Exceptional Points, Nonnormal Matrices, Hierarchy of Spin Matrices
and an Eigenvalue Problem

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Abstract Exceptional points are studied for non-hermitian Hamilton operators
given by a hierarchy of spin-operators.

1 Introduction

Kato [1] (see also Rellich [2]) introduced exceptional points for singularities appearing
in the perturbation theory of linear operators. Afterwards exceptional points
and energy level crossing have been studied for hermitian Hamilton operators
[3, 4, 5, 6, 7, 8, 9, 10] and non-hermitian Hamilton operators [11, 12, 13, 14, 15, 16]
by many authors. Here we consider the finite dimensional Hilbert space \( \mathbb{C}^n \)
and the linear operators are \( n \times n \) matrices over \( \mathbb{C} \).

For hermitian matrices the standard example in literature is

\[
H(\epsilon) = \begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix} + \epsilon \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\]

where \( \epsilon \) is real. The characteristic polynomial \( \det(H(\epsilon) - EI_2) = 0 \) is given by
\( E^2 - E - \epsilon^2 = 0 \). When \( \epsilon \) is complex, the eigenvalues may be viewed as the 2 values
of a single function \( E(\epsilon) \) of \( \epsilon \), analytic on a Riemann surface with 2 sheets joined
at branch point singularities in the complex plane. The exceptional points in the
complex \( \epsilon \) plane are defined by the solution \( \det(H(\epsilon) - EI_2) = 0 \) together with
\( d(\det(H(\epsilon) - EI_2))/dE = 0 \). One finds that the exceptional points are \( \epsilon_1 = i/2 \)
and \( \epsilon_2 = -i/2 \).
For non-hermitian systems the standard example is the matrix (Kato [1], Rotter [11], Heiss [12])

\[ \sigma_3 + z\sigma_1 = \begin{pmatrix} 1 & z \\ z & -1 \end{pmatrix} \]

where \( z \in \mathbb{C} \) and \( \sigma_1, \sigma_2, \sigma_3 \) are the Pauli spin matrices

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Let \( z = i \). Then the matrix \( \sigma_3 + i\sigma_1 \) admits the eigenvalue 0 (twice) and the only normalized eigenvector

\[
\frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \end{pmatrix}.
\]

The matrix \( \sigma_3 + i\sigma_1 \) is nonnormal. Let \( z = -i \). Then the nonnormal matrix \( \sigma_3 - i\sigma_1 \) admits the eigenvalue 0 (twice) and the only normalized eigenvector

\[
\frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix}.
\]

We extend this result to arbitrary spin. Since the matrices considered are nonnormal we summarize the properties of nonnormal matrices in section 2. In section 3 we consider the case with spin 1/2, 1, 3/2 and 2. In section 4 the general case is studied.

### 2 Nonnormal Matrices

An \( n \times n \) matrix \( A \) over \( \mathbb{C} \) is called normal if \( AA^* = A^*A \). Then for a nonnormal matrix we have \( A^*A \neq AA^* \). An example of a nonnormal matrix is the matrix given above \( \sigma_3 + i\sigma_1 \) which only admits the eigenvalue 0 (twice) and only one eigenvector. Note that not all nonnormal matrices are non-diagonalizable, but all non-diagonalizable matrices are nonnormal [17].

If \( A \) is any \( n \times n \) matrix \( A \) over \( \mathbb{C} \), then a classical result due to Schur (Roman [18]) states that there exist a unitary matrix \( U \) and a triangular matrix \( T = (t_{jk}) \) with \( t_{jk} = 0 \) for \( k < j \) such that \( A = UTU^* \). For the matrix \( \sigma_3 + i\sigma_1 \) we find

\[
\sigma_3 + i\sigma_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} 0 & 2i \\ 0 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}.
\]

Let \( A, B \) be hermitian nonzero matrices, i.e. \( A^* = A \) and \( B^* = B \). Consider the matrix \( A + iB \). What are the conditions on \( A \) and \( B \) such that \( A + iB \) is normal?
From 
\[(A + iB)^*(A + iB) = (A + iB)(A + iB)^*\]
we find that the commutator of $A$ and $B$ must vanish, i.e. $[A, B] = 0$. For the Pauli spin matrices $\sigma_1$ and $\sigma_3$ this condition is not satisfied since $[\sigma_3, \sigma_1] = 2i\sigma_2$.

Now the transition from a hermitian matrix to a non-normal matrix can be studied with the matrix 
\[\sigma_3 + e^{i\phi}\sigma_1\]
where $\phi \in [0, \pi/2]$. For $\phi = 0$ we have the hermitian matrix $\sigma_3 + \sigma_1$. For $0 < \phi \leq \pi/2$ we have a nonnormal matrix. The eigenvalues are given by 
\[\lambda_\pm = \pm \sqrt{1 + e^{2i\phi}}\]
with the eigenvectors 
\[v_\pm = \begin{pmatrix} e^{i\phi} \\ -1 + \lambda_\pm \end{pmatrix}\] 
Note that the commutator of $\sigma_3 + \sigma_1$ and $\sigma_3 + e^{i\phi}\sigma_1$ is given by 
\[[\sigma_3 + \sigma_1, \sigma_3 + e^{i\phi}\sigma_1] = 2i\sigma_2(e^{i\phi} - 1)\].

Obviously for $\phi = 0$ the commutator vanishes and for $\phi = \pi/2$ we have $2i\sigma_2(i - 1)$. The matrix $2i\sigma_2(i - 1)$ is normal, but non-hermitian.

Let $\otimes$ be the Kronecker product and $\oplus$ the direct sum. Let $A, B$ be nonnormal matrices. Then $A \otimes B$ and $A \oplus B$ are nonnormal. Let $X, Y$ be non-zero $n \times n$ matrices. We have 
\[(X^*X) \otimes (Y^*Y) = (XX^*) \otimes (YY^*)\]
if and only if $X^*X = XX^*$ and $Y^*Y = YY^*$. Note that 
\[\exp(\sigma_3 + i\sigma_1) = I_2 + \sigma_3 + i\sigma_1\].

This matrix is nonnormal. However, we cannot conclude in general that $\exp(A)$ of a nonnormal matrix $A$ is nonnormal. Consider, for example, the matrix 
\[A = \begin{pmatrix} i\pi & b \\ 0 & -i\pi \end{pmatrix}\]
with $b \neq 0$. Then $\exp(A)$ is a normal matrix. However, if a matrix $M$ is nonnormal and nilpotent, then $\exp(M)$ is nonnormal. If $N$ is a normal matrix, then $\exp(N)$ is a normal matrix.
3 Spin-$\frac{1}{2}$, 1, 3/2, 2 Cases

For the spin-$\frac{1}{2}$ case we consider the spin matrices for describing a spin-$\frac{1}{2}$ system

$$s_1 = \frac{1}{2}\sigma_1, \quad s_2 = \frac{1}{2}\sigma_2, \quad s_3 = \frac{1}{2}\sigma_3$$

with $s_1^2 + s_2^2 + s_3^2 = \frac{3}{4}I_2$. Consider the matrix $s_3 + is_1$. This is the case given above except for the factor 1/2. Obviously the matrix $s_3 + is_1$ is nonnormal and the rank is 1. Since $(s_3 + is_1)^2 = 0_2$ the matrix is nilpotent and thus the eigenvalues are 0. The trace of this nonnormal matrix is 0. The eigenvalues of the matrix are 0 (twice) and only normalized eigenvectors of the matrix is

$$\frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$  

Consider next the spin matrices for describing a spin-1 system

$$s_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad s_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad s_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

with $s_1^2 + s_2^2 + s_3^2 = 2I_3$. For spin-1 the matrix

$$s_3 + is_1 = \begin{pmatrix} 1 & i/\sqrt{2} & 0 \\ i/\sqrt{2} & 0 & i/\sqrt{2} \\ 0 & i/\sqrt{2} & -1 \end{pmatrix}$$

is nonnormal. The trace of this nonnormal matrix is 0 and the matrix is nilpotent, i.e. we have $(s_3 + is_1)^3 = 0_3$. Thus all three eigenvalues are 0 and the only normalized eigenvector is

$$\frac{1}{2} \begin{pmatrix} -1 \\ -i\sqrt{2} \\ 1 \end{pmatrix}.$$  

For spin-3/2 we have the matrices

$$s_1 = \frac{1}{2} \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0 & -i\sqrt{3} & 0 & 0 \\ i\sqrt{3} & 0 & -2i & 0 \\ 0 & 2i & 0 & -i\sqrt{3} \\ 0 & 0 & i\sqrt{3} & 0 \end{pmatrix},$$
\[
s_3 = \begin{pmatrix}
3/2 & 0 & 0 & 0 \\
0 & 1/2 & 0 & 0 \\
0 & 0 & -1/2 & 0 \\
0 & 0 & 0 & -3/2
\end{pmatrix}
\]

with \( s_1^2 + s_2^2 + s_3^2 = \frac{15}{4}I_4 \). Thus the matrix \( s_3 + is_1 \) is given by

\[
s_3 + is_1 = \begin{pmatrix}
3/2 & i\sqrt{3}/2 & 0 & 0 \\
i\sqrt{3}/2 & 1/2 & i & 0 \\
0 & i & -1/2 & i\sqrt{3}/2 \\
0 & 0 & i\sqrt{3}/2 & -3/2
\end{pmatrix}.
\]

The matrix is nonnormal and nilpotent, i.e. \((s_3 + is_1)^4 = 0_4\). Thus the trace is equal to 0 and the eigenvalues are 0 (four times). The rank of the matrix is 3. The only normalized eigenvector is

\[
\frac{1}{\sqrt{8}} \begin{pmatrix}
i \\
-\sqrt{3} \\
-\sqrt{3} \\
1
\end{pmatrix}.
\]

This eigenvector is entangled, i.e. it cannot be written as a Kronecker product of two vectors in \( \mathbb{C}^2 \). The tangle as a measure of entanglement is nonzero.

For spin-2 we have the \( 5 \times 5 \) matrices

\[
s_1 = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
1 & 0 & \sqrt{6}/2 & 0 & 0 \\
0 & \sqrt{6}/2 & 0 & \sqrt{6}/2 & 0 \\
0 & 0 & \sqrt{6}/2 & 0 & 1 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}, \quad s_2 = \begin{pmatrix}
0 & -i & 0 & 0 & 0 \\
i & 0 & -i\sqrt{6}/2 & 0 & 1 \\
0 & i\sqrt{6}/2 & 0 & -i\sqrt{6}/2 & 0 \\
0 & 0 & i\sqrt{6}/2 & 0 & -i \\
0 & 0 & 0 & i & 0
\end{pmatrix},
\]

\[
s_3 = \begin{pmatrix}
2 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -2
\end{pmatrix}
\]

with \( s_1^2 + s_2^2 + s_3^2 = 6I_5 \). Thus the matrix \( s_3 + is_1 \) is given by

\[
s_3 + is_1 = \begin{pmatrix}
2 & i & 0 & 0 & 0 \\
i & 1 & i\sqrt{6}/2 & 0 & 0 \\
0 & i\sqrt{6}/2 & 0 & i\sqrt{6}/2 & 0 \\
0 & 0 & i\sqrt{6}/2 & -1 & i \\
0 & 0 & 0 & i & -2
\end{pmatrix}.
\]
The matrix is nonnormal and nilpotent, i.e. \((s_3 + is_1)^5 = 0\). Thus the trace is equal to 0 and the eigenvalues are 0 (five times). The rank of the matrix is 4. The only normalized eigenvector is
\[
\begin{pmatrix}
1 \\
2i \\
-\sqrt{6} \\
-2i \\
1
\end{pmatrix}.
\]

4 General Case

For the general case we look at integer spin, i.e. 1, 2, 3, \ldots and half-integer spin, i.e. \(1/2, 3/2, 5/2, \ldots\) \([19]\). Let \(s\) (spin quantum number)
\[
s \in \left\{ \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots \right\}.
\]

Given a fixed \(s\). The indices \(j, k\) run over \(s, s-1, s-2, \ldots, -s+1, -s\). Consider the \((2s+1)\) unit vectors (standard basis)
\[
e_{s,s} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_{s,s-1} = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad e_{s,-s} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.
\]

Obviously the vectors have \((2s+1)\) components. The \((2s+1) \times (2s+1)\) matrices \(s_+\) and \(s_-\) are defined as
\[
s_+ e_{s,m} := \sqrt{(s-m)(s+m+1)}e_{s,m+1}, \quad m = s-1, s-2, \ldots, -s
\]
\[
s_- e_{s,m} := \sqrt{(s+m)(s-m+1)}e_{s,m-1}, \quad m = s, s-1, \ldots, -s+1
\]

The \((2s+1) \times (2s+1)\) matrix \(s_3\) is defined as (eigenvalue equation)
\[
s_3 e_{s,m} := me_{s,m}, \quad m = s, s-1, \ldots, -s.
\]

Thus \(s_3\) is a diagonal matrix with the entries \(s, s-1, \ldots, -s\) in the diagonal. Let \(s := (s_1, s_2, s_3)\), where \(s_+ = s_1 + is_2\) and \(s_- = s_1 - is_2\). Thus
\[
s_1 = \frac{1}{2}(s_+ + s_-), \quad s_2 = -\frac{i}{2}(s_+ - s_-).
\]
We have

\[(s_+)_jk = (s_-)_{kj} = \sqrt{(s-k)(s+k+1)}\delta_{j,k+1} = \sqrt{(s+j)(s-j+1)}\delta_{j,k+1}\]

and

\[(s_-)_jk = (s_+)_kj = \sqrt{(s+k)(s-k+1)}\delta_{j,k-1} = \sqrt{(s-j)(s+j+1)}\delta_{j,k-1}\]

where \(j, k = s, s-1, \ldots, -s\). Therefore

\[s_+ = \begin{pmatrix}
0 & \sqrt{2s} & 0 & 0 & \cdots & 0 \\
0 & 0 & \sqrt{2(2s-1)} & 0 & \cdots & 0 \\
0 & 0 & 0 & \sqrt{3(2s-2)} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0
\end{pmatrix}.
\]

Thus \(s_- = (s_+)^*\). We have \(s_1^2 + s_2^2 + s_3^2 = s(s+1)I_{2s+1}\). Now the \((2s+1) \times (2s+1)\) matrix \(s_3 + is_1\) is nonnormal and nilpotent, i.e.

\[(s_3 + is_1)^{2s+1} = 0_{2s+1}\]

where \(0_{2s+1}\) is the \((2s+1) \times (2s+1)\) zero matrix and \(s\) is the spin. Thus all the eigenvalues of \(s_3 + is_1\) are 0. For the eigenvectors \(v\) which is an element of \(\mathbb{C}^{2s+1}\) we set

\[v = \begin{pmatrix} v_s & v_{s-1} & \cdots & v_{-s+1} & v_{-s} \end{pmatrix}^T\]

Now the eigenvalue equation \((s_3 + is_1)v = 0\) can be easily solved. First we can set without loss of generality the last entry of the eigenvector to \(v_{-s} = 1\). Then using the last row of the matrix \(s_3 + is_1\) we obtain the equation

\[i\frac{1}{2}\sqrt{2sv_{-s+1}} - sv_{-s} = i\frac{1}{2}\sqrt{2sv_{-s+1}} - s = 0\]

with the solution \(v_{-s+1} = -i2s/\sqrt{2s}\). Then the second last row of the matrix \(s_3 + is_1\) provides the equation for \(v_{-s+2}\) and successively we can find the other entries \(v_{-s+3}, \ldots, v_s\) of the eigenvector. All the entries are nonzero. This successive construction also shows that there is only one linearly independent eigenvector.

5 Conclusion

We have studied an eigenvalue problem for a hierarchy of nonnormal matrices constructed from the spin matrices for spin \(1/2, 1, 3/2, 2\) etc. All these matrices have
only one eigenvalue and thus only one eigenvector. The matrices \((2s + 1) \times (2s + 1)\) matrices \(s_3 + is_2\) have the same properties as \(s_3 + is_1\), i.e. they are nonnormal and nilpotent for all spin \(s\).

Starting from nonnormal matrices we can construct other nonnormal matrices. Let \(c_j^\dagger, c_j\ (j = 1, 2)\) be Fermi creation and annihilation operators, respectively. Then we can form the operator

\[
\begin{pmatrix}
    c_1^\dagger & c_2^\dagger \\
    1 & i
\end{pmatrix}
\begin{pmatrix}
    c_1 \\
    c_2
\end{pmatrix} = c_1^\dagger c_1 - c_2^\dagger c_2 + i(c_1^\dagger c_2 + c_2^\dagger c_1)
\]

Using the basis \(|0\rangle, c_1^\dagger |0\rangle, c_2^\dagger |0\rangle, c_1^\dagger c_2^\dagger |0\rangle\) with \(\langle 0 | 0 \rangle = 1\) we find the matrix representation

\[
\begin{pmatrix}
    0 & 0 & 0 & 0 \\
    0 & 1 & i & 0 \\
    0 & i & -1 & 0 \\
    0 & 0 & 0 & 0
\end{pmatrix}
\]

for the operator. This matrix has the eigenvalue 0 (fourfold) and the three eigenvectors

\[
\begin{pmatrix}
    1 \\
    0 \\
    0 \\
    0
\end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.
\]

This matrix is nonnormal.

We can also consider the Kronecker product of such matrices. Consider the case of spin-\(\frac{1}{2}\). The \(4 \times 4\) matrix

\[
(\sigma_3 + i\sigma_1) \otimes (\sigma_3 + i\sigma_1)
\]

is nonnormal. However, note that the \(4 \times 4\) matrix

\[
\sigma_3 \otimes \sigma_3 + i\sigma_1 \otimes \sigma_1 = \begin{pmatrix}
    1 & 0 & 0 & i \\
    0 & -1 & i & 0 \\
    0 & i & -1 & 0 \\
    i & 0 & 0 & 1
\end{pmatrix}
\]

is normal, but non-hermitian. The eigenvalues are \((-1)^{1/4}\sqrt{2}, -(1)^{1/4}\sqrt{2}, (-1)^{1/4}i\sqrt{2}, (-1)^{1/4}i\sqrt{2}\).

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