Unified Gyro Error Model

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***Abstract*—The objective of this paper is to present a unified gyro error model. The fibre-optic gyro (FOG) will be used as the gyro of example, but with a small adjustment the model could be used for low-cost gyros as well. The error model will be used in the simulation of gyro measurements and for the compensation of the raw FOG measurements inside an Inertial Measurement Unit (IMU). Such a unified error equation is needed as most FOG design specifications and development documentation define the system performance parameters in the format of the IEEE Std 952-1997 [1]. The correspondence between this format and the more widely published gyro error models is not obvious. This paper will attempt to bridge this gap.**

I. Introduction

The field of inertial navigation is one of the key enabling technologies for the development of autonomous systems. High accuracy mechanical and ring-laser gyros have been available for many years, but it is the development of FOGs and MEMS (microelectromechanical systems) gyros (refer to references [2]–[4]) that have placed gyro technology within the reach of the average consumer market focussed system. These lower cost sensors do, however, come at a price as the cheaper sensors have significant measurement noise. The noise makes the sensors unfit for particular applications or it needs to be explicitly taken into consideration during the design of a system making use of the gyros. The intention of this paper is therefore to develop a unified sensor error model that can be used to model the gyro errors and be useful in the design of the system. The focus will be on FOGs, but could be generalized to be used for MEMS sensors as well. The unified error equation is needed as most FOG design specifications and development documentation define the system performance parameters in the format of the IEEE Std 952-1997 [1] for FOGs and IEEE Std 1431-2004 [2] for MEMS. The correspondence between the format of the error equation in the standards and the more widely published gyro error models (such as those presented in references [5]–[8]) is not obvious. This paper will attempt to bridge this gap.

II. Background Work

A. IEEE FOG Model Equation

Any discussion of FOG error modeling should start with the standard FOG model equation as defined in IEEE Std 952-1997 [1], which is

$$S_0\left[\frac{\Delta N}{\Delta t}\right] = \frac{\Omega + E + D}{1 + 10^{-6}\epsilon_K} \qquad (1)$$

where

S_0	is the nominal scale factor in units of degrees per second ($^\circ/s$),
$\Delta N/\Delta t$	is the output pulse rate in units of pulses per second,
Ω	is the inertial input terms in unit of ($^\circ/h$),
E	is the environmentally senstive drift terms in unit of ($^\circ/h$),
D	is the drift terms in unit of ($^\circ/h$),
ϵ_K	is the scale factor error terms in unit of parts per million (ppm).

Note that the IEEE Std 952-1997 [1] uses the notation I for the inertial input terms, but Ω is rather used here as I will be used to indicate an indentity matrix.

Equation (1) will be used as the baseline for our development of a unified FOG error equation. The limited space in this paper does not allow for the complete definition

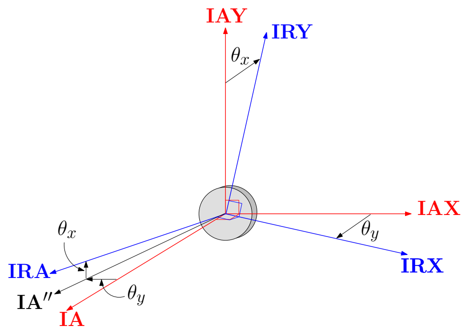

Fig. 1. Gyro axis definition with misalignment angles.

of all the terms in equation (1). The reader is referred to the IEEE Std 528-2001 [9] and the IEEE Std 952-1997 [1] for a comprehensive discussion of these error terms.

B. Axes Definition

The axes definition for a single FOG (a sensor taking rotational measurements about a single axis) is presented in figure 1. The primary sensing axis of a single axis FOG is defined as the Input Axis (IA). This axis is defined as "*the axis(es) about which a rotation of the case causes a maximum output*" [9]. This axis is perpendicular to the plane of the gyro coil while passing through the center of the coil. In terms of the axes definition presented in figure 1, IA represents the z-axis of a orthogonal triad of axes defined in terms of the right hand rule. The x-axis (defined as IAX) and y-axis (defined as IAY) are perpendicular to IA passing through the rim of the coil. IAX is not linked to a particular reference point, so a rotation of the axes around IA does not affect the axes definition.

The single-axis gyro is usually mounted as part of a triad of orthogonal gyros within the IMU casing. The Input Reference Axis (IRA) is defined as "*the direction of an axis (nominally parallel to an input axis) as defined by the case mounting surfaces, or external case markings, or both*" [9]. The IRA is therefore the mounting axis of the gyro. The IRA axis is combined with the IRX and IRY axis to complete the orthogonal set of the Reference Axis. It is assumed that the single-axis gyro (which performs its measurements in the IA) is misaligned with respect to the gyro casing axis (the RA) by the misalignment angles θ_x and θ_y as indicated in figure 1. To resolve the actual measurement in the RA, the gyro measurement needs to be rotated through the misalignment angles to transform the gyro measurement from the IA to the RA. This transformation consists of the following rotations:

1) Perform a negative rotation through θ_y around the IAY axis to change the IAX axis to the IRX axis and the IA axis to the IA'' axis.
2) Perform a negative rotation through θ_x around the IRX axis to change the IA'' axis to the IRA axis and the IAY axis to the IRY axis.

If the measurement in the IA is defined as $\boldsymbol{\omega}$ (the true rotation exerted upon the system) and the measurement as resolved in the IA'' axis is defined as $\boldsymbol{\omega}'$, the relationship between these two terms is defined by the first rotation mentioned above as[1]

$$\boldsymbol{\omega}' = \begin{bmatrix} c\theta_y & 0 & s\theta_y \\ 0 & 1 & 0 \\ -s\theta_y & 0 & c\theta_y \end{bmatrix} \boldsymbol{\omega} \qquad (2)$$

Defining the rotation as resolved in the RA as $\boldsymbol{\omega}''$, the second rotation defined above equates to

$$\boldsymbol{\omega}'' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta_x & -s\theta_x \\ 0 & s\theta_x & c\theta_x \end{bmatrix} \boldsymbol{\omega}' \qquad (3)$$

The full sequence of rotations that is necessary to transform a measurement from the IA to the RA is therefore defined through a combination of equations (2) and (3) with the full transformation being defined as

$$\boldsymbol{\omega}'' = \boldsymbol{T}_{IA}^{RA} \boldsymbol{\omega} = \boldsymbol{\omega}_{RA} \qquad (4)$$

where

$$\boldsymbol{T}_{IA}^{RA} = \begin{bmatrix} c\theta_y & 0 & s\theta_y \\ s\theta_x s\theta_y & c\theta_x & -s\theta_x c\theta_y \\ -c\theta_x s\theta_y & s\theta_x & c\theta_x c\theta_y \end{bmatrix} \qquad (5)$$

When a gyro measurement ω_m of a particular rotation ω, where the axis of rotation coincides with IA, is resolved in terms of

[1]Note that a shorthand notation is used during the development of the rotation matrices where cos is abbreviated as c and sin is abbreviated as s.

IA, it is presented by a vector only having a z-component, or

$$\boldsymbol{\omega}_m = \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} \tag{6}$$

If this rotation were to be resolved in the gyro casing axis (which is the RA), the resulting vector would be obtained through the rotation

$$\boldsymbol{\omega}_{RA} = \begin{bmatrix} \omega_{XRA} \\ \omega_{YRA} \\ \omega_{IRA} \end{bmatrix} \tag{7}$$

$$= \boldsymbol{T}_{IA}^{RA}\boldsymbol{\omega}_m \tag{8}$$

$$= \begin{bmatrix} \omega s\theta_y \\ -\omega s\theta_x c\theta_y \\ \omega c\theta_x c\theta_y \end{bmatrix} \tag{9}$$

These terms (ω_{XRA}, ω_{YRA} and ω_{IRA}) are the standard terms that are used to describe the rotation measured by the gyro in terms of the sensor axes.

C. General FOG Error Equation

Using the approach presented on page 411 of Farrell [8], we can define the gyro measurements in the RA as

$$\tilde{\boldsymbol{\omega}}_{RA} = \boldsymbol{T}_{RA}^{IA}\tilde{\boldsymbol{\omega}}_{IA} \tag{10}$$

where

$\tilde{\boldsymbol{\omega}}_{RA}$	is the computed angular rate in the RA,
$\boldsymbol{T}_{RA}^{IA}$	is the rotation through the misalignment angles from the IA to the RA and
$\tilde{\boldsymbol{\omega}}_{IA}$	is the computed angular rate in the IA.

On the level of notation it should be noted that, throughout the rest of this chapter, the "*barred*" notation (e.g. $\bar{\boldsymbol{\omega}}_{IA}$) will be used to refer to the true rate or rotation applied to the sensor, whereas the "*tilde*" notation (e.g. $\tilde{\boldsymbol{\omega}}_{IA}$) will be used to indicate the measured or computed rotational rate, i.e. the true rate polluted with noise.

$\boldsymbol{T}_{RA}^{IA}$ in equation (10) can be computed directly from equation (5), or, if we make use of the following simplifications defined on page 41 of Titterton and Weston [3], where for small angle rotations, the assumptions can be made that $sin\theta_x \rightarrow \theta_x$, $sin\theta_y \rightarrow \theta_y$, $cos\theta_x \rightarrow 1$ and $cos\theta_y \rightarrow 1$, equation (5) simplifies to

$$\boldsymbol{T}_{IA}^{RA} = \begin{bmatrix} 1 & 0 & \theta_y \\ 0 & 1 & -\theta_x \\ -\theta_y & \theta_x & 1 \end{bmatrix} \tag{11}$$

Equation (11) can be expressed as

$$\boldsymbol{T}_{IA}^{RA} = \boldsymbol{I} + \boldsymbol{\Delta}_g \tag{12}$$

where $\boldsymbol{\Delta}_g$ is defined as

$$\boldsymbol{\Delta}_g = \begin{bmatrix} 0 & 0 & \theta_y \\ 0 & 0 & -\theta_x \\ -\theta_y & \theta_x & 0 \end{bmatrix} \tag{13}$$

Using the formulation of Farrell [8] for the computed angular rate in the IA, we can express $\tilde{\boldsymbol{\omega}}_{IA}$ as

$$\tilde{\boldsymbol{\omega}}_{IA} = (\boldsymbol{I} + \delta\boldsymbol{SF}_g)(\bar{\boldsymbol{\omega}}_{IA} + \delta\boldsymbol{B}_g + \bar{\boldsymbol{\mu}}_g) \tag{14}$$

where

$\boldsymbol{I}$	is the identity matrix,
$\delta\boldsymbol{SF}_g$	is a diagonal matrix representing the gyro scale factor error,
$\bar{\boldsymbol{\omega}}_{IA}$	is the actual rate of rotation that is applied to the gyro,
$\delta\boldsymbol{B}_g$	is the combined gyro drift terms and
$\bar{\boldsymbol{\mu}}_g$	is the combined gyro measurement noise.

If we simplify equation (14) and we neglect the terms that are products of errors, we obtain the following expression which defines the general model for the gyro measurement.

$$\tilde{\boldsymbol{\omega}}_{IA} = \bar{\boldsymbol{\omega}}_{IA} + \delta\boldsymbol{SF}_g.\bar{\boldsymbol{\omega}}_{IA} + \delta\boldsymbol{B}_g + \bar{\boldsymbol{\mu}}_g \tag{15}$$

By combining equations (10), (12) and (15), we can express the computed angular rate in the RA as

$$\begin{aligned} \tilde{\boldsymbol{\omega}}_{RA} &= \boldsymbol{T}_{IA}^{RA}\tilde{\boldsymbol{\omega}}_{IA} \qquad (16) \\ &= (\boldsymbol{I} + \boldsymbol{\Delta}_g)\tilde{\boldsymbol{\omega}}_{IA} \\ &= (\boldsymbol{I} + \boldsymbol{\Delta}_g)(\bar{\boldsymbol{\omega}}_{IA} + \delta\boldsymbol{SF}_g.\bar{\boldsymbol{\omega}}_{IA} \\ &\qquad + \delta\boldsymbol{B}_g - \bar{\boldsymbol{\mu}}_g) \\ &= \bar{\boldsymbol{\omega}}_{IA} + \boldsymbol{\Delta}_g.\bar{\boldsymbol{\omega}}_{IA} + \delta\boldsymbol{SF}_g.\bar{\boldsymbol{\omega}}_{IA} \\ &\qquad + \delta\boldsymbol{B}_g + \bar{\boldsymbol{\mu}}_g \qquad (17) \end{aligned}$$

where products of error terms were once again omitted from the final expression. It should be clear from this expression that, if all the systems error components are equal to zero, then $\tilde{\boldsymbol{\omega}}_{RA}$ will be equal to $\tilde{\boldsymbol{\omega}}_{IA}$, which is to be expected.

From equation (17) we can define the gyro measurement error $\boldsymbol{\epsilon}^g_{RA}$ as resolved in the RA as

$$\begin{aligned}\boldsymbol{\epsilon}^g_{RA} &= \tilde{\boldsymbol{\omega}}_{RA} - \bar{\boldsymbol{\omega}}_{IA} \\ &= \boldsymbol{\Delta}_g.\bar{\boldsymbol{\omega}}_{IA} + \delta\boldsymbol{SF}_g.\bar{\boldsymbol{\omega}}_{IA} + \delta\boldsymbol{B}_g + \bar{\boldsymbol{\mu}}_g\end{aligned} \tag{18}$$

The general gyro measurement error equation is therefore defined as

$$\boldsymbol{\epsilon}^g_{RA} = \boldsymbol{\Delta}_g.\bar{\boldsymbol{\omega}}_{IA} + \delta\boldsymbol{SF}_g.\bar{\boldsymbol{\omega}}_{IA} + \delta\boldsymbol{B}_g + \bar{\boldsymbol{\mu}}_g \tag{19}$$

Equation (19) resembles the format of the derivations presented in the major texts on navigation error modeling. In attempting to standardize the error equation, it is necessary to determine the relationship between equation (19) and equation (1). This will be pursued in section II-D.

Space limitation in this paper does not allow it, but it is quite trivial to show that equation (19) is equal to the gyro error equation presented by Savage [7].

D. Unified Gyro Error Equation

A unified gyro error equation can be obtained if we manage to determine the relationship between equations (15) and (1). This can be achieved by first rewriting equation (1) in the following form:

$$S_0\left[\frac{\Delta N}{\Delta t}\right]\left(1 + 10^{-6}\epsilon_K\right) = \Omega + E + D \tag{20}$$

or equivalently, if we rearrange some terms, in the form

$$\Omega = S_0\left[\frac{\Delta N}{\Delta t}\right]\left(1 + 10^{-6}\epsilon_K\right) - E - D \tag{21}$$

If we take equation (14) and we multiply it out, we get

$$\bar{\boldsymbol{\omega}}_{IA} + \delta\boldsymbol{SF}_g\bar{\boldsymbol{\omega}}_{IA} = \tilde{\boldsymbol{\omega}}_{IA} - \delta\boldsymbol{B}_g - \bar{\boldsymbol{\mu}}_g \tag{22}$$

where the products of error terms have been neglected. We can also write this expression as

$$\bar{\boldsymbol{\omega}}_{IA} = (1 + \delta\boldsymbol{SF}_g)^{-1}\left(\tilde{\boldsymbol{\omega}}_{IA} - \delta\boldsymbol{B}_g - \bar{\boldsymbol{\mu}}_g\right) \tag{23}$$

or as follows if we multiply it out and neglect the products of errors

$$\bar{\boldsymbol{\omega}}_{IA} = (1+\delta\boldsymbol{SF}_g)^{-1}\tilde{\boldsymbol{\omega}}_{IA} - \delta\boldsymbol{B}_g - \bar{\boldsymbol{\mu}}_g \tag{24}$$

By direct comparison of equations (21) and (24), it should be noted that Ω and $\bar{\boldsymbol{\omega}}_{IA}$ both define the true rate applied to the sensor. By taking this into consideration we can equate the following terms

$$\delta\boldsymbol{B}_g + \bar{\boldsymbol{\mu}}_g = E + D$$

as both sides represent the cumulative drift terms,

$$\tilde{\boldsymbol{\omega}}_{IA} = S_0\left[\frac{\Delta N}{\Delta t}\right]$$

as both terms represent the inertial rate of rotation measured by the system and

$$1 + 10^{-6}\epsilon_K = (1 + \delta\boldsymbol{SF}_g)^{-1}$$

as both terms represent the measurement error due to the scale factor error. This last term may appear odd, but this apparent discrepancy can be clarified by expressing it as a set of scalar parameters (which it is in this case) as

$$1 + 10^{-6}\epsilon_K = \frac{1}{1 + \delta SF_g}$$

If the equatily is solved, the following expression for δSF_g is obtained:

$$\begin{aligned}\delta SF_g &= \frac{-10^{-6}\epsilon_K}{1 + 10^{-6}\epsilon_K} \\ &= -\frac{\Delta SF}{\hat{SF}}\end{aligned}$$

where ΔSF is the fractional scale factor error and $\hat{SF}$ is the fractionally corrected scale factor. The minus sign in this expression is somewhat arbitrary as part of the error definition. This leads to the expression

$$\frac{1}{1 + \delta SF_g} = \frac{\hat{SF}}{\hat{SF} - \Delta SF}$$

The fundamental observation to be made from these expressions is that the two scale factor formulations can be defined equivalent with the right definition of the δSF_g term.

From these expressions of equivalence, we can now write a unified gyro error equation and calibration equation as

$$\bar{\boldsymbol{\omega}}_{IA} = (1 + 10^{-6}\epsilon_K)\tilde{\boldsymbol{\omega}}_{IA} + E + D \tag{25}$$

This equation expresses the gyro error/calibration equation in the general form while making use of the standard IEEE defined parameters which one can expect to find in a system specification. Referring to references [9] and [1], one can see that the terms ϵ_K and E in equation (25) consists only of deterministic terms, meaning that, if these terms are properly defined for a given gyro, the measurement errors associated with these terms can be compensated for during the operation of the gyro. The term D consists of a number of stochastic terms, with the only deterministic terms being what IEEE Std 528-2001 [9] calls the *bias*. This is a constant, unchanging measurement error made by the gyro. This term can also be subtracted from the gyro measurements to improve the gyro measurement.

Also note that equation (25) represents a scalar measurement in the IA. To obtain the corrected gyro values in the RA, the conversion matrix $\boldsymbol{T}_{IA}^{RA}$ defined in equation (12) must be used. The corrected orientation in the IA obtained from equation (25) (ω_{IA}) is pre-multiplied with this transformation matrix with the result that the scalar valued error equation defined in equation (25) is vectorized through this operation. Following this operation, the corrected angular rate as expressed in the RA is defined as

$$\bar{\boldsymbol{\omega}}_{RA} = \begin{bmatrix} \theta_y \omega_{IA} \\ -\theta_x \omega_{IA} \\ \omega_{IA} \end{bmatrix} \tag{26}$$

where θ_x and θ_y are the misalignment angles between the IA and the RA as defined in figure 1.

Note that the gyro's output is usually used as part of a orthogonal triad of three sensors. The conversion of the corrected measurements from the IA to the RA was presented here, but the conversion of the sensor's output to the axis system of the inertial measurement unit (IMU) falls beyond the scope of the paper. Refer to Groves [10], Titterton and Weston [3] or Rogers [5] for the description of these conversions.

III. Implementation

Without going into too much detail, this section will present a brief example of the use of the gyro specifications from a typical specification to correct the raw measurements of the sensor.

Looking at the terms in equation (25), it is found from IEEE Std 952-1997 [1] that the scale factor error coefficient consists of a temperature dependant component and a rate dependant component. These components are usually not separately specified in a gyro specification, but the scale factor is presented as a single parameter. This single parameter can then be used instead of the two components presented above.

Equation (25) also contains the parameter E, which contains the environmentally sensitive parameters. E consists of the temperature dependant drift rate, drift rate attributable to the temperature gradient and the drift rate attributable to the time-varying temperature gradient (see reference [1]). One once again do not often find all these parameters to be defined in the gyro specification. If the parameters are not defined, they can simply be ignored and the ones that are provided can be included in the simulation. It is important to note that the IEEE Std 952-1997 [1] provides methods for determining all these error parameters. It could be an option to empirically determine the various components not specified on the provided system specification and then include it in the sensor compensation for a more complete sensor measurement correction.

The parameter D in equation (25) is the drift term, which consist of the bias (a constant, non-varying measurement offset) and a number of stochastic parameters. One cannot compensate for the stochastic terms, so these are ignored during the compensation and only the constant parameter is used to correct the measurement error. In the case where there is a random walk (first order Gauss-Markov process) error present in the system, the correlation time constant of the random process is usually estimated by means of a Kalman filter as part of the complete navigation system implementation. Using this estimated parameter, the stochastic errors can be corrected to some degree during the operational domain of the sensor.

Table I presents an example of a typical set of FOG specifications. Using these parameters, the corrected sensor measurement will be defined as

$$\bar{\boldsymbol{\omega}}_{IA} = (1 + 10^{-6}2000)\tilde{\boldsymbol{\omega}}_{IA} - 6 \qquad (27)$$

and the values of θ will be used in equation (26) together with equation (27) to determine the corrected sensor measurement in the RA.

TABLE I
GYRO SPECIFICATIONS.

Error term	**Value**
Bias (D)	$6°/h$
Random walk	$\leq 0.6°/\sqrt{h}$
Random walk correlation time	1000 sec
Scale factor error (ϵ_K)	2000 PPM
Axis misalignment (θ)	$\pm$5mrad

It should once again be noted that one could also determine the complete set of system parameters for a particular sensor and then use these parameters as a more comprehensive and more accurate set of system parameters to compensate the erroneous set of raw measurements presented by the sensor. Such an empirically determined set of system characteristics define a more accurate picture of the system accuracy that the more generalized parameters obtained from the sensor performance specification.

IV. Conclusion

The equivalence of the various gyro error equations was shown in this paper. A unified gyro error model was developed which can be used to compensate the raw gyro measurements. The model was formulated in such a way that the parameters that are usually used to describe the sensor's performance in the gyro specification can be used to describe the error in the sensor. As the focus of the current paper was simply to prove the equivalence of the various error formulations, extensive simulation results to support the derivation was not presented. The next step in the current research is to use the derived error model to compensate real gyro measurements and model gyro errors in a unified way.

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