Exceptional Points, Nonnormal Matrices, Hierarchy of Spin Matrices and an Eigenvalue Problem

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Abstract Exceptional points are studied for non-hermitian Hamilton operators given by a hierarchy of spin-operators.

1 Introduction

Kato [1] (see also Rellich [2]) introduced exceptional points for singularities appearing in the perturbation theory of linear operators. Afterwards exceptional points and energy level crossing have been studied for hermitian Hamilton operators [3, 4, 5, 6, 7, 8, 9, 10] and non-hermitian Hamilton operators [11, 12, 13, 14, 15, 16] by many authors. Here we consider the finite dimensional Hilbert space $\mathbb{C}^n$ and the linear operators are $n \times n$ matrices over $\mathbb{C}$.

For hermitian matrices the standard example in literature is

$$H(\epsilon) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \epsilon \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where ϵ is real. The characteristic polynomial $\det(H(\epsilon) - EI_2) = 0$ is given by $E^2 - E - \epsilon^2 = 0$. When ϵ is complex, the eigenvalues may be viewed as the 2 values of a single function $E(\epsilon)$ of ϵ, analytic on a Riemann surface with 2 sheets joined at branch point singularities in the complex plane. The exceptional points in the complex ϵ plane are defined by the solution $\det(H(\epsilon) - EI_2) = 0$ together with $d(\det(H(\epsilon) - EI_2))/dE = 0$. One finds that the exceptional points are $\epsilon_1 = i/2$ and $\epsilon_2 = -i/2$.

For non-hermitian systems the standard example is the matrix (Kato [1], Rotter [11], Heiss [12])

$$\sigma_3 + z\sigma_1 = \begin{pmatrix} 1 & z \\ z & -1 \end{pmatrix}$$

where $z \in \mathbb{C}$ and σ_1, σ_2, σ_3 are the Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $z = i$. Then the matrix $\sigma_3 + i\sigma_1$ admits the eigenvalue 0 (twice) and the only normalized eigenvector

$$\frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

The matrix $\sigma_3 + i\sigma_1$ is nonnormal. Let $z = -i$. Then the nonnormal matrix $\sigma_3 - i\sigma_1$ admits the eigenvalue 0 (twice) and the only normalized eigenvector

$$\frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix}.$$

We extend this result to arbitrary spin. Since the matrices considered are nonnormal we summarize the properties of nonnormal matrices in section 2. In section 3 we consider the case with spin 1/2, 1, 3/2 and 2. In section 4 the general case is studied.

2 Nonnormal Matrices

An $n \times n$ matrix A over $\mathbb{C}$ is called normal if $AA^* = A^*A$. Then for a nonnormal matrix we have $A^*A \neq AA^*$. An example of a nonnormal matrix is the matrix given above $\sigma_3 + i\sigma_1$ which only admits the eigenvalue 0 (twice) and only one eigenvector. Note that not all nonnormal matrices are non-diagonalizable, but all non-diagonalizable matrices are nonnormal [17].

If A is any $n \times n$ matrix A over $\mathbb{C}$, then a classical result due to Schur (Roman [18]) states that there exist a unitary matrix U and a triangular matrix $T = (t_{jk})$ with $t_{jk} = 0$ for $k < j$ such that $A = UTU^*$. For the matrix $\sigma_3 + i\sigma_1$ we find

$$\sigma_3 + i\sigma_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} 0 & 2i \\ 0 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}.$$

Let A, B be hermitian nonzero matrices, i.e. $A^* = A$ and $B^* = B$. Consider the matrix $A + iB$. What are the conditions on A and B such that $A + iB$ is normal?

From

$$(A+iB)^*(A+iB) = (A+iB)(A+iB)^*$$

we find that the commutator of A and B must vanish, i.e. $[A,B]=0$. For the Pauli spin matrices σ_1 and σ_3 this condition is not satisfied since $[\sigma_3,\sigma_1]=2i\sigma_2$.

Now the transition from a hermitian matrix to a non-normal matrix can be studied with the matrix

$$\sigma_3 + e^{i\phi}\sigma_1$$

where $\phi \in [0,\pi/2]$. For $\phi = 0$ we have the hermitian matrix $\sigma_3+\sigma_1$. For $0<\phi\le\pi/2$ we have a nonnormal matrix. The eigenvalues are given by

$$\lambda_\pm = \pm\sqrt{1+e^{2i\phi}}$$

with the eigenvectors

$$\mathbf{v}_\pm = \begin{pmatrix} e^{i\phi} \\ -1+\lambda_\pm \end{pmatrix}.$$

Note that the commutator of $\sigma_3+\sigma_1$ and $\sigma_3+e^{i\phi}\sigma_1$ is given by

$$[\sigma_3+\sigma_1, \sigma_3+e^{i\phi}\sigma_1] = 2i\sigma_2(e^{i\phi}-1).$$

Obviously for $\phi=0$ the commutator vanishes and for $\phi=\pi/2$ we have $2i\sigma_2(i-1)$. The matrix $2i\sigma_2(i-1)$ is normal, but non-hermitian.

Let $\otimes$ be the Kronecker product and $\oplus$ the direct sum. Let A, B be nonnormal matrices. Then $A\otimes B$ and $A\oplus B$ are nonnormal. Let X, Y be non-zero $n\times n$ matrices. We have

$$(X^*X)\otimes(Y^*Y) = (XX^*)\otimes(YY^*)$$

if and only if $X^*X = XX^*$ and $Y^*Y=YY^*$. Note that

$$\exp(\sigma_3+i\sigma_1) = I_2+\sigma_3+i\sigma_1.$$

This matrix is nonnormal. However, we cannot conclude in general that $\exp(A)$ of a nonnormal matrix A is nonnormal. Consider, for example, the matrix

$$A = \begin{pmatrix} i\pi & b \\ 0 & -i\pi \end{pmatrix}$$

with $b\neq 0$. Then $\exp(A)$ is a normal matrix. However, if a matrix M is nonnormal and nilpotent, then $\exp(M)$ is nonnormal. If N is a normal matrix, then $\exp(N)$ is a normal matrix.

3 Spin-$\frac{1}{2}$, 1, 3/2, 2 Cases

For the spin-$\frac{1}{2}$ case we consider the spin matrices for describing a spin-$\frac{1}{2}$ system

$$s_1 = \frac{1}{2}\sigma_1, \quad s_2 = \frac{1}{2}\sigma_2, \quad s_3 = \frac{1}{2}\sigma_3$$

with $s_1^2 + s_2^2 + s_3^2 = \frac{3}{4}I_2$. Consider the matrix $s_3 + is_1$. This is the case given above except for the factor 1/2. Obviously the matrix $s_3 + is_1$ is nonnormal and the rank is 1. Since $(s_3 + is_1)^2 = 0_2$ the matrix is nilpotent and thus the eigenvalues are 0. The trace of this nonnormal matrix is 0. The eigenvalues of the matrix are 0 (twice) and only normalized eigenvectors of the matrix is

$$\frac{1}{\sqrt{2}}\begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

Consider next the spin matrices for describing a spin-1 system

$$s_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad s_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad s_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

with $s_1^2 + s_2^2 + s_3^2 = 2I_3$. For spin-1 the matrix

$$s_3 + is_1 = \begin{pmatrix} 1 & i/\sqrt{2} & 0 \\ i/\sqrt{2} & 0 & i/\sqrt{2} \\ 0 & i/\sqrt{2} & -1 \end{pmatrix}$$

is nonnormal. The trace of this nonnormal matrix is 0 and the matrix is nilpotent, i.e. we have $(s_3 + is_1)^3 = 0_3$. Thus all three eigenvalues are 0 and the only normalized eigenvector is

$$\frac{1}{2}\begin{pmatrix} -1 \\ -i\sqrt{2} \\ 1 \end{pmatrix}.$$

For spin-3/2 we have the matrices

$$s_1 = \frac{1}{2}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0 & -i\sqrt{3} & 0 & 0 \\ i\sqrt{3} & 0 & -2i & 0 \\ 0 & 2i & 0 & -i\sqrt{3} \\ 0 & 0 & i\sqrt{3} & 0 \end{pmatrix},$$

$$s_3 = \begin{pmatrix} 3/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & -3/2 \end{pmatrix}$$

with $s_1^2 + s_2^2 + s_3^2 = \frac{15}{4} I_4$. Thus the matrix $s_3 + is_1$ is given by

$$s_3 + is_1 = \begin{pmatrix} 3/2 & i\sqrt{3}/2 & 0 & 0 \\ i\sqrt{3}/2 & 1/2 & i & 0 \\ 0 & i & -1/2 & i\sqrt{3}/2 \\ 0 & 0 & i\sqrt{3}/2 & -3/2 \end{pmatrix}.$$

The matrix is nonnormal and nilpotent, i.e. $(s_3 + is_1)^4 = 0_4$. Thus the trace is equal to 0 and the eigenvalues are 0 (four times). The rank of the matrix is 3. The only normalized eigenvector is

$$\frac{1}{\sqrt{8}} \begin{pmatrix} i \\ -\sqrt{3} \\ -i\sqrt{3} \\ 1 \end{pmatrix}.$$

This eigenvector is entangled, i.e. it cannot be written as a Kronecker product of two vectors in $\mathbb{C}^2$. The tangle as a measure of entanglement is nonzero.

For spin-2 we have the 5×5 matrices

$$s_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & \sqrt{6}/2 & 0 & 0 \\ 0 & \sqrt{6}/2 & 0 & \sqrt{6}/2 & 0 \\ 0 & 0 & \sqrt{6}/2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0 & -i & 0 & 0 & 0 \\ i & 0 & -i\sqrt{6}/2 & 0 & 1 \\ 0 & i\sqrt{6}/2 & 0 & -i\sqrt{6}/2 & 0 \\ 0 & 0 & i\sqrt{6}/2 & 0 & -i \\ 0 & 0 & 0 & i & 0 \end{pmatrix},$$

$$s_3 = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}$$

with $s_1^2 + s_2^2 + s_3^2 = 6I_5$. Thus the matrix $s_3 + is_1$ is given by

$$s_3 + is_1 = \begin{pmatrix} 2 & i & 0 & 0 & 0 \\ i & 1 & i\sqrt{6}/2 & 0 & 0 \\ 0 & i\sqrt{6}/2 & 0 & i\sqrt{6}/2 & 0 \\ 0 & 0 & i\sqrt{6}/2 & -1 & i \\ 0 & 0 & 0 & i & -2 \end{pmatrix}.$$

The matrix is nonnormal and nilpotent, i.e. $(s_3 + is_1)^5 = 0_5$. Thus the trace is equal to 0 and the eigenvalues are 0 (five times). The rank of the matrix is 4. The only normalized eigenvector is

$$\begin{pmatrix} 1 \\ 2i \\ -\sqrt{6} \\ -2i \\ 1 \end{pmatrix}.$$

4 General Case

For the general case we look at integer spin, i.e. $1, 2, 3, \ldots$ and half-integer spin, i.e. $1/2, 3/2, 5/2, \ldots$ [19]. Let s (spin quantum number)

$$s \in \left\{ \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots \right\}.$$

Given a fixed s. The indices j, k run over $s, s-1, s-2, \ldots, -s+1, -s$. Consider the $(2s+1)$ unit vectors (standard basis)

$$\mathbf{e}_{s,s} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{e}_{s,s-1} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots \quad , \mathbf{e}_{s,-s} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Obviously the vectors have $(2s+1)$ components. The $(2s+1) \times (2s+1)$ matrices s_+ and s_- are defined as

$$s_+ \mathbf{e}_{s,m} := \sqrt{(s-m)(s+m+1)}\mathbf{e}_{s,m+1}, \quad m = s-1, s-2, \ldots, -s$$
$$s_- \mathbf{e}_{s,m} := \sqrt{(s+m)(s-m+1)}\mathbf{e}_{s,m-1}, \quad m = s, s-1, \ldots, -s+1$$

The $(2s+1) \times (2s+1)$ matrix s_3 is defined as (eigenvalue equation)

$$s_3 \mathbf{e}_{s,m} := m \mathbf{e}_{s,m}, \quad m = s, s-1, \ldots, -s.$$

Thus s_3 is a diagonal matrix with the entries s, $s-1$, $\ldots$, $-s$ in the diagonal. Let $\mathbf{s} := (s_1, s_2, s_3)$, where $s_+ = s_1 + is_2$ and $s_- = s_1 - is_2$. Thus

$$s_1 = \frac{1}{2}(s_+ + s_-), \qquad s_2 = -\frac{i}{2}(s_+ - s_-).$$

We have

$$(s_+)_{jk} = (s_-)_{kj} = \sqrt{(s-k)(s+k+1)}\delta_{j,k+1} = \sqrt{(s+j)(s-j+1)}\delta_{j,k+1}$$

and

$$(s_-)_{jk} = (s_+)_{kj} = \sqrt{(s+k)(s-k+1)}\delta_{j,k-1} = \sqrt{(s-j)(s+j+1)}\delta_{j,k-1}$$

where $j, k = s, s-1, \ldots, -s$. Therefore

$$s_+ = \begin{pmatrix} 0 & \sqrt{2s} & 0 & 0 & \ldots & 0 \\ 0 & 0 & \sqrt{2(2s-1)} & 0 & \ldots & 0 \\ 0 & 0 & 0 & \sqrt{3(2s-2)} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & \sqrt{2s} \\ 0 & 0 & 0 & 0 & \ldots & 0 \end{pmatrix}.$$

Thus $s_- = (s_+)^*$. We have $s_1^2 + s_2^2 + s_3^2 = s(s+1)I_{2s+1}$. Now the $(2s+1) \times (2s+1)$ matrix $s_3 + is_1$ is nonnormal and nilpotent, i.e.

$$(s_3 + is_1)^{2s+1} = 0_{2s+1}$$

where 0_{2s+1} is the $(2s+1) \times (2s+1)$ zero matrix and s is the spin. Thus all the eigenvalues of $s_3 + is_1$ are 0. For the eigenvectors $\mathbf{v}$ which is an element of $\mathbb{C}^{2s+1}$ we set

$$\mathbf{v} = \begin{pmatrix} v_s & v_{s-1} & \cdots & v_{-s+1} & v_{-s} \end{pmatrix}^T$$

Now the eigenvalue equation $(s_3 + is_1)\mathbf{v} = \mathbf{0}$ can be easily solved. First we can set without loss of generality the last entry of the eigenvector to $v_{-s} = 1$. Then using the last row of the matrix $s_3 + is_1$ we obtain the equation

$$i\frac{1}{2}\sqrt{2s}v_{-s+1} - sv_{-s} = i\frac{1}{2}\sqrt{2s}v_{-s+1} - s = 0$$

with the solution $v_{-s+1} = -i2s/\sqrt{2s}$. Then the second last row of the matrix $s_3 + is_1$ provides the equation for v_{-s+2} and successively we can find the other entries $v_{-s+3}, \ldots, v_s$ of the eigenvector. All the entries are nonzero. This successive construction also shows that there is only one linearly independent eigenvector.

5 Conclusion

We have studied an eigenvalue problem for a hierarchy of nonnormal matrices constructed from the spin matrices for spin 1/2, 1, 3/2, 2 etc. All these matrices have only one eigenvalue and thus only one eigenvector. The matrices $(2s+1)\times(2s+1)$ matrices s_3+is_2 have the same properties as s_3+is_1, i.e. they are nonnormal and nilpotent for all spin s.

Starting from nonnormal matrices we can construct other nonnormal matrices. Let $c_j^\dagger$, c_j $(j=1,2)$ be Fermi creation and annihilation operators, respectively. Then we can form the operator

$$(c_1^\dagger \quad c_2^\dagger) \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = c_1^\dagger c_1 - c_2^\dagger c_2 + i(c_1^\dagger c_2 + c_2^\dagger c_1)$$

Using the basis $|0\rangle$, $c_1^\dagger|0\rangle$, $c_2^\dagger|0\rangle$, $c_1^\dagger c_2^\dagger|0\rangle$ with $\langle 0|0\rangle = 1$ we find the matrix representation

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & i & 0 \\ 0 & i & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

for the operator. This matrix has the eigenvalue 0 (fourfold) and the three eigenvectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

This matrix is nonnormal.

We can also consider the Kronecker product of such matrices. Consider the case of spin-$\frac{1}{2}$. The 4×4 matrix

$$(\sigma_3 + i\sigma_1)\otimes(\sigma_3 + i\sigma_1)$$

is nonnormal. However, note that the 4×4 matrix

$$\sigma_3\otimes\sigma_3 + i\sigma_1\otimes\sigma_1 = \begin{pmatrix} 1 & 0 & 0 & i \\ 0 & -1 & i & 0 \\ 0 & i & -1 & 0 \\ i & 0 & 0 & 1 \end{pmatrix}$$

is normal, but non-hermitian. The eigenvalues are $(-1)^{1/4}\sqrt{2}$, $-(-1)^{1/4}\sqrt{2}$, $(-1)^{1/4}i\sqrt{2}$, $(-1)^{1/4}i\sqrt{2}$.

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