COPYRIGHT AND CITATION CONSIDERATIONS OF A THESIS/ DISSERTATION

You should include the following information in your bibliography, the exact style will vary according to the citation system you are using:

- Name of author
- Year of publication, in brackets
- Title of thesis, in italics
- Type of degree (e.g. D. Phil.; Ph.D. or M.Sc.)
- Name of the University
- Country
- Website
- Date, accessed

Example

LEARNERS’ ERRORS WHEN SOLVING ALGEBRAIC TASKS: A CASE STUDY OF GRADE 12 MATHEMATICS EXAMINATION PAPERS IN SOUTH AFRICA

by

ANDILE MAMBA

A Masters Dissertation submitted to the University of Johannesburg, in fulfilment of the Master’s Degree in Mathematics Education

MAGISTER EDUCATIONIS

in

MATHEMATICS EDUCATION

in the

FACULTY OF EDUCATION

at the

UNIVERSITY OF JOHANNESBURG

SUPERVISOR : Dr K Luneta

June 2012
Declaration

I declare that this research report is my own, unaided work. It is being submitted for the Masters of Education degree at the University of Johannesburg, South Africa. It has not been submitted before for any degree or examination in any other university.

Signature

31st June 2012
Dedication

This study is dedicated to my late sister Lungile, my mum and dad and my family Bhekie, Nolwazi and Thabiso, and my grade 1 teacher Mrs Olga Mabuza.
Acknowledgements

My deep and sincere gratitude is hereby expressed to:

• My supervisor, Dr. Kakoma Luneta, for the guidance, support, advice and motivation he provided in the compilation of this dissertation.
• Dr. Paul Judah Makonye for all his advice, patience and feedback.
• Dr Karen Haire for all her input and for reading and giving me feedback.
• Mrs Nadine Dhasery for all her input, for reading and all the feedback.
• Ingrid for her assistance with technical issues in writing this research essay.
• My amazing husband Bhekie for his continuous love, support, encouragement and patience at all stages of my studies.
• My most loving children Nolwazi and Thabiso Mamba for their patience and for giving me the space to complete my studies.
• My wonderful parents Mary and Cranmer Magagula for their love and encouragement to continue studying and by setting a good example for me.
• Last but the most important, I acknowledge that without the strength provided by the Lord my Rock, I would not be submitting this manuscript. I also thank God for providing me with such a supportive family to surround me. I’m grateful to my encouraging parents, loyal brothers, a supportive husband and kids, and all my true, loving and caring friends.
Abstract

In spite of the efforts of the South African government, the Gauteng Department of Education and many business and private funders to place a high emphasis on mathematics performance, the mathematics achievement of South African learners is still less than desirable. In fact, the results of the three Third International Mathematics and Science Study (TIMSS) (Howie, 2001, 2003) reports in 1995, 1999 and 2003 reported South African learners as the lowest performing in those tests; performing well below the international average amongst the countries that participated. The Southern African Consortium for Monitoring Quality 2004 and Center for Development in Education 2004, 2010 and 2011 reports results show similar results (Moloi, 2005). The research study sought to get a deep understanding of why learners\(^1\) continue to perform poorly, and what the factors are which contribute to poor performance. While there are a number of factors responsible for the poor performance, one of the least researched areas is answered examination scripts. This research entailed a detailed error analysis of four items of the 2008 mathematics paper 1 senior certificate examination scripts, to see the trends and patterns of written responses with regards to the types of errors made by learners.

The study was aimed at investigating South African Grade 12 learners’ errors exhibited when solving quadratic equations, quadratic inequalities and simultaneous equations. Findings of this investigation shed light of the kind of knowledge learners bring into their learning experiences and this knowledge affects how they encode and later retrieve new information learned (Davis, 1984).

While the study was not a mixed methods one, the data was analysed quantitatively using frequency counts and qualitatively by studying selected learners’ solution of examination tasks. The study also identified common errors in the learners’ work. The four items analysed in the study comprised of questions from three important areas of algebra namely: quadratic equations, quadratic inequalities and simultaneous equations. The scripts were analysed for carelessness, conceptual and procedural errors. The learner misconceptions were discovered in learners’ work;

\(^1\) In this report the term learner and learner are used interchangeably to refer to a high school going child.
these comprised the notions of equality and inequality, the construct of the variable, order of operations, factorisation, and solution of equations instead of inequalities. From this, the researcher noted that learners' learning difficulties are usually presented in the form of errors they show. Not all the errors that learners had are the same; some errors in procedures can simply be due to learners' carelessness or overloading working memory (Davis, 1984). Some errors in procedures can be caused by faulty algorithms or "buggy algorithms". Other errors can have certain conceptual basis and can be termed as 'misconceptions'.

The results obtained indicated a number of error categories under each conceptual area, namely, quadratic equations and inequalities and simultaneous equations. Some errors emanated from misconceptions. Under the conceptual areas indicated above, the main reason for misconceptions seemed to be the lack of understanding of the basic concepts including numbers and numerical operations; functions; the order of operations; equality; algebraic symbolism; algebraic equations, expressions and inequalities; and difference between equations, expressions and inequalities. The abstract nature of algebraic expressions posed many problems to learners such as understanding or manipulating them according to accepted rules, procedures, or algorithms. Inadequate understanding of the uses of the equal sign and its properties when it is used in an equation was a major problem that hindered learners from solving equations correctly. The main difficulty in inequalities was manipulating the inequalities correctly and converting the inequality to an equation. Recommendations to the mathematics educational community based on this research were made.
# Contents

CHAPTER 1 .................................................................................................................................................................................. 1  

1.1 INTRODUCTION .................................................................................................................. 1  

1.2 STATEMENT OF THE PROBLEM .................................................................................. 4  

1.3 PURPOSE OF THE STUDY .......................................................................................... 5  

1.4 RESEARCH QUESTIONS ............................................................................................... 5  

1.5 RESEARCH METHODS ................................................................................................. 6  

1.6 SIGNIFICANCE OF THE STUDY .................................................................................. 6  

1.7 OUTLINE OF THE CHAPTERS IN THE STUDY ......................................................... 8  

CHAPTER 2    THEORETICAL FRAMEWORK AND LITERATURE REVIEW ................. 9  

2.1 INTRODUCTION ........................................................................................................... 9  

2.1 THEORETICAL FRAMEWORK .................................................................................. 9  

2.2.1 Constructivism and learner errors in mathematics ................................................. 12  

2.3 VYGOTSKY’S SOCIOCULTURAL LEARNING THEORY AND LEARNER ERRORS IN MATHEMATICS .................................................................................. 14  

CHAPTER 2 LITERATURE REVIEW .............................................................................. 16  

2.1 Definition of key terms ................................................................................................ 16  

2.2 How do we define errors in mathematics? ................................................................. 18  

2.3 Error analysis ................................................................................................................ 20  

2.4 How do learners make errors and why do they make these errors? ......................... 21  

2.5 Research on the teaching and learning of algebra .................................................... 25  

2.6 Discourse on errors and misconceptions ................................................................... 34  

2.7 What is algebra and what are the common errors on the topic? ............................... 35  

2.8 How do teachers teach algebra? .................................................................................. 42  

2.9 How do learners learn algebra? ................................................................................... 48  

2.10 Conclusion .................................................................................................................. 50  

CHAPTER 3   THE RESEARCH methodology and DESIGN OF THE STUDY ........... 52  

3.1 INTRODUCTION ........................................................................................................... 52  

3.2 QUANTITATIVE RESEARCH AND QUALITATIVE RESEARCH ............... 53  


### LIST OF FIGURES

- **Figure 1** The zone of proximal development (Adapted from Driscoll, 2005) ............. 15
- **Figure 2** The ‘research wheel’. Adapted from Johnson and Christensen (2004:18). 52
- **Figure 3** Frequency of marks scores for Items 1.1.1, 1.1.2, 1.1.3 and 1.2 ............. 68
- **Figure 4** Frequency of each type of error ............................................................... 68
- **Figure 5** Learner 1 of item 1.1.1 .............................................................................. 70
- **Figure 6** Learner 2 of item 1.1.1 .............................................................................. 72
- **Figure 7** Learner 3 of item 1.1.1 .............................................................................. 73
- **Figure 8** Learner 4 of item 1.1.1 .............................................................................. 73
- **Figure 9** Learner 5 of item 1.1.1 .............................................................................. 74
- **Figure 10** Learner 1 of item 1.1.2 ............................................................................. 76
- **Figure 11** Learner 2 of item 1.1.2 ............................................................................. 76
- **Figure 12** Learner 3 of item 1.1.2 ............................................................................. 77
- **Figure 13** Learner 4 of item 1.1.2 ............................................................................. 78
- **Figure 14** Learner 5 of item 1.1.2 ............................................................................. 78
- **Figure 15** Learner 1 of item 1.1.3 ............................................................................. 81
- **Figure 16** Learner 2 of item 1.1.3 ............................................................................. 81
- **Figure 17** Learner 3 of item 1.1.3 ............................................................................. 82
- **Figure 18** Learner 4 of item 1.1.3 ............................................................................. 82
- **Figure 19** Learner 5 of item 1.1.3 ............................................................................. 83
- **Figure 20** Learner 1 of item 1.2 ................................................................................ 85
- **Figure 21** Learner 2 of item 1.2 ................................................................................ 86
- **Figure 22** Learner 3 of item 1.2 ................................................................................ 86
- **Figure 23** Learner 4 of item 1.2 ................................................................................ 87
- **Figure 24** Learner 5 of item 1.2 .............................................................................. 88
LIST OF TABLES

Table 1 Comparison of assessed algebraic items in the 2008 NSC examinations and the required policy guidelines .......................................................... 65
Table 2 Cognitive levels and their related skills for items used for the research in terms of Subject Assessment Guidelines for Mathematics (2009). ......................... 65
The teacher pretended that algebra was a perfectly natural affair, to be taken for granted, whereas I don’t even know what numbers were. Mathematics classes became sheer terror and torture to me. I was so intimidated by my incomprehension that I did not dare ask any questions’.

Carl Jung (1875-1961)

CHAPTER 1

1.1 INTRODUCTION

Having achieved notable advances in formal access to education, South Africa urgently needs to turn its attention to the quality of its education. According Howie (2001, 2003) South Africa spends proportionately more on education than many other countries, yet in international tests South African learners’ performance is of an inferior quality compared to the performance of learners from other developing countries, including Africa. Bernstein (2007) furthermore attests that this is more pressing in the fields of mathematics and science.

Howie (2001) conducted a study in 1995 in which South Africa participated with 41 other countries, and published the Third International Mathematics and Science Study Report (TIMSS-R), in which it was reported that South African mathematics learners came last with a mean score of 351. This mean score was significantly lower than the international benchmark of 513. Less than 2% of these South African learners reached or exceeded the international mean score (Beaton, Mullis, Martin, Gonzalez, Kelly & Smith, 1996). TIMSS-R conducted in 1999 revealed that Grade 8 learners again performed poorly. The South African mean of 275 was lower than that of other developing countries (Howie, 2001; Naidoo, 2004). A later TIMSS-R conducted in 2003 indicated no improvement by South African mathematics and science learners (Reddy, 2005). Bernstein (2011) also states that South Africa is at or near the bottom in terms of learner performance in mathematics and science. These research examples paint a gloomy image of the situation of the teaching and learning of mathematics in South Africa. This country
is in desperate need of suitably qualified teachers, doctors, scientists and many other scientifically oriented professionals. With the current status of mathematical literacy being generally so poor (Howie, 2003; Bernstein, 2007)), it is difficult to believe that an adequate number of learners will qualify for further studies in science, technology, accounting, economics and medicine, to name a few where mathematics is a prerequisite. This is important because the lack of expertise impacts on the broad economic position of the country. Currently, South Africa does not have the capacity to expand economically without importing foreign scientific and technological expertise (Pratzner, 1994; Frantz, Friedenberg, Gregson & Watter, 1996). If South Africa is to participate in the technologically advancing global village, it is necessary that research should inform policy and drive transformation in order to create a mathematically literate society. Even though there has been some increase in the number of learners qualifying for university entrance, the most of schools in South Africa, (that is 90% of the schools), are still failing to meet the minimum performance standard in mathematics and science.

Bernstein (2011) still paints a very gloomy picture of the state of education in South Africa. The report states that the education system is still underperforming, especially when it comes to mathematics and science results in the National Senior Certificate (NSC) examinations. The report also alludes to the fact that South Africa’s expenditure on education compared to many other developing countries is not being matched by results. Since mathematics and science are vital for training skilled personnel to sustain and boost economic growth and national development. Improving the country’s output of good maths and science graduates has become a national imperative. Central to achieving this goal is attracting, producing, utilising, and retaining good mathematics teachers. This hopefully will result in more learners obtaining good grades in Grade 12 mathematics and hence gaining access to tertiary education.

South Africa needs to find a way of helping its learners, particularly the historically disadvantaged, to pass the National Senior Certificate mathematics examination because mathematics is a gate keeper for most programmes at universities. If
historically disadvantaged learners continue to fare poorly in mathematics examinations (mathematics is a scarce skill), then this compromises equity in education which is one of the cornerstone goals of the post-apartheid government. The research examples presented above further paint a gloomy image of the situation of the teaching and learning of mathematics in South Africa. With the current status of mathematical performance generally being so poor in the entire schooling system (Howie, 2003; Bernstein, 2007), it is doubtful that this system will be able to produce an adequate number of learners who will meet the minimum mathematics entrance requirements to enrol at institutions of higher learning to pursue further studies in not only science engineering and technology-related studies, but also the accounting, economics and medically related fields, to name a few. Currently, South Africa does not have the capacity to expand economically without importing foreign scientific and technological expertise (Pratzner, 1994; Frantz, Friedenberg, Gregson & Watter, 1996; Ramsuran, 2005). If South Africa is to participate in the technologically advancing global village, it is necessary that research should inform policy and drive transformation to a mathematically literate society. This is extremely important because the lack of expertise impacts on the broad economic position of the country. As mentioned earlier, South Africa is at or near the bottom of the all the world in terms of learner performances in maths and science.

Given the scale of the challenge represented by hundreds of thousands of teachers, and millions of learners in South Africa, one needs to recognise that the existing education programmes will be under enormous pressure to produce, for example, 1 000 more well qualified science and mathematics teachers, or to place 1 000 existing teachers in professional learning communities (see Bernstein, 2011). Such efforts might help to improve the standard of performance of South African learners.

From this researcher’s point of view garnered from many years of teaching experience in mathematics at high school and university in South Africa, one of the reasons why learners are not doing so well in mathematics is that they fail to master key mathematics topics such as quadratic equations and inequalities. The
researcher felt that often learners do not perform well in those topics because of their many misconceptions and errors on them.

1.2 STATEMENT OF THE PROBLEM

The researcher believes that by focusing on errors made by learners in the algebra section of the Grade 12 Mathematics Examination Paper 1, one gains a better understanding and appreciation of the difficulties experienced by learners in studying mathematics. In this way, a teacher is better able to recommend focused and meaningful interventions that will minimise the most prominent errors identified. It will also inform possible effective teaching approaches. Thus research on learner errors in algebra can be used as leverage to better understanding the difficulties faced by learners in learning mathematics in general. This understanding can then be replicated in other areas of mathematics, more so as mathematical concepts are closely connected and related. As stated in the National Curriculum Statement (NCS) (2003) Policy Documents on Mathematics, algebra forms the basis of most learning in mathematics. To emphasise the importance of algebra in the curriculum, it is Learning Outcome (LO) 2 out of the four mathematics LOs present in the Mathematics NCS policy documents from grade 10 to 12. Algebra has links to all mathematics including, calculus, coordinate geometry, trigonometry, arithmetic and series and sequences. The importance of algebra is such that the study of higher mathematics and science is almost impossible without it.

According to Ketterlin-Geller, Chard and Fien (2008), algebraic understanding requires a range of skills. Some skills are associated with specific constructs of algebra but others related to mathematics in general. The fundamental algebraic skills include understanding of variables and constants, decomposing and setting up word problems, symbolic manipulation, and understanding of functions (Milgram, 2005). The general mathematical skills comprise inductive reasoning, understanding of rational numbers, procedural fluency with computational skills, and advanced problem-solving skills. Extending both the fundamental and general skills into novel applications in algebra requires a firm foundation of previously
taught concepts. Ketterlin-Geller, Chard and Fien (2008) further state that many learners enter middle school lacking in this foundational knowledge on which more algebraic understanding is built. These learners are therefore at a considerable disadvantage when studying algebra.

The researcher reasons that although the study of learners’ errors as evidenced in their written work in algebra is an important mathematics education research problem, few such studies have been done in South Africa particularly those based on script analysis. A study of this nature can produce a window of opportunity to closely analyse learners’ thinking on algebra. It affords an opportunity to discover how learners think of algebraic concepts and procedures from their point of view. The researcher maintains that such a data informed study can lead to better teaching interventions that can improve learning of algebra in particular and mathematics in general.

1.3 PURPOSE OF THE STUDY
With respect to questions on quadratic equations and inequalities in the 2008 Grade 12 Mathematics Examination Paper 1, the research aims of this study are to:

1. Determine the prominent errors learners make in answering those questions.
2. Classify the identified errors into the major types of mathematical errors.
3. Provide possible reasons from the literature as to why these errors are being made.

1.4 RESEARCH QUESTIONS
In light of the above, the research question addressed in this study is:

What is the nature of the main errors made by learners in Grade 12 when answering quadratic equations and inequalities examination questions?
The above question led to the following sub-questions:

1. What are the most common errors made by learners in their scripts?
2. What are the possible reasons why learners make those errors?

1.5 RESEARCH METHODS

A brief quantitative analysis of scripts was done to establish the frequency of the various types of possible errors. However, the major method for this investigation was the qualitative approach. The research took the form of content analysis. Interpretations as those found in the literature and also to understand why the learners had made the errors as identified in the study.

Using a cognitive task analysis framework (Stein, Smith, Henningsen & Silver, 1993), 80 answer scripts from the 2008 National Senior Certificate Grade 12 Mathematics Examination Paper 1 were analysed in terms of the cognitive demands of examination items, followed by a micro-error analysis of each learner’s responses to research question 1.

The research data were collected by means of answered Grade 12 mathematics examination scripts which were readily available from a University of Johannesburg research project that commenced in 2009. Other documents that were consulted include: the National Curriculum Statement (NCS) (2003) Policy Documents on Mathematics, the Mathematics Examination Guidelines, the 2008 National Senior Certificate Grade 12 Mathematics Examination Paper 1 and the memorandum, the Revised Bloom’s Taxonomy also found in the Mathematics Examination Guideline (Anderson & Krathwohl, 2001) and Stein et al. (1993) Task Analysis Framework was used to guide analysis of the cognitive demands of examination items. A detailed error-analysis of each of the 80 learners’ responses to Question 1 followed.

1.6 SIGNIFICANCE OF THE STUDY

In South Africa, insufficient qualitative analyses are available on candidates’ errors identified in Matric (Grade 12) mathematics examination scripts. This study is
useful as it identifies common errors made by learners doing high-school algebra, with particular emphasis on quadratic equations and inequalities. The findings of this research will be helpful to mathematics educators in general in order to create an awareness of the errors that are being made by learners, and could be used as a tool to improve their own teaching. In particular, the results of this study could be used by educators to identify the pre-existing knowledge learners bring into the classroom, and in cases where there are misconceptions/ill-constructed ideas in the learners’ knowledge, to systematically help the learner to realise that their knowledge is deficient and help learners to embrace the ‘accepted’ concepts in Matric maths algebra.

The research is aimed at providing teachers, especially in South Africa, with results of an analysis of errors made by learners in the algebra section of the Grade 12 Mathematics Examination Paper 1 so that similar mistakes are anticipated as well as corrected and remedied long before learners sit for the examination. The aim can also contribute to the education of learners in such a way that both learners and teachers are more aware of common errors and misconceptions in algebra and to teach away (teach to minimise and/or amend) from the misconceptions. The research will also benefit teaching and hence learning because the results and subsequent recommendations will develop helpful ways to confront learners with counterexamples to their misconceptions and incorrect beliefs can be undone to some extent when confronted.

Although there are many reasons for learners’ difficulties in learning mathematics, the lack of support from research with respect to teaching and learning is an important one. If research could describe and distinguish learners’ learning difficulties, it would be possible to design effective guidelines to support effective learners’ learning. Research on learners’ errors and misconceptions is a means to offer such support for both teachers and learners. As Booth (1988) pointed out, ‘one way of trying to find out what makes algebra difficult is to identify the kinds of errors learners commonly make in algebra and then to investigate the reasons for these errors’ (p. 20).
A constructivist framework challenges teachers to create environments in which learners are encouraged to think and explore. The manner in which learners’ misconceptions are valued in the classroom is vital to their use as a means of advancing conceptual understanding. Therefore, it is important that methods are developed and utilised by teachers to enhance the process of using learners’ misunderstandings as a tool for learning in mathematics.

1.7 OUTLINE OF THE CHAPTERS IN THE STUDY
The programme of this study is outlined as follows:

Chapter 1. This provides the general orientation of the research. The research problem, the aim of the study, the research questions and research methods.

Chapter 2. This chapter presents the theoretical framework that guided the study, as well as a literature review. This focuses on how teachers teach; how learners learn; identifying learners’ errors and misconceptions; and why learners make errors in quadratic equations, inequalities and simultaneous equations.

Chapter 3. This outlines the research methodology and design of the study.

Chapter 4. This chapter presents the analysis of data in the study.

Chapter 5. This is the concluding chapter. It presents the findings, conclusions and recommendations for further study based on this research.
CHAPTER 2  THEORETICAL FRAMEWORK AND LITERATURE REVIEW

2.1 INTRODUCTION
This chapter explicates the theoretical framework that guided the study. It also reviews literature on mathematical errors and misconceptions in general and errors and misconceptions in algebra.

2.2 THEORETICAL FRAMEWORK
The theoretical framework of the study is informed by Pedagogical Content Knowledge (Shulman, 1986), socio-cultural theory of learning (Vygotsky, 1986) and constructivism (Piaget, 1968; von Glasserfeld, 1987). The researcher presumed that these ideas would help her to best analyse learners’ errors in answering examination tasks. Pedagogical content knowledge

The teacher’s knowledge is one of the major influences on classroom atmosphere and on what learners learn (Fennema & Franke, 1992). In a meta-analysis of 60 education production function studies, variables used to denote teacher quality (for instance teacher ability, knowledge and education level) were established to have positive effects on learner achievement (Greewald, Hedges & Laine, 1996). Teachers with better mathematical knowledge for teaching yielded substantially more gains in learner achievement, despite the fact that the study controlled for many other variables (including learner socioeconomic status, learner absence rate, teacher credentials, teacher experience, and average length of mathematics lesson). Because of its proven influence, the mathematical knowledge important for the work of teaching has become a significant issue in mathematics education (Stylianides & Ball, 2004).

Shulman (1986) presented a new way of thinking about the content knowledge necessary for teaching in his pioneering work that popularised the concept of pedagogical content knowledge (PCK). First, Shulman categorised the knowledge needed for teaching into two domains, namely content knowledge for teaching,
‘the amount and organisation of knowledge per se in the mind of the teacher’ (Shulman, 1986:9), and pedagogical knowledge, ‘the knowledge of generic principles of classroom organization and management and the like’ (Shulman, 1986:14). Shulman further divided content knowledge for teaching down into three subgroups: subject matter content knowledge, pedagogical content knowledge, and curricular knowledge. These three categories of teacher content knowledge are intertwined in practice. Pedagogical content knowledge includes an understanding of what makes the learning of specific topics easy or difficult: the conceptions, preconceptions and misconceptions that learners of different ages and backgrounds bring with them into the classroom. If those preconceptions are misconceptions, which are often the case, teachers need knowledge of the effective strategies most likely to be fruitful in reformulating the understanding of the learners (Shulman, 1986). Subject matter knowledge is more than knowledge of facts or concepts – it requires knowledge of both the substantive structure (facts and their organising principles) and syntactic structure (legitimacy principles for the rules) of a subject domain. The transformation of subject matter knowledge into pedagogical content knowledge should be a significant focus in teacher education (Goulding, Rowland & Barber, 2002).

In a seminal text, Shulman (1987) subsequently delineated seven categories of teacher knowledge:

1. Content knowledge - both ‘substantive’ and ‘syntactic’
2. General pedagogical knowledge - generic principles of classroom management
3. Curriculum knowledge - materials and programmes
4. Pedagogical content knowledge - which for a given subject area includes forms of representation of concepts, useful analogies, examples, demonstrations
5. Knowledge of learners
6. Knowledge of educational contexts, communities and cultures
7. Knowledge of educational purposes and values.

(Shulman, 1986; Wilson, Shulman & Richert, 1987:113)
In teaching mathematics most mathematics teachers would accede that there is more to the pedagogy of mathematics than simply learning or teaching the content. One of the aspects of teaching that is grossly neglected is that even in the best classrooms learners make errors/mistakes (Cite a source). Knowledge of learners’ conceptual thinking is in fact the knowledge of how learners analyse and comprehend mathematical ideas. This knowledge entails an understanding of common learner conceptions, misconceptions, difficulties, errors, and interests in the field of mathematics. Teachers’ usual reaction to these errors/mistakes is to either re-teach the lesson or to correct the error showing learners where they went wrong without questioning what the causes of those errors and mistakes are. In the teaching and learning process teachers need to carefully analyse their learners’ errors, make an effort to understand the errors, and establish the causes of these errors. Subject to the findings of the analysis, teachers need to cultivate corrective means and methods of teaching so as to deepen their learners’ understanding of mathematical concepts as well as improve their learners’ reasoning methods. Teachers require specific knowledge about errors and the methods of response to these errors in order to achieve what they set out to do. Pedagogical content knowledge can therefore help the teacher recognise whether learner’s conceptions are truly misconceptions, establish their source, and help them to unravel the patterns of their misunderstanding (Shulman, 1986).

An understanding of common learner misconceptions, and effective strategies to help learners avoid them, is an important aspect of mathematical PCK (Graeber, 1999). Over and above trying to teach in such a way that learners avoid misconceptions, teachers need to also have methodologies for dealing with those that unavoidably arise. Once the misconception is acknowledged, teachers should then decide on which strategies to use. If re-teaching takes place then decisions should be made about what to emphasise and how. Cognitive conflict is one of the strategies, in which learners have to contend with a situation that challenges their current understanding with the hope that they will then re-evaluate those beliefs and change these current beliefs (Wastson, 2002).
2.2.1 Constructivism and learner errors in mathematics

Reviewing error correction and assessing whether learners’ misconceptions have been corrected are vital elements in the constructivist view of the development of learning. Although errors are inevitable and seen as an integral part of learning, they also serve as a valuable source of information about the learners’ learning process. Moreover, persistent errors are active knowledge constructions (Resnick, 1982) and therefore point toward the existence of flawed learning rather than the absence of learning. What the learner has learnt are inappropriate procedures and/or incorrect concepts.

Cognitive development theories view individuals’ attempts to resolve inconsistencies in their thinking as a crucial and vital component of their construction of concepts (Tirosh, 1990). Therefore, teaching methods that support constructivist theoretical perspectives are more likely to minimise the formation of stable inconsistencies than methods that do not view the learner’s mathematical view as a critical factor to be taken into consideration in teaching.

According to Piaget’s (1968) theoretical model of cognitive development, humans naturally endeavour to organise/arrange their thinking processes into the simplest structures possible. These simple structures, called schemes or schemas, progressively become blended and harmonised to become more sophisticated, complex structures. When thinking and cognitive processes grow into more complex ones, new schemas develop to enable greater adaptation to the environment. Piaget suggests that, even from birth, a person starts to search for occasions to adapt to the environment in a concise and efficient manner.

In Piaget’s model, this adaptation, or intellectual growth, consists of three major processes: assimilation, accommodation, and equilibration. **Assimilation** entails the integration of new events into pre-existing cognitive structures. **Accommodation** involves existing structures being amended to accommodate to the new information. This twofold process, assimilation-accommodation, allows the child to form schemas. **Equilibration** entails the person attaining a balance between him/her and the environment, between assimilation and accommodation. According to Piaget, the delicate balancing action of organising, assimilating, and accommodating occurs when real learning and cognitive growth take place. This is
to say, the achievement of this search for equilibrium is crucial to cognitive structuring and development. When a child experiences a new event, disequilibrium sets in up until such time that s/he is able to assimilate or accommodate the new information and consequently, over time, arrive at equilibrium. When new information is too isolated or too far removed from current understandings, neither assimilation nor accommodation may happen. Therefore, *perturbation* may follow. The result of this state is that one’s schemas are incapable of reconciling what one is expecting... in other words, a state of cognitive conflict is reached. When a learner experiences perturbation, s/he may or may not seek to regulate his/her behaviour. Piaget indicated that, as an alternative to seeking to regulate behaviours following perturbations, learners frequently repeat the behaviour without making any changes. One can see that perturbations are no guarantee that regulations and compensations of learning to cognitive changes will take place. On occasions where one experiences perturbations such as a failure to understand or has a feeling that a deeper level of understanding is possible, a search for logical coherence or cognitive restructuring is only one of several possible options. Oftentimes, learners ignore or repress cognitive failures and, in such instances, there is no cognitive restructuring (Kembizky, 2009).

Repression of cognitive failures can include self-deceptions where the errors or failures are not admitted by the learner. In other cases, amendments are made that are temporary and do not result in a real change. Therefore, being corrected by a teacher, parent or fellow learner may prompt a child to adjust his answer but this may not result in an adjustment in the cognitive system that is formulating the original incorrect answer in the first place (De Lisi & Golbeck, 1999).

He advocated that equilibration follows in three stages; first, learners are content with their method of thought and are therefore in a state of equilibrium. Next, they become conscious of the deficiencies in their existing thinking and are discontented and therefore enter a state of disequilibrium as they experience cognitive conflict. Finally, they take on a more advanced approach to thought that disregards the shortcomings of the old one (i.e., attain a more stable equilibrium).
2.3 VYGOTSKY’S SOCIOCULTURAL LEARNING THEORY AND LEARNER ERRORS IN MATHEMATICS

Learner errors in mathematics can also be understood on the platform of the socio-cultural theory of learning (Vygotsky, 1986) in that if learners are taught by teachers who themselves have many errors and misconceptions in mathematics, the learners inherit those errors and misconceptions. This is because the misconceptions are communicated to them by their teachers or the learners misinterpret what the teachers communicate to them. This occurs when learners do not understand the medium of communication such as the language or symbols that are used in teaching and learning mathematics.

Communication is critical in the development of mathematical understanding. Within a sociocultural perspective, learners who share their thinking about ideas with others and listen while others reveal their thoughts and reasoning will generate understanding of culturally recognised mathematical practices. Vygotsky (1986) asserts that communication is a cultural tool. He maintained that individuals get to learn meanings of a culture by internalising the meanings and being transformed by them as they learn to be fluent in their own language. In the same way, learners craft their own knowledge and cultivate mathematical meanings as they learn to clarify and rationalise their thinking processes to others. As they learn to communicate the language of mathematics, they modify their thinking of the mathematical concepts. The language is derived from society but the thought (concept) arises from the individual.

In accordance with Vygotsky (1986), all major cognitive activities are bred in an environment of social experiences. Subsequently, the social context in which a child is brought up and the child’s particular history are crucial contributing factors to the manner in which a child will think. In this process of cognitive development, language is an essential tool for influencing how the child will acquire ways of thinking because complex modes of thought are conveyed to the child by means of words (Davydov, 1995).

One important principle in Vygotsky’s theory is the concept of the existence of what he entitled the ‘zone of proximal development.’ The zone of proximal
development (ZPD) is the difference between the child's capability to solve problems on his own, and his capability to solve them with support. In other words, the actual developmental level refers to all the functions and activities that a child can perform independently. On the other hand, the ZPD includes all the functions and activities that a child or a learner can execute only with the assistance of a more knowledgeable person. The person in this scaffolding process provides non-interfering intervention, and could be a parent or teacher or a peer who has already mastered that specific task/function. It is in this space that learning takes place. The understanding of the individual is in the process of advancing and maturing but this is only achievable with the support of others.

Figure 1 The zone of proximal development (Adapted from Driscoll, 2005)
2.3 LITERATURE REVIEW

2.3.1 Definition of key terms

The following terms and definitions will apply in this study:

Errors can be defined as mistakes, slips, blunders, oversights or deviations from accuracy that are unintended and show no pattern in their occurrence. Errors made by learners can be broadly classified as: Systematic errors, where there is a repeated incorrect process of thinking that keeps reoccurring, usually linked to misconceptions; and as non-systematic errors where random errors occur with no evidence of a recurring incorrect way of thinking (Melis, 2004; Riccomini, 2005).

Misconception: Hansen, Drews, Dudgeon & Surtees (2005: 15) define a misconception as the misapplication of a rule, an under/over generalisation or an alternate conception of a situation. Hansen et al. also state that misconceptions are not limited to children who need additional support but that more able children also make incorrect generalisations. Khazanov (2008) likewise states that a misconception is a learner's incorrect concept that produces a systematic pattern of errors and is not just a mistake, nor is it because of a learner's carelessness.

Conceptual understanding/knowledge: Understanding of ideas and generalisations that connect mathematical concepts (Ashlock, 2001) and are rich in relationships (Hiebert & Lefevre, 1986). Specifically in this study, conceptual understanding relates to the meaning of equations, inequalities, relationships between quantities, and variables.


Conceptual errors are errors that result from the misinterpretation of a mathematical rule, definition, or concept. These errors occur when a learner is unable to derive meaning from the definition or explanations provided; and, therefore, cannot adequately translate the concepts described and apply them to a given set of problems. (Cite a source)
Procedural errors are errors that occur when devising an effective course of action in an attempt to solve a problem. These errors can often result when learners are too quick to respond to a problem, consequently overlooking essential procedural steps when using a formula. (Cite a source)

The purpose of this study was to determine some of the most prominent errors made by Matric mathematics learners and hence predict the views/beliefs they hold with respect to the quadratic equation sections of Matric algebra. From a constructivist perception of learning (e.g. Piaget, 1970; Skemp, 1979) knowledge does not simply transpire from experience; a person has the capacity to learn from experience and what he/she learns from an experience is dependent on the quality of the ideas that he/she brings to that experience. That is to say that knowledge arises from the interaction between experience and the new/current knowledge structures. The learner is therefore not passively receiving knowledge from whatever the teacher is teaching but makes sense of what they are learning with reference to what they already know and believe to work for them. For the learner to understand an idea, they must assimilate the new idea into an appropriate existing schema. There are, however, times when the new idea is very different from the existing schema and therefore it is hard to link it to any existing knowledge. The consequences are that the learner might then tend to learn the idea by rote, making it an isolated piece of knowledge which is not easy to remember. Rote learning of this nature results in learners making mistakes as they tend to remember only parts of the rules/method/procedure.

Identification of errors made in matric algebra can be of extreme value to curriculum developers and educators in designing teaching and learning materials that could aid in the development of an improved understanding of the Matric algebra. A large number of educators (Drews, 2005) do not have a good understanding of why learners make certain errors and they also do not have the time and resources to establish which mathematical knowledge learners bring to the classroom. This study hopes to provide educators with quick ways of identifying the errors that learners make. With the knowledge of what the possible errors are, educators are in a better position to anticipate learner errors and will be
better able to discuss these errors at appropriate points both in their lessons and in textbooks. Mathematics education could also improve if teachers had a greater understanding of the ways in which learners learn the principles of mathematics.

Studies in error analysis in mathematics and science (mainly in science) have examined and analysed the errors learners have been making for decades (NEXUS: National Research Foundation, 2009; Luneta, 2008) but studies on error analysis in mathematics in South Africa have not been extensively done and very limited documentation of such studies is available.

For a long time mathematics teachers have been concerned about the errors and mistakes made by their learners and this has been mentioned in a number of studies that diagnosed arithmetic errors (Radatz, 1979). These concerns go back to as early as the 1960s with Roberts' (1968) study as an example, where he categorised learners’ errors into four major groups. In the 1970s with researchers such as Piaget who proposed that learners do not come into the classroom as blank slates to be filled with knowledge but that they come into the learning situation with ideas that have worked well for them in a number of situations. Sometimes there is a problem with some of the ideas that the learners come with because these very ideas/concepts compete with the concepts that are presented in class and in a number of cases ‘new’ ideas do not have a chance of succeeding against the pre-existing ideas/concepts. Later on, in the 1980s, other researchers such as Blando, Kelly, Schneider & Sleeman (1989) coined the term ‘bugs’ to describe ‘stable’ errors made by learners. These types of errors occur when a learner is faced with a difficult or unfamiliar feature in a task that leads the learner to an impasse.

2.3.2 How do we define errors in mathematics?
Errors in mathematics can be categorised into two general groups, namely systematic and non-systematic errors (Radatz, 1979; Melis, 2004; Riccomini, 2005). According to Riccomini (2005), non-systematic errors are unintentional, non-persistent incorrect responses which learners can easily correct by
themselves without much intervention from the teacher. Systematic errors on the other hand, are recurring erroneous responses methodically constructed and produced beyond space and time. Systematic errors are symptomatic of a defective course of thinking causing them to be referred to as a misconception (Green, Piel & Flowers, 2008; Nesher, 1987; Riccomini, 2005).

In addition, Drews (2005) states that these systematic errors are not just restricted to children who need further assistance but that more able children also make incorrect generalisations. She similarly agrees that it is more difficult to correct these types of errors than merely pointing out a mistake to a learner; this type of action does not usually result in the learner necessarily not making the same mistake the next time they have to answer comparable questions. The reason given by Drews (2005) is that the method used by the learners makes sense to them and has worked in other situations but the method cannot be generalised to all situations. These errors are typically due to a misconception that the learner may have.

Drews (2005) uses the word misconception to describe misapplication of a rule, under/over generalisation or an alternate conception of a situation. That is, assuming that a rule applies in all cases and overusing that rule. In other words, errors made by learners can be broadly classified into systematic (where there is a repeated incorrect process of thinking that keeps reoccurring) and non-systematic errors (random errors with no evidence of recurring incorrect ways of thinking). A misconception can be described as over- or under-generalisation of a concept or it could be a misapplication of a concept or an immature/under-developed understanding of a concept. This implies that misconceptions are not mere slips but a general misunderstanding of an existing concept.

Erbas (2004) came up with similar notions for errors but used different terminology to explain the errors. Erbas describes errors as incorrect application and conclusion of mathematical expressions and ideas, and goes on to classify errors as ‘slips’ and ‘mistakes’. Slips as described by Erbas (2004) are occurrences when you intend to execute an action but fail to do so; if the intention was wrongly formulated then the error is a mistake. The slips as described by Erbas (2004) can
be loosely seen as having similar connotations as Drews (2005) non-systematic errors and bugs in a similar way can be viewed as systematic errors or misconceptions as Drews (2005) describes them. Erbas (2004) differentiates between slips, errors and misconceptions. Errors made by learners tend to point to underlying faulty conceptual structures. The defective underlying structures which consist of beliefs and principles are the cause of systematic conceptual errors which are called misconceptions.

Drews (2005) argues that learners’ erroneous thinking is an important component of the learning process. Just as Lopez-Valero, Fernández & Clarkson (2008) also argues that learners making errors and/or mistakes is a natural part of learning; the teacher may tend to react to these mistakes in a number of ways including merely recognising the error, treating it or overlooking the error. Making mistakes plays an important role in indicating to teachers the stage learners are at as well as showing where there is a need for further teaching or study Li (2006). Learners’ mistakes help teachers to advise learners on what to do to improve and how to improve. Li (2006) alleges that examining learners’ incorrect answers may offer a way to reveal learners’ understanding of a concept but in the same breath, learners’ correct answers do not necessarily signify a sound conceptual understanding of the associated knowledge because learners could have solved the problem suitably by remembering procedures short of a good understanding.

2.3.3 Error analysis
Ketterlin-Geller & Yovanoff (2009) define error analysis as a method commonly used for identifying learners’ errors in mathematics. That is, error analysis is the practice of appraising learner’s answers to specific questions to identify a pattern of misunderstanding. Luneta (2008) uses the terms error analysis or error diagnosis to describe the activity of discovering what errors learners are making, why those errors are being made and the most suitable methods of dealing with the errors made. The focus of error analysis is on the weaknesses of learners and is meant to help teachers classify learners’ mistakes (Ketterlin-Geller & Yovanoff, 2009).
**Error analysis** is the process of reviewing learners' responses to specific questions in order to identify a pattern of misunderstanding. Errors can be categorised into two groupings, namely slips and bugs. Slips are unintentional errors in learners' procedural knowledge that are not the consequence of in-built misunderstandings in the subject/topic. Bugs represent persistent misconceptions about a subject/topic or specific knowledge or skills that consistently impede learners' ability to acquire new knowledge or skills.

Ketterlin-Geller & Yovanoff (2009) furthermore talk about cognitive diagnostic assessment as a process of identifying persistent misconceptions in order to design supplemental instruction/interventions. From this one can see that diagnosis takes on different meanings and is often approached from a number of perspectives. Significant variations exist with reference to the definition of diagnosis in education (Ketterlin-Geller & Yovanoff, 2009).

Although error analysis can offer timely information for altering teaching so as to avoid reinforcing incorrect procedures, this information may not always provide insights into the cognitive aspects learners have or have not mastered that form the basis for designing remedial instruction or supplemental interventions. It is unfortunate that teachers tend to focus on correcting the procedural errors that are evident from error analysis without realising the conceptual understanding that is responsible for the foundation for skill application (Russell & Masters, 2009).

### 2.3.4 How do learners make errors and why do they make these errors?

An error made by a learner in mathematics or in any other subject involves a complex and multifaceted process and it is not always easy to isolate a single factor that is responsible for why the error is made.

**Systematic errors** are caused by learning difficulty or difficulties, that is, by difficulties or challenges in understanding the underlying theory, concepts, or processes. According to Kempa (1991) and Furio, Calatayud, Barcenas & Padilla (2000.), a learning difficulty exists whenever a learner fails to grasp a concept or idea as a result of: (i) the nature of ideas/knowledge possessed by the learners; or (ii) the inadequacy of such knowledge in relation to the concept to be acquired.
Using more standard terminology, we can assume that what we have here have been termed as *alternative conceptions* or *misconceptions* (Kousathana & Tsaparlis, 2002).

Non-systematic or random errors are not caused by lack of relevant knowledge or by a misconception, but by an *overload of working memory, hastiness*, recklessness or field *dependence*. These errors could also be caused by a combination of the factors mentioned above. In other words, the learner should be able to correctly answer the relevant question if the question is given to him/her at a different time (Kousathana & Tsaparlis, 2002).

A note of caution should be added here: the possibility exists that what might be a random error for one learner may be a systematic one for another. Hence, the actual distinction between the two types of errors is not always clear. A qualitative study that would probe in depth the learners’ reasoning behind the errors is more appropriate for making definite the above distinction.

Learner misconceptions are established from their own personal experiences. Posamentier, Hartman & Kaiser (1998) declares that, ‘Because learners have actively constructed their misconceptions from their everyday experiences, they are very much attached to them and find them difficult to give up’. Furthermore, they state that some of learners’ misconceptions are plausible and valid while some are not plausible and valid. By identifying the origins of misconceptions, teachers can work out strategies to minimise those misconceptions that are not valid.

There are a number of strategies available to resolve learner misconceptions. Drews (2005) says that learner meta-cognition, that is, thinking about their thinking, activated learners to take charge of their learning and counteracts misconceptions. Several authors support the claim that learners must confront their misconceptions (Posamentier et al., 1998; Hale, 2000). Posamentier et al. (1998) suggest that learners receive guidance to replace their misconceptions with valid and plausible conceptions as is also seen from Vygotsky’s theory of ZPDs where Vygotsky claimed that learning most effectively takes place when a
learner/child is engaged in activities with support and guidance from a more knowledgeable other, i.e. the teachers.

Manly & Ginsburg (2010) and Radatz (1979) suggest that errors in mathematics could be as a result of but not exclusive to the following:

2.3.4.1 **Language difficulties**
Mathematics is like a ‘foreign language’ for learners who need to know and understand mathematical concepts, symbols, and vocabulary. Misunderstanding the semantics of mathematics language may cause learners’ errors at the beginning of problem solving. Rothery (1980) observed 3 groups of mathematical words:
1. Words which are unique to mathematics and not usually encountered in everyday language (e.g. hypotenuse, coefficient).
2. Words which appear in mathematics and normal everyday English, but may have different meanings in these two settings (e.g. difference, volume).
3. Words that have similar or almost the same meaning in both contexts (e.g. fewer, between).

Johnston-Wilder & Johnston-Wilder, Pimm & Westwell (2005) argue that learning to understand, appreciate and use mathematical language is a crucial part of learning mathematics. They go on to say that teachers who have become accustomed to the language used in mathematics take it for granted, and fail to realise how unfamiliar and puzzling this unique language can be to those who are still grappling to learn how to use it. Adler, Davis, Webb, Parker & Kazima (2004) reveals that learners particularly in Africa where the language of teaching and assessment is not the same as learners’ mother tongue achieve relatively lower marks/grades especially in mathematics and science than learners taught in their mother tongue. They make errors due to a lack of understanding of the item and not necessarily because they are incapable. Consequently the learners with language barriers misconstrue the question because they are unable to understand the concept.

Stacey & MacGregor (1997) looked at causes of learners’ common misunderstandings and mentioned that the grammatical rules of algebra were not
the same as ordinary language rules. For example, they concluded that some learners interpreted \( a = 28 + b \) as ‘\( a \) equals 28, then add \( b \)’; this is because they read the equation like they would ordinary English. In another example, Johnston-Wilder & Johnston-Wilder, Pimm & Westwell (2005) report that in a mathematics equation, the signals for ordering of operations is not the same as those of an ordinary language. These may include brackets, which are not used in the same way in an ordinary language, and more subtle signals that must be deduced from knowledge of formal rules for the precedence of operations. Because of the unique nature of algebraic language, teachers need to assist learners in accepting and appreciating that algebra is an atypical language that has its own conventions and uses familiar symbolisms in new ways. Furthermore, the language used by teachers and textbooks may confuse some learners.

Common everyday use of certain terms, often used in non-mathematical/scientific contexts, plays an important part in learners’ misunderstanding. Certain words have many diverse meanings in the English language and the ‘mathematical/scientific word’ can simply be confused with a common use (Stepans, 1994).

### 2.3.4.2 Inability to extract spatial information

Even though many other assortments of texts contain pictorial features, in almost all these cases, they are used to illustrate the accompanying verbal form or to demonstrate information covered elsewhere in the text (Johnston-Wilder & Johnston-Wilder, Pimm & Westwell, 2005). They go on to state that in mathematics, conversely, graphs, tables and diagrams are often used independently to convey information that may not be presented in any other format. Here we cannot presuppose that learners will intuitively develop the skills required to make sense of such diagrammatic formats. Illustrations, drawings, and 2/3-dimensional figures/models in textbooks and other teaching/learning materials can be misleading/give the wrong impression, and may result in misconceptions (Norton & Cooper, 1999; Welder, 2009).

Radatz (1979) also suggests that elementary and secondary school mathematics textbooks show an increasing tendency toward the iconic and visual
representation of mathematical information. These representations make extreme demands on learners' spatial abilities and capacity for visual discrimination. Such demands are less content-specific, for example, appropriate to the teaching and learning of geometry than they are representation specific for all the content of school mathematics. Many mathematical errors result from considerable individual variances in school-children's spatial imagery and spatial thinking (Jakimanskaya, 1976) and subsequent difficulties some children have in attaining visual or spatial information in performing a mathematical task.

2.3.4.3 Faulty understanding of prerequisite knowledge, skills, facts and concepts

There are some beliefs caused by personal experience, intuition, and ‘common sense’ that could lead learners to formulate their own ideas, concepts and models, in a number of cases, well before they are exposed to formal teaching. These experiences seem to conflict what learners learn from their textbooks and/or are taught by their teacher. Even after teaching, it is often hard for learners to relinquish these ideas, or they may regress back to them later, even though it appears they may have ‘learnt’ and accepted the correct ideas in class.

Errors can also be caused by the learners’ own formulation of rules that work well for certain questions but not in general.

2.3.5 Research on the teaching and learning of algebra

When assessing learners’ ability to solve algebraic problems, research has shown the many errors and misconceptions in their thinking are due to knowledge gaps in basic algebra (Norton & Cooper, 1999; Darr, 2003 and Warren, 2003). Wood (2005); González, Ambrose & Martínez (2004); Knuth, Alibali, Hattikudur, McNeil, & Stephens (2008) and Welder (2009) specifically highlight the difficulties experienced by learners when working with different aspects of algebra. These include errors in the concept of equality; concept of variables; order of operations; order of operations in inequalities; exponentials; simplification/ factorisation of algebraic equations; the zero product property; solving equations instead of inequalities; multiplying/dividing inequality by factors that are not necessarily positive; and forming meaningless connections with quadratic roots. If these
difficulties and challenges are experienced by learners it is possible that their teachers experience the same misconceptions as a result of their educational background and experiences. This would imply that teachers need to participate in interventions to keep them abreast and aware of the latest ideas and approaches geared towards assisting learners in facing the difficulties and challenges mentioned above and gaining a greater understanding of the algebraic concepts. Hadjidemetriou & Williams (2002) and Bernstein (2007) found that in a number of cases teachers misunderstood or could not correctly answer questions and demonstrated inadequate understanding of particular concepts. The teachers assessed in their study also overestimated the complexity of some questions because they could not envisage a solution without using advanced knowledge of gradients when this knowledge was not necessary (Hadjidemetriou & Williams, 2002).

Teachers’ knowledge of learners’ errors and misconceptions is considered by many researchers as a valuable tool to facilitate the reformulation of learners’ understandings in order for them to be able to solve mathematics problems and for making sense of the processes followed (Cornu, 1991; Even, 1998). Researchers have proposed a number of ways of distinguishing between the many facets of knowledge required by a teacher to teach effectively.

2.3.5.1 Erroneous connections and/or rigidity of thinking

It is a well-known fact that pupils who have learnt to solve quadratic equations by factoring, for example:

\[ x^2 - 5x + 6 = 0 \]
\[ \Rightarrow (x - 3)(x - 2) = 0 \]

So, either \( x - 3 = 0 \) or \( x - 2 = 0 \), tend to make the following error:

\[ x^2 - 10x + 21 = 12 \]
\[ \Rightarrow (x - 7)(x - 3) = 12 \]

So, either \( x - 7 = 12 \) or \( x - 3 = 12 \)

This type of error is very difficult to eliminate or is, at least, very hard to eradicate permanently. Even with more able pupils, receiving excellent teaching emphasising the special role of zero, this error often continues to crop up in
learners’ work. In spite of careful explanations of why it is an error and despite temporary elimination of the error, it keeps cropping up.

Matz (1982) presents a theory that clarifies why this error persists. There are two levels of procedures responsible for cognitive functioning: surface-level procedures, which are the basic rules of arithmetic and algebra, and deep-level procedures, which create, modify, control and in general guide the surface level procedures. An example of such deep-level guiding principle is the overgeneralisation of numbers, which, in effect says that ‘the specific numbers don’t matter – other numbers could be used’. This is a very important and in most cases a very necessary observation, which comes naturally to children, e.g. when learning to add, say 52 + 43 by column addition, a child will never master arithmetic if he believed the method works only for 52 + 43. He must believe that the method also works for 34 + 23 and 46 + 21 or any other sum than 52 + 43, and should also for combinations s/he has never seen before. Thus, in order to learn arithmetic a learner must have such a deep-level procedural understanding for generalising numbers.

2.3.5.2 Overgeneralisations of numbers and number properties may be the single most important underlying cause of pupil’s misconceptions.

This is exactly what happens in the case of the quadratic equation. In \((x - 3)(x - 2) = 0\), the numbers 3 and 2 are not critical to the method, but the 0 is. Learners should and can therefore generalise:

\[(x - a)(x - b) = 0\]
\[\Rightarrow x - a = 0 \text{ or } x - b = 0 \quad \text{-------------------------}(1)\]

Learners, who fail to recognise the critical nature of the 0, treat it just as they do the other numbers and over-generalise:

\[(x - a)(x - b) = c\]
\[\Rightarrow x - a = c \text{ or } x - b = c \quad \text{-------------------------}(2)\]

Equation (2) would be a correct generalisation of Equation (1) only if generalising were appropriate in this case. Regrettably it is not. This explains why the error is so stubborn and resistant to change, in spite of the teacher’s best efforts, and
despite learners’ best intentions: it is not just a matter of learning; it cannot simply be erased from memory, because it is consistently being re-created by a reasonable deep-level guiding principle. What is missing is a critic – a danger signal that in this particular case would be that the application of the deep-level procedure is wrong, which most likely only comes with experience of making such mistakes. This example shows the sensibility of learners’ errors and how learners’ misconceptions are not just random, but originate in a consistent conceptual framework based on earlier acquired knowledge.

2.3.5.3 The inability of learners to differentiate the many uses of variable (letters) in algebra

The use of letters in algebra is not the same as their uses in other contexts. Learners are normally told that in algebra, letters stand for numbers; however, they see letters used with other meanings. Letters are used in many contexts, both within and outside mathematics, as abbreviated words or as labels such as p. 6 meaning page 6 or cm meaning centimetres and \( \angle \)ABC meaning an angle in geometry. Although teachers may use the letters to stand for numbers and sometimes for names of objects, some learners see these letters as standing for the words themselves. It becomes very important for teachers to emphasise that letters in algebra stand for numbers and not necessarily for names of objects (Ryan & Williams, 2007; Usiskin, 1988; Stacey & MacGregor, 1997).

The symbolic notation for algebra, which is a meaning familiar from the arithmetic context, has at times different meanings and uses in algebra; too often children see arithmetic as an activity isolated from their ordinary concerns. Many misconceptions and faulty thinking in algebra are related to misconceptions and faulty thinking with arithmetic.

2.3.5.4 The teacher

The approach a teacher takes when teaching a concept in mathematics is influenced by their own conception of those concepts, as well as what s/he wants the learners to be able to do with those concepts (Swan, 2006). For example, if the teacher has a conception that mathematics is just about doing procedures correctly, s/he will teach a mathematical formula and show the learners how to use
it and then will expect the learners to be able to apply it. On the other hand, if the teacher has a conception that mathematics is a reasoning science, the teacher expects the learners to be able to analyse the mathematical formula, decide whether it has a solution and apply that formula only if necessary.

Sometimes even the demonstrations used by teachers usually do not involve active participation of learners but they sit back and merely observe with no opportunity to have a hands-on manipulation of materials or experiencing the phenomenon individually or in small groups Stepans (1994).

Bernstein (2011) writes that the important and major role played by teachers in the performance or non-performance of learners is pointed out. The report also confirms that the poor performance of teachers is a major reason for the poor performance of learners in the South African schooling system.

Swenson’s (1998) work on probability also showed that the teachers in the study generally (a) lacked an explicit and connected knowledge of probability content; (b) held traditional views about mathematics and the learning and teaching of mathematics; (c) lacked an understanding of the ‘big ideas’ to be emphasised in probability instruction; (d) lacked knowledge of learners’ possible conceptions and misconceptions; and (e) lacked the knowledge and skills needed to orchestrate discourse in ways that promoted learners’ higher level learning.

2.3.5.5 The curriculum
Mathematics knowledge normally found in books tends to be impersonal in the expression of theory with all the motivation, hesitation and mistakes experienced by the researcher whilst doing and writing up the research having been removed. The text is not contextualised, no history is included, neither is the problem stated that led to the results; a good example is the quadratic formula. Boaler (1998) came to the conclusion that a traditional textbook approach (which tends to be the norm in South African schools) gives emphasis to computations, rules and procedures to the detriment of deep understanding of concepts, and is disadvantageous to learners because it fosters learning that is rigid, ‘mathematics classroom bound’ and is of very limited use (Reddy, 2005).
To add to this, the report entitled: *Teaching Junior Secondary Mathematics, Module 4: Algebraic Processes* (2001) also confirms that most traditional mathematics books display algebra as no more than a manipulation of ‘letters’. The algebra presented is not realistic, practical and applicable. If the learners cannot relate to $A + B$, and $(a + b)(c + d)$, what practical situation relates to such expressions? Where do they meet $a(c + d)$, apart from in a maths book? Little to no obvious use of algebra is necessary in the majority of jobs. In the workplace and in the day-to-day situations it is not required to expand expressions (remove brackets), to collect like terms, to factorise (insert brackets) or to solve equations. Nonetheless all these do exist in most if not all syllabi in mathematics in many countries. This is extremely hard to defend; that is, the value of using letters to represent variables, because learners do not see any necessity for it. This is made worse partly because the examples used can be done without the use of letters.

Misconceptions were also found to be a product of misleading teaching materials (Welder, 2009). Stacey & MacGregor (1997) tested learners across three schools numerous times during the course of a 13-month period; results revealed that learners in one school had noticeable difficulty with letter usage in algebra and persistently misinterpreted letters as abbreviated words or labels for objects.

### 2.3.5.6 The environment

A number of learners think of mathematics as a very difficult subject; as a consequence of this belief, the learners have low anticipation of succeeding and therefore may decide to avoid taking mathematics altogether (Maseko, 2002).

Resnick (1982) attributed learners’ learning difficulties to conceptual learning: ‘difficulties in learning are often a result of failure to understand the concepts on which procedures are based.’

Similar factors were identified by M. Ed. graduates (Lim, 2000; Noridah, 1999; Radiah, 1998; Rashidah, 1997) at the Department of Science and Mathematics Education, Universiti Brunei Darussalam, and include: Learners not recognising key words or symbols; poor comprehension of the key words, symbols or the question; inability to transform sentences to mathematical forms; inability to
complete the operation accurately though the correct operation is selected; even when the correct operations are carried out, some learners write the answer incorrectly.

Having discussed the reasons why learners make errors I will go on to explain why it is important for teachers to be aware of learners’ common errors. Learners’ errors are of particular interest in the constructivist theory of learning because these deviations from accuracy tend to shed some light on how the learners, at a particular point in their conceptual development, are organising their understanding of a concept (Craighead & Nemeroff 2004). The constructivist viewpoint on learning (e.g. Piaget, 1970; Skemp, 1979) presumes that concepts are not taken directly from a given experience, but that a person’s capacity to learn from and also what he learns from an experience depends on the quality of the ideas that s/he is able to bring to that experience. Knowledge does not simply arise from experience but arises from the interaction between experience and current knowledge structures, i.e., the new knowledge that is being introduced.

This is also true for the Vygotskian view of teaching and learning where the teacher assists learners to learn conceptual ideas that are effective; helps learners to develop more sophisticated conceptual structures and methods; and forces the teachers to be critical of activities they present to learners (Maddux, Johnson & Willis, 1997). The teacher must be aware of the typical errors learners make and misconceptions they may have in algebra so that they are able to adapt their teaching practices appropriately.

The Working Group on Teachers’ Knowledge for Teaching Algebra at the 12th ICMI Study conference on The Future of the Teaching and Learning of Algebra, held at the University of Melbourne, December, 2001, suggested that appropriate teacher practices should by necessity be linked to the learner learning of algebra, which should be typified by: Reflection; knowledge about oneself as a teacher and as well as a learner; a sense of the role of time in development of concepts and in organising the teaching/learning materials; the ability to contextualise and de-contextualise an event especially in algebra; the involvement of learners in decision making; and the norm in which algebra is taught.
A number of researchers (Socas, 1997; Blanco & Garrote, 2007; Li, 2006; Erbas, 2004) have suggested various leading causes of errors in learning algebra and these include:

a. Errors that emanate from some obstacle, such as the lack of closure, that is to say, learners see algebraic expressions as statements that are at times incomplete. Hall (2002) suggests that learners tend to be reluctant to stop before getting to an answer they are comfortable with which is usually a numerical answer. In other words, if there are few signs as to when a suitable answer is reached, the learner may continue simplifying until she arrives at one acceptable to him/her. At the point of oversimplification, some errors may be identified which could have remained undetected had the learner simply stopped earlier. One reason for this type of error may lie in the unwillingness of learners to accept ‘lack of closure’ as suggested by Hoyles and Sutherland (1992) Previous studies have found that many learners cannot accept that an unclosed algebraic expression is an algebraic object. They are therefore unable to accept that an expression of the form $x + 3$ might possibly be the solution of a problem. This kind of error may suggest lack of knowledge of the difference in meaning of an expression and an equation. Such a “lack of closure” as experienced by learners may be found to be a contributing factor to the production of errors, or at least a misunderstanding of the very objective of trying to simplify an expression. As Tirosh (1990) point out, learners often face cognitive difficulty in accepting lack of closure: they conceive open expressions as incomplete and have a tendency to want to ‘finish’ them, mimicking the final number answer found in arithmetic.

b. Errors that emanate from a lack of meaning. According to Blanco & Garrote (2007) the lack of meaning is one of the primary challenges arising when working with the inequalities section of algebra. Blanco & Garrote’s (2007) study on inequalities rationalise that unless teachers provide learners with a clear idea of the concept of comparable inequalities and present the
techniques of solution with semantic content, then they are reducing the learning of inequalities to mere mechanical tasks. Reddy (2005) also concluded that learners in South Africa were not taught mathematical concepts in a purposeful manner to assist learners to understand and apply its principles learnt in their environment. If learners are taught abstract ideas without meaning, then there will be no understanding. Learners need to be provided with meaningful experiences with a concept in order to develop meaning for themselves. If teachers want learners to know and appreciate what mathematics is as a subject, then they should understand it. When teachers expect learners to memorise rules for moving symbols around on paper they may be learning something, but they are not learning mathematics (Hiebert & Wearne, 1993).

c. The complexity of the objects and of the processes of algebraic thought. Blanco & Garrote (2007) propose that learners might have not understood the mathematical entities involved in the inequalities with respect to incorporating definition, several systems of representation, properties and applications. They conclude by suggesting that this state of affairs is as a result of traditional teaching methods that put a lot of focus on developing procedures and techniques which at times do not pay attention to the meaning of the entities that are being used.

d. Learners’ affective and emotional dispositions towards algebra. There has been a prevalent perception that mathematics learning is a destiny, as one girl in the Whitebread and Chiu (2003) study stated: ‘I feel mathematics belongs to people who are really smart’. In the same study one of the four teachers also said that ‘mathematics is a very special school subject. Only in mathematics, not in other subjects, if you can do, then you can do it. If you cannot do, then you cannot.’ In the same study, some learners were classified/seen as ‘anti-mathematics’; a possible explanation for this ‘anti-mathematics’ is that their learning anticipation and styles were not rewarded by or not a good match with mathematics or the current teaching, as was revealed by their significant low disposition toward teaching. In interview data these children showed a view of mathematics as a tool only, not an aim in itself.
In a study done in South Africa by Mji & Makgato (2006) they concluded that as far as motivation and interest is concerned in the learning of mathematics and science, learners seemed not able to identify shortcomings from their side. Blame was mainly apportioned to teachers, a lack of textbooks, and even the school. For example, this learner, Kgomotso, pointed out: ‘I do not see how I can be motivated, there are few textbooks at school ... nobody cares.’ Interestingly, the teachers generally questioned the learners’ commitment. One teacher insisted ‘... if they [learners] can behave ... come to school on time, do the homework, concentrate in class ... motivation would not be a problem’.

2.3.6 Discourse on errors and misconceptions

There are very few studies on error analyses of mathematics in South Africa (NEXUS: National Research Foundation, 2009; Luneta, 2008) but even in these studies, none have focused on an error analysis to classify the errors made by South African learners in the algebra section of the Grade 12 Mathematics Examination Paper 1. Algebraic concepts form the foundation of most of the Grade 10-12 mathematics curriculum and without a reasonable understanding of algebra it is difficult to succeed in secondary school mathematics (National Curriculum Statement Grades 10-12 (Mathematics), 2003). Sound mathematical skills are crucial in literally all aspects of life and they are also a necessity in many roles played by adults. For example, research has revealed that proportion is used in virtually every workplace studied (Marr & Hagston, 2007; Selden & Selden, 2001).

Li (2006) argues that the conceptual basis for misconceptions and the errors that are associated with them could be found in the nature of mathematical knowledge, that is, the conceptual and procedural knowledge in mathematics. Rittle-Johnson, Siegler & Alibali (2001) describe conceptual knowledge as either implied or overt understanding of the principles that regulate a domain and the interconnections between units of knowledge in a domain whilst procedural knowledge is seen as the ability to perform systematic action in order to solve problems. Radatz (1979) argues that learners’ errors are as a consequence or the outcome of prior learning in the mathematics classroom. Blanco & Garrote (2007) note that research consistently indicates that misconceptions are deeply entrenched and hard to be
altered; in many cases, learners seem to have gotten rid of a misconception only to have the same misconception reappear at a later time.

Misconceptions are particularly invaluable for exposing the ways in which learners understand and represent problems or questions internally (Kembitzky, 2009). Piaget (1968) argued that by uncovering the origin of errors, one can learn as much, or even more, about the mental representations of people than by examining correct performance alone.

2.3.7 What is algebra and what are the common errors on the topic?
Algebra has been described by a number of researchers (Kriegler, 2008) as a field with several aspects including the abstract arithmetic, the language, and the tool for the study of functions and mathematical modelling aspects. Below is a description of how some researchers have made sense of algebra:

Greenes and Findell (1998) describes algebra as consisting of representation, proportional reasoning, balance, meaning of variable, patterns and functions, inductive reasoning, and deductive reasoning. Herbert and Brown (1997) describes algebra as using mathematical symbols and tools to analyse different situations by (1) extracting information from the situation; (2) representing that information mathematically in words, diagrams, tables, graphs, and equations; and (3) interpreting and applying mathematical findings, for example solving for unknowns, testing conjectures, and identifying functional relationships. Kaput (1998) describes algebra as involving the construction and representation of patterns and regularities, deliberate generalisation, and most important, active exploration and conjecture. Kaput (1998) offers five aspects of algebraic reasoning, that is; (a) the generalization and formalization of patterns and constraints, (b) syntactically guided manipulation of (opaque) formalisms, (c) the study of structures abstracted from computations and relations (but not limited to generalized arithmetic), (d) the study of functions, relations, and joint variation, (e) a cluster of modelling and phenomena-controlling languages. Herbert and Brown (1997) describe algebra as involving the development of mathematical reasoning within an algebraic frame of mind by building meaning for the symbols and operations of algebra in terms of arithmetic. Algebra is the ability to reason about functions as well as how they work and about the impact on calculations a
system’s structure has. NCTM Standards (5-8) - Algebra (NCTM, 1989) describe algebra as consisting the concept of variable, expression, and equation; represent situations and number pattern with tables, graphs, verbal rules, and equations, and explore the interrelationships of these representations; analyse tables and graphs to identify properties and relationships; develop confidence in solving linear equations using concrete, informal, and formal methods; investigate inequalities and nonlinear equations informally; apply algebraic methods to solve a variety of real-world problems and mathematical problems. Lodholz (1999) argued that in order for learners to be successful in algebra they need to have a good understanding of the following: the technical language used; the concept of variable and the concept of relationships and functions. Usiskin (1988) describe algebra is a language with five major aspects: (1) unknowns, (2) formulas, (3) generalized patterns, (4) placeholders, (5) relationships. Vance (1998) describes algebra as generalised arithmetic or as a language for generalizing arithmetic. However algebraic more than a set of rules for manipulating symbols: it is a way of thinking. Algebra is viewed as multiple strands, a symbolic, graphical, and numerical field of mathematics and of its role as a rich language for representing real-world situations and their underlying meanings. Properly understood as a mathematical language for modelling real phenomena, such aspects as patterns, generalisations of arithmetic, variables, equations, and the idea of function emerge within meaningful contexts.

From the various descriptions of algebra one sees distinctive strands of algebra; these include: the unique language of algebra, the various ways of working with variables, functions and relationships and the different representations of these and the modelling aspect of algebra.

Success in mathematics is largely affected by a good understanding of concepts found in algebra. Each division of mathematics makes use of fundamental ideas of algebra to reason about and model various phenomena. Some of the most influential and central ideas in algebra are the concept of variable, equality, functions and relationships and understanding the unique ways language is used in. Considering that these concepts play such a critical role in the learning of
algebra and many other aspects of mathematics, as well as the greater emphasis placed on passing mathematics in order to enter higher education, it becomes imperative that learners understand these concepts and are able to use and apply them correctly (Ladson-Billings, 1998).

A number of studies (Jackson & Ginsburg 2008, Kieran, 1992, Stacey & MacGregor, 1997; Brenner, Mayer, Moseley, Brar, Durán & Reed, 1997; Friel, Curcio and Bright 2001; Nathan & Kim 2007; Swafford & Langrall 2000) have been done the three most problematic algebraic concepts namely; variable, symbolic notation, and multiple representations.

2.3.7.1 Variable
Knuth et al. (2008) argued that variables are one of the core algebraic ideas and that the concepts of a variable play a very important role in problem solving as well as in thinking and communicating mathematically. Thus, understanding variables, equality, functions and relationships and the technical language of algebra are key prerequisites for success in the subject (Lodholz, 1999). The use of variable is especially important in algebra as it form the basis of generalisations. Learners find it difficult to discriminate between the various ways letters are used; from letters used in formulae, to letters representing a specific number to be determined, to letters representing a general number which is not one particular value to letters representing variables which could represent a range of unspecified values together with a systematic relationship between them; a variable used as a label (Kieran, 1992 & Li, 2006). Understanding the different use of variables is crucial for learners’ success in algebra without which could lead to learners making errors when solving problems. Secondly, some learners experience difficulty understanding that some of the symbolic notation familiar from arithmetic has different implications and uses in algebra. For example, the plus (+) and minus (−) symbols signify executable operations (addition, subtraction) in arithmetic, but they also show negative and positive numbers as well as operations in algebra (Gallardo 2002; Kieran 1992 as cited in Kieran 2007). Lastly, multiple representations are used to describe, understand, and
communicate generalizations algebraically; these comprise symbols, tables, graphs, and verbal descriptions.

Usiskin (1988) further points to problems caused when some teachers try to teach an oversimplified conception of variable. He delineates the importance of granting learners the opportunity to explore the various ways in which variables are used. This is to offer learners opportunities to develop an understanding that the use of variables is inextricably related to the conceptions that are held of algebra and the use of algebra. The different uses of variables create a conceptual dilemma, the continuous-variable struggle, which most algebra learners need to come to terms with. Schoenfeld (1985) referred to a conception of a variable as an unknown, which implies that a variable has a fixed value that is unknown. Hence, the viewpoint of variable as an unknown and a constant are closely related. Philipp (1999) indicated that the viewpoint of variable as a variable quantity suggests that learners have the conception that variables represent many values and that they may assume an unlimited number of values for a particular variable. In order to address these difficulties the researchers all suggest that learners be given opportunities to explore the different uses of variables and that they are given opportunities to consider their own conceptions of variables. It therefore becomes important that when teaching algebra teachers need to be aware of the different uses and meaning of variables and that they are discussed explicitly with learners. Such discussions should be related to the context in which the variables were explored. Teachers need to be aware of the different ways these symbols are used and recognize the particular characteristics they exhibit in various contexts. (Wagner & Kieran, 1989). Also the process of resolving these natural differences in such representations can offer rich learning opportunities for learners. When learners are asked to explain the meanings behind their mathematical terms and procedures they are being afforded an opportunity to think more deeply about the underlying mathematical concepts, and to clarify what the symbols represent and defend why their actions make sense.

2.3.7.2 Equality
Even though learners encounter the equals sign from early in their primary school education, many learners misunderstand its meaning (Behr, Erlwanger, & Nichols,
1980; Falkner, Levi, & Carpenter, 1999). Research (Falkner, Levi & Carpenter, 1999; Ketterlin-Geller, Jungjohann, Chard, & Baker (2007) Knuth, et al., 2008) has shown that both younger and older learners alike have serious difficulties understanding the meaning of the equals sign. Learners need to understand that equality is a relationship expressing the idea that two mathematical expressions hold the same value. A lack of such an understanding is one of the major stumbling blocks for learners when they move from arithmetic towards algebra (Falkner, Levi & Carpenter, 1999; Kieran, 1981; Matz, 1982).

Learners see the equal sign as a procedural marking that tells them “to do something,” or as a symbol that separates a problem from its answer, rather than a relational symbol of equivalence (Falkner, Levi, & Carpenter, 1999; Kieran, 1989). This incorrect way of thinking could be made worse when using calculators because the “=” button on a calculator is used as an operational key that always returns an answer.

The correct interpretation of the equal sign is crucial to the learning of algebra, since algebraic reasoning is depended on fully understanding of equality and appropriately using the equal sign for expressing generalisations (Carpenter, et al., 2000). Additionally, in order to manipulate and solve equations learners need to understand that the equal sign represents a relation, and that the two sides of an equation are equivalent expressions, and that every equation can be substituted by an equivalent equation (Carpenter, et al., 2003; Kieran, 1981). Thus, Carpenter et al. (2003) consider that having limited understanding of what the equal sign means is one of the major challenges learners face in learning algebra.

2.3.7.3 Functions and relationships

Li (2006) states that by the time learners reach the middle school level; they should at least understand how the change in one variable causes change in another. Furthermore, learners are expected to know linear or nonlinear functions, be able to use a graph to represent a function, and be able to compare different functions. Li goes on to mention that one of the learners’ misconceptions of
function identified was that learners only consider the dependent variable without
the independent variable. Li furthermore mentions that the learners' errors related
to the above misconception did not change even after one year of instruction.

Adams (1997) in Eraslan (2005) did a study on learners' difficulties with the
concept of function and came to the conclusion that the learners' definition of
function was dominated by the ordered pair representation and the vertical line
representation. Consequently, learners viewed functions as collections of points or
ordered pairs.

In a different study on the concept of functions by Even (1998) where she studied
learners’ flexibility in moving from one representation to another and other aspects
of knowledge and understanding. She concluded that learners deal with functions
point-wise; for instance, they could plot and read points, but could not think of a
function as it behaves over intervals or in a global way. Some of the findings of
this study showed that subjects which can use a global analysis of changes in the
graphic representation have an improved and enhanced understanding of the
relationships between graphic and symbolic representations. These findings are in
line with the findings by Eraslan (2005), which tells us that learners tend to take a
point-wise approach in their understanding of functions.

As far as quadratic equations, Eraslan (2005) referred to obstacles to
understanding of the quadratic function. The first obstacle identified has to do with
the relationship between a quadratic function and a quadratic equation. Learners
treated a quadratic function as if it were a quadratic equation. The second
obstacle is related to the analogy between a linear function and a quadratic
function. When learners were asked to find the equation of a quadratic function on
the basis of three points on the graph, they first used two points to calculate the
slope between those points, and then inserted this value into the equation as the
leading coefficient.

The third obstacle showed up when one of the parameters in a quadratic function
was equated to zero. Learners rejected this kind of equation as an equation of
quadratic functions. In effect, this is a special case but it still belongs to the
quadratic family. The last obstacle identified was reported with respect to overemphasis on only one coordinate of special points.

Mathematics, algebra in particular, consists of structures which are concerned with the relationships between quantities (for example, are the quantities equivalent, is one less than or greater than the other); group properties of operations (for example, is the operation associative and/or commutative, do inverses and identities exist); relationships between the operations (for example, does one operation distribute over the other); and relationships across the quantities (for example, transitivity of equality and inequality) (Warren, 2003).

Learners often fail to understand the meaning linked with the formal symbols they use, these include operational symbols (Lannin, 2005). Researchers believe that this is because, just like the equal sign, some operational symbols are used in arithmetic in ways that are inconsistent with their meanings in algebra (Kieran, 1992; Kuchemann, 1978).

2.3.7.4 Unique language of algebra

Stacey and MacGregor (1997) noted that algebra is a unique language with its own rules, conventions and practices. Mathematical ideas often need to be reformulated before they can be represented as algebraic statements into symbolic notation. One of the difficulties for learners is to interpret these symbols correctly. The rules for interpreting and manipulating mathematical symbols are not always in accord with the way relationships are conveyed through the English language. This tends to be a cause of difficulties for learners. Another challenge is that of language-related translations in the form of converting real-world situations into equations, tables, and graphs (and vice versa) though an important way to understand and solve problems, it is not always straightforward. Houssart & Weller (1999) also noted that many of the learners in their study saw language used as a source of their difficulty, they did not all see the use of correct mathematical language as the solution.
Wilson (2009) believes and claims that, “the proper use of language is essential to the learning process.” He goes on to warn that, “the meaning of terms, operations, and symbols must be completely unambiguous or communication is lost and meaning slips away.” If learners are using terms, operations, or symbols that have a meaning in standard mathematics, we expect them to be using those terms correctly. Certainly, the incorrect use of mathematical terminology could develop or reinforce misconceptions about mathematical ideas and cause impasse in future learning.

According to Sönnerhed (2009), the algebraic problem solving process shows that learners first ought to translate problems communicated in daily words into algebraic structures by making use of algebraic symbols. After that the learners have to formulate the algebraic structures with specifically given rules. Then they have to solve the problems. All three steps necessitate that the learners are able to handle symbols and concepts and that they have the required skills needed for operations and also they must understand the contents of the problems.

### 2.3.8 How do teachers teach algebra?

The primary intention for teaching mathematics is to assist learners to learn and appreciate mathematics in the best way possible. With some innovation, passion, and resources available to them, teachers are able to put into practice various techniques and strategies in the classroom to make learning more meaningful and stimulating to their learners. In many classrooms, the typical way of teaching is the chalk-and-talk method. Teachers give the input verbally or write on the board and the learners follow their instructions (Chavez, White & Hock, 2009). Similarly research by Manly & Ginsburg (2010) suggests that the predominant mode of teaching numeracy to adults continues to be one of transmission where the teacher demonstration to learners the procedures, breakdown concepts down into smaller units and explain examples. The most dominant forms of organisation are whole class discussion and learners working individually through worksheets. Teachers seldom to ask higher-order questions, and there is minimum group or collaborative work, or is there much use of practical resources or ICT. The purposes for teaching and learning algebra are regulated by, or are connected to, the different conceptions of algebra. These are associated with the various uses of
variables. Evidence has shown that the transmission approach to teaching does not stimulate robust, transferable learning that is sustained over time nor does it result in knowledge and skills that can be used in non-routine conditions outside the classroom (Swan, 2006; Ofsted, 2006).

According to the report by Kesianye, Durwaarder and Sichinga (2001) the traditional, formal approach to teaching algebra is to look at algebra as a purely mathematical discipline with no emphasis on linking algebra to day-to-day circumstances. At the end of the algebra course, learners will have ‘done algebra’ without really realising a necessity for it, resulting in numerous errors. The report further notes that the contemporary trend is to place all mathematics including algebra, in a context. The emphasis here is on doing mathematics, recognising connections and modelling real-life situations and noting that algebra is a powerful tool in the hands of the learners on condition that they understand its uses and the limitations of the tool at hand.

Likewise Manly & Ginsburg (2010) state that algebra teaching is likely to focus on fundamental issues of symbol manipulation, simplifying expressions, and solving equations. This ‘quick fix’ approach is largely reliant on rote learning of progressions of actions and does not deliberately represent a coherent picture of algebra. Learners scarcely ever reach the kind of conceptual understanding and reasoning competence essential for the successful search of further goals. It is not unexpected then that this minimal teaching style is also inadequate to provide learners with enough information to select being in/out of developmental algebra. In addition to that, a number of teachers in South Africa are not appropriately qualified to teach mathematics (2011 CDE report) and are furthermore not comfortable teaching algebra to learners. Their teaching emulates their own experience with algebra, in which they focused on symbols and the apparently incomprehensible rules that show procedures using them in the abstract domain. It is a theoretical approach in which ‘real’ problems are inserted into some procedural lessons to provide practice in the ‘new skill’; this unfortunately serves mainly as puzzle-like addition and not ‘serious problem solving’.
The nature of typical algebra teaching is not surprising, given that the rote-learning features of algebra described above are also the main elements of assessments currently used in determining the competence level of learners for educational advancement. The outcome of such teaching is predictable — to help learners advance, teachers teach what they think will be tested, and the curricula narrow to meet the demands of tests/examinations Reddy, 2005).

Effective mathematics teachers know the key role played by mathematics in influencing how individuals manage the different areas of their private, social and civil lives. The mathematics teacher therefore plays a vital role in shaping learners’ thinking where mathematics is concerned. One of the main complaints about mathematics is the absence of meaning when doing algebra; learners often ask the question, ‘Where will I use this?’ (Anthony & Walshaw, 2008)

Following the framing of the South African Constitution, proposed approaches to teaching in South Africa shifted from being teacher-centred to being learner-centred; the teachers are facilitators of learning, in charge of developing teaching strategies in response to what they want their learners to achieve at the end of a teaching experience. This approach to teaching was meant to make mathematics more relevant to learners’ lives and for them to see where and how they would use the concepts learnt in class (Reddy, 2005). In her study, Reddy found that the main emphasis on teaching, in South Africa, is on content and preparation for the examination; most of the time learners are addressed through whole-class presentations and the teachers acknowledged that they used the textbook as their major resource when preparing questions instead of drawing up their own original mathematics questions related to the learners’ environment. Reddy (2005) further observed that the teachers did not utilise any other teaching resources when teaching mathematics and its use in their environment.

Anthony & Walshaw (2008) argue that effective mathematics teachers appreciate learner misconceptions and how learners build their thinking. Skilled mathematics teachers also incorporate meaningful experiences in the classroom to involve learners. By using these methods, learners should achieve more and take responsibility for their own learning.
In a research study conducted by Swan (2006) on the most common methods teachers used, his results showed the following:

Most common teaching methods (the statements are rank-ordered from most common to least common):

1. Learners are given work starting with simple questions and work up to harder questions.
2. The teacher tells learners which questions to attempt.
3. The teacher teaches the whole class at once.
4. The teacher knows exactly what mathematics the lesson will have.
5. Learners learn through doing exercises.
6. The teacher tries to cover everything in a topic.
7. The teacher avoids learners making mistakes by explaining things carefully first.
8. Learners work on their own, consulting a neighbour from time to time.
9. The teacher teaches each topic from the beginning, assuming they know nothing.
10. The teacher tends to teach each topic separately.
11. Learners use only the methods the teacher teaches them.
12. The teacher draws links between topics and move back and forth between topics.
13. The teacher tends to follow the textbook or worksheets closely.

On the other hand, he found the following least common teaching methods (the statements are rank-ordered from most common to least common):

1. The teacher only goes through one method for doing each question.
2. The teacher encourages learners to make and discuss mistakes.
3. Learners work collaboratively in pairs or small groups.
4. Learners learn through discussing their ideas.
5. The teacher jumps between topics as the need arises.
6. The teacher finds out which parts learners already understand and don’t teach those parts.
7. The teacher teaches each learner differently according to individual needs.
8. Learners compare different methods for doing questions.
9. The teacher is surprised by the ideas that come up in a lesson.
10. The teacher encourages learners to work more slowly.
11. Learners choose which questions they tackle.
12. Learners invent their own methods.

In the same research study conducted by Swan (2006) to establish which strategies learners used when learning concepts in mathematics, he came up with the following strategies: Most common learning strategies (the statements are rank-ordered from most common to least common):

1. I listen while the teacher explains.
2. I copy down the method from the board or textbook.
3. I only do questions I am told to do.
4. I work on my own.
5. I try to follow all the steps of a lesson.
6. I do easy problems first to increase my confidence.
7. I copy out questions before doing them.
8. I practise the same method repeatedly on many questions.
9. I ask the teacher questions.
10. I try to solve difficult problems in order to test my ability.
11. When work is hard I don’t give up or do simple things.

On the other hand, he found the following least common learning strategies (the statements are rank-ordered from most common to least common):

1. I discuss my ideas in a group or with a partner.
2. I try to connect new ideas with things I already know.
3. I am silent when the teacher asks a question.
4. I memorise rules and properties.
5. I look for different ways of doing a question.
6. My partner asks me to explain something.
7. I explain while the teacher listens.
8. I choose which questions to do or which ideas to discuss.
9. I make up my own questions and methods.

From these data one can deduce that the teachers’ tendency is more towards the ‘transmission’ mode of teaching where mathematics is seen as a body of knowledge and procedures to be ‘covered’. Learning is seen as an individual activity based on listening and imitating. Teaching is seen as structuring a linear curriculum for the learner; giving explanations and checking that these have been understood through practice questions and ‘correcting’ misunderstandings when learners fail to ‘grasp’ what is taught (Swan, 2006). This also shows little evidence of negotiation of meaning with the learners. The learners did the bare minimum but looking at the least common strategies for learning, the learners seem to lean towards a ‘collaborative’ mode of learning where mathematics is seen as a network of ideas which teacher and learners construct together; learning is seen as a social activity in which learners are challenged and arrive at understanding through discussion; and teaching is seen as non-linear dialogue in which meanings and connections are explored and recognising misunderstandings, making them explicit and learning from them.

Swan (2006) proposed that teaching is more effective when it:

*Builds on the knowledge learners already have* – This means developing formative assessment techniques and adapting our teaching to accommodate individual learning needs (Black & William, 1998).

*Exposes and discusses common misconceptions* – Learning activities should exposing current thinking, create ‘tensions’ by confronting learners with inconsistencies, and allow opportunities for resolution through discussion (Askew & William, 1995).

*Uses higher-order questions* – Questioning is more effective when it promotes explanation, application and synthesis rather than mere recall (Askew & William, 1995).

*Uses cooperative small group work* – Activities are more effective when they encourage critical, constructive discussion, rather than argumentation or uncritical
acceptance (Mercer, 2000). Shared goals and group accountability are important (Askew & William, 1995).

Encourages reasoning rather than ‘answer getting’ – Often, learners are more concerned with what they have ‘done’ than with what they have learnt. It is better to aim for depth than for superficial ‘coverage’.

Uses rich, collaborative tasks – The tasks we use should be accessible, extendable, encourage decision making, promote discussion, encourage creativity, encourage ‘what if’ and ‘what if not?’ questions (Ahmed, 1987).

Creates connections between topics – Learners often find it difficult to generalise and transfer their learning to other topics and contexts. Related concepts (such as division, fraction and ratio) remain unconnected. Effective teachers build bridges between ideas (Askew, Brown, Rhodes, Johnson, & Wiliam 1997).

Uses technology in appropriate ways – Computers and interactive whiteboards allow us to present concepts in visual dynamic and exciting ways that motivate learners.

2.3.9 How do learners learn algebra?

Shulman (1999) emphasised the fact that learning is a twofold process, with influences from both internal and external forces to the individuals interacting with one another. Furthermore, he reasoned that it was the learning already existing in the learner, not so much the teaching that had a crucial impact on new learning. Equally, Ausubel (1968) commented: ‘If I had to reduce all of educational psychology to just one principle, I would say this: The most important single factor influencing learning is what the learner already knows. Ascertain this and teach him accordingly’ (also cited by Shulman, 1999). There are three research approaches to learners’ conceptions: Piagetian approaches to cognitive development (Piaget, 1970) with an emphasis on the development of conceptions with time; the application of the philosophy of science on education research with a focus on learners’ perception, misconception, and conceptual change; and research on systematic errors (Confrey, 1990).

Some researchers agree with this notion claiming that ‘earlier learning constrains later learning’ (McNeil & Alibali, 2005), while other researchers (Anderson et al.,
have attributed some of the learning difficulties sometimes experienced by learners to the lack of the necessary proficiency or knowledge.

Conclusive research on how algebra is optimally learnt and taught is not readily available; at the same time, studies on different approaches have been diverse and at times conflicting (National Advisory Panel, 2008). However, various learning theories exist that describe how learners attain knowledge. The WNCP Mathematics Research Project (2004) suggests that these theories fall into two main conceptual categories, namely the ‘acquisitionist’ group that uses cognitive approaches to explain learning and knowledge in terms of notions such as schemes, models, concept images, or misconceptions; and the ‘participationist’ group that includes the theories that view learning as an activity integrating the learner with a community of practice. These two groups are not mutually exclusive of each other but aspects of each are often present in every theory; however, one tends to be more prominent than the other (Sfard, 1998).

Kieran (2008) argues that conceptual understanding of algebraic techniques should enable learners to do the following:

a. See a certain form in algebraic expression and equation, for example, the linear and/or quadratic form;

b. See relationships that are, for instance, equivalent in their different forms; and

c. See through algebraic transformations to the actual variations in form of the algebraic object and are able to give an explanation for these changes.

Traditionally, algebra has been ordered around the concept and methods for solving equations. Teaching concentrated on becoming skilled at symbolic manipulation, which Skemp (1979) calls ‘instrumental understanding’. Skemp describes instrumental understanding as the ability to recall an algorithm and having the capacity of executing it; while relational understanding of an algorithm is when one knows the purpose of the algorithm and why the algorithm works. This is to say that relational understanding comes about when a learner is able to both solve a problem and understand why the procedure used works. Instrumental
understanding occurs when a learner knows how to attain a correct answer without necessarily understanding why the process used works. Skemp reasoned that while it is easier to acquire an instrumental understanding, it is more challenging to remember what is learnt. Skemp thus concluded that repetitive computation is a necessary foundation from which children move from simple to more difficult problem-solving. Following his research, a number of research studies (e.g., Haapasalo & Kadijevich, 2000; Kilpatrick & Findell, 2001; Rittle-Johnson, Siegler, and Alibali (2001); Star, 2005) have investigated the effect of instrumental and relational numeric understanding on learner learning.

Past research on learners’ errors regarding subtraction and addition has produced impressive results. For example, the ‘buggy’ theory (Brown & Burton, 1978) can predict approximately 50% of learners’ errors even prior to learners actually doing the calculation (Resnick, 1982). Other studies (Chi, 2005; Slotta & Chi, 2006) in the field of science education also make available necessary research methods and theoretical frameworks about robust misconceptions and conceptual change, which provide clues for investigating errors and misconceptions in mathematics education.

Mathematical structure is concerned with the (i) relationships between quantities (for example, are the quantities equivalent, is one less than or greater than the other); (ii) group properties of operations (for example, is the operation associative and/or commutative, do inverses and identities exist); (iii) relationships between the operations (for example, does one operation distribute over the other); and (iv) relationships across the quantities (for example, transitivity of equality and inequality). In the traditional approach to algebra, it is implicitly assumed that learners are familiar with these concepts from their work with arithmetic. (Kieran, 1992).

2.3.10 Conclusion
I started this chapter with a discussion of the theoretical framework for the study. In this I discussed how learners’ errors and misconceptions can be understood and explained in their lenses. This was followed by a literature review that disused mathematical errors; as well as how and why they are made by learners. Later,
the nature of algebra and the common errors made by learners in algebra were discussed. Finally I discussed research on the teaching and learning of algebra.

Having outlined my theoretical framework and literature review, I now proceed to articulate my research methodology and design.
CHAPTER 3  THE RESEARCH methodology and DESIGN OF THE STUDY

3.1 INTRODUCTION

In the previous chapter a literature review was presented, situating the research in the context of what is already known about this topic. In this chapter I briefly discuss some research paradigms and situate my study in one of them. I also discuss the research methodologies employed to answer the research question: What is the nature of the prominent errors made by learners in Grade 12 when answering algebra examination questions? This will be achieved by identifying and analysing grade 12 learners’ errors and determining the possible misconceptions underlying the errors that learners make in quadratic equations, inequalities and simultaneous equations. In the main, the study did not use mixed methods although, I used qualitative methods do draw frequencies of the various types of errors in the scripts. At most, the study was done qualitatively in order to obtain full and balanced interpretations to the research questions.

Quantitative and qualitative methods are essentially in most studies and are not necessarily opposed to each other. They can be complementary as the former can be verified by the later (Sciarra, 1999). For this research, I have chosen both paradigms to provide a more inclusive picture of learners’ algebraic errors and misconceptions in quadratic equations, inequalities and simultaneous equations. Johnson & Christensen (2004) suggests that the only difference between qualitative and quantitative researcher is the starting point of the research on a ‘research wheel’. Figure 2 explains this diagrammatically.

![Research Wheel Diagram](image)

Figure 2 The ‘research wheel’. Adapted from Johnson and Christensen (2004:18)
In this way quantitative research is deductive and aims at proving or disproving pre-determined theories and hypothesis. On the other hand, qualitative research is grounded and seeks to generate and induced theories. Qualitative research is confirmatory whereas qualitative research is conjunctural.

### 3.2 QUANTITATIVE RESEARCH AND QUALITATIVE RESEARCH

There are two main research paradigms, the quantitative and the qualitative (Creswell, 2003). The author offers a discussion concerning the differences between the two models. **Quantitative research** has its genesis in the biological and physical sciences and measurement is primarily concerned with ‘*statistical analyses of numerical data*’. Quantitative methods are a good fit for deductive approaches, in which a theory or hypothesis justifies the variables, the purpose statement, and the direction of the narrowly defined research questions. The purpose of quantitative studies is for the researcher to project his or her findings onto the larger population through an objective process. Data is often collected through surveys administered to a sample or subset of the entire population and this allows the researcher to generalise or make inferences. Results are interpreted to determine the probability that the conclusions found amongst the sample can be replicated within the larger population. Conclusions are derived from data collected and measures of statistical analysis (Creswell, 2003).

**Qualitative research** owes its genesis to the behavioural and sociological sciences (anthropology, linguistics, history, philosophy, psychology, and sociology) and measurement is primarily concerned with *verbal and written descriptions and interpretations* (Olds, Moskal & Miller, 2005). Qualitative research is characterised by the collection and analysis of textual data from surveys, interviews, focus groups, conversational analysis, observation, ethnographies, and by its emphasis on the context within which the study occurs (Olds et al., 2005). The research questions that can be answered by qualitative studies are questions such as: What is occurring? Why does something occur? How does one phenomenon affect another? While numbers can be used to summarize qualitative data, answering these questions generally requires rich, contextual descriptions of the data, what is often called “thick” description. Several texts
provide descriptions and examples of qualitative research in the social sciences (Creswell, 2003; Denzin and Lincoln, 2005; Merriam, 1998 & Patton, 1990).

What is most fundamental in the choice of methods is the research question; the research methods should follow the research questions in a way that offers the best chance to obtain useful answers. Many research questions and combinations of questions are best and most fully answered through mixed research solutions. The qualitative approach takes precedence over the quantitative in this study in process of exploring learners’ errors and misconceptions in solving quadratic equations, and inequalities and simultaneous equations. This is because quantitative methods cannot explain the reasons why a learner thinks that a certain answer is right when in fact it is not correct. Such reasons can only be explained through use of words and not numbers, the main means parameter in quantitative research.

A qualitative approach was chosen as the main research method because this research paradigm allows the researcher to extract complex information from a number of sources using an assortment of methods to understand the perceptions and perspectives of participants in the research setting. This method suited the research since it concerns descriptions of thinking exhibited by learners’ answers to specific questions. The qualitative research approach is a method that provides a way to ‘figure out’ the understanding of the learners selected for the study and its measurement is concerned with verbal and written descriptions and interpretations (only written descriptions will be discussed). When considering an epistemological perspective, qualitative research can be described as ‘interpretive’ or ‘constructivist’ with an emphasis on understanding the world through the perspective of its participants (Bryman, 2004; Hatch, 2002; Yin, 1994). Merriam (1998) highlights the characteristics of qualitative research as follows:

1. It has to do with the meaning that individuals have constructed in their contexts. The contexts are found in an empirical field (Brown & Dowling, 1998) which incorporates the space where data is collected. Furthermore, Merriam (1998) contends that attaining a deep understanding of a particular phenomenon, to investigation beneath the surface of a situation and to offer a
rich context for understanding the phenomena under study is the intention of qualitative research.

2. When doing qualitative research the researcher is the primary ‘instrument’ for data collection and analysis. The data are therefore mediated through this human instrument and not through inventories, questionnaires or machines. This means that qualitative research depends heavily on the ‘integrative powers of the researcher’ (Benbasat, Goldstein & Mead, 1987). The researcher ensured care in the design of the research, collecting an appropriate sample, and analysing the data of personal biases. Being aware of personal biases and their impact on the validity of the research findings, the researcher analysed the data over and over each time scrutinising the prior analysis. One of the points of qualitative methodology is to view the world through the eyes of the researcher, to describe and take into account the context, to emphasise the process and not only the final results, to be flexible and to develop the concepts and theories as outcomes of the research process (Bryman, 2004).

3. Qualitative research typically involves fieldwork with the emphasis on studying people, settings, sites or institutions to observe or record behaviour in its natural setting. In this study, the fieldwork was substituted by documents, actual 2008 Mathematics Paper 1 examination scripts (answered).

4. The process of analysing qualitative data is inductive in that the researcher builds hypotheses or theories rather than testing hypotheses.

The product of qualitative research is a rich description (Cohen & Manion, 1994); thus, it is exploratory with a primary intention to explain and understand processes in depth, the meaning gained through words or pictures rather than focusing on the outcome of the products. I chose a qualitative methodology because I wanted to understand the thinking processes that led to learners making errors and hence to be able to capture the essence of what the learners have written in response to
algebra questions in the 2008 Grade 12 Mathematics Examination Paper 1. Even though I conducted a minor quantitative analysis, the qualitative approach was more suitable because the intention was to thoroughly understand the learners’ errors and misconceptions. The minor quantitative component was merely to complement the qualitative study by exploring the prevalence of errors in the scripts.

3.3 SAMPLING

Sampling refers to the process of selecting the sample from a population to obtain information regarding a phenomenon in a way that ensures that the population will be well represented (Brink, 1991). The objective in qualitative research is to select participants who are best able to give the researcher access to a special perspective, experience or condition which the researcher wishes to understand (Morse, 1994; Yegidis & Weinbach, 1996).

Regarding sampling, Merriam (1998) explains that, ‘once the general problem has been identified, the task becomes to select the unit of analysis’. The unit of analysis is the object which is to be studied in terms of research variables that constitute the constructs of interest (Brown & Dowling, 1998; McMillan & Schumacher, 2001; Yin, 1994). In this study, the learners’ errors and misconceptions in the examination items quadratic and simultaneous equations and inequalities are the units of analysis. The unit of analysis, a construct, is located in the sample. The written work provides evidence of the thinking processes of the learners. Qualitative sampling is theory-driven because the selection of participating entities, settings and interactions are determined by the conceptual question and not a concern for representativeness (Miles & Huberman, 2004; Yin, 1994).

The participants in this research were purposively selected (De Vos, 1998). Purposive sampling techniques are typically used in qualitative studies and may be defined as selecting units (e.g., individuals, groups of individuals etc.) based on
the specific purposes associated with answering a research study’s questions. Maxwell (1997) further defined purposive sampling as a type of sampling in which, “particular settings, persons, or events are deliberately selected for the important information they can provide that cannot be gotten as well from other choices” (p. 87). Purposive sampling techniques involve selecting certain units or cases “based on a specific purpose rather than randomly” (Tashakkori & Teddlie, 2003, p. 713). Purposive sampling leads to greater depth of information from a smaller number of carefully selected cases, whereas other types of sampling like probability sampling may lead to greater breadth of information from a larger number of units selected to be representative of the population (e.g., Patton, 1990). Purposive sampling frames are typically informal and based on the expert judgment of the researcher or some available resource identified by the researcher and the selected cases offers rich data from which the most can be learned (Teddlie & Yu, 2007).

The sample consisted of 80 scripts and consisted of very high performing learners making few errors and very low performing learners with many errors and misconceptions. The sample was therefore able to provide the researcher with rich information on the topic of investigation (Patton, 1990).

3.4 INSTRUMENTATION AND DATA COLLECTION METHODS
Using three multidimensional conceptual frameworks, namely the Revised Bloom’s Taxonomy (Anderson & Krathwohl, 2001) (see Appendix 1), and Stein, Smith, Henningsen & Silver, (1993) Mathematical Task Demand Levels in the National Curriculum Statement Mathematics Subject Assessment Guidelines (2009) (see Appendix 2), I analysed the following items in the question paper (see Appendix 3) and memorandum (see Appendix 4) for their academic and mathematical demand: quadratic and simultaneous equations and inequalities. These frameworks were used to determine the cognitive skills expected of the learners in the study. The three frameworks were used to ensure that where one was lacking the others would compensate and vice versa, and also to strengthen
the analysis and to ensure that the mathematics tasks were regarded from all angles.

The main method of data collection in this study focused on content analysis. Content analysis is a method that may be used with either qualitative or quantitative data; furthermore, it may be used in an inductive or deductive way. The purpose of the study determines which of these is used. In the case where there is not enough former knowledge about the phenomenon or when this knowledge is fragmented, the inductive approach is recommended (Lauri & Kyngäs, 2005). The categories are derived from the data in inductive content analysis. Deductive content analysis is used when the structure of analysis is operationalised on the basis of previous knowledge and the purpose of the study is theory testing. An approach based on inductive data moves from the specific to the general, so that particular instances are observed and then combined into a larger whole or general statement (Chinn & Kramer, 1999). A deductive approach is based on an earlier theory or model and therefore it moves from the general to the specific (Burns & Grove, 2005). These approaches have similar preparation phases.

3.5 DATA COLLECTION

Marshall & Rossman (1993) discuss several methods of qualitative data collection – such as participant observation, observation, interviewing, questionnaires and surveys, and document analysis. For the purposes of this research the focus will be on document analysis, that is, ‘gathering and analysing of documents produced in the course of everyday events’; and ‘the approach used to review the documents is called content analysis – a systematic examination of forms of communication to document patterns objectively’ (Marshall & Rossman, 1993). The specific kind of document analysis for this research is content analysis which falls in the interface of observation and document analysis. Content denotes what is contained and content analysis is the analysis of what is contained in a message. Broadly content analysis may be seen as a method where the content of the message forms the basis for drawing inferences and conclusions about the content.
The objective of content analysis is to convert recorded ‘raw’ phenomena into data, which can be treated in essentially a scientific manner so that a body of knowledge may be built up. In fact, the researcher who wishes to undertake a study using content analysis must deal with four methodological issues, namely selection of units of analysis, developing categories, sampling appropriate content, and checking reliability of coding (Stempel, 1989).

In qualitative research, it is characteristic that data are collected in a verbal and visual form rather than in a numeric form. Another aspect of qualitative research is that the sampling is not random but the sample is selected through purposeful sampling given the purpose of the study (Borg & Gall, 1989). ‘Purposeful sampling can be pursued in ways that will maximise the investigator’s ability to devise ground theory that takes adequate account of local conditions, local mutual shapings, and local values (for possible transferability)’ (Lincoln & Guba, 1985). In this study, I purposely selected 80 answered Grade 12 mathematics examination scripts which were already available from a research project that commenced in 2009. The project was undertaken by staff at the University of Johannesburg analysing learners’ answers to the questions set in the 2008 National Senior Certificate Grade 12 Mathematics Examination Paper 1. The GDE/EAB/UJ/SAP made available approximately 5 000 scripts for the research.

Data collected in research can be classified as primary or secondary data (Opie, 2004). Primary data are described as first-hand data collected by the researcher specifically for the purpose of the research. Glaser & Strauss (1967) discussed the use of documentary materials could be as valuable for generating theory as the observations and interviews usually used in qualitative, sociological studies. Secondary data, however, are described as data obtained from information collected by someone else in the past for some other purpose. Even though secondary data may be ‘old’, it may be the only possible source of desired data as recreating the past and the conditions that existed is impossible.
In this research, primary-secondary data was used. Although learners answered the examination questions for the purpose of passing the mathematics examination, the data to this study are primary as they are precisely linked to the intentions of this research. The data were found to be appropriate for the purposes of this study by the researcher as many scripts had been randomly selected from about 60 schools in the Gauteng Province. Moreover, due to the high importance of these examinations, these data are real and useful since learners were very serious when they took the high-stakes examinations set to determine their educational futures.

### 3.6 ACADEMIC BACKGROUND OF THE LEARNERS

All learners in the South African schooling system are expected to do some form of mathematics; either mathematics (pure) or mathematical literacy (National Curriculum Statement (NCS), 2003). Mathematics, unlike Mathematical Literacy, is taken by more mathematically able learners in the Further Education and Training Phase. Mathematics is taken by those learners who wish to engage in mathematically focused studies at tertiary level meaning that Mathematics is more demanding than mathematical Literacy. On the other hand, Mathematical Literacy is taken by those learners who could not do well mathematically in General Education and Training and who in the past usually stopped studying Mathematics after Grade 9.

### 3.7 LIMITATIONS OF THE STUDY

Finally, it is appropriate to discuss the limitations of this study. First, I assumed that the implementation of the secondary mathematics curriculum in Gauteng classrooms is in line with the principles of quality learning and teaching, as laid out by the Department of Education. These assume that learners were given adequate opportunities to learn using constructivist approaches. Nevertheless, if learners learned the required skills traditionally without using innovative methods including problem solving and inquiry approaches and are separate from learners’ context, then my assumption may possibly be inappropriate. This concern of the teaching and learning approaches however may not be important in this study as it
mainly aimed at the achieved outcomes of the curriculum; to study what errors in algebra learners still held even as they exited the school system. Second, because I did not observe the actual activities of classroom teaching in this study, the findings may need to be verified with observation of actual classroom teaching. Observing teaching has the advantage of checking if errors emanate from teachers’ teaching approaches or teachers’ deficient mathematical knowledge. Interviews with learners are another important limitation of this study. However despite these limitations, I strongly believe that literature helped to explain the errors I found in learners’ work. Beside the limitations here outlined, the error analysis provided important insight on the prominent errors that learners have on the investigated topics.

3.8 RELIABILITY AND VALIDITY
Reliability concerns the ability of different researchers to make the same observations of a given phenomenon if and when the observation is conducted using the same method(s) and procedure(s) (Geertz, 1973, 17). In this research actual authentic learners’ senior certificate examination scripts were used to do a detailed analysis of errors to see what could be learned through an analysis of the trends and patterns of written responses and errors.

Brinberg & McGrath (1985) put forward that validity is not a commodity that can be purchased with techniques instead; validity is, like integrity, character and quality, to be assessed relative to purposes and circumstances. Validity, in a broad sense, is concerned with the relationship between an account and something external to it, that is, the phenomenon that the account is about and whether that phenomenon is interpreted as objective reality or a variety of other possible interpretations for the same phenomena. In this research validity concerns were addressed by the thick descriptions provided as an essential component of the qualitative research enterprise. The examination written by the learners in this research went through a rigorous process as stipulated by the Department of Education (2002) and verified by Umalusi (2008). In this study error analysis aimed at answering the main research question; what is the nature of the
prominent learner errors in solving equations and inequalities? In this way the validity of the research was maintained as analysis was always focused on answering the main research question. Also the researcher asked five mathematics teachers to verify if her analysis was consistent with they own analysis of several items. The teachers discussed the analysis with the researcher and agreed that the analysis was quite good. Therefore the error analysis in this study was checked against the thinking of practicing mathematics teachers. The analysis then can be regarded as reliable and valid.

3.9 RESEARCH ETHICS
Research results obtained without complying with proper ethical considerations are considered invalid (Makonye, 2011). The ethical principles for research are; participation must be voluntary, participants must be informed and consent to participation, as well as ensuring confidentiality and anonymity of participants. These were adhered to, for the protection of participants against invasion of privacy or harm (physical and/or psychological) resulting from their participation.

The research data was obtained in the context of the Gauteng Department of Education, Examinations and Assessment Board, University of Johannesburg, Script Analysis Project (GDE/EAB/UJ/SAP). This project was being undertaken by staff at the University of Johannesburg analyses learners’ answers to the 2008 Senior Examination questions to find out what can be learnt from learners’ responses. The scripts for analysis were therefore sourced from within GDE/EAB/UJ/SAP with their full consent and approval.

3.10 CONCLUSION
In this chapter the methodology of the study was explicated and discussed. Both quantitative and qualitative methods were used for the study though more detailed qualitative analysis was done. The chapter also discussed the sampling of the scripts used in the research as well described the analysis of the data.
CHAPTER 4 DATA ANALYSIS

4.1 INTRODUCTION

This analysis sought to explain the data in an attempt to address the following research questions: What are the most common errors that learners in Grade 12 make when answering algebra examination questions? Why do learners make these errors? The answers to the two research questions are interwoven together into each of the learner’s vignette discussion.

While the study was not a mixed methods one (as explained in the research design), I used both qualitative and quantitative data analysis though most of the analysis in this chapter was done qualitatively; the essence of which is seeking for patterns in the analysed materials (Bryman, 2004). The central role of the qualitative analysis of the material is influenced by the coding process, i.e. interpreting the analysed text and ascribing the meaning (of key words, notions, codes) to its distinct parts (Charmaz, 2006; Bryman, 2004; Flick, 1998), respectively.

I analysed the data using the methods as explained by Johnson and Onwuegbuzie (2004) namely; discovery of patterns (induction), analysis of scripts to identify errors and the testing of the researcher's assumptions (deduction). I also aimed to uncover the best set of explanations for the findings (abduction). According to Rothchild, (2006) the two main data analysis approaches used for the mathematical error analysis are deduction and induction. Deductive reasoning works from the more general to the more specific; sometimes this is informally called a “top-down” approach and conclusions follows logically from available facts. Inductive reasoning works the other way, moving from specific observations to broader generalizations and theories; this is informally called a "bottom-up" approach and conclusions are likely based on premises involving a degree of uncertainty (Hatch, 2002; Yin, 1994; Miles & Huberman, 2004). Babbie (1998) describes induction as moving from the specific to the general, while deduction begins with the general and ends with specific. Arguments based on laws, rules
and accepted principles are generally used for deductive reasoning. Observations tend to be used for inductive arguments. My focus was mainly on learners’ errors, wrong procedures, and the possible reasoning processes behind them. Since the goal of this study was to identify learners’ misconceptions underlying their errors, I justified, whenever possible, how learners’ wrong responses expose their misconceptions. The data analysis is discussed with an emphasis on the following conceptual areas: equality, variable, order of operation, equation versus expression and equation versus inequality.

Four items were selected from the 2008 National Senior Certificate Mathematics Paper 1 Examination. The four items were questions on quadratic equations, quadratic inequalities and simultaneous equations. The items selected for the study were then mapped out against the cognitive and mathematical demand of those items as compared to the National Curriculum Statement Mathematics Subject Assessment Guidelines (2009) see Appendix 2; that is the knowledge, skills and values that learners should show to achieve the Learning Outcome 2 in each of the Further Education and Training grades (Department of Education, 2002)-Appendix 5. Lastly, the actual error analysis of learners work was carried out. This involved scanned vignettes from actual learners’ scripts and analysed using vignettes. The aim of this analysis is to describe, discuss and identify the errors in those learners’ micro-worlds (Davis, 1984).

4.2 DOCUMENT ANALYSIS

In the study I began by analysing documents such as the National Curriculum Statement (NCS) (2003) policy documents on Mathematics, the Mathematics Examination Guidelines, the 2008 National Senior Certificate Grade 12 Mathematics Paper 1 examination question paper and the corresponding memorandum were analysed first.

4.2.1 The findings from the document analysis were as follows: The curriculum was sufficiently covered; reasonable percentage of Grades 11 and 12 work; all questions were within the scope of the algebra section of the syllabus. The researcher noted that the percentage of coverage of Algebraic manipulation
and Equations in the 2008 examination was higher (17.3%) than the one suggested in the Guidelines (13.3%); higher by 4% see Table 1 below. This table shows the suggested weighting of each of the content topics of the NSC against the actual weighting of the same content topics in the 2008 mathematics examination.

Table 1 Comparison of assessed algebraic items in the 2008 NSC examinations and the required policy guidelines

<table>
<thead>
<tr>
<th>Topic/Content</th>
<th>Suggested in Guidelines</th>
<th>Actual in 2008</th>
</tr>
</thead>
<tbody>
<tr>
<td>Patterns &amp; Sequences</td>
<td>30/150=20%</td>
<td>28/150=18.7%</td>
</tr>
<tr>
<td>Annuities &amp; Finance</td>
<td>15/150=10%</td>
<td>17/150=11.3%</td>
</tr>
<tr>
<td>Functions &amp; Graphs</td>
<td>35/150=23.3%</td>
<td>28/150=18.7%</td>
</tr>
<tr>
<td><strong>Algebraic manipulation, Equations</strong></td>
<td>20/150=13.3%</td>
<td>26/150=17.3%</td>
</tr>
<tr>
<td>Calculus</td>
<td>35/150=23.3%</td>
<td>35/150=23.3%</td>
</tr>
<tr>
<td>Linear Programming</td>
<td>15/150=10%</td>
<td>16/150=10.7%</td>
</tr>
</tbody>
</table>

All the items selected for this study were either at cognitive level 1 or 2 as shown in table 2 below. This meant that the higher percentage of marks allocated for the algebraic manipulation and equations section made the items easier than they should have been (this is as set by the Mathematics Subject Assessment Guidelines) see table 2 and Appendix 2.

Table 2 Cognitive levels and their related skills for items used for the research in terms of Subject Assessment Guidelines for Mathematics (2009).

<table>
<thead>
<tr>
<th>ITEM</th>
<th>CONCEPTS AND PROCEDURES REQUIRED TO ANSWER THE ITEM AND COGNITIVE DEMAND OF ITEM</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1.1</td>
<td>Concept: Solving quadratic equations by factorisation; Procedure to Answer: This question required the use of algorithms; it required straight recall of the algorithm.</td>
</tr>
<tr>
<td>Concept</td>
<td>Cognitive Demand: <strong>Level 1</strong> (Knowledge)[see Appendix 2 - Subject Assessment Guidelines (SAG) for Mathematics (2009)]</td>
</tr>
<tr>
<td>---------</td>
<td>---------------------------------------------------------------------------------------------------------------</td>
</tr>
<tr>
<td><strong>1.1.2</strong></td>
<td>Concept: Solving quadratic equations by using the quadratic formula. This question required the use of algorithms. Procedure to Answer: It required straight recall of the algorithm and the knowledge and use of the quadratic formula. Cognitive Demand: <strong>Level 1</strong> (Knowledge)[see Appendix 2 - Subject Assessment Guidelines (SAG) for Mathematics (2009)]</td>
</tr>
<tr>
<td><strong>1.1.3</strong></td>
<td>Concept: Solving quadratic inequalities in one variable and interpreting the solution graphically; Procedure to Answer: This question required learners to perform well-known procedures as well as do simple calculations which have a few steps and require interpretation of the solution. Cognitive Demand: <strong>Level 2</strong> (Routine Procedures)[see Appendix 2 - Subject Assessment Guidelines (SAG) for Mathematics (2009)]</td>
</tr>
<tr>
<td><strong>1.2</strong></td>
<td>Concept: Solving equations in two unknowns, one of which is linear and one of which is quadratic, algebraically or graphically; Procedure to Answer: This question required learners to perform well-known procedures and to do simple applications and calculations which have a number of steps and require interpretation from given information and of the solution obtained. Cognitive Demand: <strong>Level 2</strong> (Routine Procedures)[see Appendix 2 - Subject Assessment Guidelines (SAG) for Mathematics (2009)]</td>
</tr>
</tbody>
</table>

Table 2 above explains the cognitive levels that each of the items in the research were mapped to. The requirements for cognitive demand in level 1 include being able to carry out: algorithms, estimation; appropriate rounding of numbers, theorems, straight recall, identifying from data sheet, simple mathematical facts, know and use of appropriate vocabulary, and knowledge and use of formulae. Cognitive demand level 2 requires learners to have competence in: solving problems that are not necessarily unfamiliar and can involve the integration of different Learning Outcomes, perform well-known procedures, simple applications...
and calculations which must have many steps and may require interpretation from
given information and identifying and manipulating of formulae.

From the information above, one noted that the items selected for this study were
all relatively easy, that is, the questions were not as cognitively demanding as they
needed to be. They only required recall of basic knowledge learned or doing
routine procedures (Stein et al., 1993).

4.3 QUANTITATIVE ANALYSIS OF THE ALGEBRA ITEMS
This section reports on the quantitative variation in the tasks performance from the
80 sample scripts.
Error analysis grids (see Appendix 6) were prepared to categorise the error types
for each algebra item of question 1.1.1 to 1.2. The errors were classified by the
error type into the following categories: unintended/careless, procedural and
conceptual. The causes of the errors were not analysed at this time but were
inferred later from the learners’ answers as well as informed by literature. This was
assisted by the method of “constant comparison” where the researcher uses the
data to generate concepts, categories and/or prepositions emerging as data
‘speaks’ for itself (Strauss & Corbin, 1990).

As a first step, mean percentage errors for each item were calculated. There were
two steps to this process. Firstly, the overall performance of the learners in the
sample on the algebra tasks was established. Item 1.1.1 and 1.1.2 were
attempted by all learners in the study. Item 1.1.1 was by far the best in terms of
learner performance, with 87.5% of the learners obtaining a mark of above 50%
and 72.5% of the learners obtaining full marks on this item. The highest mark
possible for item 1.1.1 was 3. In Item 1.1.2 more errors were made but 66.25% of
the learners obtained a mark of above 50% and 46.25% of the learner got full
marks for this item. The highest mark possible for item 1.1.2 was 5. Item 1.1.3
showed the largest number of errors with 32.5% of the learners obtaining a mark
that is above 50% and only 11.25% of the learners getting full marks for this item.
With item 1.1.3 of the learners in the study did not attempt this question. The
highest mark possible for item 1.1.3 was 5. In item 1.2 there was a wide spread in
the marks obtained by learners, and the number of learner that did not attempt is
question was four. In this item, 67.5% of the learners obtained a mark of above 50% and 26.25% of the learners got full marks for this item. The highest mark possible for item 1.2 was 8. (See figure 3 below).

![Figure 3 Frequency of marks scores for Items 1.1.1, 1.1.2, 1.1.3 and 1.2](image)

Secondly, the number of erroneous responses for each question under each item was calculated and classified into conceptual, procedural and carelessness errors (see figure 4 below).

**The frequency of each type of error per item is shown in Fig. 4 below.**

![Figure 4. Frequency of each type of error.](image)

Quadratic inequalities had the highest number (40.33% of total number of errors) of errors both conceptual and procedural followed by simultaneous equations (30.04% of total number of errors. Solving inequalities could be difficult for learners because there were many steps (both conceptual and procedural) involved in the solving process. These steps include but are not limited to reading
the question given in algebraic notation, understanding it, formulating a method, and solving it using an algorithm or any other method and also remembering the rules for multiplying and dividing inequalities.

4.4 CONCEPTUAL, PROCEDURAL AND CARELESSNESS ERRORS

In addition to this analysis, the errors observed were further classified into careless, procedural, conceptual and application errors. Several learners in the sample showed a number of errors including errors in the concept of equality, concept of variable, order of operations, order of operations in inequalities, exponentials, simplification/factorisation of algebraic equations; order of operations in inequalities, the zero product property; solving equations instead of inequality, multiplying/dividing inequality by factors that are not necessarily positive, and forming meaningless connections with quadratic roots. Of the 80 learners, 20.99% of them displayed carelessness errors, 32.92% of the learners displayed conceptual errors and 43.21% of the learners displayed procedural errors. Only 2.88% of the learners did not show errors in their work.

4.5 QUALITATIVE ANALYSIS OF LEARNERS’ ERRORS AND MISCONCEPTIONS: VIGNETTES OF LEARNERS’ ANSWERS

In the following section, learners’ work is presented to analyse how their errors and misconceptions were expressed. The 20 vignettes covering the four items are analysed.

4.5.1 Item by item analysis of selected vignettes

Five learner’s answers were used for each of the selected items to illustrate the errors made for each item.

4.5.1.1 Item 1.1.1

Item 1.1.1 was related to finding the roots of a quadratic equation given in the standard format i.e. \( ax^2 = bx - c \) where \( a, b, c \in \mathbb{R} \). Nearly all of the learners
solved this equation by factorisation using the FOIL method, only a few learners used the quadratic formula. Even though 87.5% of the learners got above 50% in this item, 12.5% of the learners exhibited either conceptual or procedural errors.

Question:
Solve for \( x \), rounded off to TWO decimal places where necessary:

\[ \text{1.1.1 } x^2 = 5x - 4 \]  

National Memorandum:

<table>
<thead>
<tr>
<th>1.1.1</th>
<th>( x^2 = 5x - 4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( x^2 - 5x + 4 = 0 )</td>
</tr>
<tr>
<td></td>
<td>( (x - 4)(x - 1) = 0 )</td>
</tr>
<tr>
<td></td>
<td>( x = 4 ) or ( x = 1 )</td>
</tr>
</tbody>
</table>

\( -1 \) for not equal to zero in this question only. If \( = 0 \) appears once in this question then full marks

\( \text{OR} \)

- standard form = 0
- factorisation
- both answers

By the formula
- standard form = 0
- substitution
- both answers

Item 1.1.1 required learners to re-write the equation in the form \( x^2 + bx + c = 0 \) and then use the method of factorisation by inspection.

Even though the learners’ performance on this item was good with 72.5% of the 80 learners getting full marks the remaining 27.5% of the learners made errors when answering this item. The errors in this item varied from changing the equation to an expression to incorrect factorisation to factorising the wrong coefficient to incorrect transposing of terms from one side of the equals sign to the other. Below are examples of learners’ written solutions:

Learner 1 of item 1.1.1

\[ \begin{align*}
\text{Figure 5 Learner 1 of item 1.1.1}
\end{align*} \]
The learner did not distribute correctly, s/he may not have understood the distributive laws. This is an example out of 80, 8 learners wrote similar responses to this question in this manner. This means the errors in the above vignette on item 1.1.1 represented 8 such learners (10% of the learners). This is an indication of a possible conceptual error. The learner is supposed to express the question in the format of a general quadratic equation $x^2 + bx + c = 0$ before factorising. It is possible that the learner’s procedural understanding is flawed and hence made a procedural error.

The learner ignored the equals sign and ended up with an expression. The learner also incorrectly distributed the terms; s/he multiplied -4 by -1 to get -5, thereby adding instead of multiplying the numbers. This learner seems to interpret the equal sign as an indication for action or a need to do something as well as indicating the next step of the solution.

The learner did not write the equation in the standard format of $x^2 + bx + c = 0$ in order to factorise the equation. Another error was that of adding $-x$ to $-4x$ and getting $-4x^2$ as well as adding $x^2$ to $-4x^2$ to get $4x^2$.

The learner lost the equals sign and ended up with an expression showing that there is lack of both conceptual and procedural knowledge. This shows a lack of understanding of equality, a critical concept for solving equations (Kieran, 1992, Linchevski, 1995, McNeil & Alibali, 2005). Norton & Cooper, (1999) also agreed that this kind of error shows poor understanding of the concept of equality. The learner skipped a step, that is, rewriting the equation such that all the terms are on one side of the equation and the other side has zero. Again here as mentioned by Norton & Cooper, (1999) the learner’s understanding of equal seems to be inadequate. Even though the learner did not do the second step of the solution correctly, the learner factorised correctly. The problem now is that the learner has an expression instead of an equation. After having factorised, the learner incorrectly distributes the brackets, not following basic arithmetic laws which Norton & Cooper (1999) state are crucial in algebra. Gardella (2009) concurs with
the claim that Kieran (1979) made, that is, the order of operations is still being so poorly developed that learners do not see the meaning behind the concept. The next steps are not necessary but the learner made a mistake by adding \(-x\) to \(-4x\) and ends up with \(-4x^2\). This is possibly both conceptual and procedural ill-thinking and goes on to add \(x^2\) to \(-4x^2\) to get \(4x^2\) which is incorrect (Li, 2006). This learner started with an equation and ended with an expression indicating that the learner may have problem understanding the difference between and expression and an equation; also it is possible that this learner does not understand the concept of equality. The learner might also have not been able to understand the terminology; ‘solve for x’; thus the issue of language plays a big role in the learners’ errors.

Learner 2 of item 1.1.1

![Figure 6 Learner 2 of item 1.1.1](image)

The learner worked out the factors of 5 instead of 4. This error was made by 14 learners out of the 80 learners in the sample. This could be a careless or arbitrary error or a procedural error. The first step was incorrectly done, the order of operation when transposing ‘5x-4’ to the left-hand-side of the equation is not adhered to, the second step should have been \(x^2 - 5x + 4 = 0\). The \(-4\) is not correctly transposed to the left hand side. The learner might have forgotten to add “invisible” bracket around \((5x-4)\) so that the second term is transposed correctly. The learner has two “equals” signs which could be attributed to either the teaching or the learner felt a need to show that the next step was coming up by putting the “equals” sign, what (Norton & Cooper, 1999) calls “and then” showing the next step in the solution.

The next step is incorrect, instead of finding the factors of 4 the learner worked out the factors of 5; i.e., the learner did not factorise the correct coefficient, the
factorisation is incorrect showing possible conceptual and may also be a procedural error.

Learner 3 of item 1.1.1

![Figure 7 Learner 3 of item 1.1.1](image)

The first step has been correctly done. The next step is incorrect; the factorisation is partly correct but the learner mixed up the signs and ended up with an incorrect answer. This is possibly an error due to carelessness. The learner did not consider the signs (negative and positive) when factorising. The learner seems to have neglected to consider all the factors of 4; s/he did not consider that there are a number of combinations i.e. 1 and 4; -1 and -4 in that specific combination and that you cannot take one from the first set and the other from that second set.

Learner 4 of item 1.1.1

![Figure 8 Learner 4 of item 1.1.1](image)

This learner made several errors; firstly, instead of factorising the constant term, the learner worked out two numbers that add up to 5 (this error may be due to both conceptual and procedural understanding). The learner should have re-written the equation in the standard format as \( x^2 - 5x + 4 = 0 \) and then factorise this equation. The factorisation is incorrect as the FOIL rules are not adhered to, again showing lack of conceptual understanding. When doing the last step, the learner seem to have not acknowledged that \((x+2) (x-3) = 0\) (they lost the equality); and just concluded (brought back the equality) that \(x=2\) or \(x=-3\) and not \(x=-2\) or \(x=3\). Again here one supposes that the learner may not have a good
understanding of equality with regards quadratic equations. These errors seem to stem from the learner not having a good conceptual understanding of concepts of equality, factorisation and solving linear equations. The procedure for equation resolving is depended on the principle that adding the same number to or subtracting the same number from both sides of the equation preserves the equality (Filloy, Rojano, & Solares, 2003; Filloy & Rojano, 2007). Also the many related and similar mathematical objects seem to have clouded the learners' capacity to differentiate between the different sets of mathematical objects. Kostsopoulos (2007) suggests that “learners’ problems with factorisation and with identifying varied representations of the same quadratic relationship” may be associated with the ways in which the brain constructs cognitive representations. Factorising of quadratics involved the rewriting of polynomials as a product of polynomials. This requires learners to have both a conceptual understanding of multiplication of polynomials and the procedural knowledge to retrieve basic multiplication facts effectively.

The learner ignored the fact that this is an equation; and went further to assume that $x^2\neq 2x$. The learner then added the like terms, didn’t seem to know what to do with the remaining term (-4) and simply dropped the term ending with an answer which is an expression rather than an equation (Welder, 2009). This learner seems not to understand a number of mathematical concepts showing conceptual misunderstanding in the difference between equations and expressions and the addition and subtraction of like and unlike terms.

4.5.1.2 Item 1.1.2

Item 1.1.2 of the examination was related to finding the roots of a quadratic equation given in the form, i.e. $x(b-x)=c$ where $a$, $b$, $c \in R$. Nearly all of the
learners that made errors tried to solve this equation by factorisation using the FOIL method instead of using the quadratic formula. Even though 66.25% obtained a mark of above 50% for this item, 38.75% of the learners exhibited either carelessness, conceptual or procedural errors.

Solve for \( x \), rounded off to TWO decimal places where necessary:

\[ 1.1.2 \quad x(3 - x) = -3 \quad (5) \]

Solution (memo):

The second item 1.1.2 was related to finding the roots of a quadratic equation given in the form \( ax^2 + bx + c = 0 \) where \( a, b, c \in \mathbb{R} \). Nearly all of the learners that made errors tried to solve this equation by using the FOIL method of factorisation instead of using the quadratic formula. Even though 66.25% of the sample obtained a mark of above 50% in this item, 38.75% of the learners exhibited either carelessness, conceptual or procedural errors.

Below are examples of learners’ written solutions:

Learner 1 of item 1.1.2
The learner lost the “equals” sign and end up with an expression showing lack of understanding of the difference between equations and expressions (Welder, 2009; Norton & Copper, 1999; Darr, 2003). The learner distributed x (3-x) correctly but somehow added -3x-9, it looks like the learner “made up” -3(3-x) on the right hand side of the equation; s/he imposed “invisible” brackets on what should have been the “right hand side” of the equation. The learner did not add like terms correctly. The errors identified looked like they were based on both procedural and conceptual problems.

Learner 2 of item 1.1.2

The learner expanded the brackets correctly, but then did two steps in the next step of solving the question and was not careful to change all the signs correctly when adding or subtracting different terms, showing carelessness errors.

When using the quadratic equation, the learner substituted ‘correctly’ (using the incorrect numbers obtained in the previous step), but later realised that s/he
couldn’t solve the equation. Instead of going back to check, the learner just ignored the fact that he could not find the square root of \(-3\) \((\sqrt{-3})\) the learner ends up with a solution; s/he also ignored the \(\pm\) sign \((\frac{\sqrt{-3}}{2})\) but somehow ended with two solutions. The errors identified are more of a careless mistake, after the first error, the learner did not check why they had a \((\sqrt{-3})\) and that this could not be solved. The procedure used is basically correct and the conceptual understanding seems to be in place, it looks like the learner got confused as to why they could not work out the square root of -3.

Learner 3 of item 1.1.2

![Figure 12 Learner 3 of item 1.1.2](image)

All the steps done by this learner are incorrect. When solving equations, whatever is done to one side must also be done on the other side of the equal sign (this shows a lack of understanding of equality, (Welder, 2009; Norton & Copper, 1999; Knuth, et al., 2008). The learner might have thought that because there in \((3-x)\) on the left hand side then the right hand side 'must' also have it. The learner ignored the distributive laws when removing the brackets; somehow gets \(3x^2\) from \(3x-x\). This work shows poor understanding of the concepts of equality \((-3=-3(3-x)), variable(x (3-x) =3x-x)\), distribution laws \((-3(3-x) -9x)\).

Learner 4 of item 1.1.2

![Learner 4 of item 1.1.2](image)
Figure 13 Learner 4 of item 1.1.2

The learner distributes the terms in the brackets correctly but did not correctly transpose all terms in the next step. There is also a procedural mistake in that the learner factorised terms that are on either sides on the equals sign. Since the learner did not write the equation in the standard format $x^2 + bx + c = 0$, s/he did not factorise correctly. The learner might not fully understand the rules of factorising. Even after having factorised the incorrect equation, one of the final solutions is still “incorrect”.

Learner 5 of item 1.1.2

Figure 14 Learner 5 of item 1.1.2

The learner did not distribute correctly, ignored some of the terms, when s/he got a term that could not fit into what s/he wanted to do, s/he just crossed it out and “imposed” 3x which could be added to the first term. After changing the equation to an expression, the learner performed an incorrect distribution and added a term to the expression. There seems to be conceptual knowledge gaps as well as procedural inaccuracies. This learner seems not to understand a number of mathematical concepts including the difference between equations and expressions and the laws of distribution and addition of terms containing variables.

What were the most common errors in quadratic equations?

A number of learners changed the quadratic equation to an expression. Learners tended to ignore equivalence in equations. Many learners conjoined unlike terms; addition of terms like ($-x-4x$) and ($x^2-4x^2$) to each other. Some learners factorised the wrong coefficient and some did that incorrectly. The use of the equals sign (=) as a symbol for the word ‘then’ or the phrase ‘the next step is’ was very common amongst the learners.
When learners transposed terms, the overlooked that fact that they should change the sign of the term(s) sometimes one of the terms was correctly transposed and the other term was incorrectly transposed. The distributive laws were not always adhered to. Also learners had great difficulties in dealing with adding and subtracting integers. They had difficulties in expanding terms in brackets as well as simplifying terms using indexes.

4.5.1.3 Item 1.1.3

Item 1.1.3 of the examination, was related to finding the solution of a quadratic inequality given in the form, i.e. \( c - bx < ax^2 \) where \( a, b, c \in \mathbb{R} \). The learners were expected to solve the quadratic inequality with one variable and to be able to interpret the solution preferably graphically. Nearly all of the learners solved this inequality by factorisation. A large number of learners ignored the fact that this is not an equation but an inequality and solved it as though it were and equation. Only 32.5% of the learners obtained marks of above 50% in this item; however, 66.25% of the learners exhibited either carelessness, conceptual or procedural errors.

Question:
1.1.3 Solve for \( x \), rounded off to TWO decimal places where necessary:

\[
3 - x < 2x^2
\]  

National memorandum:
The third item 1.1.3 of the examination was related to finding the solution of a quadratic inequality given in the form \( ax^2 + bx + c < 0 \) where \( a, b, c \in \mathbb{R} \). The learners were expected to solve the quadratic inequality with one variable and to be able to interpret the solution graphically. Nearly all of the learners solved this inequality by factorisation. A large number of learners ignored that fact that this is not an equation but an inequality. Only 32.5% of the learners obtained marks of above 50% in this item, however, 66.25% of the learners exhibited either carelessness, conceptual or procedural errors.

\[
\begin{align*}
3 - x &< 2x^2 \\
-2x^2 - x + 3 &< 0 \\
2x^2 + x - 3 &> 0 \\
(2x + 3)(x - 1) &> 0 \\
x &< -\frac{3}{2} \text{ or } x > 1 \\
\text{OR} \\
x \in \left(-\infty; -\frac{3}{2}\right) \cup \left(1; \infty\right)
\end{align*}
\]
Below are examples of learners' written solutions:

Learner 1 of item 1.1.3

All steps in the solution are incorrect and in the second step the learner introduces an “equals” sign most likely to indicate the next step. The factorisation is incorrect and the learners ends up with a solution that has both equality and inequality. The learner seems to have not established the difference between equation and inequality, this is seen in the introduction of the “equals” sign and in the final answer which has both an equation and an inequality. The learner did not seem to grasp the concept of inequality. This is a topic that learners tend to have difficulties with possibly because of its similarities to equations but has unique ways and properties. (Tsamir & Almog, 2001)

Learner 2 of item 1.1.3

The learners conjoined 3-x to get 3x (Bosse & Faulconer, 2008) and then decided to divide the terms by 2 to get an incorrect answer; (Schechter, 2000). The learner went on to do an incorrect cancellation 3 divide by 2 is not equals to 1. The operational sign was changed twice in the solution and the reasons are not clear. The learner seems to have not understood inequalities.

Learner 3 of item 1.1.3
Figure 17 Learner 3 of item 1.1.3

The learner introduces a second inequality in the second step and then an equals sign adjacent to the inequality. The presence of both inequity and equality signs next to each other indicates that the learner may have not grasped the concept of inequalities. The learner then drops the inequality and ends up with an equation. The learner added unlike terms (conjoined) \( 3-x \) to get \( 3x \) (Bosse & Faulconer, 2008) and also changed \( 2x^2 \) to \( 2x \). The final answer of \( 3x = 2x \) indicates that the learner may not fully understand the implications of the equals sign. This solution shows conceptual and procedural gaps.

Learner 4 of item 1.1.3

Figure 18 Learner 4 of item 1.1.3

The learner starts off by making an error with working out that \( 3-x \) is equals to 3 what Bosse & Faulconer (2008) called conjoining. The learner goes on to introduce the equality at the same time does some unknown operation ending with \( 5 \leq x^2 \) and then goes on to solving this as though it were a quadratic equation and gets a solution of \( x=2 \) or \( x=3 \). These roots do not come from \( 5 \leq x^2 \) nor does it come from \( 5=x^2 \). This learner seems to lack both conceptual and procedural understanding of inequalities.

Learner 5 of item 1.1.3
Figure 19 Learner 5 of item 1.1.3

The learner disregarded all mathematics rules of inequalities; got rid of the inequality, worked out 3-x and got 3x; and 3x-2x² to get -1x i.e. conjoined different terms (Bossé & Faulconer, 2008). The learner seems not to have an understanding of what procedure(s) to follow when solving for inequalities, they also do not show evidence of conceptual understanding of inequalities.

What were the most common errors in quadratic inequalities?

A number of learners introduced the ‘equals’ sign and then solved the question as if it were an equation. Incorrect factorisation was very common in the learners’ work. Also learners had much difficulty in transposing terms. The learners were unable to correctly interpret their solutions. Failure to establish the difference between equation and inequality was the most common error/misconception noted from the learners’ work. Some of the learners had difficulty in understanding the two inequalities together (in the solution), and decided to change one of the inequalities ending with one inequality and one equation.

4.5.1.4 Item 1.2

1.2 Determine the values of x and y if they satisfy both the following equations simultaneously: (8)

\[2x + y = 3\]
\[x^2 + y + x = y^2\]

National Memorandum:
Below are examples of learners’ written solutions:

<table>
<thead>
<tr>
<th>Problem</th>
<th>Solution</th>
</tr>
</thead>
</table>
| 12) $y = 3 - 2x$  
$x^2 + (3 - 2x) + x = (3 - 2x)^2$  
$y^2 + 3 - 2x + x = 9 - 12x + 4x^2$  
$3x^2 - 11x + 6 = 0$  
$(3x - 2)(x - 3) = 0$  
$x = \frac{2}{3}$ or $x = 3$  
$\therefore y = \frac{5}{3}$ or $y = -3$  

OR

$x = \frac{3 - y}{2}$  
$(\frac{3 - y}{2})^2 + y = \frac{3 - y}{2} + y^2$  
$9 - 6y + y^2 + y = \frac{3 - y}{2} + y^2$  
$9 - 6y + y^2 + 4y + 6 - 2y = 4y^2$  
$0 = 3y^2 + 4y - 15$  
$0 = (3y - 5)(y + 3)$  
$y = \frac{5}{3}$ or $y = -3$  
$\therefore x = \frac{2}{3}$ or $x = 3$  

OR

$y = 3 - 2x$  
$x^2 - x + y + y = 0$  
$(x + y)(x - y) + (x + y) = 0$  
$(x + y)(x - y + 1) = 0$  
$y = x + 1$  
$3 - 2x = -y$  
$x = 3$ or $x = \frac{2}{3}$  
$y = -3$  
$y = \frac{5}{3}$  

OR

$x = \frac{3 - y}{2}$  
$(\frac{3 - y}{2})^2 + y = \frac{3 - y}{2} + y^2$  
$9 - 6y + y^2 + y = \frac{3 - y}{2} + y^2$  
$9 - 6y + y^2 + 4y + 6 - 2y = 4y^2$  
$0 = 3y^2 + 4y - 15$  
$0 = (3y - 5)(y + 3)$  
$y = \frac{5}{3}$ or $y = -3$  

OR

$x = \frac{3 - y}{2}$  
$(\frac{3 - y}{2})^2 + y = \frac{3 - y}{2} + y^2$  
$9 - 6y + y^2 + y = \frac{3 - y}{2} + y^2$  
$9 - 6y + y^2 + 4y + 6 - 2y = 4y^2$  
$0 = 3y^2 + 4y - 15$  
$0 = (3y - 5)(y + 3)$  
$y = \frac{5}{3}$ or $y = -3$  
$\therefore x = \frac{2}{3}$ or $x = 3$  

<table>
<thead>
<tr>
<th>Problem</th>
<th>Solution</th>
</tr>
</thead>
</table>
| 84) $y = 3 - 2x$  
$y^2 + 3 - 2x + x = 9 - 12x + 4x^2$  
$3x^2 - 11x + 6 = 0$  
$(3x - 2)(x - 3) = 0$  
$x = \frac{2}{3}$ or $x = 3$  
$\therefore y = \frac{5}{3}$ or $y = -3$  

OR

$x = \frac{3 - y}{2}$  
$(\frac{3 - y}{2})^2 + y = \frac{3 - y}{2} + y^2$  
$9 - 6y + y^2 + y = \frac{3 - y}{2} + y^2$  
$9 - 6y + y^2 + 4y + 6 - 2y = 4y^2$  
$0 = 3y^2 + 4y - 15$  
$0 = (3y - 5)(y + 3)$  
$y = \frac{5}{3}$ or $y = -3$  
$\therefore x = \frac{2}{3}$ or $x = 3$  

OR

$x = \frac{3 - y}{2}$  
$(\frac{3 - y}{2})^2 + y = \frac{3 - y}{2} + y^2$  
$9 - 6y + y^2 + y = \frac{3 - y}{2} + y^2$  
$9 - 6y + y^2 + 4y + 6 - 2y = 4y^2$  
$0 = 3y^2 + 4y - 15$  
$0 = (3y - 5)(y + 3)$  
$y = \frac{5}{3}$ or $y = -3$  
$\therefore x = \frac{2}{3}$ or $x = 3$  

<table>
<thead>
<tr>
<th>Problem</th>
<th>Solution</th>
</tr>
</thead>
</table>
| 84) $y = 3 - 2x$  
$y^2 + 3 - 2x + x = 9 - 12x + 4x^2$  
$3x^2 - 11x + 6 = 0$  
$(3x - 2)(x - 3) = 0$  
$x = \frac{2}{3}$ or $x = 3$  
$\therefore y = \frac{5}{3}$ or $y = -3$  

OR

$x = \frac{3 - y}{2}$  
$(\frac{3 - y}{2})^2 + y = \frac{3 - y}{2} + y^2$  
$9 - 6y + y^2 + y = \frac{3 - y}{2} + y^2$  
$9 - 6y + y^2 + 4y + 6 - 2y = 4y^2$  
$0 = 3y^2 + 4y - 15$  
$0 = (3y - 5)(y + 3)$  
$y = \frac{5}{3}$ or $y = -3$  
$\therefore x = \frac{2}{3}$ or $x = 3$  

OR

$x = \frac{3 - y}{2}$  
$(\frac{3 - y}{2})^2 + y = \frac{3 - y}{2} + y^2$  
$9 - 6y + y^2 + y = \frac{3 - y}{2} + y^2$  
$9 - 6y + y^2 + 4y + 6 - 2y = 4y^2$  
$0 = 3y^2 + 4y - 15$  
$0 = (3y - 5)(y + 3)$  
$y = \frac{5}{3}$ or $y = -3$  
$\therefore x = \frac{2}{3}$ or $x = 3$  

Below are examples of learners’ written solutions:
Learner 1 of item 1.2

![Image of learner's work](image)

Figure 20 Learner 1 of item 1.2

The learner does undistributed cancellations, that is, did not divide one of the terms by 2 (Schechter, 2000). 19 learners in the sample of 80 learners made an error of this nature. This could be because the learner does not have a good understanding why all terms must be divided by the same number. After doing this, the learner dropped the equality and assumes an expression; this was done for both equations. The learner also seems to forget that when dealing with equations the equivalence of both sides must be maintained throughout, in this case the learner forgot to divide the y by 2 in the first equation. In the second equation, the learner conjoined terms. After conjoining some of the terms, the learner decided to separate some of the variables. From the learner’s solution, there seems to be a poor understanding of the procedures for solving simultaneous equations because from the memo, the learner should have made one of the variables the subject of the formula, after that to substitute into the other equation.

Learner 2 of item 1.2
The learner changed the equations to expressions this was done by replacing an equals sign by a subtraction sign. 42 learners out of 80 learners in the sample change the equations to expressions. This shows a misunderstanding of what equality means. The addition of the expressions shows that the learner may not know the procedures used to solve simultaneous equations, this also demonstrations that the learner may not understand the concept of simultaneous equations.

Learner 3 of item 1.2

In one of the equations, the learner makes y the subject of the equation, in the other equation, the learner make x the subject. In the first equation, the learner made an error when making y the subject of the equation. The learner also conjoined unlike terms. In the second equation, the learner y+x to x^2 and y^2 to x, and then tried to make x the subject of that equation at the same time dropped x^2. In this equation the learner again made a transposing error and in some instances seemed to have assigned a value (most likely 1) to one of the variables.

The learner seems not to understand the nature of the solution associated with simultaneous equations. The learner does not show any understanding of what
the procedure are in solving simultaneous equations and there is also no evidence of any conceptual understanding of simultaneous equations.

The learner knows that they should isolated the y in the first equation (this was incorrectly done). For some reason, s/he did not proceed further with the solution and this was considered to be an incomplete answer. One of the objectives of this question was to identify that the y is the same as the y in equation 2, and it can be substituted into equation 2 in order to work out the value of x then later the value of y.

Learner 4 of item 1.2

![Learner 4 of item 1.2][1]

The learners knows that they should make y the subject of the equation 1; then to use that to substitute into equation 2 but makes a mistake of not squaring on the right hand side. This is most probably a carelessness error. The learner then multiplies the brackets by x instead of adding the x; again I would assume that this is an error due to carelessness. The learner also made the mistake of not subtracting a 3 instead added the 3 to the left hand side but subtracted 3 from the right hand side. After this the learner made more mistakes, collecting unlike terms, incorrect factorisation and forgets to use both solutions to find y. This learner showed understanding of the procedures and seems to have the appropriate conceptual understanding but made too many carelessness errors.

Learner 5 of item 1.2

---

[1]: Figure 23 Learner 4 of item 1.2
The learner decided to add the two equations to each other even though the terms involved were unequal. The solution show a poor understanding of procedures used for solving simultaneous equations and a poor conception of what simultaneous equations are and the kinds of solutions they yield. The learner dropped the equality and remains with an expression. This learner does not seem to know that two equations cannot just be added to one equation unless one has shown that one or two of the terms are equal to each other. The learner seems not to understand when and how substitution is done. Both conceptual and procedural knowledge seems to be lacking in this example.

**What were the most common errors in simultaneous equations?**

Some learners displayed uneven distribution and cancellation errors. Some learners seemed to confuse equations and expressions. The addition of unlike terms was a problem for a number of learners. Nature of the solution associated with simultaneous equations seemed not to be clear for many learners. Many learners showed that they did not appreciate that substitution was a leeway for solving the simultaneous equations. In their reasoning they showed that deficit. We can say that the learners had not yet achieved substitution, a key skill in doing algebra and the solution of equations.

**4.5.2 Summary of errors from learners’ written work**

I identified learner errors pertaining to four main areas in algebra: *equality, variables, substitution, order of operations and mix-up between solving equations and not inequalities*. In addition to this, one main observation is that learners did not check their work or see the need to check their work. They did not appreciate that algebra does make sense. The data analysis contained three stages:
discovery of patterns (induction), testing of the researcher’s assumptions (deduction), and uncovering the best set of explanations for the findings. My focus was on learners’ conceptions, procedures, algorithms, possible misconceptions. Since the goal of this study was to identify learners’ misconceptions underlying their errors, I justified, whenever necessary, how learners’ wrong answers expose their misconceptions.

The quantitative analysis of the data showed that the learners had most difficulties in answering questions on inequalities with the highest percentage of errors followed by simultaneous equation. Solving quadratic equations showed more learner errors than solving the same using inspection but these showed less learner errors.

In the qualitative phase, 20 learners’ written work was analysed in detail. This analysis showed that learners misunderstood a number of notions due to a variety of reasons. Among these are incorrect uses of equality, variable and order of operations.

4.6.1 Errors made by learners in this study regarding the concept of equality included:

For the learners, the equal and plus signs are typically interpreted as actions to be performed; however, that is not how they are always used in algebra (Behr, et al., 1976, 1980). This may be because in traditional equality problems, the equal symbol generally comes on the right side of the equation and usually only one number comes after it.

Learners interpret the equals sign as simply a command to perform an operation or a series of operations in order to provide an answer, they are denied this insight. That is, as an indication for action (e.g., “makes or gives” – Stacey & MacGregor, 1997, p. 113)

As a way of showing the place where the answer should be written (Filloy & Rojano, 2007); they used the equals sign to say, “and then”. Some of the learners had two different numbers on either side of the equals as a solution or the final answer (Darr, 2003). Some of the learners used the equal sign as a step marker.
This misuse led them to misconstrue the equation as an expression and may have got confused as to where the equals sign should be placed.

4.6.2 Errors associated with the variable:
Some of the learners simply ignored the presence of some of the variables. They also made the error of interpreting variables, for example, that $x^2$ to be equal to $2x$, such an error is an arbitrary error. They added unlike terms to each other. They seem to have assigned values to some of the variables and simplified “the expression” which was meant to be an equation. Some of the learners did not seem to understand how they were meant to substitute in the case of the simultaneous equations.

4.6.3 Errors associated with the order of operations:
The learner took the plus sign as a signal to conjoin two terms together irrespective of whether they are like or unlike terms. Some learners ignored order of convention, operational laws and directed numbers (Norton & Cooper, 1999). Combining letters and numbers incorrectly because they think that operation symbols cannot be part of an answer. Learners tended to disregard signs when manipulating equations. Some of the learners seemed to have difficulty with the problems in which closure was not possible or not easily obtained. This was a problem when learners converted equations to expressions then tried to solve the expression. Some of the learners showed improper understanding of negative numbers.

4.7 CONCLUSION
From the analysis, learner errors and misconceptions pertaining to four main areas in algebra were identified: equality, variables, order of operations and solving for equations instead of solving for inequalities. There were three stages of data analysis as explained by Johnson and Onwuegbuzie (2004): discovery of patterns (induction), analysing of scripts to identify errors and the testing of the researcher’s assumptions (deduction), and uncovering the best set of
explanations for the findings (abduction). While the focus of the study was on learners’ errors and the possible misconceptions that brought about these errors, it also identified learners’ conceptions, procedures and algorithms. Since the goal of this study was to identify learners’ misconceptions underlying their errors, I justified, whenever necessary, how learners’ wrong responses expose their misconceptions.

The quantitative analysis of the data showed that the learners had most difficulties in answering the question on inequalities with 88.75% of the learners making errors, followed by simultaneous equations (73.75%). Solving quadratic equations using the quadratic formula and then using the FOIL method were the next two subdivisions with percentage errors of 53.75% and 27.5% respectively. The errors identified were classified into carelessness, arbitrary, conceptual and procedural errors. Within these classifications the error types of each conceptual area (equality, variable, order of operation and equation versus inequality) were identified.

The qualitative phase showed that learners may have misconceived some important concepts used in algebra due to a variety of reasons. Among them, misuse of rules, confusion with previously learned concepts, problems with the syntax of algebra, problems with the structure of algebra, not identifying arithmetic-algebraic connections and possibly not knowing the core concepts. Thus learners showed that they had a nexus of errors which were conceptual, procedural, arbitrary and random, which often occurred during answering a single question. In some cases learners’ errors had rudiments of correct algebraic results, but these were often followed by many errors in simplifying out terms by addition and subtraction wrongly. It became clear that one of the greatest cause of errors in this study was the failure of learners to add and subtract integers and also simplification of algebraic terms. Learners also had errors due to failure to group like terms. They added and subtracted unlike terms at will. They had problems of multiplying terms, and of removing brackets. Such errors are both conceptual and procedural. Whenever such errors occur, they snowball and
escalate rendering doing algebra a mightily meaningless and tormentous enterprise instead of an enjoyable and enlightening experience that it really is.
CHAPTER 5 FINDINGS, DISCUSSIONS AND RECOMMENDATIONS

5.1 INTRODUCTION

In the following chapter, I discuss the errors that I found when I analysed the learners’ written work, discussed in chapter 4 in section 4.5. The aim of this research was to investigate learners’ errors and the misconceptions that belie them in three main areas of algebra namely: quadratic equations, quadratic inequalities and simultaneous equations. So the study thus also aimed to reveal learners’ reasoning for their errors. It was hoped that by so doing a better understanding of the cognitive difficulties, in the form of errors, of how learners’ learn algebra may be isolated. Regardless of the difficulty of directly gaining access to learners’ mathematical thinking and reasoning behind their answers, it may be argued that one can have some access to their thinking through a detailed analysis of their written work. The findings, discussion, conclusions and recommendations of this research arise from the following research question:

What is the nature of the main errors that mathematics learners in Grade 12 make when answering quadratic equations and inequalities and simultaneous equations examination questions?

The sub-questions to this question were:

1. What are the most common errors that learners show in their scripts?
2. What are the possible reasons why learners make those errors?

5.2 RESEARCH FINDINGS

Exploring learners’ incorrect answers provides one way to display learners’ different levels of understanding of a concept. Learners’ correct answers, on the other hand, may not necessarily reveal a good conceptual understanding of associated knowledge because learners may have solved the problem correctly by merely memorising procedures or definitions without a real understanding (Nesher, 1987). Such type of understanding is called instrumental understanding (Skemp, 1979). For example, learners may use formulae without understanding
how and why the formula works the way it works. Moreover, learners’ correct answers are normally homogeneous, and do not provide appropriate data for research. Usually learners who understand a concept converge to a correct answer that tends to be exactly the same, but those who do not understand produce a variety of incorrect answers, even though sometimes the reasoning for correct answers may be similar. Hiebert & Carpenter (1992) argue that research on learners’ errors exposes specific deficits in the way learners’ knowledge is linked. In that way instructional intervention may be designed to cater for the specific connections learners lack. These can also be used to discuss with the learners why their linking of old and new concepts are not viable.

The findings with respect to the learners’ errors are generally similar to the findings of previous studies such as Herscovics & Linchevski (1994), Kieran (1992), Olivier (1999), Tsamir & Almog (2001) and Blanco & Garrote (2007). In this study, the errors made by learners were varied. The analysis showed that the learners had a poor understanding of the concept of equality; concept of variable; order of operations; order of operations in inequalities; simplification/factorisation of algebraic equations; the zero product property; solving an equation instead of an inequality; notation errors, errors due to misunderstanding language and terminology; multiplying/dividing inequality by factors that are not necessarily positive; and forming meaningless connections with quadratic roots.

The findings of the data analyses for the research questions are reported below.

5.2.1 The nature of common errors and possible reasons explanations for their causes

The errors found were not merely the absence of correct answers or the result of unfortunate coincidences. Many of these errors were as a result of robust misconceptions partially because they arose several times and not just from one learner but from a number of the learners. Interestingly, they were the
consequences of certain processes whose nature needed to be discovered. It is for this reason that it was important to analyse these errors and misconceptions in such a way as to expose the underlying reasoning. In the next section, I will elaborate those errors under each conceptual area (concept of equality; concept of variable; order of operations; order of operations in inequalities; notation errors, simplification/factorisation of algebraic equations; the zero product property; solving an equation instead of an inequality; multiplying/dividing inequality by factors that are not necessarily positive; and forming meaningless connections with quadratic roots; brackets and errors due to misunderstanding language and terminology) and also relate them to various findings in literature. Whenever possible, I will explain how learners’ errors will allow us to determine their misconceptions.

Many learners struggled with the following concepts:

5.2.1.1 Concept of equality
The learners seemed not to always have a good understanding of the meaning of the equals sign and therefore of the equality relation. Research has shown that learners starting to learn algebra tend to struggle with misconceptions about the meaning of the equals sign (Kieran, 1992; Radatz, 1979). In working with algebra, Saenz-Ludlow & Walgumuth (1998) found that learners need to see the equals sign as relational; indicating that either side of the equals sign has equal value. For example learners’ work in the following vignettes seemed confused with the equality:

It is clear that these learners had difficulty in distinguishing the difference between the equality and inequality sign. In the above example they treated them interchangeably, as if they were the same.
Also, very few of the learners displayed a good understanding that the concept of equality as was shown in the relationship between different parts of an equation and meaning ‘the same as’ as illustrated in the examples above (Lee, Park, Shim & Vu, 2006). This could be the reason why they make errors in their work.

5.2.1.2 Concept of variable
The concept of variable is of crucial importance in algebra, because it forms the foundation of generalisations. Research (Welder, 2009; Lannin, 2005; Sleeman, 1984) has identified some reasons why learners have difficulty understanding and using variables. Learners have difficulty distinguishing between the various ways that letters are used. Kuchemann (1978) noted that when examining algebraic expressions, learners may assign a numerical value to the variable, ignore the variable, view the variable as a specific unknown, or see the variable only consisting of whole numbers. In addition to the above misconceptions more are identified in the literature. These are (1) changing the variable symbol as changing the referent. That is, different variables must take on different values. For example, the expression $3m$ is not the same concept as $3x$ as $m$ and $x$ could never be equal (Booth, 1988; Chalouh & Herscovics, 1988). (2) Assigning numerical values to letters according to their rank in the alphabet, for example, $a=1$ and $z=26$, or if $x=3$ then $y=4$ and $z=5$ (Booth, 1988; Chalouh & Herscovics, 1988). (3) Assigning the letter as a sub divisional label, for example, $3a$ refers to the first part of the problem (Chalouh & Herscovics, 1988). Further research is needed to ascertain whether these misconceptions are also present in South African learners since a good understanding the various uses and presentation of variables is crucial for learners’ success in algebra.

5.2.1.3 Order of operations
Learners have a tendency to conjoin algebraic expressions. For example such as when writing the expression $3\cdot x$ as 3. For instance this learner:
regards 3-x as the same as 3, while this learner regards 3-x as 3x. Literature suggests several different reasons for this error. One of the reasons associated with conventions which do not differentiate between conjoining and adding. Stacey and MacGregor (1997) state that learners may draw on prior learning from other fields to their work with algebraic symbols, e.g., in chemistry, adding oxygen to carbon produces CO₂. Tall & Thomas (1991) point out that due to similar meanings of ‘and’ and ‘plus’ in ordinary language, it is not uncommon for learners to regard ‘ab’ to mean the same as ‘a+b’ because the symbol ‘ab’ is read as ‘a and b’ and may be interpreted as ‘a+b’, so 3+x may be taken as 3x. Alternate thinking is that learners often disregard a difficult question and reformulate it to another easier question such as changing 3-x to 3. Such an error is called an arbitrary error.

Other researchers (Booth, 1988; Collis, 1975; Davis, 1995) have come up with another explanation often given for this kind of error; learners often face cognitive difficulties in accepting lack of closure and have a tendency to understand open expressions as incomplete. The second explanation still leaves room for the question: Why is it that learners have this feeling? A standard rationalisation is that learners anticipate the ‘behaviour’ of algebraic expressions to be similar to that of arithmetic expressions. At times they expect a specific answer, that is, a final answer with one term (e.g., Booth, 1988; Tall & Thomas, 1991); at other times, they interpret symbols such as ‘+’ only in terms of actions to be performed, as is usually done in arithmetic, and therefore conjoin the terms (Davis, 1995).

Additionally, a rather broader explanation for the similar behaviour is linked to the dual nature of mathematical notations of process and object (Davis, 1995; Sfard, 1991; Tall & Thomas, 1991). Sfard (1991) proposed another method to research mathematics knowledge: the dual nature of mathematics knowledge. She laid emphasis on the fundamental difference between the duality and dichotomy methods. Sfard stressed that actually refer to facets of the same thing.
Most mathematical concepts express such duality. The mathematical concept ‘number’ will be discussed to demonstrate the meaning of process and object. When children study the concept of ‘number,’ they start with ‘counting’ which is natural and is done with relative ease. This is the ‘process’ stage. Thereafter, children need to translate the counting process to an abstract concept of ‘number.’ This is the ‘object’ stage as Sfard and Linchevski (1994) reasoned that learners need to make the transition from process-like thinking to object-like thinking so as to understand a concept completely. Possible connections between Sfard’s theories present the potential to ‘borrow’ research on learners’ misconceptions in school science to develop a theory to explain why some misconceptions in school mathematics are robust to change (Li, 2006). Certainly, it is still important to keep in mind the differences between mathematical concepts and scientific concepts. For example, almost all mathematical concepts in numbers and algebra were constructed by mathematicians and are at the abstract level and thus have no concrete representations in daily life. In contrast, almost all school science concepts, such as heat, stem from learners’ lives.

5.2.1.4 Order of operations in inequalities

Learners had more challenges solving the inequality item than any of the other items. In fact the question on inequalities was the worst in terms of performance as shown in the analysis of Item 1.1.3. Many learners seem not to have grasped that there is a semantic difference between equation and inequality, and changed the inequality to an equality, for example, this learner’s work: . For some of the learners, their conception of interval was as ‘a set of natural numbers, or at best a set of integers between some two integers’, for example: and Blanco & Garrote (2007) also came up with similar results. Very few learners in the study were able
to give a satisfactory interpretation of their solution, for example:

\[
\begin{align*}
3x + 2x^2 &< 2x^2 \\
3x - 2x^2 &< 2x^2 \\
-1 &< 0
\end{align*}
\]

In fact, the majority of learners in the sample could not make sense of their answers, for example:

\[
\begin{align*}
3 - x < 2x^2 \\
3 - x > 2x^2 \\
3 - x = 2x^2 \\
3 - x = -2x^2 \\
3 - x = -2x
\end{align*}
\]

and to show a few cases. These results are similar to the findings by Socas (1997) that the complexity of the objects and processes of algebra is a source of the learners’ difficulties.

For a number of the learners in the sample, algebra is seen as merely ‘operating’ with numbers and letters, with no other goal other than obtaining their specific values by applying algorithms without a good understanding of the algorithms. Consequently, in dealing with an expression of the form \(3 - x < 2x^2\), their objective is to re-write the inequality as \(ax + bx + c < 0\). The learners tended to ignore the ‘negative signs but just transposed terms to the other side of the inequality as a divisor’ just as though the relationship was the same as that of an equation. In this they showed basic procedural errors, failure to balance an equation/inequality. The objective of finding values of the unknown that make the inequality true was grossly neglected. Here is an example of learner’s work to illustrate this:

\[
\begin{align*}
3 - x &< 2x^2 \\
2 - 3x &< 2x^2 \\
-1 &< 0 \\
2x^2 + 3x - 2 &< 0 \\
-5 &< 3 \quad x = -1 \\
2 &< 3 \\
x = -3
\end{align*}
\]

This learner had a least a good knowledge of the procedures of solving an equation but failed to factorise correctly and be able to indicate the solution
intervals. Such a learner may not have been aware that solution of an equation is a point but the solution of an inequality is an interval.

Since many learners may not have fully mastered arithmetic, they experience difficulties in handling of the distributive property, and in their use of brackets, handling of negative numbers, and the value ascribed to the equals sign when doing algebra. These results are in agreement with those indicated by Collis (1975); Behr et al. (1980); Kieran (1979, 1981), and Enfedaque (1990). These challenges make it even more difficult for the learners to acquire a new concept that requires the appropriate use of these prerequisite skills.

5.2.1.5 Notation

Some of the learners have trouble with the correct interpretation of notation. For example, when asked the question: Is $a \times a \times a$ the same as $3a$? Most learners say Yes. Likewise when asked if $a^3$ is the same $a \times 3$, they answer Yes. A number of learners in this study showed similar thinking processes like this example: and they may not have understood the correct mathematical meaning of various notations. In this case the learner completely ignores the number -4 while concentrating on terms with x only.

Thus in this case we see that the learners construct their own errors using prior knowledge. Also this is because they construct knowledge without enough information, without being thorough. Multiple representations of the same entity may also be partly responsible for this line of thinking. This misconception could be to some extent attributed to learners not fully understanding and not being functionally fluent in manipulating algebraic equations/expression (Schechter, 2000).
5.2.1.6 Simplification and factorisation of algebraic equations

Learners abandoned the rules or misinterpreted them in many types of simplification problems. For example, when simplifying a rational expression such as $\frac{x+2}{x}$; learners often think this expression simplifies to 2 just like one of the learners did here: $\frac{a}{a} + \frac{y}{y} = \frac{2}{x+2}$. Also when doing factorisation, many learners do not easily see the common factor in expressions such as $2 \times 3 \times 4 \times 25 + 5$. This could be the reason why this learner had difficulty seeing the common factor in $2 \times 3 \times 4 \times 25$ (Schechter, 2000).

Kotsopoulos (2007) adds that difficulty in retrieving multiplication facts, for example the learner's factors may have been a difficulty with remembering the factors of either 4 or 5: $x^2 + xy + y^2 = x^2 + y^2 + x^2 + y^2$. This directly affects learners' ability to engage effectively with factorisation of quadratics, since factorisation is a process of finding products within the multiplication table. Kotsopoulos adds that learners also find it challenging to recognise and understand varied representations of the same quadratic relationship. This was seen in this research when some learners worked out the factors of 5 instead of 4 in the equation, $x^2 = 5x - 4$. The quadratic equation in this case was written in the form $ax^2 = bx + c$ and not in the usual format of $ax^2 + bx + c = 0$. 
5.2.1.7 The zero product property

This type of error as can be seen in this example and is very difficult to eradicate or is, at least, very difficult to eliminate permanently. Regardless of whether one is dealing with more mathematically able learners, even when they are receiving excellent teaching which emphases the special properties/role of zero in the zero product principle, this kind of error continues to appear in learners’ work (Olivier, 1989).

Matz (1982) offers a theory that explains the persistence of this error. There are two levels of procedures guiding cognitive functioning, namely surface-level procedures, which are the familiar rules of arithmetic and algebra, and deep-level procedures, which create, modify, control and typically guide the surface-level procedures. An example of such deep-level guiding principle is the overgeneralisation of numbers, which in effect says that ‘the specific numbers do not matter – you could use other numbers in its place.’

Hence, in order to learn algebra a learner should have such a deep-level procedure to overgeneralise numbers; that is, the learner must believe that certain procedures work irrespective of the numbers used.

Olivier (1989) explains that this works very well; as a matter of fact too well: learners have the natural inclination to overgeneralise. Because learners are so accustomed to overgeneralise numbers, one can predict that errors may be made for any type of problem whose particular numerical values are significant. Overgeneralisation of number and number properties could be the single most important fundamental reason for learners’ misconceptions.
As Olivier (1989) explains it, this is precisely what is likely to happen in the instance of the quadratic equation as was also seen in this research. In \((x - 3) (x - 2) = 0\), the numbers 3 and 2 are not critical to the method, but the 0 is! Learners can and should therefore generalise:

\[(x - a)(x - b) = 0\]
\[\Rightarrow x - a = 0 \text{ or } x - b = 0 \quad \text{--------------------------(1)}\]

Learners who are unable to realise the critical nature of the 0, tend to handle it as they do any other numbers and overgeneralise to:

\[(x - a)(x - b) = c\]
\[\Rightarrow x - a = c \text{ or } x - b = c \quad \text{--------------------------(2)}\]

Equation (2) would be a correct generalisation of Equation (1) only if generalising were appropriate in this case. Regrettably it is not true. This is perhaps one of the first important rules that learners meet where a specific number ought not to be generalised.

Teachers are aware of this rule, that the guiding deep-level procedure to overgeneralise numbers is the reason for the error; in this case, the surface-level procedures are operating correctly. This offers an explanation as to why the error is so obstinate and resistant to change, notwithstanding the teacher’s best efforts, and despite learners’ best intentions: this is not merely a problem of learning; it cannot be easily be erased from memory, because it is incessantly being recreated by a reasonable deep-level guiding principle. What is missing is a danger signal that in such unique cases will warn the learner that the application of the deep-level procedure is wrong; this awareness most probably only comes with experience of making similar mistakes.

The zero product error shows that that errors may be mediated to learners by their teachers. In this case the learners are taught a correct fact but they generalise it wrongly. This is because human are prone to generalising what they learn and in some cases they generalise wrongly as exemplified in this case.
This example below shows again the sensibility of learners’ errors and how learners’ misconceptions are not random, but originate in a consistent conceptual framework based on earlier acquired knowledge. Example from this research:

\[
\begin{align*}
&x(3-x) = -3 \\
&3x^2 = -3 \\
&-3 = 3x + x^2 \\
&x^2 + 3a = -3 \\
&(x-1)(x+2) = 0 \\
&x = 1 \text{ or } x = -2.
\end{align*}
\]

5.2.1.8 Solving an equation instead of an inequality

More than half of the learners in the study changed the given inequality symbol to an equal sign in Item 1.1.3. For example,

\[
\begin{align*}
&3 - x < 2a^2 \\
&\frac{3a}{x} > 4a^2, \\
&\frac{3 - x}{2} = 2a^2, \\
&2a - 2a.
\end{align*}
\]

Tsamir & Almog (2001) deduced from their findings that many learners drew inappropriate analogies between the solution of inequalities and those of equations. They suspect that this may be because the remarkable similarities between equations and inequalities create a convincing instinctive feeling that similar strategies hold for solving the two kinds of mathematical entities. The results of this study are similar to those found in other studies (Tsamir & Almog, 2001; Fischbein, 1987) and showed that these instinctive beliefs successfully compete with formally acquired knowledge. Boero, Bazzini & Garuti (2001) discussing this phenomena stated that learners’ tendencies to make irrelevant connections between equations and inequalities as problematic. Boero, Bazzini & Garuti (2001) also attribute these types of errors to the traditional, algorithmic teaching approaches as the main cause of these errors. Kieran (2004) on the other hand says that this connection is an important step in solving algebraic problems with the use of non-algebraic methods.
Boero & Bazzini (2004) suggest that teachers investigate the use of various representations of equations and inequalities in different contexts. This approach is also supported by Blanco & Garrote (2007) when they recommended the use of different strategies to approach questions related to inequalities both enriches the learning process and allows more learners to acquire the concept. The also advocate the use of different languages: 'everyday' language, visual geometric language, and algebraic language when teaching inequalities. Translating from one to another favours a better understanding of the concept. Blanco & Garrote (2007) and Tsamir & Almog (2001) also put forward the suggestion that learners are more successful when they solve inequalities using graphs.

5.2.1.9 Multiplying and dividing inequality by non-positive factors
Some of the learners inappropriately multiplied both sides of the rational inequality by the denominator, without taking into account non-positive cases. For example, and . These are cases where learners do not abide by the rules for manipulating inequalities and instead treat them as though they were equations, as was also found by Tsamir & Almog (2001).

5.2.1.10 Forming meaningless connections with quadratic roots
Some of the learners worked out the solutions to the inequality and gave an answer in the form of an equation rather than an inequality. For example, and . This is imaginable because of their familiarity with equations.
5.2.1.11 Brackets

Brackets are a vital feature of mathematical symbolisation in both arithmetic and algebra. When working with algebra, learners are required to have a much more flexible understanding of brackets (Linchevski, 1995). In arithmetic, brackets are generally used as an unchanging signal informing learners which operation should be done first. Although brackets symbolise the grouping of two terms (in an additive situation) and that is only one important usage, learners seemed not to have understood that brackets can also be used as a multiplicative operator, for example,

\[ \begin{aligned}
5 - & (3 - x) = -3 \\
5x - & (3 - x)^2 = 0 \\
& (x - 3)(x - 1) = 0 \\
& x = -3 \text{ or } x = 1
\end{aligned} \]

The above vignettes show that learners can make many different errors even in solving one mathematics problem. So in most cases, errors occur in lumps where it may be very difficult to isolate them.

5.2.1.12 Language of instruction different to mother tongue and mathematical terminology

In some cases it could be inferred that learners had errors and misconceptions in answering the items because they did not understand the language in which the items were set. They did not understand what was required of them. A major part of learning algebra is learning the language of how to write down consistently and precisely what you mean when you are trying to express or represent something numerically; this is about learning a precise mathematical language (Wilson, 2009). Research (Setati, & Adler, 2001; Adler et al., 2004) has shown that learners, especially in South Africa where the language of teaching and assessment is not the same as learners' mother tongue, achieve relatively lower grades particularly in mathematics and science than learners who are taught in their mother tongue. These learners at times make errors as a result of lack of understanding of the question and not necessarily because of incompetence. Consequently, the learners with language barriers misinterpret the question because they are unable to understand the linguistics
Setati, & Adler (2001) emphasise that language is a vital aspect for developing the necessary mathematical reasoning and conceptual understanding.

When mathematics is taught in English, learners have to learn more than just English but should also learn specific English words used uniquely in the mathematical context. This sometimes leads to misconceptions due to the possible lack of understanding of English (Goolamally & Ahmad, 2010). Consequently, mathematics has the potential of being especially challenging for learners who are learning English as their second or third language. Mathematics has its own specialised language, grammatical patterns and rules, and it comprises formulas, relationships, applications, and explanations (Short & Spanos, 1989). Owing to this complexity, learners develop a certain kind of attitude (often negative) and thinking about mathematics. As mentioned by MacGregor & Moore (1991), language plays a vital role in shaping and organising knowledge, logical thinking, giving explanations, and presenting results.

Concurrently with learning English as a second language, second-language learners are expected to also learn the meanings of some English words which have their own precise meaning in the mathematical context (Bossé & Faulconer, 2008). Second-language learners need to learn and master many content-specific vocabulary words such as quotient, equivalent, divisor, numerator, denominator, variable, equality, inequality, solve, factorise, quadratic and so on. Additionally, they have to grasp the meaning of many complex phrases such as least common factor and greatest common factor (Bossé & Faulconer, 2008).

Goolamally & Ahmad (2010) propose that another problem that lead to serious learning difficulties in mathematics is the occurrence of misconceptions which may be an outcome if learners do not have appropriate or adequate teaching, informal thinking processes or poor memory. These misunderstandings may cause learners endless trouble in grasping mathematics from the most elementary concepts.
5.2.2 General reasons for learners’ errors and misconceptions

Although in many cases, learners manufacture their own errors by overgeneralising what they learn, in some cases, the errors emanate from their teachers. So errors can be a result of a learner’s own constructions or simply communicated from teacher to learner by an unknowledgeable teacher or a teacher who does not appropriately teach a mathematical concept. However it is intriguing that different learners may construct very similar errors.

Drews (2005) suggests that at times errors might be worsened by teachers making suppositions about their learners’ experiences; this is particularly significant for teachers of younger learners. She furthermore points to the inappropriate use of resources by the teacher which can also cause learners to make errors.

Errors can also be caused by teachers who are not knowledgeable in those concepts. The activities and tasks that teachers and learners do together in the mathematics classes are at the heart of mathematics education. If teachers harbour misconceptions about certain areas in mathematics, they will pass those misconceptions on to their learners. Research (Stewart, 2009) has given attention to the distinctive blend of mathematics content and the pedagogical knowledge needed to teach it effectively. Holloway (1999) further argues that teachers with inappropriate conceptions of how learning occurs can distort the ideas presented by curriculum designers. Moreover, during curriculum reforms and/or when teachers are issued with new curriculum materials aligned to reform approaches, like it has been the case in South Africa, the teachers’ knowledge and beliefs continue to influence how they are interpreted and the use of those materials (Collopy, 1999; Remillard, 1999). Kennedy (1997) suggests that in order to efficiently help learners develop their mathematical understanding, it is imperative to help the teachers build their own understanding of the relevant ideas and their relationships.

With respect to the different systems of representation, ideally the use of more than one system would favour the understanding of algebra since different
systems provide alternative and complementary strategies (Palarea & Socas, 1999-2000). At least 90% of the learners in the sample only used algebraic language to approach the different problems they had to answer. This may be due to the fact that many teachers only use and encourage this approach in the teaching and learning of algebra. In developing the content of algebra, teachers only use limited algebraic language and do not provide learners with any other tools to represent concepts and hopefully enable easier learning, understanding and transferability of the concepts. Studies over the past 20 years consistently reveal that the mathematical knowledge of many teachers is dismaying thin (Ma, 1999).

As far as inequalities are concerned, the main problem with working with inequalities is the lack of meaning as was also related by Blanco and Garrote (2007) in their study. The majority of the learners in this study had the most difficulties answering the question on quadratic inequalities.

Tsamir & Almog (2001) advises that intuitive beliefs about inequalities sometimes successfully compete with formally acquired knowledge. It therefore seems sensible to explicitly discuss the similarities and differences between equations and inequalities, as it was clearly seen in this study that learners somehow solve for equations in place of inequalities. It is also important to make learners aware of the role of intuitive beliefs and how it affects learning new concepts.

The findings of this study, in agreement with those reported in the Boero and Bazzini, (2004) research study, indicate that the method most prevalent for solving inequalities was the algebraic manipulation. This approach usually means a ‘trivialisation’ of the topic, resulting in a series of routine procedures, which cannot easily be understood, interpreted and controlled by learners. As a result of this approach, learners are unable to cope with inequalities as they do not fit into the schemas taught (Boero and Bazzini, 2004). The few learners that used the graphic method were more successful, that is, got the correct answer and were able to correctly interpret the solution. Because of the success learners had with the graphic method, it seems natural to conclude that the graphic method of solving
inequalities may be the answer to the difficulties experienced by learners. The graphic method provides learners with visual images of the solution and therefore facilitates interpretation of the answer (Tsamir & Almog, 2001).

There is evidence that traditional teaching, which is by elicitation or in expository mode, separates concepts, ideas and skills and does not pay attention to connections between concepts (Vaiyavutjamai & Clements, 2006). They went on to state that the over-emphasis on skills and procedure was to the detriment of understanding and of construction of conceptual knowledge. This makes it difficult even at a later stage to understand the reasons why a certain rule was followed or why a given step was taken. There is therefore an urgent need to change the way teaching of concepts is done, particularly in South Africa where the pass rate of mathematics is so low.

Von Glasersfeld (1995) argues that as constructivist epistemology suggests, the learner is an active participant in the construction of his/her own mathematical knowledge, and through the reception and the interaction of new ideas with the learners’ existing ideas, misconceptions may arise. Learners see new ideas through the lens of their pre-existing knowledge; therefore learners may not see the need to change their misconceptions if their ideas are not challenged to a point that they see the crucial need to reconsider them. This realisation does not happen in a void; rather it occurs in an environment where the use of the concept is verified. This implies that learners tend not to realise the need to reconsider their misconceptions unless they are convinced that their conceptions are inadequate for application on new tasks (von Glasersfeld, 1995).

5.3 CONCLUSIONS

The fundamental goal of this research was to investigate the nature of the most common errors made by learners when answering questions on quadratics equations and inequalities and simultaneous equations in the 2008 National Senior Certificate Grade 12 Mathematics Examination Paper 1. Such an investigation would help to get a better understanding and appreciation of the
difficulties experienced by learners in learning mathematics. In this study I assumed that a better understanding of learners’ errors in some basic algebraic concepts has the potential of leading to a better understanding of learners’ general mathematics learning processes, especially in those learners who fail to grasp those basic mathematics concepts. This section rounds up the study. It includes conclusions on misconceptions shown by learners in and recommendations for future research emanating from this study.

The theoretical framework for this study was rooted in the work by constructivist and socio-cultural learning theorists, such as Piaget and Vygotsky. They acknowledge that learner errors are an important component of the learning process and that they are an invaluable source of information for both the learner and the teacher. The role of the teacher is to provide an environment and opportunities for the learner to accept that misconceptions are a natural by-product of the learning process. Teachers and learners can work within a scaffolding mechanism to analyse and address/amend the misconceptions.

It is important for mathematics teachers to examine why some concepts in mathematics are so difficult for learners to learn. Brown and Burton (1978) argue that, “one of the greatest talents of teachers is their ability to synthesize an accurate ‘picture,’ or model, of a learner’s misconceptions from the meagre evidence inherent in his errors”. However, to recognise the existence of misconception is only part of the battle. It is even more critical for mathematics teachers to be empowered with effective strategies to help learners overcome these misconceptions.

The results of this study show that learners have several misconceptions concerning the meaning of equality, variables, order of operations, exponents, factorisation of algebraic equations, differentiating between equations, expressions and inequalities. It may be that these misconceptions are instilled upon learners at the very beginning of their experiences with algebra. Previous research has shown that these misconceptions occur not only at college level (Trigueros, Reyes, Ursini & Quintero, 1996; Rosnick, 1981), but have also been identified at the secondary level as well (Kuchemann, 1978). This research
supports these previous findings, as it seems learner understanding of the algebraic concepts mentioned above seem to be a challenge even in the South African situation.

Learning algebra continues to be a struggle for a large number of learners in South Africa as can be seen by the very low pass rate of mathematics in the past 12 years (CDE, 2010). The learners’ written answers on quadratic equations and inequalities and simultaneous equations did provide clues of learners’ difficulties in working with these basic algebra concepts. Learners showed major difficulties in interpreting quadratic inequalities and simultaneous equations in particular. The most common errors identified in this study were learners’ failure to accurately carry out simplifications and/or factorisations of equations and/or inequalities.

The findings show that learners’ knowledge and understanding rests largely on isolated facts and procedures and that their conceptual understanding of quadratic equations and inequalities and simultaneous equations is deficient. Previous research affirms these. These concepts include: (c) The order of operations (Kieran, 1979, 1988; Pinchback, 1991); (d) Equality (Behr, Erlwanger, & Nichols, 1976, 1980; Falkner, Levi, & Carpenter, 1999); Algebraic symbolism (Kieran, 1992; Küchemann, 1981) including letter usage (Booth, 1988; Küchemann, 1978, 1981; Macgregor & Stacey, 1997; Sleeman, 1984; Usiskin, 1988; Watson, 2002); (g) Algebraic equations (Clement, Narode, & Rosnick, 1981; Wollman, 1983); and (i) Graphing (Brenner et al., 1997, Chazan & Yerushalmy, 2003; Kieran, 1992)

In summary, in response to the research question: What is the nature of the main errors made by learners in Grade 12 when answering quadratic equations and inequalities and simultaneous equations examination questions?

The most prominent errors that were identified in this study were:

a. Learners converting inequalities to equations and then solving the equation instead of the inequality;

b. Learners converting equations to expressions and then trying to solve the expression;

c. Learners failure to accurately carry out simplifications and/or factorisations;
d. Learners ignoring the order of operation especially where there were
inequalities;
e. Learners ignoring brackets and
f. Learners incorrect interpretation of notation.

These were mostly as a result of:

a. Inequalities tend to be taught and learned as a subordinate subject (usually
in relationship with equations) and are dealt with in a purely algorithmic
manner. This approach often results in a sequence of routine procedures
that are not easy for learners to understand, interpret and manipulate. As a
consequence of this approach, learners are unable to manage
inequalities and solve problems related to the concept. In addition, the
apparent similarities in finding the solution between equations and
inequalities are confusing to learners.

b. Learners tend to add “= 0” to expressions and therefore confuse equations
and expressions. This confusion may be attributed to the failure to fully
understand the concept of equality.

c. – f. Learners failure to understand and accurately use basic mathematics
laws/rules (including numbers and numerical operations; ratios/proportions;
the order of operations; equality; patterning; algebraic symbolism including
letter usage; algebraic equations; functions; and graphing) which tend to be
very specific to the question posed.
This study revealed that the majority of the learners in this study had the most difficulties answering the question on quadratic inequalities. Therefore, further research is necessary to determine the reasons for the difficulties experienced by learners and to develop interventions to help overcome these obstacles as well as to develop a deep mathematical understanding of inequalities. It would probably be important to investigate the instructional approaches teachers use to teach these topics, it would also be vital to know from the learners through an extensive investigation to interview learners on the difficulties they encounter when solving inequalities.

This study furthermore revealed that learners experienced more success in answering the question on inequalities when using the graphic method; it therefore seems natural to conclude that the graphic method of solving inequalities may be the answer to the difficulties experienced by learners.

As shown in this study, another important finding is related to learners’ difficulties in interpreting quadratic inequalities and simultaneous equations. It would be valuable to do further research to find out what learners’ interpretation and representation of quadratic inequalities and simultaneous equations is like.

Lastly, the study did not focus on the actual reasons (from the learners themselves) why South African learners make the errors they do in quadratic equations and inequalities and simultaneous equations. Thus, it would be worthwhile to find out from interviews and other long term studies with South African learners why the learners make the errors they display in their written work in these topics.
REFERENCES


Appendix 1

REVISED BLOOM'S TAXONOMY

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>A. Factual Knowledge</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>B. Conceptual Knowledge</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C. Procedural Knowledge</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>D. Metacognitive Knowledge</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The Revised Bloom's Taxonomy (Adapted from Anderson and Krathwohl, 2001)
Appendix 2
NATIONAL CURRICULUM STATEMENT MATHEMATICS SUBJECT ASSESSMENT GUIDELINES
Appendix 4
NATIONAL SENIOR CERTIFICATE MATHEMATICS: GRADE 12
MATHEMATICS PAPER 1: NOVEMBER 2008: MEMORANDUM
Appendix 5

LEARNING OUTCOME 2 FOR EACH OF THE FURTHER EDUCATION AND TRAINING GRADES

Learning Outcome 2: Functions and Algebra

The learner is able to investigate, analyse, describe and represent a wide range of functions and solve related problems.

A fundamental aspect of this outcome is that it provides learners with versatile and powerful tools for understanding their world while giving them access to the strength and beauty of mathematical structure. The language of algebra will be used as a tool to study the nature of the relationship between specific variables in a situation. The power of algebra is that it provides learners with models to describe and analyse such situations. It also provides them with the analytical tools to obtain additional, unknown information about the situation. Such information is often needed as a basis for reasoning about problem situations and as a basis for decision making.

Learners should:
- understand various types of patterns and functions;
- investigate the effect of changing parameters on the graphs of functions;
- use symbolic forms to represent and analyse mathematical situations and structures; and
- use mathematical models and analyse change in both real and abstract contexts.

The mathematical models of situations may be represented in different ways – in words, as a table of values, as a graph, or as a computational procedure (formula or expression). The information needed is mostly acquired in the following ways:
- finding values of the dependent variable (finding function values);
- finding values of the independent variable (solving equations);
- describing and using the behaviour of function values, periodicity;
- considering the increasing and decreasing nature of functions, rates of change, gradient, derivative, maxima and minima;
- finding a function rule (formula); and
- transforming to an equivalent expression (‘manipulation’ of algebraic expressions).

It is important that the Learning Programme provides for appropriate experiences of these problem types, and that it develops the underlying concepts and techniques to enable learners to experience the power of algebra as a tool to solve problems. The emphasis is on the objective of solving problems and not on the mastery of isolated skills (such as factorisation) for their own sake.
Proposed content

Grade 10

Solution of:
- linear equations;
- **quadratic equations by factorisation**;
- exponential equations of the form $kax+p = m$ (including examples solved by trial and error);
- linear inequalities in one variable and graphical illustration of the solution;
- linear equations in two variables simultaneously (numerically, algebraically and graphically).
- the discrete or continuous nature of graph.

Grade 11

Algebraic manipulation:
- **completing the square**;
- simplifying algebraic fractions with binomial denominators

Solution of:
- **quadratic equations (by factorisation, by completing the square, and by using the quadratic formula)**;
- **quadratic inequalities in one variable and graphical interpretation of the solution**;
- **equations in two unknowns, one of which is linear and one which is quadratic, algebraically or graphically.**
Appendix 6
ERROR ANALYSIS GRIDS

Frequency of marks scores for each of the Items: 1.1.1, 1.1.2, 1.1.3 and 1.2

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>No attempt</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1.1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(possible 3)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.1.2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(possible 5)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.1.3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(possible 5)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(possible 8)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The frequency of each type of error per item: 1.1.1, 1.1.2, 1.1.3 and 1.2

<table>
<thead>
<tr>
<th></th>
<th>CARELESSNESS ERRORS</th>
<th>CONCEPTUAL ERRORS</th>
<th>PROCEDURAL ERRORS</th>
<th>LEFT BLANK</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1.1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.1.2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.1.3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>