

# Hyperdeterminant and an integrable partial differential equation

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**Abstract.** We discuss an integrable partial differential equation arising from the hyperdeterminant.

*Key words:* Hyperdeterminant, Legendre Transformation, Integrable Partial Differential Equation

It is well known [1,2,3,4] that the Bateman equation in two dimensions

$$\frac{\partial^2 u}{\partial x_1^2} \left( \frac{\partial u}{\partial x_2} \right)^2 + \frac{\partial^2 u}{\partial x_2^2} \left( \frac{\partial u}{\partial x_1} \right)^2 - 2 \frac{\partial u}{\partial x_1} \frac{\partial u}{\partial x_2} \frac{\partial^2 u}{\partial x_1 \partial x_2} = 0$$

can be derived from the condition

$$\det \begin{pmatrix} 0 & \partial u / \partial x_1 & \partial u / \partial x_2 \\ \partial u / \partial x_1 & \partial^2 u / \partial x_1^2 & \partial^2 u / \partial x_1 \partial x_2 \\ \partial u / \partial x_2 & \partial^2 u / \partial x_1 \partial x_2 & \partial^2 u / \partial x_2^2 \end{pmatrix} = 0.$$

It is also well known (see for example [5,6]) that the Bateman equation can be linearized by a Legendre transformation and the general implicit solution is given by

$$x_1 f(u(x_1, x_2)) + x_2 g(u(x_1, x_2)) = c$$

where  $f$  and  $g$  are smooth functions. This partial differential equation plays a central role in studying the integrability of partial differential equation using the Painlevé test [5,7]. For example the differential equation appears in the Painlevé analysis of the inviscid Burgers equation, double sine-Gordan, discrete Boltzmann equation.

Here we generalize the condition given above from the determinant to the  $2 \times 2 \times 2$  hyperdeterminant of a  $2 \times 2 \times 2$  hypermatrix and derive the nonlinear partial differential equation and discuss its properties. The extension to  $2 \times 2 \times 2 \times 2$  hyperdeterminants will be straightforward.

Cayley [8] in 1845 introduced the hyperdeterminant. Gelfand et al [9] give an in debt discussion of the hyperdeterminant. The hyperdeterminant arises as entanglement measure for three qubits [10,11,12], in black hole entropy [13]. The

Nambu-Goto action in string theory can be expressed in terms of the hyperdeterminant [14]. A computer algebra program for the hyperdeterminant is given by Steeb and Hardy [11]

Let  $\epsilon_{00} = \epsilon_{11} = 0$ ,  $\epsilon_{01} = 1$ ,  $\epsilon_{10} = -1$ , i.e. we consider the  $2 \times 2$  matrix

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then the determinant of a  $2 \times 2$  matrix  $A_2 = (a_{ij})$  with  $i, j = 0, 1$  can be defined as

$$\det A_2 := \frac{1}{2} \sum_{i=0}^1 \sum_{j=0}^1 \sum_{\ell=0}^1 \sum_{m=0}^1 \epsilon_{ij} \epsilon_{\ell m} a_{i\ell} a_{jm}.$$

Thus  $\det A_2 = a_{00}a_{11} - a_{01}a_{10}$ . In analogy the hyperdeterminant of the  $2 \times 2 \times 2$  array  $A_3 = (a_{ijk})$  with  $i, j, k = 0, 1$  is defined as

$$\text{Det}A_3 := -\frac{1}{2} \sum_{ii'=0}^1 \sum_{jj'=0}^1 \sum_{kk'=0}^1 \sum_{mm'=0}^1 \sum_{nn'=0}^1 \sum_{pp'=0}^1 \epsilon_{ii'} \epsilon_{jj'} \epsilon_{kk'} \epsilon_{mm'} \epsilon_{nn'} \epsilon_{pp'} a_{ijk} a_{i'j'm} a_{npk'} a_{n'p'm'}.$$

There are  $2^8 = 256$  terms, but only 24 are nonzero. We find

$$\begin{aligned} \text{Det}A_3 &= a_{000}^2 a_{111}^2 + a_{001}^2 a_{110}^2 + a_{001}^2 a_{101}^2 + a_{100}^2 a_{011}^2 \\ &\quad - 2(a_{000} a_{001} a_{110} a_{111} + a_{000} a_{010} a_{101} a_{111} \\ &\quad + a_{000} a_{100} a_{011} a_{111} + a_{001} a_{010} a_{101} a_{110} \\ &\quad + a_{001} a_{100} a_{011} a_{110} + a_{010} a_{100} a_{011} a_{101}) \\ &\quad + 4(a_{000} a_{011} a_{101} a_{110} + a_{001} a_{010} a_{100} a_{111}). \end{aligned}$$

The hyperdeterminant  $\text{Det}(A)$  of the three-dimensional array  $A_3 = (a_{ijk}) \in \mathbb{R}^{2 \times 2 \times 2}$  can also be calculated as follows

$$\begin{aligned} \text{Det}(A_3) &= \frac{1}{4} \left( \det \left( \begin{pmatrix} a_{000} & a_{010} \\ a_{001} & a_{011} \end{pmatrix} + \begin{pmatrix} a_{100} & a_{110} \\ a_{101} & a_{111} \end{pmatrix} \right) \right. \\ &\quad \left. - \det \left( \begin{pmatrix} a_{000} & a_{010} \\ a_{001} & a_{011} \end{pmatrix} - \begin{pmatrix} a_{100} & a_{110} \\ a_{101} & a_{111} \end{pmatrix} \right) \right)^2 \\ &\quad - 4 \det \begin{pmatrix} a_{000} & a_{010} \\ a_{001} & a_{011} \end{pmatrix} \det \begin{pmatrix} a_{100} & a_{110} \\ a_{101} & a_{111} \end{pmatrix}. \end{aligned}$$

If only one of the coefficients  $a_{ijk}$  is nonzero we find that the hyperdeterminant of  $A_3$  is 0.

Given a  $2 \times 2 \times 2$  hypermatrix  $A_3 = (a_{jkl})$ ,  $j, k, \ell = 0, 1$  and the  $2 \times 2$  matrix

$$S = \begin{pmatrix} s_{00} & s_{01} \\ s_{10} & s_{11} \end{pmatrix}.$$

The multiplication  $A_3 S$  which is again a  $2 \times 2$  hypermatrix is defined by

$$(A_3 S)_{jkl} := \sum_{r=0}^1 a_{jkr} s_{rl}.$$

If  $\det(S) = 1$ , i.e.  $S \in SL(2, \mathbb{C})$ , then  $\text{Det}(A_3 S) = \text{Det}(A_3)$ .

In analogy with the Bateman equation we set

$$\begin{aligned} a_{000} &\mapsto 0, & a_{001} &\mapsto \frac{\partial u}{\partial x_3}, & a_{010} &\mapsto \frac{\partial u}{\partial x_2}, & a_{100} &\mapsto \frac{\partial u}{\partial x_1}, \\ a_{011} &\mapsto \frac{\partial^2 u}{\partial x_2 \partial x_3}, & a_{101} &\mapsto \frac{\partial^2 u}{\partial x_1 \partial x_3}, & a_{110} &\mapsto \frac{\partial^2 u}{\partial x_1 \partial x_2}, & a_{111} &\mapsto \frac{\partial^3 u}{\partial x_1 \partial x_2 \partial x_3} \end{aligned}$$

and obtain the nonlinear partial differential equation

$$\begin{aligned} &\left(\frac{\partial u}{\partial x_1}\right)^2 \left(\frac{\partial^2 u}{\partial x_2 \partial x_3}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2 \left(\frac{\partial^2 u}{\partial x_1 \partial x_3}\right)^2 + \left(\frac{\partial u}{\partial x_3}\right)^2 \left(\frac{\partial^2 u}{\partial x_1 \partial x_2}\right)^2 \\ &- 2 \left( \frac{\partial u}{\partial x_1} \frac{\partial u}{\partial x_2} \frac{\partial^2 u}{\partial x_1 \partial x_3} \frac{\partial^2 u}{\partial x_2 \partial x_3} + \frac{\partial u}{\partial x_1} \frac{\partial u}{\partial x_3} \frac{\partial^2 u}{\partial x_1 \partial x_2} \frac{\partial^2 u}{\partial x_2 \partial x_3} + \frac{\partial u}{\partial x_2} \frac{\partial u}{\partial x_3} \frac{\partial^2 u}{\partial x_1 \partial x_2} \frac{\partial^2 u}{\partial x_1 \partial x_3} \right) \\ &+ 4 \frac{\partial u}{\partial x_1} \frac{\partial u}{\partial x_2} \frac{\partial u}{\partial x_3} \frac{\partial^3 u}{\partial x_1 \partial x_2 \partial x_3} = 0. \end{aligned}$$

The partial differential equation is invariant under the permutations of  $x_1, x_2, x_3$ . The group of symmetries is  $SL(3, \mathbb{R})$ . The equation can also be linearized by a Legendre transformation.

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