An Integrated First-Order Theory of Points and Intervals: Expressive Power in the Class of All Linear Orders

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***Abstract*—There are two natural and well-studied approaches to temporal ontology and reasoning, that is, point-based and interval-based. Usually, interval-based temporal reasoning deals with points as a particular case of duration-less intervals. Recently, a two-sorted point-interval temporal logic in a modal framework in which time instants (points) and time periods (intervals) are considered on a par has been presented. We consider here two-sorted first-order languages, interpreted in the class of all linear orders, based on the same principle, with relations between points, between intervals, and inter-sort. First, for those languages containing only interval-interval, and only inter-sort relations we give complete classifications of their sub-fragments in terms of relative expressive power, determining how many, and which, are the different two-sorted first-order languages with one or more such relations. Then, we consider the full two-sorted first-order logic with all the above mentioned relations, restricting ourselves to identify all expressively complete fragments and all maximal expressively incomplete fragments, and posing the basis for a forthcoming complete classification.**

***Keywords*-First-order logic, definability, interval, point and mixed relations.**

I. Introduction

The relevance of temporal logics in many theoretical and applied areas of computer science and AI, such as theories of action and change, natural language analysis and processing, and constraint satisfaction problems, is widely recognized. While the predominant approach in the studies of temporal reasoning and logics has been based on the assumption of time points (instants) as the primary temporal ontological entities, there have also been active studies of interval-based temporal reasoning and logics over the past two decades. The variety of binary relations between intervals in linear orders (now known as Allen's relations) was first studied systematically by Allen [1], who explored their use in systems for time management and planning. Allen's work, and other relevant followups, are based on the assumption that time can be represented as a dense line, and points are excluded from the semantics. Various modal and first-order formalisms for reasoning about Allen's relations have been studied in the literature. At the modal level, Halpern and Shoham [2] introduced the multi-modal logic HS that comprises modal operators for all possible relations between two intervals in a linear order, and it has been followed by a series of publications studying expressiveness and decidability/undecidability and complexity of the fragments of HS, e.g., [3], [4], [5]. At the first-order level representation theorems have been a major concern: for a given set of interval relations, is it possible to write down a set of first-order formulas in the corresponding signature which would constrain relational structures to be isomorphic to concrete one consisting of intervals over a linear order together with the appropriate interval relations? Results here include van Benthem [6], Allen and Hayes [1], Ladkin [7], Venema [8], Goranko, Montanari, and Sciavicco [9], and Coetzee [10]. Many studies on interval logics have considered the so-called 'non-strict' interval semantics, allowing point-intervals (with coinciding endpoints) along with proper ones, and thus encompassing the instant-based approach, too; see e.g., [2], [3], [4]. Yet, little has been done so far on formal treatment of both temporal primitives, points and intervals, together in a two-sorted framework. A detailed philosophical study of both approaches, point-based and interval-based, can be found in [6]. A similar mixed approach has been studied in [11]. In [12] both sorts are used and the relations between them is studied, but this only under the hypothesis of denseness. More recently, a modal logic that includes different operators for points and interval was presented in [13].

The present paper provides a systematic treatment of point and interval relations at the first-order level, with equality not necessarily assumed in the language. We study the relative expressive power of first-order languages containing interval relations. Our work is motivated by, among other observations, the fact that natural languages incorporate both ontologies on a par, without assuming the primacy of one over the other, and have the capacity to shift smoothly the

$[a,b]$ 34_{ii} $[c,d] \Leftrightarrow b = c$	
$[a,b]$ 44_{ii} $[c,d] \Leftrightarrow b < c$	
$[c,d]$ 14_{ii} $[a,b] \Leftrightarrow a = c, d < b$	
$[c,d]$ 03_{ii} $[a,b] \Leftrightarrow b = d, a < c$	
$[c,d]$ 04_{ii} $[a,b] \Leftrightarrow a < c, d < b$	
$[a,b]$ 24_{ii} $[c,d] \Leftrightarrow a < c < b < d$	

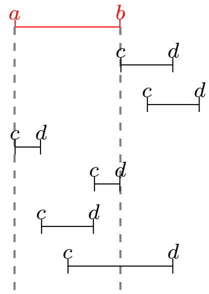

Table I
INTERVAL-INTERVAL RELATIONS (A.K.A. ALLEN'S RELATIONS).

perspective from instants to intervals and vice versa within the same discourse, and that there are various temporal scenarios which neither of the two ontologies alone can grasp properly (that is, when treatment of intervals as sets of their internal points, nor the treatment of points as 'instantaneous' intervals, is really adequate). In this way, we pose the basis for a further study of point-interval temporal modal logics, and, on a longer perspective, for a proper definition and study of point-interval (first-order and modal) logics over non-linear orders. We therefore ask the question: how many and which expressively different first-order languages can be obtained by varying the combinations of intervals and points relations in the signature? Since, as we will see, there are 26 different relations (including equality of both sorts) between points, intervals, and points and intervals, 2^{26} is an upper bound to this number[1]. However, since certain relations are definable in terms of other ones, the actual number is less and in fact, as we will show, much less. The answer will also depend on our choices of certain semantic parameters, specifically, the class of linear orders over which we construct our interval structures. In this paper we consider the classification problem in the class of all linear orders. Preliminary work that provides a similar classifications, based on intervals only, appeared in [14].

II. First Order Logic and Point / Interval Relations

Given a linear order $\mathbb{D} = \langle D, < \rangle$, we call the elements of D *points* (denoted by $a, b, \ldots$) and define an *interval* as an ordered pair $[a, b]$ of points in D, where $a < b$ (and endpoints are included in the interval). Abstract intervals will be denoted by $I, J, \ldots,$ and so on. There are 12 possible relations, excluding equality, between any two intervals, which we call *interval-interval* relations. Besides equality, there are 2 different relations that may hold between any two points (*before* and *after*), called hereafter *point-point* relations, and 5 different relations between a point and an interval and vice-versa: we call those *interval-point* and *point-interval* relations, respectively, and we use the term *mixed* relations to refer to them indistinctly. Interval-interval relations are exactly Allen's relations [1]; point-point relations are the obvious one on a linear order, and mixed relations will be explained below. Traditionally interval relations are represented by the initial letter of the description of the relation, like m for *meet*. However, when one considers more relations (like point-point and point-interval relations) this notation becomes confusing, and even more so in the presence of more relations, e.g. when one wants to consider interval relations over a *partial order*[2]. We introduce the following notation to resolve this issue: an interval $[a, b]$ induces a partition of $\mathbb{D}$ into five regions (see [15]): region 0, which contains all points less than a, region 1 which contains a only, region 2 which contains all the points strictly between a and b, region 3 which contains only b and region 4 which contains the points greater than b. Likewise, a point c induces a partition of $\mathbb{D}$ into 3 pieces: region 0 is all the points less than c, region 2 contains only c and region 4 contains all the points greater than c. Interval-interval relations will be denoted by $I k k'_{ii} J$, where $k, k' \in \{0, 1, 2, 3, 4\}$, and k represent the region of the partition induced by I in which the the left endpoint of J falls, while k' is the region of the same partition in which the right endpoint of J falls; for example, $I 34_{ii} J$ is exactly Allen's relation *meets*. Similarly, interval-point relations will be denoted by $I k_{ip} a$, where k represents the position of a with respect to I; for example, $I 4_{ip} a$ is the relation *before*. Symmetrically, point-point relations will be denoted by the symbol k_{pp}, and point-interval relations by the symbol $k k'_{pi}$. For point-point relations it is more convenient to use $<$ instead of 4_{pp}, and $>$ instead of 0_{pp}. In Tab. I we show six of the interval-interval relations, and in Tab. II we show the interval-point relations. Finally, we consider a sorted equality; the symbol $=_i$ will denote the equality between intervals, and $=_p$, as mentioned above, the equality between points, substituting 2_{pp}. Now, given any one of the mentioned relations r, its inverse $\bar{r}$ can be obtained by inverting the roles of the objects in the case of non-mixed relations; therefore, for example, the inverse of the relation 22_{ii} (Allen's relation *during*) is the relation 04_{ii} (*contains*). On the other hand, mixed relations present a different situation: the inverse of a point-interval relation is an interval-point relation; thus, for example, the inverse of 3_{ip} is 02_{pi}. Finally, notice that some combinations, such as 22_{pi}, makes no sense, and are forbidden.

We will denote by $\mathfrak{R}$ the set of all above described relations; by $\mathfrak{I} \subset \mathfrak{R}$ the subset of all interval-interval relations (Allen's relations); by $\mathfrak{M} \subset \mathfrak{R}$ the subset of all

[1]It is worth noticing that in [12] the authors distinguish 30 relations, instead of 26; this is due to the fact that the concepts of the point *a starting* the interval $[a, b]$ and *meeting* it are considered to be different.

[2]This paper is focused on linearly ordered sets only; nevertheless, it is our intention to complete this study to include the treatment of partial orders also, and, at this stage, we want to make sure that we will be able to maintain the notation consistent.

$[a,b]$ 3_{ip} $c \Leftrightarrow b = c$

$[a,b]$ 4_{ip} $c \Leftrightarrow b < c$

$[a,b]$ 2_{ip} $c \Leftrightarrow a < b < c$

$[a,b]$ 1_{ip} $c \Leftrightarrow a = c$

$[a,b]$ 0_{ip} $c \Leftrightarrow c < a$

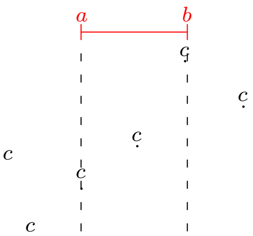

Table II
INTERVAL-POINT RELATIONS.

MIS($\mathfrak{I}^+$)s	mcs($\mathfrak{I}^+$)s
$\{14_{ii}, =_i\}$	$\{34_{ii}\}$
$\{03_{ii}, =_i\}$	$\{24_{ii}, 14_{ii}\}$
$\{44_{ii}, 04_{ii}, 24_{ii}, =_i\}$	$\{24_{ii}, 03_{ii}\}$
	$\{04_{ii}, 14_{ii}\}$
	$\{04_{ii}, 03_{ii}\}$
	$\{14_{ii}, 03_{ii}\}$
	$\{14_{ii}, 44_{ii}\}$
	$\{03_{ii}, 44_{ii}\}$

Table III
MINIMAL $\mathfrak{I}^+$-COMPLETE AND MAXIMAL $\mathfrak{I}^+$-INCOMPLETE SETS.

mixed relations; and, finally, by $\mathfrak{P} \subset \mathfrak{R}$ the subset of all point-point relations. Clearly, $\mathfrak{R} = \mathfrak{I} \cup \mathfrak{M} \cup \mathfrak{P}$. As we work within first-order logic, all inverses of relations are explicitly definable; accordingly, let $\mathfrak{I}^+$ be the set of interval-interval relations given in Tab. I together with $=_i$, $\mathfrak{M}^+$ be the set of interval-point relations given in Tab. II, and $\mathfrak{P}^+ = \{<, =_p\}$. Lastly, let $\mathfrak{R}^+ = \mathfrak{I}^+ \cup \mathfrak{M}^+ \cup \mathfrak{P}^+$. Given a subset $S = \{r_1, \ldots, r_n\} \subseteq \mathfrak{R}^+$, a *concrete point-interval structure of signature* S is a two-sorted relational structure $\mathcal{F} = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), r_1, r_2, \ldots, r_n\rangle$, where $\mathbb{D}$ is a linear order, $I(\mathbb{D})$ is the set of all intervals in $\mathbb{D}$, and each r_i is defined on $\mathbb{D} \cup \mathbb{I}(\mathbb{D})$ according to Tab. I and Tab. II. Since all relations are already implicit in $\mathbb{D} \cup \mathbb{I}(\mathbb{D})$, we will often simply write $\langle \mathbb{D}, \mathbb{I}(\mathbb{D})\rangle$ for a concrete point-interval structure $\langle \mathbb{D}, \mathbb{I}(\mathbb{D}), r_1, r_2, \ldots, r_n\rangle$, and we denote by $FO(S)$ the two-sorted language of first-order logic (without equality on both sorts) with relation symbols corresponding to the relations in S; such definitions are readily adaptable to the single-sort cases. In the two-sorted context, we will use different symbols for variables that are supposed to be interpreted over different sorts; in particular, $x_p, y_p, \ldots$ will denote point variables, $x_i, y_i, \ldots$ interval variables, and $x, y, \ldots$ will be used when we do not want to specify the sort.

Definition 1: Let $S \subseteq \mathfrak{R}^+$ (resp., $S \subseteq \mathfrak{I}^+$). We say that $FO(S)$ *defines* $r \in \mathfrak{R}$ (resp., $r \in \mathfrak{I}$) over all linear orders, denoted by $FO(S) \rightarrow r$, if there exists $FO(S)$-formula $\varphi(x,y)$ such that $\varphi(x,y) \leftrightarrow r(x,y)$ is valid on the class of all concrete point-interval structures (resp., concrete interval structures) of signature $(S \cup \{r\})$ based on any linear order. Obviously, $FO(S) \rightarrow r$ for all $r \in S$.

Definition 2: Let $S \subseteq \mathfrak{R}^+$ (resp., $S \subseteq \mathfrak{I}^+$). We say that S is $\mathfrak{R}^+$*-complete over all linear orders* (resp. $\mathfrak{I}^+$*-complete over all linear orders*), if and only if $FO(S) \rightarrow r$ for all $r \in \mathfrak{R}^+$ (resp., $r \in \mathfrak{I}^+$); and, *minimally complete over all linear orders*, denoted by mcs (resp., *maximally incomplete over all linear orders*, denoted by MIS) if and only if it is complete (resp., incomplete) over all linear orders, and, every proper subset (resp., every strict superset) of S is incomplete (resp., complete) over all linear orders.

In what follows, in order to prove that $FO(S) \not\rightarrow r$ for some r, we will make use of the notion of strong surjective homeomorphism and its properties w.r.t. the satisfiability preserving of FO-formulas (see, e.g. [16]). A *strong surjective homomorphism* from $\mathcal{F} = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), S\rangle$ to $\mathcal{F}' = \langle \mathbb{D}', \mathbb{I}(\mathbb{D}'), S\rangle$, where $S = \{r_1, r_2, \ldots, r_n\}$ and $\mathbb{D}, \mathbb{D}'$ are linear orders, is a map $\zeta : \mathbb{D} \cup \mathbb{I}(\mathbb{D}) \rightarrow \mathbb{D}' \cup \mathbb{I}(\mathbb{D}')$ such that: 1) it respects sorts and is surjective; 2) $r_i(a,b)$ iff $r_i(\zeta(a), \zeta(b))$ for all point-point relations $r_i \in S$; 3) $r_i(a,I)$ (resp., $r_i(I,a)$) iff $r_i(\zeta(a), \zeta(I))$ (resp., $r_i(\zeta(I), \zeta(a))$) for all point-interval (resp., interval-point) relations $r_i \in S$, and 4) $r_i(I,J)$ iff $r_i(\zeta(I), \zeta(J))$ for all interval-interval relations $r_i \in S$. If $=_i$ and $=_p$ are in S then it follows that ζ will be injective, and, thus, an isomorphism. If, also $\mathcal{F} = \mathcal{F}'$ then ζ is an *automorphism* of $\mathcal{F}$. For the sake of clarity we will define ζ as a pair (ζ_i, ζ_p), where $\zeta_i : \mathbb{I}(\mathbb{D}) \rightarrow \mathbb{I}(\mathbb{D}')$ and $\zeta_p : \mathbb{D} \rightarrow \mathbb{D}'$. If ζ is a strong surjective homomorphism, then $\mathcal{F}$ and $\mathcal{F}'$ satisfy the same $FO(S)$ formulas. Thus, to show that $FO(\{r_1, r_2, \ldots, r_n\}) \not\rightarrow r$ it is sufficient to find two structures $\mathcal{F}$ and $\mathcal{F}'$ and a strong surjective homomorphism $\zeta : \mathcal{F} \rightarrow \mathcal{F}'$ which *breaks* r, i.e., such that there are two objects of the right type in S related through r and such that their images are not.

III. $\mathfrak{I}^+$-COMPLETENESS AND INCOMPLETENESS

We start with the expressive power of relations in $\mathfrak{I}^+$. This has been studied in [14] for different classes of linear orders but the difference here is that equality is treated at the level of the other relations.

Theorem 3: [14] The minimal $\mathfrak{I}^+$-complete and maximal $\mathfrak{I}^+$-incomplete sets over the class of all linear orders are those and only those shown in Tab. III.

Proof: For each minimal complete set S, we prove that S can express all interval-interval relations. We first show that $=_i$ can be expressed in terms of 34_{ii}, by using the following definition:

$$x_i =_i y_i \quad \leftrightarrow \quad \forall z_i(z_i 34_{ii} x_i \leftrightarrow z_i 34_{ii} y_i) \wedge \forall z_i(x_i 34_{ii} z_i \leftrightarrow y_i 34_{ii} z_i).$$

Referring ourselves to [1] for the remaining definitions in the case of 34_{ii}, we can easily prove that 34_{ii} is indeed complete. From there, the definability equations in [14] give us the

MIS($\mathfrak{M}^+$)s	mcs($\mathfrak{M}^+$)s
$\{\mathit{3}_{ip}, \mathit{4}_{ip}\}$	$\{\mathit{1}_{ip}, \mathit{3}_{ip}\}$
$\{\mathit{0}_{ip}, \mathit{1}_{ip}\}$	$\{\mathit{1}_{ip}, \mathit{2}_{ip}\}$
$\{\mathit{0}_{ip}, \mathit{2}_{ip}, \mathit{4}_{ip}\}$	$\{\mathit{2}_{ip}, \mathit{3}_{ip}\}$
	$\{\mathit{1}_{ip}, \mathit{4}_{ip}\}$
	$\{\mathit{0}_{ip}, \mathit{3}_{ip}\}$

Table IV
MINIMAL $\mathfrak{M}^+$-COMPLETE AND MAXIMAL $\mathfrak{M}^+$-INCOMPLETE SETS.

entire result, with the exception of the case of $\{\mathit{14}_{ii}, \mathit{44}_{ii}\}$, as equality plays a central role in the definitions. On the other hand, $=_i$ can be actually defined in terms of $\mathit{14}_{ii}$ as follows:

$$x_i =_i y_i \quad \leftrightarrow \quad \forall z_i(z_i \mathit{14}_{ii} x_i \leftrightarrow z_i \mathit{14}_{ii} y_i) \wedge \forall z_i(x_i \mathit{14}_{ii} z_i \leftrightarrow y_i \mathit{14}_{ii} z_i),$$

and, so, we are done. Incompleteness results are not affected by the presence of the equality: the proofs in in [14] use automorphisms which apply unchanged in our current setting. Finally, minimality of complete sets is proven by observing that each proper subset of a complete set is contained in an incomplete one, and maximality of the incomplete ones by observing that each proper extension of an incomplete set contains a complete one. ∎

Theorem 4: The expressively different fragments of $FO(\mathfrak{I}^+)$ are exactly those depicted in Fig. 1.

Proof: In [14] it was shown that, when equality is always assumed in the language, the expressively different fragments of $FO(\mathfrak{I}^+)$ are exactly those corresponding to $\mathfrak{I}^+$, $\{\mathit{14}_{ii}, =_i\}$, $\{\mathit{03}_{ii}, =_i\}$, and all subsets of $\{\mathit{44}_{ii}, \mathit{04}_{ii}, \mathit{24}_{ii}, =_i\}$ which contain $=_i$. In other words, the picture looks exactly like the sublattice of the one in Fig. 1 above $\{=_i\}$. To complete the picture, it suffices to make the following observations: (1) $FO(\{\mathit{14}_{ii}\}) \rightarrow =_i$, as was shown in the proof of Theorem 3. (2) Symmetrically, $FO(\{\mathit{03}_{ii}\}) \rightarrow =_i$. (3) The fragments $\{\mathit{14}_{ii}, =_i\}$, $\{\mathit{03}_{ii}, =_i\}$, and $\{\mathit{44}_{ii}, \mathit{04}_{ii}, \mathit{24}_{ii}, =_i\}$ are pairwise incomparable in terms of expressive power, for assuming that they are not leads to a contradiction with the fact that they are incomplete. (4) No relation in $\{\mathit{44}_{ii}, \mathit{04}_{ii}, \mathit{24}_{ii}, =_i\}$ is definable in terms of any set of the others. Indeed to see that none of $\mathit{44}_{ii}$, $\mathit{04}_{ii}$, $\mathit{24}_{ii}$ can be defined in terms of any of the others one can use the simple automorphism arguments used for this purpose in the proof of Theorem 5 of [14] unchanged. Lastly, to see that $FO(\{\mathit{44}_{ii}, \mathit{04}_{ii}, \mathit{24}_{ii}\}) \not\rightarrow =_i$, consider the structures $\mathcal{F} = \langle \mathbb{I}(\mathbb{D}), \mathit{44}_{ii}, \mathit{04}_{ii}, \mathit{24}_{ii} \rangle$ with $\mathbb{D}$ the order $\{0 < 1 < 2\}$, and $\mathcal{F}' = \langle \mathbb{I}(\mathbb{D}'), \mathit{44}_{ii}, \mathit{04}_{ii}, \mathit{24}_{ii} \rangle$ with $\mathbb{D}'$ the order $\{a < b\}$. The map $\zeta : \mathbb{I}(\mathbb{D}) \rightarrow \mathbb{I}(\mathbb{D})$ which sends every interval in $\mathbb{I}(\mathbb{D})$ to $[a, b]$ is a strong surjective homomorphism, since the relations $\mathit{44}_{ii}$, $\mathit{04}_{ii}$, $\mathit{24}_{ii}$ are empty in both structures. However, ζ breaks $=_i$. ∎

IV. $\mathfrak{M}^+$-COMPLETENESS AND INCOMPLETENESS

The situation for relations in $\mathfrak{M}^+$ over linear orders is depicted in Tab. IV; it is worth to observe that that there is no equality involved.

Lemma 5: Each set in the rightmost column of Tab. IV is $\mathfrak{M}^+$-complete over the class of all linear orders.

Proof: Let us work case-by-case. In the case $\{\mathit{1}_{ip}, \mathit{3}_{ip}\}$ we exploit the result recalled in Theorem 3 concerning the $\mathfrak{I}^+$-completeness of the interval-interval relation $\mathit{34}_{ii}$. First, we prove that the latter can be expressed in the fragment $\{\mathit{1}_{ip}, \mathit{3}_{ip}\}$:

$$x_i \mathit{34}_{ii} y_i \quad \leftrightarrow \quad \exists z_p(x_i \mathit{3}_{ip} z_p \wedge y_i \mathit{1}_{ip} z_p),$$

whose correctness is immediate. Then, we can use any interval-interval relation to express the remaining interval-point relations:

$$\begin{aligned} x_i \mathit{0}_{ip} y_p &\quad \leftrightarrow \quad \exists z_i(x_i \mathit{03}_{ii} z_i \wedge z_i \mathit{1}_{ip} y_p), \\ x_i \mathit{2}_{ip} y_p &\quad \leftrightarrow \quad \exists z_i(x_i \mathit{24}_{ii} z_i \wedge z_i \mathit{1}_{ip} y_p), \\ x_i \mathit{4}_{ip} y_p &\quad \leftrightarrow \quad \exists z_i(x_i \mathit{14}_{ii} z_i \wedge z_i \mathit{3}_{ip} y_p). \end{aligned}$$

Again, the correctness of the above is straightforward. In the case of $\{\mathit{1}_{ip}, \mathit{2}_{ip}\}$, we first define $\mathit{0}_{ip}$, as follows:

$$x_i \mathit{0}_{ip} y_p \quad \leftrightarrow \quad \exists z_i \exists w_p(z_i \mathit{1}_{ip} y_p \wedge x_i \mathit{1}_{ip} w_p \wedge z_i \mathit{2}_{ip} w_p).$$

Then, we observe that the interval-interval relation $\mathit{14}_{ii}$ can be defined in this fragment:

$$x_i \mathit{14}_{ii} y_i \quad \leftrightarrow \quad \exists z_p \exists w_p(y_i \mathit{1}_{ip} w_p \wedge x_i \mathit{1}_{ip} w_p \wedge \neg(x_i \mathit{2}_{ip} z_p) \wedge y_i \mathit{2}_{ip} z_p).$$

Finally, from the above we easily get $\mathit{4}_{ip}$, and then $\mathit{3}_{ip}$ by difference:

$$x_i \mathit{4}_{ip} y_p \quad \leftrightarrow \quad \neg(x_i \mathit{0}_{ip} y_p \vee x_i \mathit{1}_{ip} y_p \vee x_i \mathit{2}_{ip} y_p) \wedge \exists z_i(z_i \mathit{14}_{ii} x_i \wedge \exists w_p(\neg(y_i \mathit{2}_{ip} w_p) \wedge z_i \mathit{2}_{ip} w_p)).$$

Again, it is rather straightforward to prove that the above definitions are correct. ∎

Lemma 6: Each set in the leftmost column of Tab. IV is $\mathfrak{M}^+$-incomplete over the class of all linear orders.

Proof: As in the previous lemma, we proceed case-by-case. Incompleteness of $\{\mathit{0}_{ip}, \mathit{1}_{ip}\}$ can be shown as follows. Consider the point-interval structure $\mathcal{F} = \langle \mathbb{Q}, \mathbb{I}(\mathbb{Q}), \mathit{0}_{ip}, \mathit{1}_{ip} \rangle$, where $\mathbb{Q}$ is the set or rational numbers with their usual ordering. Define ζ as a pair of functions $\zeta = (\zeta_i, \zeta_p)$, where $\zeta_i : \mathbb{I}(\mathbb{Q}) \rightarrow \mathbb{I}(\mathbb{Q})$ such that $\zeta : [a, b] \mapsto [a, a + 2 \cdot |b - a|]$, and where ζ_p is the identity function on $\mathbb{Q}$. In other words, the image of any interval $[a, b]$ under ζ has the same beginning point, but double the length of $[a, b]$. We claim that ζ is an automorphism of the structure $\mathcal{F}$. It is clear that ζ is a bijection. Further, $[a_1, b_1] \mathit{0}_{ip} c_1$ if and only if $c_1 < a_1$, that is, if and only if $\zeta([a, b]) = [a_1, a_1 + 2 \cdot$

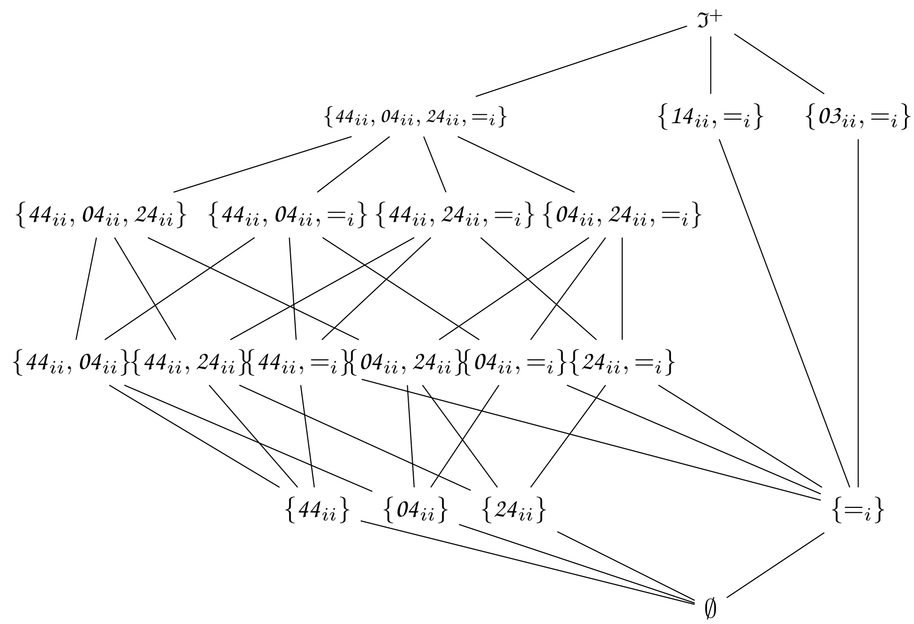

Figure 1. The lattice of expressively different fragments of $FO(\mathfrak{I}^+)$

$|b_1 - a_1|]0_{ip}c_1 = \zeta(c_1)$, and similarly for 1_{ip}. Now, as $\neg[0,1]2_{ip}1$ and $\zeta([0,1])2_{ip}\zeta(1)$ (i.e., $[0,2]2_{ip}1$), we have that ζ breaks 2_{ip}, that is, $FO(\{0_{ip}, 1_{ip}\}) \not\to 2_{ip}$. Finally, in the case of $\{0_{ip}, 2_{ip}, 4_{ip}\}$, consider the structure $\mathcal{F} = \langle \mathbb{D} = \{a < b\}, \mathbb{I}(\mathbb{D}), 0_{ip}, 2_{ip}, 4_{ip}\rangle$. Define $\zeta = (\zeta_i, \zeta_p)$ as a pair of functions $\zeta_i : \mathbb{I}(\mathbb{D}) \to \mathbb{I}(\mathbb{D})$ such that $\zeta([a,b]) \mapsto [a,b]$, and $\zeta_p : \mathbb{D} \to \mathbb{D}$ such that $\zeta(a) \mapsto b$ and $\zeta(b) \mapsto a$. We claim that ζ is an automorphism of $\mathcal{F}$. Indeed, ζ is clearly a bijection, and, further the relations 0_{ip}, 2_{ip}, and $4_{ip}\}$ are empty and hence respected. It is also clear that ζ breaks 3_{ip}, for example. Indeed $[a,b]\ 3_{ip}\ b$, while it is not the case that $(\zeta([a,b])3_{ip}\zeta_p(b)$, i.e., that $[a,b]\ 3_{ip}\ a$. Therefore 3_{ip} cannot be expressed in this language, and thus the considered set must be incomplete. ■

Now, Theorem 7 follows from lemmas 5 and 6 together with the observation that each proper subset of a minimally complete set is contained in an incomplete one and each proper extension of an incomplete set contains a complete set.

Theorem 7: The minimal $\mathfrak{M}^+$-complete and maximal $\mathfrak{M}^+$-incomplete sets over the class of all linear orders are those and only those shown in Tab. IV.

Theorem 8: The lattice of expressively different fragments of $FO(\mathfrak{M}^+)$ is depicted in Fig. 2.

Sketch: Given Theorem 7, it only remains to classify the subsets of $\{3_{ip}, 4_{ip}\}$, $\{0_{ip}, 1_{ip}\}$, and $\{0_{ip}, 2_{ip}, 4_{ip}\}$. Moreover, by their maximal incompleteness, these three sets must be incomparable. Let us work case-by case. First, consider $FO(\{3_{ip}\}) \not\to 4_{ip}$. Consider the structure $\mathcal{F}_1 = \langle \mathbb{R}, \mathbb{I}(\mathbb{R}), 3_{ip}\rangle$ with $\mathbb{R}$ the reals with their usual ordering, and the automorphism $\zeta = (\zeta_i, \zeta_p)$ where ζ_i is given by $\zeta_i([a,b]) = [a-1, b-1]$ if b is rational and $\zeta_i([a,b]) = [a+1, b+1]$ otherwise, and ζ_p is given by $\zeta_p(a) = a-1$ if a is rational and $\zeta_p(a) = a+1$ otherwise. This breaks 4_{ip} since, e.g., $[0,\sqrt{2}]4_{ip}2$ but it is not the case that $\zeta_i([0,\sqrt{2}])4_{ip}\zeta_p(2)$, i.e., that it is not the case that $[1, \sqrt{2}+1])4_{ip}1$. As for the case $FO(\{4_{ip}\}) \not\to 3_{ip}$, let $\mathcal{F}_2 = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), 4_{ip}\rangle$ with $\mathbb{D}$ the ordering $0 < 1 < 2$. Consider the automorphism $\zeta = (\zeta_i, \zeta_p)$ where ζ_i is the identity on $\mathbb{I}(\mathbb{D})$, and ζ_p swaps 0 and 1 leaving 2 fixed. This breaks 3_{ip} since $[0,1]3_{ip}1$ but it is not the case that $\zeta_i([0,1])3_{ip}\zeta_p(1)$, i.e., it is not the case that $[0,1]3_{ip}0$. The cases $FO(\{1_{ip}\}) \not\to 0_{ip}$ and $FO(\{0_{ip}\}) \not\to 1_{ip}$ are symmetric to the above two cases. The case $FO(\{0_{ip}, 2_{ip}\}) \to 4_{ip}$ is dealt with as follows. Consider any structure $\langle \mathbb{D}, \mathbb{I}(\mathbb{D}), 0_{ip}, 2_{ip}\rangle$. Define $R_1 \subseteq \mathbb{I}(\mathbb{D}) \times \mathbb{D}$ such that $x_i R_1 y_p$ iff $\neg(x_i 0_{ip} y_p) \wedge \neg(x_i 2_{ip} y_p)$. Thus $[a,b]R_1 c$ iff $c = a$ or $b \leq c$. Next define $R_2 \subseteq \mathbb{I}(\mathbb{D}) \times \mathbb{D}$ such that

$$x_i R_2 y_p \quad \leftrightarrow \quad (x_i R_1 y_p) \wedge \exists w_p(x_i 2_{ip} w_p) \wedge \exists z_i \exists w_p(x_i R_1 w_p \wedge z_i 0_{ip} w_p \wedge z_i R_1 y_p).$$

So $[a,b]R_2 c$ iff $[a,b]$ is not a unit interval and $b \leq c$. We also have that $FO(\{0_{ip}, 2_{ip}\}) \to 14_{ii}$. Indeed

$$x_i 14_{ii} z_i \quad \leftrightarrow \quad \forall w_p(x_i 0_{ip} w_p \leftrightarrow z_i 0_{ip} w_p) \wedge \exists w_p(z_i 2_{ip} w_p \wedge \neg x_i 2_{ip} w_p).$$

Putting all this together we can define 4_{ip}:

$$x_i 4_{ip} y_p \quad \leftrightarrow \quad \exists z_i(x_i 14_{ii} z_i \wedge z_i R_2 y_p).$$

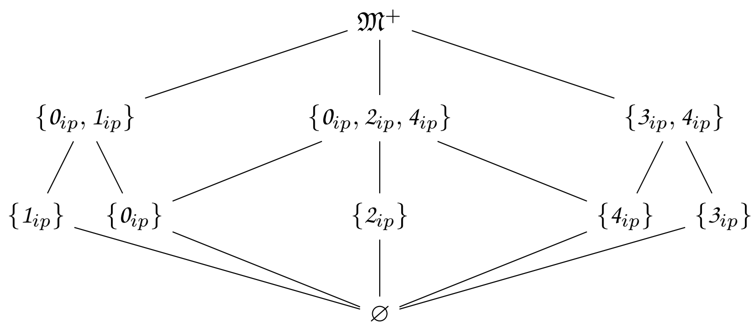

Figure 2. The lattice of expressively different fragments of $FO(\mathfrak{M}^+)$

A symmetric argument shows that $FO(\{4_{ip}, 2_{ip}\}) \to 0_{ip}$. As for the case $FO(\{0_{ip}, 4_{ip}\}) \to 2_{ip}$, consider any structure $\langle \mathbb{D}, \mathbb{I}(\mathbb{D}), 0_{ip}, 4_{ip} \rangle$. Define $R_3 \subseteq \mathbb{I}(\mathbb{D}) \times \mathbb{D}$ such that $x_i R_3 y_p$ iff $\neg(x_i 0_{ip} y_p) \wedge \neg(x_i 4_{ip} y_p)$. Thus $[a,b] R_3 c$ iff $a \leq c \leq b$. It is then not difficult to see that

$$\begin{aligned} x_i 2_{ip} y_p \quad \leftrightarrow \quad & x_i R_3 y_p \wedge \\ & \exists z_i \exists k_p (x_i R_3 k_p \wedge z_i R_3 y_p \wedge z_i 0_{ip} k_p) \\ & \wedge \exists z'_i \exists k'_p (x_i R_3 k'_p \wedge z'_i R_3 y_p \wedge z'_i 4_{ip} k'_p). \end{aligned}$$

The cases $FO(\{0_{ip}\}) \not\to 2_{ip}$ and $FO(\{0_{ip}\}) \not\to 4_{ip}$ can be solved together. Consider the structure $\mathcal{F}_3 = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), 0_{ip} \rangle$ with $\mathbb{D}$ the ordering $0 < 1 < 2$. Consider the automorphism $\zeta = (\zeta_i, \zeta_p)$, where ζ_i is the identity on $\mathbb{I}(\mathbb{D})$ and ζ_p swaps 1 and 2 leaving 0 fixed. This breaks 2_{ip} since $[0,2] 2_{ip} 1$ but it is not the case that $\zeta_i([0,2]) 3_{ip} \zeta_p(1)$, i.e., that $[0,2] 3_{ip} 2$. It also breaks 4_{ip} since $[0,1] 4_{ip} 2$ but it is not the case that $\zeta_i([0,1]) 4_{ip} \zeta_p(2)$, i.e., that $[0,1] 4_{ip} 1$. A symmetric argument shows that $FO(\{4_{ip}\}) \not\to 2_{ip}$ and $FO(\{4_{ip}\}) \not\to 0_{ip}$. Finally, as for the cases $FO(\{2_{ip}\}) \not\to 0_{ip}$ and $FO(\{2_{ip}\}) \not\to 4_{ip}$, consider the structure $\mathcal{F}_4 = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), 2_{ip} \rangle$ with $\mathbb{D}$ the ordering $0 < 1 < 2$. Consider the automorphism $\zeta = (\zeta_i, \zeta_p)$ where ζ_i is the identity on $\mathbb{I}(\mathbb{D})$ and ζ_p swaps 0 and 2 leaving 1 fixed. This breaks 0_{ip} since $[1,2] 0_{ip} 0$ but it is not the case that $\zeta_i([1,2]) 0_{ip} \zeta_p(0)$, i.e., that $[1,2] 0_{ip} 2$. It also breaks 4_{ip} since $[0,1] 4_{ip} 2$ but it is not the case that $\zeta_i([0,1]) 4_{ip} \zeta_p(2)$, i.e., that $[0,1] 4_{ip} 0$. ■

V. $\mathfrak{R}^+$-Completeness and Incompleteness

We can now turn our attention to the set $\mathfrak{R}^+$. As before, due to space constraints, the results will only be sketched. The situation for arbitrary linear orders is depicted in Fig. V.

Lemma 9: Each set in the rightmost column of Tab. V is $\mathfrak{R}^+$-complete over the class of all linear orders.

Proof: We proceed case by case. To prove the completeness of $\{1_{ip}, 3_{ip}\}$ we recall Lemma 5 (specifically the proof of the case for $\{1_{ip}, 3_{ip}\}$) and Theorem 3: as we already know that this fragment can express 34_{ii}, we can deduce that it is both $\mathfrak{M}^+$- and $\mathfrak{I}^+$-complete; therefore, by simply observing that we can express $<$ and $=_p$ in the fragment, we are done:

$$\begin{aligned} x_p < y_p \quad &\leftrightarrow \quad \exists z_i (z_i 1_{ip} x_p \wedge z_i 3_{ip} y_p), \\ x_p =_p y_p \quad &\leftrightarrow \quad \forall z_i (z_i 1_{ip} x_p \leftrightarrow z_i 1_{ip} y_p). \end{aligned}$$

Next, we observe that the sets $\{2_{ip}, 3_{ip}\}, \{1_{ip}, 4_{ip}\}$, $\{0_{ip}, 3_{ip}\}$, and $\{1_{ip}, 2_{ip}\}$ are $\mathfrak{M}^+$-complete by Lemma 5, and, thus, 1_{ip} and 3_{ip} can be expressed, which implies that they are all $\mathfrak{R}^+$-complete thanks to the previous result. Consider, now, the case $\{34_{ii}, 3_{ip}\}$. From the previous results, we know it suffices to express 2_{ip} in this fragment to prove its completeness. Recalling that 34_{ii} is $\mathfrak{I}^+$-complete, we can use any $\mathfrak{I}^+$-relation, and so:

$$x_i 2_{ip} y_p \quad \leftrightarrow \quad \exists z_i (z_i 14_{ii} x_i \wedge z_i 3_{ip} y_p).$$

Bearing in mind that 01_{ii} is definable in terms 34_{ii}, being its inverse, the case $\{34_{ii}, 1_{ip}\}$ can be treated symmetrically. The case $\{14_{ii}, 3_{ip}\}$ can be sorted out by simply defining 2_{ip} by means of the same equation as in the previous case, and the case $\{03_{ii}, 1_{ip}\}$ comes by symmetry. The set $\{34_{ii}, 0_{ip}, <\}$ can be proved to be $\mathfrak{R}^+$-complete by observing that 1_{ip} is expressible, and, then, recalling that we have already shown the $\mathfrak{R}^+$-completeness of $\{34_{ii}, 1_{ip}\}$:

$$x_i 1_{ip} y_p \quad \leftrightarrow \quad \forall z_p (x_i \ 0_{ip} \ z_p \leftrightarrow z_p < y_p).$$

Symmetrically, from $\{34_{ii}, 4_{ip}, <\}$ we express 3_{ip}, and exploit the completeness of $\{34_{ii}, 3_{ip}\}$. In the case $\{34_{ii}, 2_{ip}, <\}$, we can define one of 0_{ip}, 1_{ip}, or 3_{ip} to reduce to one of the previous cases:

$$\begin{aligned} x_i \ 0_{ip} \ y_p \quad \leftrightarrow \quad & \exists z_i \exists w_p ((x_i 03_{ii} z_i) \wedge (z_i 2_{ip} w_p) \wedge \\ & (\neg x_i 2_{ip} w_p) \wedge (y_p < w_p)) \end{aligned}$$

As for the case of $\{14_{ii}, 4_{ip}, <\}$, we can exploit the completeness of $\{14_{ii}, 3_{ip}\}$ shown above. This can be done by means of the following equation:

$$x_i 3_{ip} y_p \quad \leftrightarrow \quad \forall z_p (x_i 4_{ip} z_p \leftrightarrow y_p < z_p).$$

Symmetrically, $\{03_{ii}, 0_{ip}, <\}$ can be proven complete by exploiting the completeness of $\{03_{ii}, 1_{ip}\}$. In the case of

MIS($\mathfrak{R}^+$)s	mcs($\mathfrak{R}^+$)s
$\mathfrak{I}^+ \cup \{\mathit{0}_{ip}, \mathit{2}_{ip}, \mathit{4}_{ip}, =_i, =_p\}$	$\{\mathit{1}_{ip}, \mathit{3}_{ip}\}, \{\mathit{34}_{ii}, \mathit{0}_{ip}, <\}$
$\mathfrak{I}^+ \cup \{<, =_i, =_p\}$	$\{\mathit{2}_{ip}, \mathit{3}_{ip}\}, \{\mathit{34}_{ii}, \mathit{2}_{ip}, <\}$
$\{\mathit{14}_{ii}, \mathit{0}_{ip}, \mathit{1}_{ip}, <, =_i, =_p\}$	$\{\mathit{1}_{ip}, \mathit{4}_{ip}\}, \{\mathit{34}_{ii}, \mathit{4}_{ip}, <\}$
$\{\mathit{03}_{ii}, \mathit{3}_{ip}, \mathit{4}_{ip}, <, =_i, =_p\}$	$\{\mathit{0}_{ip}, \mathit{3}_{ip}\}, \{\mathit{14}_{ii}, \mathit{4}_{ip}, <\}$
$\{\mathit{04}_{ii}, \mathit{24}_{ii}, \mathit{44}_{ii}, \mathit{2}_{ip}, <, =_i, =_p\}$	$\{\mathit{1}_{ip}, \mathit{2}_{ip}\}, \{\mathit{03}_{ii}, \mathit{0}_{ip}, <\}$
$\{\mathit{24}_{ii}, \mathit{04}_{ii}, \mathit{44}_{ii}, \mathit{0}_{ip}, \mathit{1}_{ip}, <, =_i, =_p\}$	$\{\mathit{34}_{ii}, \mathit{1}_{ip}\}, \{\mathit{14}_{ii}, \mathit{2}_{ip}, <\}$
$\{\mathit{24}_{ii}, \mathit{04}_{ii}, \mathit{44}_{ii}, \mathit{3}_{ip}, \mathit{4}_{ip}, <, =_i, =_p\}$	$\{\mathit{34}_{ii}, \mathit{3}_{ip}\}, \{\mathit{03}_{ii}, \mathit{2}_{ip}, <\}$
	$\{\mathit{14}_{ii}, \mathit{3}_{ip}\}, \{\mathit{0}_{ip}, \mathit{2}_{ip}, <\}$
	$\{\mathit{03}_{ii}, \mathit{1}_{ip}\}, \{\mathit{2}_{ip}, \mathit{4}_{ip}, <\}$

Table V
MINIMAL $\mathfrak{R}^+$-COMPLETE AND MAXIMAL $\mathfrak{R}^+$-INCOMPLETE SETS.

$\{\mathit{14}_{ii}, \mathit{2}_{ip}, <\}$, we define $\mathit{4}_{ip}$ and exploit the completeness of $\{\mathit{14}_{ii}, \mathit{4}_{ip}, <\}$:

$$x_i \ \mathit{4}_{ip} \ y_p \quad \leftrightarrow \quad \exists z_i \exists w_p((x_i \mathit{14}_{ii} z_i) \wedge (z_i \mathit{2}_{ip} w_p) \wedge (\neg x_i \mathit{2}_{ip} w_p) \wedge (w_p < y_p)).$$

Once again, we can solve the case of $\{\mathit{03}_{ii}, \mathit{2}_{ip}, <\}$ by symmetry, making use of the completeness of $\{\mathit{03}_{ii}, \mathit{0}_{ip}, <\}$. Finally, the set $\{\mathit{0}_{ip}, \mathit{2}_{ip}, <\}$ can be proven to be complete by expressing $\mathit{1}_{ip}$:

$$x_i \mathit{1}_{ip} y_p \quad \leftrightarrow \quad \forall z_p(x_i \mathit{0}_{ip} z_p \leftrightarrow z_p < y_p),$$

and, symmetrically, the set $\{\mathit{2}_{ip}, \mathit{4}_{ip}, <\}$ can be proven to be complete by expressing $\mathit{3}_{ip}$. ■

Lemma 10: Each set in the leftmost column of Tab. V is $\mathfrak{R}^+$-incomplete over the class of all linear orders.

Proof: Let us work case-by case, starting with the R^+-incompleteness of $\mathfrak{I}^+ \cup \{\mathit{0}_{ip}, \mathit{2}_{ip}, \mathit{4}_{ip}, =_i, =_p\}$. Consider the structure $\mathcal{F} = \langle \mathbb{D} = \{a < b\}, \mathbb{I}(\mathbb{D}), S \rangle$, where $S = \mathfrak{I}^+ \cup \{\mathit{0}_{ip}, \mathit{2}_{ip}, \mathit{4}_{ip}, =_i, =_p\}$. Define $\zeta = (\zeta_i, \zeta_p)$ as a pair of functions $\zeta_i : \mathbb{I}(\mathbb{D}) \to \mathbb{I}(\mathbb{D})$ such that $\zeta([a, b]) \mapsto [a, b]$, and $\zeta_p : \mathbb{D} \to \mathbb{D}$ such that $\zeta_p(a) \mapsto b$ and $\zeta_p(b) \mapsto a$. We claim that ζ is an automorphism of $\mathcal{F}$. Indeed, ζ is clearly a bijection, and, further, no pair of intervals is r-related, for any $r \in \mathfrak{I}^+$; moreover, as there are only two points, and no interval in the structure is related to any point which is before, after, or inside it, ζ respects the three interval-point relations; finally, equality of both sorts is respected too. It is also clear that ζ does not respect, for example, the relation $\mathit{3}_{ip}$, as $[a, b]\mathit{3}_{ip}b$, while it is not the case that $\zeta([a, b])\mathit{3}_{ip}\zeta(b)$. Therefore $\mathit{3}_{ip}$ cannot be expressed in this language, and thus the considered set must be incomplete. In the case of $\mathfrak{I}^+ \cup \{<, =_i, =_p\}$, we consider the structure $\mathcal{F} = \langle \mathbb{Z}, \mathbb{I}(\mathbb{Z}), S \rangle$, where $S = \mathfrak{I}^+ \cup \{<, =_i, =_p\}$, and define $\zeta = (\zeta_i, \zeta_p)$ such that ζ_i is the identity function over $\mathbb{I}(\mathbb{D})$, and $\zeta_p(n) = n + 1$ for all $n \in \mathbb{Z}$. Then ζ is an automorphism of $\mathcal{F}$, clearly being a bijection respecting all $\mathfrak{I}^+$-relations and the ordering among the points. At the same time, the relations between the points and the intervals are not respected by ζ, which shows that no interval-point relation can be expressed, making S incomplete. As for the case $\{\mathit{14}_{ii}, \mathit{0}_{ip}, \mathit{1}_{ip}, <, =_i, =_p\}$, we consider the structure $\mathcal{F} = \langle \mathbb{Q}, \mathbb{I}(\mathbb{Q}), S \rangle$, where $\mathbb{Q}$ is the set or rational numbers with their usual ordering, and $S = \{\mathit{14}_{ii}, \mathit{0}_{ip}, \mathit{1}_{ip}, <, =_i, =_p\}$. Define ζ as a pair of functions $\zeta = (\zeta_i, \zeta_p)$, where $\zeta_i : \mathbb{I}(\mathbb{Q}) \to \mathbb{I}(\mathbb{Q})$ such that $\zeta : [a, b] \mapsto [a, a + 2 \cdot |b - a|]$, and where ζ_p is the identical function over $\mathbb{Q}$. In other words, the image of any interval $[a, b]$ under ζ has the same beginning point, but double the length of $[a, b]$. We claim that ζ is an automorphism of the structure $\mathcal{F}$. It is clear that ζ is a bijection. Further, $[a_1, b_1] \ \mathit{14}_{ii} \ [a_2, b_2]$ if and only if $a_1 = a_2$ and $b_1 < b_2$, that is, if and only if $a_1 = a_2$ and $a_1 + 2 \cdot |b_1 - a_1| < a_2 + 2 \cdot |b_2 - a_2|$, which happens if and only if $\zeta([a_1, b_1]) \ \mathit{14}_{ii} \ \zeta([a_2, b_2])$. Finally, points are identically related to each other under ζ, implying that $<$ is respected, and if a point is the beginning (resp., before the beginning) point of an interval, it remains so under ζ, proving that ζ respects $\mathit{0}_{ip}$ and $\mathit{1}_{ip}$. Now, we show that $\mathit{44}_{ii}$ cannot be defined in this language, for which it is enough to observe that, since $\zeta([0, 1]) = [0, 2]$ and $\zeta([2, 3]) = [2, 4]$, for all formulas $\varphi(x, y)$ of this language, we have that $\mathcal{F} \models \varphi([0, 1], [2, 3])$ if and only if $\mathcal{F} \models \varphi([0, 2], [2, 4])$, but, at the same time, $[0, 1]\mathit{44}_{ii}[2, 3]$ and $\neg([0, 2]\mathit{44}_{ii}[2, 4])$. The case $\{\mathit{03}_{ii}, \mathit{3}_{ip}, \mathit{4}_{ip}, <, =_i, =_p\}$ is symmetric. Following the same idea, the set $\{\mathit{04}_{ii}, \mathit{24}_{ii}, \mathit{44}_{ii}, \mathit{2}_{ip}, <, =_i, =_p\}$ is shown incomplete by considering a structure $\mathcal{F} = \langle \mathbb{D} = \{a < b < c\}, \mathbb{I}(\mathbb{D}), S \rangle$, where $S = \{\mathit{04}_{ii}, \mathit{24}_{ii}, \mathit{44}_{ii}, \mathit{2}_{ip}, <, =_i, =_p\}$, and by defining $\zeta = (\zeta_i, \zeta_p)$ as a pair of functions $\zeta_i : \mathbb{I}(\mathbb{D}) \to \mathbb{I}(\mathbb{D})$ is such that $\zeta([a, c]) \mapsto [a, c]$, $\zeta([a, b]) \mapsto [b, c]$, and $\zeta([b, c]) \mapsto [a, b]$, while $\zeta_p : \mathbb{D} \to \mathbb{D}$ is the identical function. As before, ζ is an automorphism of $\mathcal{F}$, as it is a bijection, it respects all interval-interval and interval-point relations, and it does not involve equalities. Obviously, $\mathit{34}_{ii}$ is not respected, and therefore the set is incomplete. Finally, a very similar construction shows that $\{\mathit{24}_{ii}, \mathit{04}_{ii}, \mathit{44}_{ii}, \mathit{0}_{ip}, \mathit{1}_{ip}, <, =_i, =_p\}$ is incomplete; it suffices to define, over the same structure as before, the automorphism $\zeta = (\zeta_i, \zeta_p)$, where $\zeta_i : \mathbb{I}(\mathbb{D}) \to \mathbb{I}(\mathbb{D})$ is such that $\zeta([a, b]) \mapsto [a, c]$, $\zeta([a, c]) \mapsto [a, b]$, and $\zeta([b, c]) \mapsto [b, c]$, while $\zeta_p : \mathbb{D} \to \mathbb{D}$ is the identical function; as the initial point of the intervals is respected, $\mathit{0}_{ip}$ and $\mathit{1}_{ip}$ are respected as well, and, again, 34_{ii} is not. Modulo taking inverses of some relations, a symmetrical argument proves the incompleteness of $\{24_{ii}, 04_{ii}, 44_{ii}, 3_{ip}, 4_{ip}, <, =_i, =_p\}$. ■

Now, Theorem 11 follows from lemmas 9 and 10 together with the observation that each proper subset of a minimally complete set is contained in an incomplete one and each proper extension of an incomplete set contains a complete set.

Theorem 11: The minimal $\mathfrak{R}^+$-complete and maximal $\mathfrak{R}^+$-incomplete sets over the class of all linear orders are those and only those shown in Tab. V.

VI. Conclusions and Future Work

We considered here two-sorted first-order temporal languages including relations between intervals, points, and inter-sort. We analyzed the fragments of this language containing respectively, only interval-interval relations and only inter-sort relations, giving complete classifications of their sub-fragments w.r.t. expressive power. We have found that there are 19 expressively different extensions of first-order logic with interval relations, and 10 with inter-sort relations. Lastly, we have considered the extension of two-sorted first-order logic with all the above mentioned relations, restricting ourselves to the identification of all expressively complete fragments and all maximal expressively incomplete fragments, laying some foundations for a forthcoming complete classification. We plan to complete this classification, as well as to consider this problem for different classes of linear orderings and of branching orderings.

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