurement equation

The case of interest is where the quantity Y being measured, called the *measurand*, is not measured directly, but is determined from *N* other quantities $X_1, X_2, \ldots, X_N$ through a functional relation *f*, often called the measurement equation.

$$Y = f(X_1, X_2, \cdots, X_N)$$

Equation 71: Measurement equation

Included among the quantities X_i are corrections or correction factors, as well as quantities that take into account other sources of variability, such as different observers, instruments, samples, laboratories and times at which observations are made. Therefore the function f of the above equation should express not simply a physical law, but a measurement process, and in particular, should contain all quantities that can contribute a significant uncertainty to the measurement result.

An estimate of the measurand or output quantity Y denoted by y, is obtained from the above equation using input estimates $x_1, x_2, \ldots, x_N$ for the values of the N input quantities $X_1, X_2, \ldots, X_N$. Therefore the output estimate y, which is the result of the measurement, is give by the following equation.

$$y = f(x_1, x_2, \cdots, x_N)$$

Equation 72: Output estimate

Appendix 3: Statistical functions

The normalised probability density function (*pdf*) $f(x)$ of a random variable is defined as follows.

$$\int_{-\infty}^{\infty} f(x) = 1$$

Equation 73: Normalised probability density function

The form of $f(x)$ as shown above depends on the nature of the variation of the random variable. In particular, for a uniform deviate random number whose value is selected from the range [0,1] with each number in that range having equal probability of occurrence:

$f(x) = 1$ for $0<x<1$
$f(x) = 0$

Given a random variable x with a *pdf* $f(x)$, its mean or average (m), and its variance which is the square of its standard deviation s are given by:

$$m = \int_{-\infty}^{\infty} x f(x) dx$$

Equation 74: Mean

$$s^2 = \int_{-\infty}^{\infty} (x-m)^2 f(x) dx$$

Equation 75: Standard deviation

Appendix 4: Case 1

Uncertainty Equations for the Overall Heat Transfer Coefficient in A Tube-in-Tube Heat Exchanger

The experiment was conducted by Dirker (1999) using R22 and water in a tube-in-tube heat exchange system. Using the values from the experiment, the uncertainty for the overall heat transfer coefficient was calculated.

The mass flow rate for this experiment is given by the following equation.

$$\dot{m} = rq$$

Equation 76: Mass flow rate

Therefore the uncertainty associated with the mass flow rate can be derived based on the Cole equation (1953).

$$U_{\dot{m}} = \sqrt{\sum_{i=1}^{n}\left(\frac{\partial \dot{m}}{\partial V_i}U_{Vi}\right)^2} = \sqrt{\left(qU_r\right)^2 + \left(rU_q\right)^2}$$

Equation 77: Uncertianty of the mass flow rate

The uncertainties present in the above equation are further derived by use of the Cole formula (1953). Each of them is discussed separately. The heat transfer for the experiment is given by the following equation.

$$Q_w = \dot{m}C_p\Delta T$$

Equation 78: Heat transfer equation

Therefore the uncertainty for the change in temperature must also be derived as shown below.

$$\Delta T = T_{wi} - T_{wo}$$

Equation 79: Change in temperature

Since the uncertainties of the change in temperature of the inside wall and the outside wall are the same, and the uncertainty of the C_p value is approximately zero, the uncertainty for the heat transfer can be calculated, using appropriate values.

$$U_{Q_w} = \sqrt{\sum_{i=1}^{n}\left(\frac{\partial Q_w}{\partial V_i}U_{Vi}\right)^2}$$

Equation 80: Uncertainty of the heat transfer

$$U_{Q_w} = \left\{ \left(C_p \Delta T U_{\dot{m}} \right)^2 + \left(\dot{m} C_p \sqrt{2 (U_{Tw})^2} \right)^2 \right\}^{\frac{1}{2}}$$

Equation 81: Derived uncertainty of the heat transfer

Substituting the above uncertainty for the mass flow rate into the uncertainty equation for the uncertainty of heat transfer enables a more complete equation to be obtained.

$$U_{Q_w} = \left\{ (C_p \Delta T)^2 \left[(q U_r)^2 + (r U_q)^2 \right] + \left(\dot{m} C_p \right)^2 2 U_{Tw} \right\}^{\frac{1}{2}}$$

Equation 82: Total heat transfer uncertainty

The other factor that affects the uncertainty equation is the logarithmic mean temperature difference.

The logarithmic temperature difference is calculated from the following equation.

$$\Delta T_{LMTD} = \frac{(T_{wi} - T_{ro}) - (T_{wo} - T_{ri})}{\ln \left(\frac{T_{wi} - T_{ro}}{T_w - T_{ri}} \right)}$$

Equation 83: Logarithmic mean temperature difference

The uncertainty for the logarithmic mean temperature difference was derived using Cole's formula (1953), which is shown below.

$$U_{\Delta T_{LMTD}} = \left\{ \begin{array}{l} \left(\frac{\left[\left(\ln\left(\frac{T_{wi}-T_{ro}}{T_{wo}-T_{ri}} \right) \right) - \left(\frac{1}{\frac{T_{wi}-T_{ro}}{T_{wo}-T_{ri}}} \right) \left(T_{wi}-T_{ri}-T_{wo}+T_{ri} \right) \left(\frac{1}{T_{wo}-T_{ri}} \right) \right] U_{Twi}}{\left[\ln\left(\frac{T_{wi}-T_{ro}}{T_{wo}-T_{ri}} \right) \right]^2} \right)^2 \\ + \left(\frac{\left[-\left(\ln\left(\frac{T_{wi}-T_{ro}}{T_{wo}-T_{ri}} \right) \right) + \left(\frac{1}{\frac{T_{wi}-T_{ro}}{T_{wo}-T_{ri}}} \right) \left(T_{wi}-T_{ro}-T_{wo}+T_{ri} \right) \left(\frac{1}{T_{wi}-T_{ro}} \right) \right] U_{Tro}}{\left[\ln\left(\frac{T_{wi}-T_{ro}}{T_{wo}-T_{ri}} \right) \right]^2} \right)^2 \\ + \left(\frac{\left[-\left(\ln\left(\frac{T_{wi}-T_{ro}}{T_{wo}-T_{ri}} \right) \right) - \left(\frac{1}{\frac{T_{wi}-T_{ro}}{T_{wo}-T_{ri}}} \right) \left(T_{wi}-T_{ro}-T_{wo}+T_{ri} \right) \left(T_{wi}-T_{ro} \right) \left(\frac{1}{\left(T_{wo}-T_{ri} \right)^2} \right) \left(\frac{1}{T_{wo}-T_{ri}} \right) \right] U_{Two}}{\left[\ln\left(\frac{T_{wi}-T_{ro}}{T_{wo}-T_{ri}} \right) \right]^2} \right)^2 \\ + \left(\frac{\left[\left(\ln\left(\frac{T_{wi}-T_{ro}}{T_{wo}-T_{ri}} \right) \right) - \left(\frac{1}{\frac{T_{wi}-T_{ro}}{T_{wo}-T_{ri}}} \right) \left(T_{wi}-T_{ro}-T_{wo}+T_{ri} \right) \left(T_{wi}-T_{ro} \right) \left(\frac{1}{\left(T_{wo}-T_{ri} \right)^2} \right) \left(\frac{1}{T_{wo}-T_{ri}} \right) \right] U_{Tri}}{\left[\ln\left(\frac{T_{wi}-T_{ro}}{T_{wo}-T_{ri}} \right) \right]^2} \right)^2 \end{array} \right\}^{\frac{1}{2}}$$

Equation 84: Uncertainty of the logarithmic mean temperature difference

The other uncertainty factor that needs to be derived is the heat transfer area. The heat transfer area can be calculated as follows.

$$A = pDL$$

Equation 85: Heat transfer area

Therefore the respective uncertainty would be as follows.

$$U_A = \sqrt{(pLU_D)^2 + (pDU_L)^2}$$

Equation 86: Uncertainty of heat transfer area

The heat transfer coefficient can be calculated by the following equation.

$$U_i = \frac{Q_w}{A\Delta T_{LMTD}}$$

Equation 87: Heat transfer coefficient

Therefore the uncertainty for the heat transfer coefficient is shown below.

$$U_{U_i} = \left\{ \left(\frac{1}{A\Delta T_{LMTD}} U_{Q_w} \right)^2 + \left(\frac{Q_w}{\Delta T_{LMTD} A^2} U_A \right)^2 + \left(\frac{Q_w}{A(\Delta T_{LMTD})^2} U_{\Delta T_{LMTD}} \right)^2 \right\}^{\frac{1}{2}}$$

Equation 88: Uncertainty for the overall heat transfer coefficient

After substituting all the uncertainties that have been derived in this appendix into the above equation, the complete uncertainty for the heat transfer coefficient can be determined. It can be seen that the uncertainty factor for the overall heat transfer coefficient is 0.038 or 3.8%.

Uncertainty Equations for the Convection Heat Transfer Coefficient in A Tube-in-Tube Heat Exchanger

The experiment was conducted by Dirker (1999) using R22 and water in a tube-in-tube heat exchange system. Using the values from the experiment, the uncertainty for the convection heat transfer coefficient was calculated.

The convection heat transfer coefficient is calculated form the equation formed by equating the Nusselt number and the inside heat transfer correlation.

$$Nu_i = \frac{h_i d_i}{k_i} = C_i \,\mathrm{Re}_i^{\,Pi}\,\mathrm{Pr}_i^{\frac{1}{3}}\left(\frac{T}{T_{wall}}\right)_i^{0.14}$$

Equation 89: Nusseldt number

However, in order to find this uncertainty, the uncertainty for C_i, the constant for inside heat transfer correlation, must be calculated. This correlation value is found by applying the Wilson Plot techniques. The values of the Wilson Plots lead to the following equation being formed.

$$\frac{1}{U_o} = \frac{\frac{k_i}{d_i}\mathrm{Re}_i^{0.8}\,\mathrm{Pr}_i^{\frac{1}{3}}\left(\frac{T}{T_{wall}}\right)_i^{0.14}}{\left(\frac{1}{C_o}\right)\frac{k_o}{d_o}\mathrm{Re}_o^{Pi}\,\mathrm{Pr}_o^{\frac{1}{3}}\left(\frac{T}{T_{wall}}\right)_o^{0.14}} + \frac{A_o}{A_i C_i}$$

Equation 90: Linear regression line based on the Wilson Plots

The uncertainty for the constant for inside heat transfer was derived and then calculated as shown in the tables. The uncertainty value was calculated as 2.02E-06 or 0.000202%.

The uncertainty equation for the Nusselt number is then derived as shown below.

$$U_{Nui} = \left\{ \begin{array}{l} \left(\mathrm{Re}_i^{Pi}\,\mathrm{Pr}_i^{\frac{1}{3}}\left(\frac{T}{T_{wall}}\right)^{0.14} U_{Ci}\right)^2 + \left(P_i C_i \mathrm{Re}_i^{Pi-1}\left(\frac{T}{T_{wall}}\right)^{0.14} U_{\mathrm{Re}}\right)^2 + \left(C_i \mathrm{Re}_i^{Pi} \ln \mathrm{Re}\,\mathrm{Pr}_i^{\frac{1}{3}}\left(\frac{T}{T_{wall}}\right)^{0.14} U_{Pi}\right)^2 \\ + \left(\frac{1}{3} C_i \mathrm{Re}_i^{Pi}\,\mathrm{Pr}_i^{-\frac{2}{3}}\left(\frac{T}{T_{wall}}\right)^{0.14} U_{\mathrm{Pr}i}\right)^2 + \left(0.14 C_i \mathrm{Re}_i^{Pi}\,\mathrm{Pr}_i^{\frac{1}{3}} T^{-0.86} T_{wall}^{-0.14} U_{Ti}\right)^2 \\ + \left(-0.14 C_i \mathrm{Re}_i^{Pi}\,\mathrm{Pr}_i^{\frac{1}{3}} T^{0.14} T_{wall}^{-1.14} U_{Twall}\right)^2 \end{array} \right\}^{\frac{1}{2}}$$

Equation 91: Uncertainty equation for the Nusselt number

The uncertainty equation for the convection heat transfer coefficient, shown below, is derived as well as the associated uncertainty factor.

$$U_{hi} = \left\{ \left(\frac{k_i}{d_i} U_{Nui} \right)^2 + \left(\frac{Nu_i}{d_i} U_{ki} \right)^2 + \left(-\frac{Nu_i k_i}{d_i^2} U_{di} \right)^2 \right\}^{\frac{1}{2}}$$

Equation 92: Uncertainty equation for the convection heat transfer coefficient

Uncertainty Equations for the Overall Heat Transfer Coefficient in A Tube-in-Tube Heat Exchanger

The experiment was conducted by Coetzee (2000) using R22 and water in a tube-in-tube heat exchange system. Using the values from the experiment, the uncertainty for the overall heat transfer coefficient was calculated.

The overall heat transfer is given in the equation below.

$$U = \frac{Q}{A\Delta T_{LMTD}}$$

Equation 93: Overall heat transfer coefficient

Values and Results for the Overall Heat Transfer Coefficient in A Tube-in-Tube Heat Exchanger

	1
L	12.5
Dii	0.00811
A	0.318478955
UL	0.01
Ud	0.00001
	0.000392699
	0.000254783
UA	0.00046811
Twi	53.5
Tro	53.1
Two	40
Twi	12.1
	-27.5
(/)	0.014336918
(ln)	-4.244917421
Tlmtd	6.478335683
Ut	0.1
(ln)	-4.244917421
(1/())	69.75
(add)	-27.5
(ln2)	18.01932391
(Tw-Tr)	27.9
(1/)	0.035842294
	64.50508258
(Twi)	3.579772632
(*.1)	0.357977263
(Utwi)	0.128147721
	-64.50508258
(Tro)	-3.579772632
(*.1)	-0.357977263
(Utro)	0.128147721
(Tw-Tr)	0.4
(tTw-Tr2)	0.00128467
	4.280245847
(Two)	0.237536429
(*.1)	0.023753643
(Utwo)	0.000564236
	-4.280245847

(Tri)	-0.237536429
(*.1)	-0.023753643
(Utri)	0.000564236
Utlmtd	0.1
m	0.083
Cpw	4179.98
(t-t)	-13.5
Qw	-4683.66759
p	1.000E+03
q	300
Up	1
Uq	1
Um	0.001
UQ	44.01080938
U	1942.6
(Q)	21.33119411
(T)	-35.04115634
UU	41.02319441
uncertainty	0.978882326

Values and Results for the Reynolds Number in the Annulus of a Tube-in-Tube Heat Exchanger

V	0.311
Dh	0.00477
u	1.80E-05
p	1.000E+03
Up	1
Uv	0.001
Udh	0.00001
Uu	1.80E-07
Up	82.415
Uv	265
Uu	8.24E+02
Ure	869.6208684
Reo	1949
uncertainty	0.553811766

Values and Results for the Reynolds Number in the Inner Tube of a Tube-in-Tube Heat Exchanger

V	1.613
Dh	0.00477
u	1.80E-05
p	1000
Up	1
Uv	0.001
Udh	0.00001
Uu	1.80E-07
Up	427.445
Uv	265
Uu	4274.45
Ure	4303.935063
Rei	22386
uncertainty	0.807739879

Values and Results for the Constant for the Inside Heat Transfer Coefficient in a Tube-in-Tube Heat Exchanger

di	0.00811
L	12.5
Ai	**0.318478955**
Udi	0.00001
Ul	0.01
Uai	**0.019818274**
do	0.00953
L	12.5
Ao	**0.374242225**
Udo	0.00001
Ul	0.01
Uao	**0.019818898**
Uo	**1942.6**
Uuo	**0.055545588**
ko	**621.28**
Uko	**0.0002**
do	**0.00953**
Udo	**0.00001**
Reo	1949
Reo0.8	**428.400009**
Ureo	**1**
Pro	**5.1072**
Pro1/3	**1.722110264**
Upro	**0.1**
To	**53.1**
Uto	**0.1**
Ti	319.9
Uti	0.1
To	12.1
Q	4758
hoe	551.9316
Ao	0.374242225
Twallo	**309.811**
Uto	0.1

Uq	0.088653302
Uhoe	0.0001
Uao	0.019818898
	0.01
	1.8421E-07 1.74182E-11
	1.488079782
Utwallo	**0.1**
di	0.008111
l	12.5
Ai	**0.318518225**
Udi	0.00001
Ul	0.01
Uai	**0.019818275**
do	0.00953
l	12.5
Ao	**0.374242225**
1/Ao	**2.672066201**
Udo	0.00001
Ul	0.01
Uao	**0.019818898**
Uo	**1942.6**
1/Uo	**0.000514774**
Uuo	**0.055545588**
ki	**639.88**
Uki	**0.0002**
di **Udi**	**0.008111** **0.00001**
Rei	22386
Rei0.8	**3019.820991**
Urei	**1**
Pri	3.7778
Pri1/3	**1.557445963**
Upri	**1.547196278**
Ti	**53.5**
Uti	**0.1**
Ti	53.5

Q	4702
hoe	551.9316
Ai	0.318478955
Twall	**330.514**
Uto	0.1
Uq	0.088653302
Uhoe	0.0001
Uai	0.019818274
	0.01
	2.54365E-07
	9.95253E-13
	2.77078891
Utwalli	**0.1**
Co	**0.000114**
outside	
1st term	125979.3569
2nd term	0.00335301
final	6.30088E-11
Ai	
1st term	395516.95
2nd term	0.010526901
final	7838.463355
	2.4393E-13
Ao	
1st term	336625.1817
2nd term	0.008959464
final	-6671.539965
	1.76707E-13
Uo	
1st term	64.85089927
2nd term	1.72604E-06
final	-3.60218122
	5.1515E-20
ki	
1st term	196.8796601
2nd term	5.24006E-06
final	0.039375931
	6.15551E-24
di	
1st term	15531914.3

2nd term	0.413390422
final	-155.3191389
	9.57751E-17
Rei	
1st term	4.502076545
2nd term	1.19825E-07
final	4.502076425
	8.04689E-20
Pri	
1st term	11115.76022
2nd term	0.000295852
final	17198.26239
	1.17428E-12
Ti	
1st term	329.6656069
2nd term	8.77423E-06
final	32.96655981
	4.31469E-18
Twalli	
1st term	0.715279243
2nd term	1.90375E-08
final	-0.071527922
	2.0312E-23
ko	
1st term	125979.3569
2nd term	7.08387E-16
3rd term	60475.1845
final	1.07939E-09
	4.62549E-39
do	
1st term	125979.3569
2nd term	7.08387E-16
3rd term	3942499751
final	3.51837E-06
	4.91458E-32
Reo	
1st term	125979.3569
2nd term	7.08387E-16
3rd term	15422.07188
final	1.3763E-06
	7.52018E-33

Pro	
1st term	125979.3569
2nd term	7.08387E-16
3rd term	2452225.788
final	2.18842E-05
	1.90136E-30
To	
1st term	125979.3569
2nd term	7.08387E-16
3rd term	21143.0254
final	1.88685E-07
	1.41344E-34
Twallo	
1st term	-125979.3569
2nd term	7.08387E-16
3rd term	16978.36154
final	-1.51519E-07
	9.11454E-35
Ci	**0.016295**
Uci	**1.26294E-06**
error	0.999922495

Values and Results for the Nusselt Number in a Tube-in-Tube Heat Exchanger

Ci	0.016295
Uci	1.26294E-06
Rei	22386
Urei	0.807739879
Pi	1.66594
Upi	0.019979793
Pri	3.7778
Pri1/3	1.557445963
Upri	0.0001
Ti	53.5
Uti	0.1
Twall	33.514
Utwall	0.1
1st term	1374.253047
2nd term	340.7947505
3rd term	9162147436
4th term	17.8110562
5th term	15666.01779
6th term	39922.12487
final	95719.40638
Nui	478305.1467
uncertainty	0.79987795

Values and Results for the Convection Heat Transfer Coefficient in a Tube-in-Tube Heat Exchanger

Nui	478305.1467
Unui	0.79987795
ki	639.88
Uki	0.0002
di	0.00811
Udi	0.00001
1st term	3982931285
2nd term	139132436.6
final	64203.29993
hi	6480.511
uncertainty	8.907135398

Appendix 5: Case 2

Uncertainty Equations for the Overall Heat Transfer Coefficient in A Tube-in-Tube Heat Exchanger

The experiment was conducted by Coetzee (2000) using R22 and water in a tube-in-tube heat exchange system. Using the values from the experiment, the uncertainty for the overall heat transfer coefficient was calculated.

The overall heat transfer is given in the equation below.

$$U = \frac{Q}{A\Delta T_{LMTD}}$$

Equation 94: Overall heat transfer coefficient

The equation for the uncertainty of the overall heat transfer coefficient, derived from the general uncertainty equation by Cole (1953), is given below.

$$U_U = \sqrt{\left(\frac{1}{A\Delta T_{LMTD}} U_Q\right)^2 + \left(\frac{Q}{A^2 \Delta T_{LMTD}} U_A\right)^2 + \left(\frac{Q}{A\Delta T_{LMTD}^2} U_{\Delta T_{LMTD}}\right)^2}$$

Equation 95: Uncertainty for the overall heat transfer coefficient

Uncertainty Equations for the Heat Transfer in A Tube-in-Tube Heat Exchanger

The experiment was conducted by Coetzee (2000) using R22 and water in a tube-in-tube heat exchange system. Using the values from the experiment, the uncertainty for the heat transfer was calculated.

The equation for the heat transfer is given below.

$$Q = m\dot{C}p\Delta T$$

Equation 96: Heat transfer

The derived uncertainty for the heat transfer is given below.

$$U_Q = \sqrt{(Cp\Delta TU_m)^2 + \left(\dot{m}\Delta TU_{Cp}\right)^2 + \left(\dot{m}CpU\Delta T\right)^2}$$

Equation 97: Uncertainty for the heat transfer

The average heat transfer is calculated from the hot water and cold water heat transfers.

$$Q_{ave} = \frac{Q_h + Q_c}{2}$$

Equation 98: Average heat transfer

The uncertainty for the average heat transfer can then be derived.

$$U_{A_{ave}} = \sqrt{\left(\frac{1}{2}U_{Q_h}\right)^2 + \left(\frac{1}{2}U_{Q_c}\right)^2}$$

Equation 99: Uncertainty for the average heat transfer

Uncertainty Equations for the Convection Heat Transfer Coefficient in the Inner Tube of A Tube-in-Tube Heat Exchanger

The experiment was conducted by Coetzee (2000) using R22 and water in a tube-in-tube heat exchange system. Using the values from the experiment, the uncertainty for the convection heat transfer coefficient for the inner tube was calculated.

The convection heat transfer coefficient is given in the equation below.

$$h_i = C_i \frac{k_i}{d_i} \mathrm{Re}_i^{0.8} \mathrm{Pr}_i^{\frac{1}{3}} \left(\frac{m}{m_w} \right)_i^{0.1}$$

Equation 100: Convection heat transfer coefficient

The uncertainty for the convection heat transfer coefficient was derived from the formula of Cole (1953).

$$U_{h_i} = \left\{ \begin{array}{l} \left(\frac{k_i}{d_i} \mathrm{Re}_i^{0.8} \mathrm{Pr}_i^{\frac{1}{3}} \left(\frac{m}{m_w} \right)_i^{0.14} U_{C_i} \right)^2 + \left(C_i \frac{1}{d_i} \mathrm{Re}_i^{0.8} \mathrm{Pr}_i^{\frac{1}{3}} \left(\frac{m}{m_w} \right)_i^{0.14} U_{k_i} \right)^2 \\ + \left(C_i \frac{k_i}{d_i^2} \mathrm{Re}_i^{0.8} \mathrm{Pr}_i^{\frac{1}{3}} \left(\frac{m}{m_w} \right)_i^{0.14} U_{d_i} \right)^2 + \left(0.8 C_i \frac{k_i}{d_i} \mathrm{Re}_i^{-0.2} \mathrm{Pr}_i^{\frac{1}{3}} \left(\frac{m}{m_w} \right)_i^{0.14} U_{\mathrm{Re}_i} \right)^2 \\ + \left(\frac{1}{3} C_i \frac{k_i}{d_i} \mathrm{Re}_i^{0.8} \mathrm{Pr}_i^{-\frac{2}{3}} \left(\frac{m}{m_w} \right)_i^{0.14} U_{\mathrm{Pr}_i} \right)^2 + \left(0.14 C_i \frac{k_i}{d_i} \mathrm{Re}_i^{0.8} \mathrm{Pr}_i^{\frac{1}{3}} \left(\frac{m^{-0.86}}{m_w^{0.14}} \right)_i U_{m_i} \right)^2 \\ + \left(-0.14 C_i \frac{k_i}{d_i} \mathrm{Re}_i^{0.8} \mathrm{Pr}_i^{\frac{1}{3}} \left(\frac{m^{0.14}}{m_w^{1.14}} \right)_i U_{m_w} \right)^2 \end{array} \right\}^{\frac{1}{2}}$$

Equation 101: Uncertainty of the convection heat transfer coefficient

Uncertainty Equations for the Convection Heat Transfer Coefficient in the Annulus of A Tube-in-Tube Heat Exchanger

The experiment was conducted by Coetzee (2000) using R22 and water in a tube-in-tube heat exchange system. Using the values from the experiment, the uncertainty for the convection heat transfer coefficient for the annulus was calculated.

The convection heat transfer coefficient for the annulus is given in the equation below.

$$h_a = \left[\frac{1}{U_a} - \frac{1}{h_i}\frac{A_a}{A_i} - \frac{A_a \ln\left(\frac{D_o}{D_i}\right)}{2pk_{Cu}L} \right]^{-1}$$

Equation 102: Convection heat transfer coefficient for the annulus

$$U_{h_i} = \left\{ \left(-\left(\frac{1}{U_a} - \frac{1}{h_i}\frac{A_a}{A_i} - \frac{A_a \ln\left(\frac{D_o}{D_i}\right)}{2\pi k_{Cu} L} \right)^{-2} \right)^2 \left(\begin{array}{l} \left(\frac{1}{U_a^2} U_{U_a} \right)^2 + \left(-\frac{1}{h_i^2}\frac{A_a}{A_i} U_{h_i} \right)^2 \\ + \left(-\frac{1}{h_i}\frac{1}{A_i} U_{A_a} \right)^2 + \left(-\frac{1}{h_i}\frac{A_a}{A_i^2} U_{A_i} \right)^2 \\ + \left(-\frac{\ln\left(\frac{D_o}{D_i}\right)}{2\pi k_{Cu} L} U_{A_a} \right)^2 + \left(-\frac{A_a\left(\frac{D_i}{D_o}\right)\left(\frac{1}{D_i}\right)}{2\pi k_{Cu} L} U_{D_o} \right)^2 \\ + \left(-\frac{A_a\left(\frac{D_i}{D_o}\right)\left(\frac{D_o}{D_i^2}\right)}{2\pi k_{Cu} L} U_{D_i} \right)^2 + \left(\frac{A_a \ln\left(\frac{D_o}{D_i}\right)}{2\pi k_{Cu}^2 L} U_{k_{Cu}} \right)^2 \\ + \left(\frac{A_a \ln\left(\frac{D_o}{D_i}\right)}{2\pi k_{Cu} L^2} U_L \right)^2 \end{array} \right)^{\frac{1}{2}} \right\}$$

Equation 103: Uncertainty equation for the convection heat transfer coefficient for the annulus

Uncertainty Equations for the Nusselt Number in the Inner Tube of A Tube-in-Tube Heat Exchanger

The experiment was conducted by Coetzee (2000) using R22 and water in a tube-in-tube heat exchange system. Using the values from the experiment, the uncertainty for the Nusselt Number in the inner tube was calculated.

The Nusselt Number for the inner tube is given in the equation below.

$$Nu_i = 0.0257\,\mathrm{Re}_i^{0.8}\,\mathrm{Pr}_i^{\frac{1}{3}}\left(\frac{m}{m_w}\right)_i^{0.14}$$

Equation 104: Nusselt Number for the inner tube

The uncertainty equation for the Nusselt Number was derived from the formula of Cole (1953). 0.02056

$$U_{Nu_i} = \left\{\left(0.02056\mathrm{Re}_i^{-0.2}\,\mathrm{Pr}_i^{\frac{1}{3}}\left(\frac{m}{m_w}\right)_i^{0.14}U_{\mathrm{Re}_i}\right)^2 + \left(0.008567\mathrm{Re}_i^{0.8}\,\mathrm{Pr}_i^{-\frac{2}{3}}\left(\frac{m}{m_w}\right)_i^{0.14}U_{\mathrm{Pr}_i}\right)^2 + \left(0.003598\mathrm{Re}_i^{0.8}\,\mathrm{Pr}_i^{\frac{1}{3}}\left(\frac{m^{-0.86}}{m^{0.14}}\right)_i U_{m_i}\right)^2 + \left(-0.003598\mathrm{Re}_i^{0.8}\,\mathrm{Pr}_i^{\frac{1}{3}}\left(\frac{m^{0.14}}{m^{1.14}}\right)_i U_{m_i}\right)^2\right\}^{\frac{1}{2}}$$

Equation 105: Uncertainty equation of the Nusselt Number of the inner tube

Uncertainty Equations for the Nusselt Number in the Annulus of A Tube-in-Tube Heat Exchanger

The experiment was conducted by Coetzee (2000) using R22 and water in a tube-in-tube heat exchange system. Using the values from the experiment, the uncertainty for the Nusselt Number in the annulus was calculated.

The Nusselt Number for the annulus is given in the equation below.

$$Nu_a = 0.936\,\mathrm{Re}_a^{0.742}\,\mathrm{Pr}_a^{\frac{1}{3}}\left(\frac{m}{m_w}\right)_a^{0.14}$$

Equation 106: Nusselt number for the annulus

The uncertainty for the Nusselt number in the annulus is therefore shown by the equation below.

$$U_{Nu_i} = \left\{ \begin{array}{l} \left(0.0694512\mathrm{R}\,\mathrm{e}_i^{-0.2}\,\mathrm{Pr}_i^{\frac{1}{3}}\left(\frac{m}{m_w}\right)_i^{0.14} U_{\mathrm{Re}_i} \right)^2 + \left(0.0312\mathrm{Re}_i^{0.8}\,\mathrm{Pr}_i^{-\frac{2}{3}}\left(\frac{m}{m_w}\right)_i^{0.14} U_{\mathrm{Pr}_i} \right)^2 \\ + \left(0.013104\mathrm{Re}_i^{0.8}\,\mathrm{Pr}_i^{\frac{1}{3}}\left(\frac{m^{-0.86}}{m^{0.14}}\right)_i U_{m_i} \right)^2 + \left(-0.013104\mathrm{Re}_i^{0.8}\,\mathrm{Pr}_i^{\frac{1}{3}}\left(\frac{m^{0.14}}{m^{1.14}}\right)_i U_{m_i} \right)^2 \end{array} \right\}^{\frac{1}{2}}$$

Equation 107: Uncertainty equation for the Nusselt number in the annulus

Values and Results for the Heat Transfer in the Inner Tube of a Tube-in-Tube Heat Exchanger

m	0.111413
Um	0.08
Cp	4180
Ti	340.2637
To	301.853
Ut	0.1
delta T	38.4107
Qh	17888.08
1st term	12844.53808
2nd term	46.570634
final	164984327.3
Uqh	12844.62251
error	0.718054845
error %	71.80548447
error	0.281945155

Values and Results for the Heat Transfer in the Annulus of a Tube-in-Tube Heat Exchanger

m	0.123932
Um	0.08
Cp	4180
Ti	286.4959
To	320.6104
Ut	0.1
delta T	34.1145
Qc	17668.3
1st term	11407.8888
2nd term	51.803576
final	130142610.5
Uqc	11408.00642
error	0.645676518
error %	64.56765178
error	0.354323482

Values and Results for the Average Heat Transfer of a Tube-in-Tube Heat Exchanger

Qave	17778.19
1st term	113471321.9
2nd term	102033665.7
Uqave	14680.08813
error	0.825735811
error %	82.57358108
error	0.174264189

Values and Results for the Overall Heat Transfer Coefficient in the Inner Tube of a Tube-in-Tube Heat Exchanger

Qh	17888.08
Uqh	12844.62251
Ai	0.26124
delta Th	38.4107
Ut	0.1
U	1782.673594
1st term	1280.057411
2nd term	4.641085932
final	1638568.514
Uu	1280.065824
error	0.718059564
error %	71.80595643
error	0.281940436

Values and Results for the Overall Heat Transfer Coefficient in the Annulus of a Tube-in-Tube Heat Exchanger

Qc	17668.3
Uqh	11408.00642
Ao	0.26124
delta Tc	34.1145
Ut	0.1
U	1982.513162
1st term	1280.062195
2nd term	5.811350487
final	1638592.995
Uu	1280.075386
error	0.645683172
error %	64.56831717
error	0.354316828

Values and Results for the Reynolds Number in the Inner Tube of a Tube-in-Tube Heat Exchanger

p	988.7342
V	0.001213205
Uv	0.00001
Dh	0.00774
Udh	0.00001
uh	0.000564
Reh	31021.21
1st term	0.135687991
2nd term	0.021268387
final	0.018863575
Ure	0.137344731
error	4.42745E-06
error %	0.000442745
error	0.999995573

Values and Results for the Reynolds Number in the Annulus of a Tube-in-Tube Heat Exchanger

p	995.5955
V	0.001340227
Uv	0.00001
Dh	0.00774
Udh	0.00001
uc	0.000785
Rec	7504.507
1st term	0.098164448
2nd term	0.016997756
final	0.009925183
Ure	0.099625211
error	1.32754E-05
error %	0.001327538
error	0.999986725

Values and Results for the Convection Heat Transfer Coefficient of the Inner Tube of a Tube-in-Tube Heat Exchanger

Ci	0.01998
ki	0.64106
Uki	0.0002
di	0.00811
Udi	0.00001
Rei	31021.21
Urei	0.137344731
Pr	3.175826
Upr	0.1
u	0.000564
Uu	0.000005
uw	0.000629
Uuw	0.000005
hi	9442.94
1st term	2.796309017
2nd term	11.05179937
3rd term	0.031746592
4th term	94.07536068
5th term	11.12430231
6th term	-5650.172228
final	31933550.1
Uhi	5650.9778
error	0.598434153
error %	59.84341529
error	0.401565847

Values and Results for the Convection Heat Transfer Coefficient of the Annulus of a Tube-in-Tube Heat Exchanger

Ua	1982.513162
1/ua	0.00050441
Uua	1280.075386
hi	9442.94
1/hi	0.000105899
Uhi	5650.9778
Aa	0.26124
Ai	0.26124
1/ai	3.827897719
Do	0.00953
Udo	0.00001
Di	0.01727
1/di	57.90387956
Udi	0.00001
k	0.64106
Uk	0.0002
L	12
ha	4452.867
1st term	11104.9756
2nd term	-0.000325689
3rd term	6.33737E-05
4th term	-0.019015262
5th term	0.010493077
6th term	1.00249E-06
7th term	58181.9794
final	241.2094099
error	0.054169462
errror %	5.416946202
error	0.945830538

Values and Results for the Convection Heat Transfer Coefficient of the Annulus of a Tube-in-Tube Heat Exchanger

Rei	31021.21
Urei	0.137344731
Pri	3.175826
Upri	0.1
ui	0.000564
Uui	0.000005
uw	0.000629
Uuw	0.000005
Nui	145.8524273
1st term	0.000516603
2nd term	1.530920381
3rd term	0.181022516
4th term	0.162315897
final	1.550107442
error	0.010627917
error %	1.062791666
error	0.989372083