LEARNER MATHEMATICAL ERRORS IN INTRODUCTORY DIFFERENTIAL CALCULUS TASKS: A STUDY OF MISCONCEPTIONS IN THE SENIOR SCHOOL CERTIFICATE EXAMINATIONS

BY:

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JOHANNESBURG

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DECLARATION

I declare that this research report is my own, unaided work. It is being submitted for the Doctor of Philosophy at the University of Johannesburg, South Africa. It has not been submitted before for any degree or examination in any other university.

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Signature

25th Day of March, 2011.
This research report is the culmination of a protracted, arduous and hard fought effort to acquire this lofty and veritable qualification. For this reason and many others, I am ever thankful to the Lord Almighty; who granted me his grace and protection so that I could triumphantly complete these studies.

I will not forget my parents, Samson and Hilda who have shown me love and the value of perseverance even in the midst of the greatest adversity. For them I pray that the Almighty grant them His peace and His grace.

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ABSTRACT

The research problematised the learning of mathematics in South African high schools in a Pedagogical Content Knowledge context. The researcher established that while at best, teachers may command mathematics content knowledge, or pedagogic knowledge, that command proves insufficient in leveraging the learning of mathematics and differentiation. Teachers' awareness of their learners' errors and misconceptions on a mathematics topic is critical in developing appropriate pedagogical content knowledge. The researcher argues that the study of learner errors in mathematics affords educators critical knowledge of the learners' Zones of Proximal Development. The space where learners experience misconceptions as they attempt to assign meaning to new mathematical ideas to which they may or may not have obtained semiotic mediation. In their Zones of Proximal Development learners may harbour concept images that are incompetition with established mathematical knowledge. Educators need to study and understand those concept images (amateur or alternative conceptions), and how learners come to have them, if they are to help learners learn mathematics better.

Besides the socio-cultural view, the study presumed that the misconceptions formed by learners in mathematics may also be explained within a constructivist perspective of learning. The constructivist perspective of learning assumes that learners interpret new knowledge on the basis of the knowledge they already have. However, some of the knowledge that learners construct though meaningful to them may be full of misconceptions. This may occur through overgeneralisation of prior knowledge to new situations. The researcher presumed that the ideas that learners have of particular mathematical concepts were concept images they construct. Though some of the concept images may be deficient or defective from a mathematics expert's point of view, they are still used by the learners to learn new mathematics concepts and to solve mathematics problems. The lack of success in mathematics that results in the application of erratic concept images ultimately leads to unsuccessful learning of mathematics with the danger of snowballing if there are no practicable interventions.
Differentiation is a new topic in the South African mathematics curriculum and most teachers and learners have registered problems in teaching and learning it. Hence it was imperative to do research on this topic from an angle of learner errors on that topic. The significance of the study is that this research isolated the differentiation learner errors and misconceptions that teachers can focus on for the improvement of learning and achievement in the topic of introductory differentiation.

The research focused on the nature of errors and misconceptions learners have on introductory differentiation as exhibited in their 2008 examination scripts. It sought to identify, categorise (form a database) and discuss the errors and their conceptual links. A typology of errors and misconceptions in introductory calculus was constructed. The study mainly used qualitative methods to collect and analyse data. Content analysis techniques were used to analyse the data on the basis of a conceptual framework of mathematics and calculus errors obtained from literature. One thousand Grade 12, Mathematics Paper 1 examination scripts from learners of both sexes emanating from diverse social backgrounds provided data for the study. The unit of analysis was students’ errors in written responses to differentiation examination items.

The findings of the study were various. About 65% of the students utterly failed to distinguish between calculus concepts and algebra concepts. Students lacked competency with algebra concepts and procedures, which concepts and procedures is the language with which calculus is conveyed. Students often could not distinguish calculus from algebra; it was very common to find students bearing all the etiquette of calculus in their writings yet deeper analysis revealed that they in substance never engaged to real calculus concepts. Students avoided as much as possible to deal with central calculus concepts such as tangent per se. As most students still had difficulties with marginal but essential algebraic concepts they minimally showed any engagement with central calculus concepts such as that of a limit. Besides algebra, students had difficulties with the concept of a function. Students’ handling of the function concept and its operationalisation directly affected their understanding of differentiation and its applications. Further, students were confused by the specialist terminology of calculus. Students had much misunderstanding and confusion on many terms such as function, co-ordinate, tangent, limit, secant, distance, midpoint, equation of line,
variable, gradient, maximum point, minimum point and point of inflexion. Students held competing meanings of these terms that led them to improperly understand what was being required. Thus students apparently did not understand the questions not least because for most of them, mathematics is taught and studied in a second language which learners are not fluent at. This finding is supported by the fact that students who did mathematics in their home language such as Afrikaans performed quite well and had fewer errors. In addition, analysis showed most students had not mastered the important mathematical 'construct' of substitution. Students had many errors related to unawareness of substitution as a leeway in mathematics and so lost important opportunities to understand calculus. It appeared that the majority of students held extremely weak concept images of substitution.

There were errors which were quite common across different students. There were also errors which were peculiar to individual students and not shared with others. Students also had errors because of lack of metacognition. They lacked skills to think about their thinking in order to verify and justify their work. Further, the errors that learners made in calculus were often closely intertwined and connected to the errors that learners had in other areas of mathematics. The research suggested a framework for error analysis in mathematics and calculus, as well as categorising the types of the errors and misconceptions. The research recommends carrying out pedagogical interventions using the errors described in its error analytical protocol to determine to what extent such interventions impact learning and achievement in introductory differentiation.
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CHAPTER 1: INTRODUCTION AND MOTIVATION FOR THE STUDY

1.1 Introduction

This chapter introduces the study. It begins by outlining the literature from which the research problem of this study is articulated. The chapter discusses the nature of errors and misconceptions in mathematics and locates the study within the South African curriculum focussing on learner errors and misconceptions in calculus; differentiation and its applications in particular. Then the aims and research questions are presented. The context of the study is also discussed as are the limitations, de-limitations and the significance of the study. It concludes by describing how the rest of the thesis is structured.

1.2 Context of the study

Perspectives on reform in mathematics education have been articulated in a number of papers and policy documents aiming at increasing each learner's mathematical understanding and achievement (for example National Council of Teachers of Mathematics (NCTM), 1989, 2000; National Curriculum Statement (NCS), 2003, and National Mathematics Advisory Panel (NMAP), 2008). These suggest a shift from mainly teaching and learning mathematics procedural knowledge to emphasis on developing learner competences such as conjecturing, argumentation, multiple representations, communicating mathematically and problem-solving. Also, there is increasing emphasis on the didactic shifts from teacher-centred to learner-centred approaches and from viewing mathematics as a finished product (see Ernest, 1985 on the absolutist philosophy of mathematics) to mathematics as a process, often called thinking mathematics (Rand Mathematics Study Panel, 2002).

In line with aforementioned paradigm shifts on the new directions that mathematics teaching and learning should now take, the researcher has been intrigued by how learner errors and misconceptions in mathematics in school exit examinations affect their mathematics performance in general and in certain tasks (introductory calculus) in particular. In their responses to mathematics tasks and items, learners often manifest many difficulties as
evidenced by low achievement in mathematics in both national examinations and international comparison tests (Howie, 2001). These challenges often underlie learners' mathematics errors and misconceptions related to answering examined tasks. Misconceptions are epistemological obstacles (Sierpinska, 1992), amateur/unscientific concepts (Vygotsky, 1986), non-standard conceptions (Ely, 2010) or alternative conceptions (Driver & Easly, 1978) that learners hold about mathematics. Despite their obvious negative effects on mathematics learning and performance, regarded from a different perspective, the mathematical errors of learners reveal important hints that could help educators pinpoint the nature of learners' cognition as they attempt to make sense of mathematics in their own, albeit inappropriate way. The premise is that learners try to make sense of mathematics. This implies that mathematical knowledge cannot be transmitted from the teacher to the students without their understanding, rather, it must be dynamically re-interpreted, re-organised and reconstructed in each learner's mind (Cobb & Bauersfeld, 1995; Hatano, 1996). Unfortunately, some of the understandings that learners construct are misconceptions that hinder their progress in learning mathematics. This is because mathematics is a hierarchical subject, meaning that learning at higher levels is dependent on appropriately attained requisite knowledge and skills at lower levels.

After a needs-analysis, on the direction education should take in the new democracy, the new democratic government of South Africa in 1998 introduced a reform curriculum set to achieve the new educational goals for the new society, named Curriculum 2005 (C2005), the New Curriculum, or National Curriculum Statement (NCS) (Taylor, Muller, & Vinjevold, 2003).

1.3 The South African Curriculum

In 1994, transformation and many changes were made in South Africa, in virtually all government departments including the Department of Education (DoE). According to (Taylor, Muller, & Vinjevold, 2003), the transformation aimed to meet the ideals of the new constitution of the Republic of South Africa Act 108 of 1996. The South African National Curriculum Statement (Grades R-9) was introduced to schools in 1998. Schooling was structured into progressive bands and phases. These were Foundation Phase (Grades 0-3),
General Education and Training Phase (GET) (Grades 4-6), Senior Phase (Grades 7-9) and the Further Education and Training Phase (FET) (Grades 10-12).

The National Curriculum Statement (NCS) (2003) is an outcomes-based curriculum (Jansen, 1999b). It is outcome based in that the NCS emphasises that learners explicitly demonstrate acquisition of pre-determined curricula knowledge and skills when they leave school. However, the NCS has weak framing and weak classification (Bernstein, 2000). Framing of the curriculum relates to the control that the teacher has on learners and the learning process while classification relates to how far curricula subjects stand distinct from other subjects or are integrated with them. Weak framing means that teaching is learner-centred and teachers are regarded as facilitators of learning rather than authoritative subject experts. Weak classification means that the curriculum avoids over specialisation and curricula subjects tend to overlap; the boundaries between different subject disciplines are indistinct and are collapsed. This curriculum is opposed to strong framing in which there is strong differentiation among subjects as they stand in isolated silos of specialist knowledge. Often strong classification has strong framing leading to strong teacher-directed lessons. Methodologically, Curriculum 2005 hinges on the constructivist philosophy to learning, with teachers as facilitators in student-centred learning approaches (Donnelly, 2002).

The NCS replaced the apartheid era curriculum; implemented differentially on the basis of the racial and ethnic difference of learners. The apartheid curriculum had separate curricula for Blacks, Coloureds, Indians and Whites. To underpin the low expectations for Bantu education, leading apartheid architect and philosopher Hendrick Verwoord remarked in 1953 that:

> What is the use of teaching the Bantu child mathematics when it cannot use it in practice? That is quite absurd. Education must train people in accordance with their opportunities in life, according to the sphere in which they live? (Clark & Worger, 2004, p.50)

Verwoord's philosophy which argued for systemic inferior education for black children, particularly in mathematics met its nemesis when the 1990s democratic process led to the drawing up of the new constitution of the Republic of South Africa. One of the kingpins and central tenets of the new constitution is equality of all South Africans in all spheres of life, including education. The old curricula that had differentiated school children by race and
ethnic origin was torn down and replaced by a brand new curriculum; that demanded high knowledge and high skills for all learners (NCS, 2003), irrespective of their racial or ethnic background.

The NCS highlights seven generic critical outcomes and four developmental outcomes applicable to all learning areas and all the grades (Department of Education, 2002). Critical outcomes are long-term, but developmental outcomes are mid-term involving the cultivation of positive and progressive outlook of life. The critical outcomes involve the development of reflexive and reflective thinking and problem solving, effective communication, group and self-responsibility, appreciation and utilization of science and technology as well as propensity to suspend judgement prior to critical analysis of information. Developmental outcomes envisage capabilities to develop strategies for learning effectively, the ability to learn and work with others, appreciation and tolerance of diversity, and awareness of opportunities that education opens up including careers, as well as healthy appreciation of the value of entrepreneurship to personal and national development (National Curriculum Statement, 2003).

The Revised National Curriculum Statement (RNCS) Grades R-9 (Schools) is an amendment and reorganisation of Curriculum 2005. It has the foundation, intermediate and senior phases under the GET band. The first school exit examinations under the NCS were written in 2008, thus completing the eleven year journey started in 1998 when the new curriculum was launched at Grade 1. The 2008 National Senior Certificate (NSC) qualification replaced the old Matriculation certificate. It equates to Level 4 of the National Qualifying Framework (NQF), administered by the South African Qualification Authority (SAQA). Under the NQF, the Foundation Phase offers a Level 1 qualification, the Intermediate Phase, Level 2 and Senior Phase, Level 3. Level 3, is equivalent to the General Education Certificate. The Further Education and Training Phase (FET) stands at level 4. The NCS has thirty nine subjects and mathematics is one of the core subjects studied by learners from Grade 0 to matriculation. It is core because it is one of the compulsory subjects that are studied by every child throughout the whole schooling system from Grade 0 to Grade 12.
1.4 Mathematics in the South African Curriculum

Reform mathematics as prescribed by the NCS and NCTM is about learners developing mathematical power; the ability to investigate and formulate conjectures that help to perceive and generalise patterns needed in solving problems (NCTM, 2000; Laridon, 1991; Van de Walle, 2004). The primacy of the mathematics learning processes that lead to achieving the required learning outcomes is emphasised. A reform mathematics curriculum has problem solving, communication, representations, reasoning, and connections as important outcomes in addition to acquisition of content (NCS, 2003; NCTM, 2000). Ball (2002) emphasised that "the reform curriculum places emphasises on explicit mathematical know-how called "mathematical practices" (p.24). In the NCS, mathematics is viewed as a human activity that involves observing, representing and investigating patterns and quantitative relationships in physical and social phenomena and between mathematical objects themselves(NCS, 2003; p.9). It is also argued that, "competence in mathematical process skills such as investigating, generalising and proving is more important than the acquisition of content knowledge for its own sake" (NCS, 2003; p.9). It is through this process that new mathematical ideas and insights are generated. Mathematical concepts build on each other, thereby creating a coherent structure (Ball, 2003). The NCS emphasises the fallibilist/constructivist approach which the researcher has alluded to earlier. Similar to NCTM (1989), the new curriculum assumes that reform is much more about how children learn and how they achieve the desired content and process goals. The new curriculum views the teaching and learning of mathematics as a two-pronged approach; learner acquisition of mathematical knowledge and learner participation and engagement in mathematical practices that can lead to that acquisition (Sfard, 1997). Sfard contrasts these as the "acquisition metaphor" and the "participation metaphor" (Sfard, 1997, p. 39). Developing learners who have mathematics sense making and critical thinking skills is central to the NCS. The researcher presumes that as all learners engage in mathematical learning practices and mathematics sense making they become exposed to various important facets of mathematical knowledge.

Reform in mathematics education revolves around the nature of mathematics (see for example Ernest, 1991), mathematics teaching (for example Sfard, 1997), and mathematical knowledge and understanding (for example Hiebert & Wearner, 1986; Skemp, 1976). In the NCS, mathematics is viewed as a cultural enterprise comprising investigation and
representation of patterns and quantitative relationships that occur in physical and social phenomena and between mathematical objects themselves. The fallibilist philosophy rather than the absolutist philosophy of mathematics (see Chapter 2) has been embraced in that mathematical concepts are viewed as having no existence on their own but that the concepts can be imposed on and reorganise the physical world by a thinking mind (Van de Walle, 2004). The meaning of mathematical concepts is primarily cogitated from reflected relationships between concrete-manipulative objects. Thereafter, learners develop other concepts; the secondary concepts that are de-situated from the physical situation from which they are originally abstracted. These secondary concepts represented by symbols, become themselves the requisite objects for the formation of yet other tertiary mathematics concepts. This abstracted nature of mathematical knowledge enables the power of generalisability and re-application of the concepts to new and novel problem contexts (Freudenthal, 1991; Treffers, 1991). The abstractness and power of generalisability over superficially dissimilar contexts is due to mathematics' underlying logical structure.

The Mathematics and Mathematical Sciences Learning Area (MMSLA) is one of the eight learning areas (NCS, 2003). Because competency in mathematics is regarded as beneficial to both the individual learner and society, in terms of the personal, social and national empowerment it bestows, it is a compulsory subject at all levels of South African schooling. Mathematical Literacy is offered from grade 10-12 as an alternative to Mathematics for those learners who would not wish to pursue a career in the future for which a mathematics qualification is a requirement, and for those who find 'pure' mathematics academically too challenging.

The Mathematics Learning Area statement is described in the four Learning Outcomes (LOs) at FET level of which Functions and Algebra is the Learning Outcome 2 (NCS, 2003). The main topic of this research; Differential Calculus, falls under this LO and is studied at Grade 12 level. Learning outcomes tend to be stable and incremental from grade to grade, stipulating assessment standards that entail the knowledge, values and skills that learners have to demonstrate at the end of an education and training band. Assessment standards complement learning outcomes and are more detailed and specific. Assessment standards are also incremental; describing the degree and activities, grade by grade. Learners must
demonstrate sufficient progress towards achieving learning outcomes. The learning outcomes spell out the content themes and topics such as Space and Shape or Data Handling that serve as the means for attaining the crucial and strategic critical outcomes espoused in the democratic Constitution of the Republic of South Africa. The summative assessment of how far learners have achieved the learning outcomes at the end of grade 12 is done through the National Senior Certificate Examinations (NSCE). Assessment of mathematics in the NSCE is composed of internal (school) continuous assessment (CASS) weighing 25% of the final mark and the “external” national examination weighing 75% of the final mark, through at least two three hour examinations.

1.5 Motivation of the study

It is in tests and examinations, the usual measuring rods for educational performance and achievement; that South Africa is found lagging in mathematics competitiveness at home and internationally (Howie, 2001). As a result of the low attainment of the South African learners in attaining educational goals, many debates have arisen on the suitability of the National Curriculum Statement (Jansen, 1999a; Potenza & Monyokolo, 1998). Despite these debates on Curriculum 2005, learners are neither learning nor performing well in Mathematics (see Taylor, Muller, & Vinjevold, 2003). Performance in national school exit examinations such as Matriculation and Independent Examination Board (IEB), as well as in international comparisons tests (for example RAND Mathematics Study Panel, 2002; Trends in Mathematics and Science Study (TIMMS), (Howie, 2001; Reddy, 2006)), find South African learners’ achievement in mathematics at the bottom of the pile. For instance, in the TIMSS 2003 mathematics (and science) international comparisons tests, South Africa was ranked the lowest performing country out of fifty countries that entered the competition. It even performed worse than poorer African countries with much fewer resources (Reddy, 2006). This implies that poor performance does not lie in lack of at least (financial) resources but could be due to structural problems. One problem of the South African education system seems to be toleration of low standards. For example, a 30% mark is regarded as pass mark in the NSC examinations. As a result, the mathematics competence of matric graduates is held in doubt by many people even in South Africa itself. This has resulted in some South African university faculties requiring student mathematics pass at Grade 12, conducting their own selection examinations to determine the actual mathematical preparedness of applicants.
The researcher wondered why mathematics is so barely achieved to many learners\(^1\) in South Africa.

### 1.6 Pedagogical Content Knowledge

The implementation of the official and intended curriculum on mathematics occurs daily in mathematics classrooms across the nation. But the teaching and learning of mathematics is fraught with problems. In many cases, learners fail to achieve curricula objectives in national public examinations (see Umalusi, 2005). My argument is that it is critical and urgent for educators to particularly understand why learners perform poorly in mathematics which results in them failing to achieve the NCS objectives. The researcher hypothesised that the central problem of learner mathematics underachievement rests in learners' mathematical errors and misconceptions. The researcher presumed that within learners' errors and misconceptions in mathematics lay seeds of successful mathematics teaching and learning. The errors and misconceptions held by some learners in mathematics have so far been regarded negatively as ever-present but unwanted obstacles to learning and achievement (Smith et al., 1993). The errors and misconceptions are now regarded as in reality the key to greater understanding of how learners learn mathematics (Riccomini, 2005). This is because as learners learn mathematics, they naturally make mathematical errors and misconceptions (see Cox, 1975; Smith et al., 1993; Green, Piel, & Flowers; 2008; Hatano, 1996) in their attempt to build meaning to new mathematics concepts and procedures.

Errors and misconceptions constitute challenges to reaching curriculum outcomes. But they can also be viewed in a positive light. It is crucial that if there should be effective teaching of mathematics, teachers understand the nature of learner mathematical errors so that they can design appropriate strategies to help learners understand mathematics better. This quest for understanding learners' errors and misconceptions in order to teach them more effectively falls under Pedagogical Content Knowledge (PCK) (Shulman, 1986) (see fig. 1 below).

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\(^1\)In this thesis, the terms learner and student are used interchangeably. However, in general the term learner is used for school children attending high schools and student for young adults attending university and other higher education institutions.
Shulman argued that a teacher's subject matter knowledge and/or pedagogic knowledge are not sufficient to help learners to understand mathematics. He argued that it is important for teachers to understand among other factors, the common difficulties learners encounter when learning a particular subject, such as their errors and misconceptions. PCK is knowledge for teaching of a particular subject that is quite different from the subject matter itself or pedagogic knowledge per se. It is the knowing ways of formulating and presenting subject matter so that it is comprehensible to learners. Thus, Schulman (1986) argued that besides content knowledge, like that possessed by mathematics majors, teachers need to possess Pedagogical Content Knowledge and curricula knowledge. Curricula knowledge encompasses knowledge of the mathematics that has been taught to learners before teaching the present grade as well as the mathematics that will later be taught to the same learners in the future, and its possible application in contexts in and out of the school. Thus PCK is knowledge that goes beyond subject matter content knowledge. A teacher who has PCK has the wisdom of practice.

This study sought to research on the errors and misconceptions that learners face in a first course in calculus. It cannot be overemphasised that the calculus is one of the most important
branches of mathematics; it is studied in high schools, tertiary institutions and universities. This is because calculus or analysis is the language of higher mathematics, the sciences and technology (Shuard & Neill, 1986). Calculus is the supreme tool for analysis of phenomenon involving change, and change of change; from analysis of the motion of heavenly bodies moving with variable or invariable acceleration to rates of change of chemical reactions for example. Calculus is a required course for students aiming for many important careers such as in science, engineering, medicine, technology, business and so on. In the South African scene, calculus is important in helping build the skills base of the country which at present suffers critical skills shortages in all sectors of the economy including in industry, commerce and education (Taylor, Müller, & Vinjevold, 2003). This is well illustrated by the employment of many expatriate engineers, doctors, accountants and, mathematics and science teachers among other professionals from neighbouring Southern African Development Community (SADC) countries as well as other countries as far afield as India and Cuba.

Calculus is one of the more modern inventions in mathematics. It was invented in the 17th century by Isaac Newton and Gottfried Leibniz (Eves, 1990). However, earlier in the 15th century Fermat’s work had seeds of the differential calculus in what he called ‘methods of differences’ in optimisation problems. Still in Greek antiquity, Archimedes suggested finding areas of curved surfaces through “methods of exhaustion” thereby preordaining the dawn of calculus more than two millennia before Newton and Leibniz (Eves, 1990). The method of finding areas of curved surfaces falls under the Integral Calculus. The discovery that finding areas of curved surfaces is intimately connected to differentiation was made by Newton and Leibniz themselves. It is surprisingly pleasant and has had the most important repercussions in mathematics. This relationship is embodied in the Fundamental Law of Calculus stating that differentiation and integration are inverse operations. Calculus has many branches, such as Differential Equations which is a bridge between calculus and integration. Calculus is the language of Physics. In Statistics; probability is studied in terms of calculus. Aerodynamics, the study of flying is impossible without calculus and so are many branches of engineering, economics and mathematics itself. The researcher views calculus as the epicentre of scientific knowledge.
Differential calculus, a branch of mathematics, helps in the analysis of how functions change as their inputs change (Stewart, 2004). Differential calculus has roots from the humble concept of the gradient of a straight line but together with the extremely complicated, perplexing and quite mundane limit concept. The idea of a limit which gives birth to calculus is quite difficult to hold in some learners’ minds and even to some experienced mathematics teachers. It is difficult to think about and quite intractable, thereby causing much difficulty in learning it (Artigue, 1996; Bezuidenhout, 1998; Tall & Vinner, 1981). The limit and associated concepts of introductory differentiation, offers a rich study on learner errors and misconceptions in the South African context given that South African learners’ competency in mathematics is way lower than most countries (Howie, 2001). The researcher presumed that because of that fact, South African learners would make errors in introductory calculus which are quite different from those found from other studies carried out in the rest of the world in the past decades (for example Orton, 1983b; Tall & Vinner, 1981).

1.7 Comparisons and contrasts of learner mistakes, errors and misconceptions

While mistakes, errors and misconceptions all indicate that something is not quite right to the solution of a mathematics task; they are actually different in one way or another (Erlwanger, 1975). All three are characterised by deviations from accuracy, and can be placed in a continuum. Errors can be systematic or non-systematic.

On one end of the continuum, are mistakes or slips. On the opposite end lie misconceptions. Mistakes or slips can result from misreads or instantaneous lapses in memory, for example, a learner may write $3 + 2 = 6$, when she would have misread a plus sign for a multiplication sign. This can happen when learners rapidly work through an otherwise familiar mathematical procedure. Learners and teachers make slips unintentionally and can readily correct them by themselves once they become conscious of them. Such mistakes are due to inadvertence or inattention. They are not intended, non-recurring wrong answers but are due to carelessness, sloppiness or oversight. Slips are errors of performance and may be due to tiredness, complacency or lack of cross-checking. Mistakes can be easily rectified by their owners because there are no underlying and flawed conceptual structures associated with them. So the resolution of a mistake is easy and often uneventful. Notwithstanding that slips are unintentional, they nonetheless undermine and frustrate learners’ performance when
committed. This is because in mathematics part solutions to a question are often inputted for finding other solutions in multi-step tasks. So, if the answer to the first part is wrongly done even because of a slip, the rest of the solution to the question is affected. This can insidiously dent the learners’ confidence of learning and doing mathematics in that learners worry why they are not getting it when they have done everything “correctly”. If learners cannot identify and sort out their mistakes in time, the mistakes can snowball and make mathematics seem difficult because learners can lose the meaning when working out long mathematics problems. For example, they may find a large value when their instinct shows that a small value should be expected. Such nonsensical answers however can show that something is wrong and trigger metacognition – thinking about one’s own thinking, what Davis (1984) called ‘inside critics’ (p.37). As the researcher referred to earlier, there are two types of errors. Slips and mistakes discussed in the foregoing paragraph are referred to as unsystematic errors.

At the other end of the spectrum, systematic errors are more serious in that these indicate lack of competence rather than performance. Lack of performance occurs when one has the conceptual knowledge but fails to use it for one reason or another, while lack of competence denotes lack of understanding or grasp of the underlying concepts. Systematic errors are evidenced by recurrent wrong answers methodically re-applied across space and time in answering particular types of mathematical questions. These incorrect answers are not recognized yet by learners as wrong. Indeed learners may defend these answers because they make sense to them. They are symptomatic of a deep-seated but faulty hypothesis in the thinking of the learner that causes them, referred to as misconceptions (Olivier, 1992; Nesher, 1987). Nesher (1987) explains that, “the notion of misconception denotes a line of thinking that causes errors, all resulting from an incorrect underlying premise, rather than sporadic, unconnected and non-systematic errors” (p. 35). A misconception is a knowledge structure that is activated in a wide variety of contexts, is stable and can be resistant to change, and competes with accepted cannons of scientific knowledge (Erlwanger, 1975).

Misconceptions could be over-generalisations of earlier acquired valid knowledge wrongly applied to an extended domain (Erlwanger, 1975; Smith, Di Sessa & Roschelle, 1993; Nesher, 1987). For instance, based on comparison of whole numbers, learners often have the
misconception that a decimal number with more digits is larger than one with less digits or that multiplication always results in bigger numbers (Smith, Di Sessa & Roschelle, 1993; Nesher, 1987). They then use these conclusions to interpret mathematical situations. Also learners at grade 12, generalising from the differentiation of polynomials might prematurely conclude that if \( f(x) = x^x \), then \( f'(x) = x\cdot x^{x-1} \). In addition, misconceptions are amateur and naive preconceptions intuitively sensible to learners (Smith, Di Sessa & Roschelle, 1993; Vygotsky, 1986). Ely (2010) refers to them as non-standard conceptions. These misconceptions or alternative conceptions often compete with scientific concepts. For example, the conjoining misconception occur when learners think that \( 3x + 2 = 5x \). Learners add 3 and 2 to get 5 (from the maths they first encountered at school), and the x is put beside 5; everything is there. Since to them, addition conceptually refers to putting together available quantities. So misconceptions often make sense from the point of view of the learners. Therefore we need to get inside the learner’s mind to understand how and why they hold this misconception. Misconceptions are also transitional and perturbable in that they are often thoroughfares to the formation of valid concepts (Ely, 2010). As learners’ misconceptions lie in a spectrum, where on one end there are deep-seated, resilient, tenacious and belief-like misconceptions which can resist change at all costs; defying situations designed to dislodge them, such as open confrontation or negotiation. Such misconceptions are well rationalised by learners. On the other end of the spectrum, lies some very perturbable and transitional misconceptions that learners give up fairly easily if someone facilitates this. They could be referred to as amateur concepts (Vygotsky, 1978) or preceptions (Chi, 2005). All these level of misconceptions are important in mathematics teaching and learning.

As referred to earlier, learners who commit systematic errors may continue to commit them, even when the errors are pointed out to them (Cox, 1975; Riccomini, 2005) as the errors are sensible to their constructors. These systematic errors are usually exhibited in learners’ writings or speech. The alternative conceptions or misconceptions are usually generalizations of specific mathematical knowledge in which learners fail to negotiate the limits of specific knowledge. These occur when there are over-generalisations. For example, having noted that \( 3 + 5 \) gives the same answer 8, as \( 5 + 3 \) (commutativity of the addition operation); learners might go on to think that the subtraction operation is also commutative. They would incorrectly deduce that \( 5 - 8 \) give the same answer 3, just as \( 8 - 5 \). (But in another context the sum or difference of 5 and 8 or 8 and 5 will give the same respective answers!). This limited
usefulness and meaning of mathematical processes often puzzles learners.

Misconceptions can be due to faulty learner inferences or the nature of mathematics itself as the above discussed example on addition and subtraction shows. The main point as far as this research is concerned is that learners often have defensible, though incorrect reasons for their errors, hence it is important to note that learners have integrity in the answers they provide to mathematics tasks. It is up to teachers and researchers to understand/interpret the lines of thinking learners draw upon when they answer mathematics questions incorrectly. Even in learners’ correct responses may reside the seeds of a misconception (Nesher, 1987). This is because some learners may have correct mathematical answers for the wrong reasons. This occurs when learners use flawed reasoning (a misconception) and so accidentally gets a correct answer. For that reason, it is always important to elicit and probe learners to justify and defend their answers even if they are correct. It is important then to firstly identify and classify the common errors the learners make in answering particular questions and then secondly, examine why learners make the errors they make. This helps to find out and reveal their reasons for the answers they give. This grounded knowledge helps teachers to formulate appropriate interventions that intercept deficient learners’ reasoning and to engage with them to see for themselves why their understanding is incomplete.

Confrey (1987) reported that misconceptions are resilient even in the face of instruction meant to overcome them and that the attraction of misconceptions is compelling to students. However, even though they are tenacious, misconceptions are putrable concepts which suggest that they are potentially viable and challengeable, that means that they could eventually change. Basically, errors and misconceptions are linked to procedural and conceptual knowledge (Hiebert & Wearner, 1986) or instrumental and relational understanding of mathematics (Mellin-Olsen,1970; Skemp, 1976). Learners’ misconceptions in mathematics often arise out of a backdrop of mis-understanding of how new concepts link with the old. The construct of understanding can also be explained in Piagetian terms. As Grossman (1986) argues, to understand a concept is to assimilate it to an appropriate existing schema. This implies that in mathematics, students learn through connecting and incorporating new concepts to concepts they already possess. As Piaget has taught, assimilation is the use of an existing schema to give meaning to new ideas. If new ideas
cannot be assimilated to an existing schema, then a cognitive shift is required to restructure the schema to accommodate the new ideas. As Van de Walle (2004) argues, understanding is qualitative as it involves the linking of new ideas to the ones a learner already has. Understanding is also quantitative in that the amount of ideas a learner understands increases. Understanding is qualitative in that existing ideas become more comprehensive, enduring, richer and more powerful. For instance, mature mathematics students realise that calculus provides a universal language for studying any phenomena involving rate of change and optimisation. Misconceptions in the topic of differential calculus at grade 12 then, infer that learners have a lack of breadth and depth of the required mathematical concepts and procedures.

Sometimes errors are due to learners' intuition rather than misapplication of previously acquired knowledge (Chi, 2005). Errors then point to an underlying misconception responsible for their occurrence. For teaching to be effective, it is important that error analysis be always carried out so that teaching strategies engage the thinking patterns that result in learners committing the mathematical errors they commit. When the underlying notions that learners resort to are not engaged in pedagogy, teaching continues to be a 'hit or miss affair' as it only takes cognisance of mathematics logical structure with little regard of the psychological standpoint of the learners. This neutralises and prejudices their teaching efforts.

1.8 Differential Calculus in the South African school curriculum

According to Clark & Worger (2004), calculus is a new topic in the NCS and was introduced to black learners who were not allowed to study such advanced mathematics before. Calculus is introduced to South Africa learners at the FET band (see NCS, 2003). Calculus is introduced at this level to create a bridge for further education, where it is demanded in many tertiary courses. In the final NSC Mathematics examination, calculus is awarded 35 marks out of 150 in paper 1 (see Appendix, C). It carries 35 marks out of 300 for paper 1 and 2 which is 11.7% for the whole examination. Hence, the calculus theme is quite significant in the South African mathematics curriculum as it is internationally (Artigue, 2000).
An important aspect in mathematics in the Further Education and Training band is the establishment of proper connections between mathematics as a discipline and other curricular subjects such as science and the application of mathematics in real-world contexts. The application of mathematical knowledge and procedures to solve problems inside and outside mathematics is called mathematical modelling. Mathematical modelling provides learners with the means to analyse and describe their world mathematically, and so allows learners to deepen their understanding of mathematics while adding to their mathematical tools for solving real-world problems (Stewart, 2004). Learners are expected to develop capacity to use the principles of differential calculus to determine the rate of change of a range of simple, non-linear functions and to solve simple optimisation problems. The learning achieved in mathematics in the General Education and Training band is expected to provide an essential base from which to proceed into the demands of mathematics in the Further Education and Training band. Mathematics, in particular calculus, is a cognitive discipline that requires understanding before competence in it can be achieved (Davis, 1984). These competences progress as learners’ errors and misconceptions are analysed, diagnosed and reconciled.

1.9 Error Analysis and Diagnosis

In mathematics education, it is often helpful to establish the breadth and depth of learners’ understanding of mathematical concepts through error analysis and diagnosis (EAD), so as to commence teaching at a learner’s conceptual level. EAD aims to analyse, expose and interpret error patterns for the purpose of modifying instruction (Borasi, 1994; Nyaumwe, 2008) in reaction to them. By assessing and evaluating the students’ independent work to identify specific error types, error analysis informs priorities for teaching (Riccomini, 2005). As learner errors and misconceptions are often consistent rather than random, this suggests instantiation or launching of necessary instructional strategies that can begin to resolve them. As a result, investigating misconceptions afford researchers uncanny insight. It gives them the opportunity to question the learner’s perceptions of mathematical ideas so that educators can rectify any misconceptions. Errors are used as springboards of enquiry (Borasi, 1994), as they give insight into learner difficulties and feedback to teaching. They help researchers to discover learners’ strengths and weaknesses in mathematical thinking.
As Leikin & Zaslavsky (1994) argue, effective critical analysis of learner reasoning restrains under-hearing or over-hearing of learners' work. Under-hearing learners occurs when teachers fail to acknowledge the learners' strengths because they think that learners have not understood, while over-hearing occurs when teachers think the learners understand when they actually do not. EAD promotes heuristic listening when teachers withhold prejudging the learner, as they capture the threshold of the learners' mathematical thoughts, and empathise with them. Consequently, EAD helps the teacher to proffer appropriate academic support to learners on the basis of actual learner strengths and weaknesses in mathematics (Green, Flower, & Piel, 2008). I propose to articulate my research problem with the above background in mind.

1.10 Statement of the problem

As has been discussed earlier, South Africa has fared abysmally in international comparison tests (Reddy, 2006). In this context, learner mathematics errors and misconceptions are not generally perceived as an important factor in this under-achievement, yet researchers have identified that learners' misconceptions are tenacious and resistant to change (for example Smith et al., 1993; Nesher, 1987). They have argued that unless learners come to realize the inadequacy of their conceptions in their experiential contexts they would not give up their misconceptions; not least because they have conceived the misconceptions themselves and that these misconceptions are at times productive, in certain mathematical situations.

Early research on mathematical errors and misconceptions focused on arithmetic operations on whole numbers and fractions; from devising diagnostic models explaining learners subtraction errors (Brown & Burton, 1978), to examining the patterns of the most frequent errors on whole number arithmetic operations (Cox, 1975). More recently, Riccomini (2005) researched systematic errors in subtraction and described how subtracting “small-from-larger” (SFL) and “borrowing-across-zero” (BAZ) (p. 235) were common errors among primary school children. However, research has also lately been done on other topics of mathematics besides arithmetic. Melis (2003) studied how learners learn from wrongly worked mathematical examples to enhance understanding of problematic mathematics concepts, while Borasi (1994) used learner errors to stimulate mathematical inquiry to
enhance mathematical thinking and learning.

Lately, studies of misconceptions in algebra (Quinn & Brown, 2006) showed that learner inability to perform basic operations on common fractions culminated in the emergence of error patterns in algebra. Closer to this study, Monson and Kevin (2001) designed a diagnostics system that assessed students' misconceptions in calculus, while Khazanov (2008), reported that instructors who targeted students' errors and misconceptions in teaching probability achieved better results in the resolution of learner misconceptions than instructors who used traditional instructional methods. This Khazanov (2008) finding is a prototype of the instrumental role that the use of learners' misconceptions has in improving the learning of mathematics. It highlights the potential of learner errors and misconceptions in raising learning and achievement to a new level.

The researcher argues that if students' flawed mathematical reasoning and thinking is not deconstructed, unpacked and unveiled to both teachers and learners for re-consideration and reflection, then little progress in learning will result, because teaching will continue to be misdirected away from the most important challenges facing learners. Teaching mathematics will continue to be irrelevant when it does not connect with learners' needs. Sometimes, it may be an appropriate teaching strategy for learners to be directed to reflect on their own errors and misconceptions or that of their peers. Consider Melis (2003), who argued that just as in computer programming where students learn by de-bugging their errors, the same approach helps if learners are exposed to wrongly worked mathematical examples and they correct them themselves. Consequently awareness of learner erroneous thinking empowers teachers and learners alike to apply their academic effort on really identified learning problems. Focusing on identified learning problems such as mathematical misconceptions learners harbour, feed-forwards the teaching and learning process. This occurs when teachers, learners and other key stakeholders address misconceptions in mathematics, firstly by anticipating them before they emerge; secondly when they elicit and accelerate their occurrence in learners so that they are promptly processed, and thirdly to alert teachers and learners of their propensity and how to deal with them should these occur. The researcher argues that these interventions; prophylactic or remedial, can make mathematics instruction not only economic but effective. Mathematics instruction becomes effective and economic
because it will be targeted at the real problems learners face in understanding mathematics. As problematic comprehension is tackled, the ground is prepared for learning more concepts more effectively. This is because a single misconception reproduces a cluster of errors (Nesher, 1987), since mathematical ideas are hierarchical and closely inter-connected. Therefore, practical knowledge of learner erratic thinking is a portent resource for leveraging teaching.

As has been alluded to in the above, the vast body of research in error analysis and students’ misconceptions in mathematics, computer science, natural sciences and language is evidence that researchers have long recognized the value of examining students’ errors in education (Confrey, 1990; Graeber & Johnson, 1991). Within these studies, which are consistent with a constructivist view of learning (Cobb & Bauersfeld, 1995; Hatano, 1996), errors are seen not only as natural, inevitable and integral part of learning but are also regarded as valuable sources of information about the learning process; providing clues that researchers and teachers should take advantage of in order to uncover current students’ knowledge and how they come to construct such knowledge. Constructivist perspectives on errors and misconceptions represent a considerable step forward when compared with Behaviourism; a learning theory perspective that recommended negative reinforcement to errors or ignoring them so that they become extinct (Skinner, 1984). Behaviourism posited that as errors were liabilities to learning they were to be routed out and eliminated. The researcher disagrees with this stance on the basis that learners’ errors often have roots in valid underlying understandings on which teachers must build rather than destroy. Completely destroying underlying understandings disturbs the learner who will then lose confidence in learning mathematics, because every misconception has a partially correct part and partially wrong part. I argue that the correct part must not be destroyed because of the wrong part; rather the incorrect part must be understood and integrated into learners’ viable cognitive schema.

Most teachers are unaware of the mathematical misconceptions held by their pupils. This evidenced by the absence of error analysis and diagnosis courses in the pre-service teacher education programmes at least in South Africa (Luneta, 2008). As a result, teachers are surprised to find how commonplace misconceptions are if prompted to probe for them in their learners by researchers. Mostly, teachers teach mathematics in line with the scope and
sequence of its logical structure, quite oblivious of the psychological standpoint from which learners ascribe their mathematical meanings (Nesher, 1987). This unfamiliarity with the frames of reference learners ascribe their logic constitute a critical weakness in pedagogy resulting in mathematics lessons being misdirected to learners’ needs. As a result students’ mathematical thinking is poorly supported since instruction rarely connects with their underlying logic which has some misconceptions. It is this inattention to learner conceptual meaning that Thompson (2008) identified as the root problem of mathematics teaching and learning. As argued, the central problem of ineffective mathematics instruction is exacerbated firstly by teacher unawareness and unfamiliarity with general or specific learner errors and misconceptions on key mathematics concepts and competences. If they do, the teachers cannot explain the underlying reasons to which learners attribute these errors (Nesher, 1987). This double challenge of unawareness of learner misconceptions and failure to interpret them indicates that in-depth study is needed if learners’ error making expertise (Nesher, 1987) is to be exploited by teachers for improving learning of mathematics. Nesher argued that teachers must embrace the fact that learners are experts at making errors. What is important is to take advantage of those errors and leverage them to help learners understand mathematics from their own perspectives. The researcher chose to do in-depth study of learner errors and misconceptions on the topic of Introductory Differential Calculus at grade 12 level.

The advancement of knowledge through error identification has been established in research. For example, Pierce (1958/1887) argued that errors are useful for the advancement of knowledge as they force people to reconsider and relook at the beliefs that influence their thought. Also Popper (1963) intimated that the duty of science is to seek errors, as error detection is the powerful strategy that drives the development of knowledge in most fields of knowledge including mathematics. Even the most eminent mathematics thinkers had errors in their most important results. For example, the famous Euclidian parallel postulate is disproved in spherical geometry (Eves, 1990). The implication drawn from the above is that what is perceived as valid knowledge could be a misconception.

An internet search in South Africa’s NEXUS research databases shows that error analysis has mostly been done in languages (see National Research Foundation (NRF), 2009). Research
based on error analysis of mathematics examination scripts has not yet been done. In this
vein, Luneta (2008) comments that it is surprising that research on error analysis in
mathematics (and physics) has not taken centre-stage given its potential to help educators to
identify students’ skills and knowledge acquisitions that would guide the teaching and
learning of these subjects. Also in South Africa, Engelbrecht, Harding, & Potgieter (2005)
failed to find enough evidence to support the claim that students entering South African
universities had more mathematical procedural understanding than conceptual understanding
of calculus as this was not a study on error analysis. But earlier, Olivier (1992) had
categorised prototypes of learner mathematics errors in South African high schools. Also in
South Africa, Brodie (2005) used the cognitive and socio-cultural perspectives of learning to
explain learner reasoning that causes misconceptions during classroom mathematics
discourses. International research involving South Africa; (see for example TIMSS, 1999,
2003,(Reddy, 2006)) focused on mathematics and science tests, and the performance results
were mainly used for statistical comparison purposes and not for analysing errors in learners’
scripts. In addition, TIMMS and EAA were mainly quantitative studies focussing on
learners’ achievement, quite different from this study that used mixed methods to study
learner errors and misconceptions in answering calculus examination questions.

In particular, all the above research did not do error analysis of students’ texts written under
the tension of high-stakes terminal examinations when mastery of all the accumulated
mathematics knowledge, competences and skills acquired throughout a student’s school
career is tested. This study investigated the nature of the epistemological obstacles faced by
students in responding to introductory differential calculus items in the South African NSC
2008 examination. Although many researchers report that students harbour many errors and
misconceptions in calculus (Artigue, 1996; Monson & Kevin, 2001; Orton, 1983a, 1983b;
Tall & Vinner, 1981), these studies have been done in countries different from South Africa.
Thus, I declare that my research problem here is; What is the nature of learner errors and
misconceptions of South African learners in the Grade 12 topic of Introduction to
Differential Calculus? The problem then concerns an analysis of the achieved curriculum on
the calculus theme against the examined curriculum, of the intended curriculum set out in the
NCS FET assessment standards Learning Outcome 2.
The researcher problematised learners’ errors and misconceptions in calculus and argued that if these are studied, their nature can be understood and known and that such understanding can produce new knowledge that can begin to shed light to boost the learning of calculus. The researcher further argues that before a problem can be solved, before calculus learning should be improved, the nature of the problem of learning calculus should first be understood through studying the errors that learners show in examination scripts. These must first of all be fully understood and their characteristics well documented before teaching of calculus can be effective.

1.11 Aims, objectives and purpose of the study

With respect to learners’ answers to calculus tasks in the 2008, South African National Senior Certificate examinations, the purpose of this research was to:

1. Determine the most common learners’ errors and misconceptions in learners’ examinations scripts.

2. Classify learners’ errors and misconceptions in answering Grade 12 calculus examination questions.

3. Formulate an analytical protocol for the identification of errors and misconceptions in introductory calculus grounded from the study.

1.12 Research questions

In view of the articulated aims and objectives above, the research sought to investigate the major research question and sub-questions spelt out below.

What is the nature of the most common errors that students display in answering Grade 12 mathematics examination questions on introductory differential calculus?
The sub-questions to this question were:-

(a) What are the types of errors made by learners in response to Grade 12 examination Differential Calculus tasks?

(b) How would the common errors and misconceptions the learners made be described?

(c) What typology of errors and misconceptions in calculus is grounded from the study?

1.13 Context and of the study

This research project is an offshoot of the Script Analysis Project (SAP) of the University of Johannesburg. SAP aimed to investigate the academic difficulties learners experienced in sitting for the National Senior Certificate (NSC) examinations of South Africa. This was done through the analysis of learners’ scripts for the 2008 NSC examinations. The year 2008 was significant in the South African educational landscape because that was the year when the first National Curriculum Statement based examinations were written. Also students did not do well in those mathematics examinations. In the university project, script analysis was done in six subjects namely; English, Computer Applications Technology (CAT), History, Physical Sciences, Life Sciences, Mathematics and Mathematical Literacy. The project analysed examination scripts for these six subjects to find out what can be learnt from them that could be communicated to stakeholders such as educators and policy makers in order to enlighten them about the teaching of these learning areas. Learner errors and misconceptions inherent in the scripts were also studied. My own research centred on analysing examinee errors and misconceptions in introductory differentiation.

Gauteng Department of Education [GDE] is one of the thirteen educational provinces of South Africa. By area, it is the smallest province in the Republic yet it contributes to more than 50% of the Gross Domestic Product. The Gauteng Provincial Department of Education (GDE) falls under the Ministry of Basic Education headed by the Minister assisted by the Director General in Pretoria. At the top of provincial education is the Member of the Executive Council (MEC) for Education. The Examinations and Assessment Board (EAB) falls under the MEC (Education) and is responsible for examinations and assessments in schools. The SAP research at the University of Johannesburg was funded by EAB.
The scripts used were from learners from different home backgrounds and social statuses. Learners' home languages were isiZulu, Pedi, Xhosa, Ndebele, Tsonga, Venda, Tswana, Sotho, Swazi, Indians, Afrikaans and English. But there were also other groups of European, Asian and other African migrants. The different social backgrounds of these learners and their effect on mathematical performance are not a subject of study in this research, although this might be interesting research on its own. I did not study the language and cultural impact on learners' errors and misconception per se although I commented on them in my recommendations. As only candidate numbers were used, it was not possible to determine the sex of the learners. Neither was it possible to determine the location or status of their schools as only centre numbers were used.

1.14 Limitations and delimitations of the study

In doing the study, a number of factors constrained the study. I discuss these and explain how I tried to negotiate and minimise them. Firstly, the main source of data was the examination scripts of Grade 12 learners obtained from Gauteng Department of Education (GDE), Johannesburg. These scripts were part of the University of Johannesburg, Examination and Assessment Board Script Analysis Project (UJ/EAB/SAP). However, only scripts from the national Department of Education (DoE), Senior Certificate Examinations (SCE) for the year 2008 were analysed in the research. Scripts from the Independent Examination Board (IEB) or other examination boards such as University of Cambridge Local Examination Syndicate (UCLES), General Certificate of Education examinations were not used in this study, although these examinations are also written in South Africa as Grade 12 exit examinations. That left a considerable sample (around 8%) of grade 12 learners work unresearched. This limitation would have been great in that students who attend independent schools mostly come from the upper middle classes and upper classes of South Africa, who historically perform well in mathematics. The sampling was random, in a case where stratified sampling would have been ideal. Despite these limitations in the sampling, the scripts in this study represented the mainstream South African learners (about 92% of the learners). Hence the results of this study took into account data from the bulk of the learners and therefore increased the reliability and validity of the study. Also being mainly a
qualitative study, the representativeness of the sample was not actually important. Hence, the issue of a non-representative sample no longer became an important limiting factor.

Secondly, time and resources were limited so that the study could be not longitudinal. This limitation occurred in that the analysis of the examination scripts was only done on the single 2008 cohort; incidentally the first year the NCS examinations were written. This however was not so negative an occurrence as the study also studied in some way the impact of the new curriculum in the learning of mathematics.

Thirdly, there was the constraint of a single researcher on the phenomenon of learner errors and misconceptions in calculus. As usual in most PhD studies, the student was the sole researcher, directed by his supervisor and to a limited extent by the doctoral committee. Also the literature sources reviewed made the researcher feel that he was being aided by other mathematics education researchers from across the globe. As such the researcher never really felt alone in this study. The pilot study done with the help of the Script Analysis Project (see Section 5.13) also increased the reliability and validity of the analysis of this study as four teachers who currently teach mathematics in Gauteng's high schools were involved in the error analysis.

Fourthly, the study concentrated on errors and misconceptions in introductory differentiation and no other topic, although other topics closely allied to differentiation and its applications, such as algebra and functions were studied in reference to calculus. The study concentrated on categorising the errors and misconceptions. The fact that the study was carried out solely on examination scripts implies that learners were putting their best efforts into answering the questions. Learners were assumed to have been sincere in their responses in order to be awarded better marks to earn a passing grade. This scenario, was therefore more likely to be conducive to generating reliable and valid data than when learners were given some other tasks not as high-stakes as the examinations. However, the strength in the type of data for this research could also have been its weakness in that learners wrote the examination under real examination tension. It is possible that some learners might not have given their usual performance due to the fact that they were cracking under examination pressure, and
forgetting how to think as they usually do in a more relaxed situation. However, such intervening factors always occur in qualitative research and have to be negotiated, since it can be very difficult to have optimal conditions for qualitative research. From the discussions given, I argue that the study was done in a way that minimised its limitations so that the validity and reliability of this research were not compromised.

This research was based on script analysis. Therefore it mainly used content analysis methodology. This limitation is a significant one though not insurmountable. It is brought about by the fact that the students who wrote the examination in 2008 had already left school by the time this study was launched in 2009. As such, it was impractical to hunt them down in order to identify and interview them. The lack of the interviewing method to collect data deprived the research of the students’ voices that could have illuminated this study a great deal on how students would explain the reasons behind their errors. This important limitation on data collection methods has however been addressed by linking the errors and misconceptions observed in learners’ scripts with literature. Learners throughout the world generally make similar misconceptions and errors in mathematics and calculus (Davis, 1984). To this end, the researcher obtained an extensive corpus of learner errors and misconceptions on calculus and how learners formulate them from literature. I obtained expert findings from literature on how and why learners commit errors and misconceptions in calculus and how researchers think about this phenomenon. However, what is new is to find the errors and misconceptions that learners in South Africa have and if they are related or similar to those common to the rest of the world. Moreover, the explanation of the causes of the errors is not really the major thrust of my study. My study had the focus of creating a protocol for South African educators to help with the determination of learner calculus errors and misconceptions. What is important is how learners think and what they think in answering calculus questions wrongly. This thinking can generally be reconstructed from what learners write. Therefore, the researcher believed that scripts produce and reveal a great deal about how learners think. I argue that through comparing learners’ responses, general patterns become clear, clear enough to enable me to come up with an analytical protocol applicable in teaching and learning calculus in the South African context and beyond.
1.15 Significance of the study

As argued in National Mathematics Advisory Panel (NMAP) (2008), mathematics education researchers need to be responsive to classroom challenges faced by teachers and learners through undertaking grounded research that produces scientific-based evidence helping to determine concrete decisions to improve learning and performance. In this regard, this study hoped to add expertise to mathematics education professional practice through investigating, examining, exposing and evaluating prominent and also unusual errors and misconceptions that come out of the Grade 12 mathematics examination on the topic differential calculus. It brought to light and explored the submerged errors and misconceptions in calculus that many educators, teachers and learners took for granted. It contributes to an awareness of what it is that learners are grappling with on calculus examination items. The study posits that research that problematises learner errors and misconceptions is learner-centred and holds promise for upgrading teaching and learning of mathematics in general and calculus in particular (Khazanov, 2008). One product of this research is a register and database of errors and misconceptions in calculus. This database is important in helping teachers to dispel learners’ mathematics misconceptions. Inferences from literature review will contribute to the nature of the errors and misconceptions that are found from the analysis. The results of the study will be available to education policy makers and educationists.

1.16 Conclusion

This chapter introduced the study. It contextualised the teaching and learning of mathematics in general, and calculus in particular, and hypothesised that learner errors are springboards of enquiry (Borasi, 1994) in the teaching and learning of calculus. The study of learner errors contribute to the growth of mathematics PCK fundamental to effective mathematics teaching. The researcher has argued that understanding the nature of errors and misconceptions in calculus is vital to teaching the subject more effectively than is presently the case in South Africa. In closing this chapter, I outline the structure of the rest of the study.
1.17 Structure of the research report

The structure of the study is as follows:

Chapter 1 Introduction and motivation of the study
Chapter 2 Theoretical framework
Chapter 3 Conceptual framework
Chapter 4 Literature review
Chapter 5 Research methodology
Chapter 6 Data analysis
Chapter 7 Conclusions, findings and recommendations

References

Appendices
CHAPTER 2: THEORETICAL FRAMEWORK

2.1 Introduction

In this chapter, this researcher presents a theoretical framework for this thesis. The theoretical framework specifies the researcher’s assumptions and beliefs about how the main ideas of the research may be viewed. The main concern of this thesis is learner errors and misconceptions in mathematics in general and calculus in particular. Also, the theoretical framework informs how and why learners interpret mathematics in ways that result in them making errors and misconceptions.

I start by sketching and reflecting on this theoretical framework. The word: theory derives from Greek *theorein* which means to gaze-upon (Klein, 1998, 2005). The term theory refers to ways of seeing things and how they may be explained. A theory is a system of interpretation focussing on certain events and putting others on the peripheral. So my theoretical framework discusses and gives rationale for my choice of what I regard as the main and most useful ideas impinging on my research which centres on learner errors and misconceptions and learning mathematics in ways that learners encounter these barriers. I discuss the features of the theoretical framework and justify its relevance in studying aspects of learner errors and misconceptions in calculus.

In discussing the theoretical framework, I first begin with philosophies on the nature of mathematical knowledge. Ernest (1991), Lakatos (1976), Polya (1973) and others have contemplated that the belief on the nature of mathematics that a teacher or learner has, produces a profound bearing on how they handle mathematical knowledge. This notion embraces one of my assumptions on the cause of learner mathematical errors and misconceptions namely, how a lack of total comprehension of learning of mathematics occurs and why learners make errors and misconceptions in mathematics. This is a most formidable task in education research.
Secondly, as regards the making of learner errors, I drew mainly from cognitive constructivism (Cobb & Bauersfeld, 1995; Hatano, 1996; Piaget, 1968; von Glasserfeld, 1990) as well its corollary notions of concept image and concept definition (Tall & Vinner, 1981), to explain the state of learner thinking and how they make sense of mathematical concepts. The theoretical framework also embraces the Activity, Process, Object, Schema (APOS) theory (Dubinsky, Assiala, Schwingendorf, & Cotrill, 1997) for learning calculus.

Although the foregoing frameworks are valuable in analysing error genesis in calculus, they do not offer sufficient guidelines on how to deal with the pedagogical and the learning challenge arising from working with learner errors and misconceptions. For the purposes of didactics to remodel errors and misconceptions presented from my analysis, I assume that Vygotsky's socio-cultural theory has the greatest potential for reducing them. It also explains the basis of misconceptions.

I discuss in depth, one by one, the above outlined theoretical underpinnings. I also discuss how in this study, they relate to each other as well as how they bear on analysis of learners' answers to calculus questions. Later, I discuss how these elements illuminate teaching, learning, and assessment of mathematics in relation to my research questions and methodology. These principles are discussed in depth in the following sections beginning with the philosophy of mathematics.

### 2.2 Philosophies of mathematics

Mathematics education scholars hold quite divergent views on the nature of mathematics (for example Ernest, 1995; Hersh, 1997; Lakatos, 1976; Russell, 1919). The study of the nature of mathematics falls in the realm of the philosophy of mathematics. The philosophy of mathematics is a reflection upon mathematics and does not add or increase the amount of mathematical knowledge per se; rather it gives an account of mathematics. While some scholars believe that mathematical knowledge has objective truth and its statements have universal validity as those in the Absolutist school (for instance, early Pythagoreans), others keenly dismiss that notion and argue that mathematics exists only in the minds of human beings and cannot exist apart from the intellects of people who created it (Hersh, 1997;
Lakatos, 1976; Polya, 1973). Between these extremist positions lie many other philosophies and orientations. Ernest (1991) regards the philosophy of mathematics as an important educational discipline that attempts to explain the epistemological and ontological basis of mathematics. Epistemology is regarded as the study of what knowledge is and how we come to know, whereas ontology concerns the study of the nature of reality; the nature of being (Ernest, 1991). The philosophy of mathematics then helps us to explore what mathematical knowledge is, what its nature is and how we come to know it. The philosophy of mathematics influences the type of mathematical knowledge to be included or excluded in the curriculum, and how it should be taught and learnt.

Ernest (1985) also agrees that there are two main movements in the philosophy of mathematics: Absolutists on one extreme; and Fallibilists on the other. The absolutist philosophies of mathematics include logicism, formalism and Platonism. These assert that mathematical knowledge is absolute, immutable and has its own existence independent of the knower. This group argues that mathematical knowledge is certain, objective and true, thus it is the duty of humans to discover its truth. However, the Fallibilists or constructivists argue that mathematics is a human creation, cogitated by man to solve the practical and theoretical problems met in his/her existence. Indeed mathematics is not perceived as the absolute truth. Being made by man it is prone to error as humans are by nature prone to error. Constructivist epistemology explains the nature of knowledge and how human beings construct that knowledge. So constructivism stresses that mathematical knowledge is flawed. Closely allied to Fallibilism is Intuitionism. In this research I assume the Constructivist perspective that mathematics is a fallible process, but at the same time I presume that some of the absolutist principles on the nature of mathematics are viable. I particularly assume that learning mathematics is a fallibilist process and that though mathematical knowledge is created by people, it is also absolute at times. If all mathematical results were fallibilist, then people would always get different answers to the same question such as $3x + 5x$.

2.2.1 Logicism

Logicism is the view that mathematics is a part of logic (Russell, 1919). Russell argued that mathematics can be reduced to simple laws of logic. As such, mathematics can be derived
from rules, axioms and postulates, from which results can be obtained through careful and rigorous and logical reasoning. However, Russell (1919) discovered that these simple logical laws were sometimes inconsistent and contradictory leading to the third crisis in mathematics. This crisis led to a revision of logicism as the nature of mathematics. To some extent, this view of mathematics is applicable in this study, particularly on the definition of the derivative although no axioms or postulates are involved. The basis of the derivative, and hence calculus emanates from rigorous reasoning, in the definition of the derivative that involves the very abstract concept of a limit. Calculus then is built on the foundations of logic being applied to functions that are geometrically represented.

2.2.2 Formalism

Critics of Formalism argue that, “Formalism is the view that mathematics is a meaningless game played with marks on paper, following rules” (Ernest, 1985, p. 606). To formalists, mathematics is a strictly formal logical system, without a concern for meaning. To formalists, axioms and postulates are the basis of mathematics, not contexts. The formalists deny meaning to all mathematics claiming that mathematics is all about symbols and rules for manipulating them. This philosophy of mathematics urges for form without substance and seems anathema to the teaching and learning of mathematics with understanding as we know it. No lasting mathematical learning can be obtained from such a philosophy where mathematics is studied independent of contexts and meaning. Mathematics problem solving is impossible with this philosophy. Teachers who are poorly educated in mathematics often subscribe to this philosophy of mathematics. Formalism can be compared to procedural teaching and learning, which is concerned with use of “rules without understanding” (Skemp, 1976) or use of procedures without connection to meanings or contexts (Stein et al., 1993). I argue that learners who have this philosophy of mathematics often harbour many misconceptions in mathematics because they hold a very superficial understanding of mathematics concepts. The researcher does not subscribe to this elitist view of mathematics.

2.2.3 Platonism

Ernest (1985) observes that Platonists (a term used after Plato, an important Greek classical philosopher) regard mathematical objects as having an ideal existence in some abstract world.
For example numbers are regarded as existing in their own somewhat metaphysical world. The Pythagoreans were early Platonists and regarded numbers mythically (Eves, 1990). Platonists discard as nonsensical the suggestion that human beings construct mathematical knowledge, rather, they argue, humans discover mathematical knowledge. Platonists emphasise a rigid and fixed body of knowledge as opposed to the dynamic nature of mathematical knowledge argued for by constructivist philosophers. Platonists argue that knowledge can be reduced to form different disciplines. Platonists then assume that mathematics is a product. Platonism has been the main philosophy of mathematics up to the 20th century despite the blows shaking it to its very foundations through the discovery of irrational numbers (first crisis of mathematics) (Eves, 1990), the invention of non-Euclidean geometries by Gauss, Bolyai, Lobachevsky and Riemann (second crisis of mathematics) and contradictions discovered when mathematics was reduced to laws of logic by Frege (third and greatest crisis of mathematics). Although there are still many scholars subscribing to Platonism, they encounter problems in explaining how mathematics should be taught; primarily in that this philosophy seems to imply that teaching mathematics ought to be through telling. If this view is to be of help to mathematics education, then it is necessary for teachers to diagnose what problems learners are meeting to acquire that fixed knowledge waiting to be discovered.

2.2.4 Fallibilism

Fallibilism is one of the latest philosophies on the nature of mathematics that counters the absolutist view of mathematics (Hersh, 1997; Lakatos, 1976; Polya, 1973). Fallibilism regards the socio-cultural and historical development of mathematics as paramount in explaining the nature of mathematics. Fallibilists argue that knowledge does not need perfect evidence. Fallibilists regard the development of mathematics as spurred on by the recognition that some long established results in mathematics had errors (Eves, 1990). A case in point is Pythagorean discovery that square root of 2 is not rational. This crisis clearly showed that results in mathematical knowledge are prone to errors even by the most talented mathematicians. Also Euclid’s postulate that the shortest distance between any two points is a straight line was rebuffed by Lobachevsky (Eves, 1990). For centuries, mathematicians held fast to this view as incorrigible and unquestionable until Lobachevsky introduced lunar geometry. Then this Euclid’s postulate was at once exposed. Similarly, Gödel’s theorems
proved that mathematics is not ontologically incorrigible and Russell (1919) showed that mathematical truths are essentially contradictory in the third crisis of mathematics. Radical fallibilists argue that mathematics is as fallible as science.

Fallibilism values how mathematical knowledge was historically created through informal means such as trial and error (see Realistic Mathematics Education (RME); Freudenthal, 1991; Gravemeijer, 1994; Treffers, 1991). The RME theory regards mathematics as an activity that humans create and have created to solve problems in their lives. It teaches that all mathematics teaching should emanate from realistic contexts that appeal to learners, so that learners come to realise mathematics as a useful tool for solving real problems. Fallibilism assumes constructivism as the way learners build mathematical knowledge. Fallibilists’ views of mathematics encompass ethno-mathematics and are increasingly powerful in the teaching and learning of mathematics, as regards equitable access to mathematics (Boaler, 1997). They take into account the historical and cultural identities of learners, which mean that learners’ diversities are respected. This research to some extend embraces Fallibilism and acknowledges learners’ mathematical errors acceptable in learning mathematics.

2.2.5 Intuitionism

This philosophy opposes Platonism. Intuitionists base mathematics on subject’s beliefs in the mind. They regard intuition as critical in the formation of mathematical proofs for example. Their argument is that mathematics exists in the minds of people. The intuitionists do not believe in the classical view of mathematics which they regard as quite unsafe. Intuitionists believe that no mathematical knowledge could exist beyond that which the mathematician has proved. I do not subscribe to this subjective view of the nature of mathematical knowledge. To suggest no mathematics exists beyond what one has proven is to ignore the social nature of mathematical knowledge where mathematical claims are subjected to rigorous tests by other mathematicians in a sphere of sharing knowledge. The importance of Intuitionism to this study is that often the errors and misconceptions learners make emanate from their intuitions. Intuition then is very important as it helps to explain how learners come to have alternative but sensible conceptions.
I presume that adopting intuitionism is good to begin with as teachers suspend the conclusive mathematical results while they wait for the learners to re-examine their intuitions before accepting publicly validated mathematical truths. Intuition in mathematics helps in the formulation of conjectures. From the above discussions, the nature of mathematics then lies in a continuum. From Platonists on the extreme right who hold/conclude that mathematics is an objective, immutable metaphysical truth on one end and on the other extreme; that mathematics is intuitive and that no mathematical truth can exist beyond what an individual has proved by himself/herself.

My position is that these philosophies of mathematics have their strengths and weaknesses as regards my study. I believe that proper teaching and learning of mathematics depends on contexts at hand and that at one time or another each philosophy of mathematics could help in the understanding of learner errors and misconceptions in mathematics.

2.2.6 Mathematics as mathematical practices

Besides the philosophies of mathematics outlined above, there are recent philosophies that argue that mathematics also concerns the practices that mathematics practitioners do in their daily work (Ball, 2002). These mathematical practices include problem solving, conjecturing mathematical results, proving and disproving them, communicating mathematically, argumentation, and representation as well as involvement in mathematical discourses with other mathematicians. The mathematics as mathematics practices stance argues that mathematics is composed not only of mathematical knowledge but these practices are also akin to the daily work of professional mathematicians.

In spite of the macro-epistemologies of what mathematics and mathematical knowledge is, there are yet more dissensions on what is the most useful type of mathematical knowledge. This dissension is between procedural knowledge, conceptual knowledge (Hiebert & Lefevre, 1986) as well as between instrumental and relational understanding (Skemp, 1976).
2.3 Theories of learning

This section discusses the theories of learning that I assumed affect learners’ difficulties in calculus. I consider the constructivist and socio-cultural theories and some bridging theories that shed light in the teaching and learning of mathematics.

Cognitivists (for example Sfard, 1997), regard learning as a matter of acquiring content – a notion that assumes that knowledge is an object which is separate from and can be apprehended by the learner. Knowledge takes objective form and exists outside of knowing agents and learning is the process of internalising this knowledge. Sfard (1997) refers to this as the acquisition metaphor for learning. Cognitivists maintain that acquired knowledge and understanding resides in the minds of knowers. How knowledge is acquired is different according to the main cognitive theories of constructivism (for example Piaget, 1968; Siegler, 1995; von Glasserfeld, 1989) and socio-cultural perspective (Vygotsky, 1978). In particular, the socio-cultural theory regards knowledge as historical and cultural. Knowledge is transferred to learners through semiotic mediation during social interaction in the classroom set up for example.

2.3.1 Constructivism

Constructivism posits that knowledge is constructed by each person through cognition driven by mental self-regulation. As the individual encounters circumstances that are at variance with their current understanding, they develop tension and anxiety, called cognitive conflict (Piaget, 1968). This perturbation of thought drives learners to explore this problem in relation to their prior understanding. This revision forces the learners to think in order to reconcile and settle their disturbed state. Thus while the socio-cultural perspective regards knowledge as apriori and historical (Vygotsky, 1978), constructivism assumes that knowledge is fallible and subjective as constructed by the person (von Glaserfelt, 1989). In cognitive constructivism, social factors are peripheral to the learning process. The key mechanisms for learning and development in constructivism are assimilation, accommodation and equilibration (Siegler, 1995). Assimilation is an active incorporation of experience into a representation already available to the child. Yet that assimilation is not always
unproblematic. Misconceptions that occur because of assimilation essentially emanate from over-generalising. When a learner meets a new mathematical object, he/she might hurriedly think that the mathematical object belongs to a class he/she already has. The learner then assimilates the new mathematical object in the existing class, that is, operates on it as he/she does objects in that class. This could well be a misconception of inclusion as the object might have subtle features that make it distinct from the class of concepts that the learner wishes to include it. It is necessary that the learner becomes aware that the mathematical object does not belong to the class he/she puts it. This creates a discrepancy. When the discrepancies between a new concept and the child's current cognitive structure become too great, and it is important for the learner to recognise this, the learner will actively reorganize his or her thoughts through accommodation.

Cobb & Bauersfeld (1995) argue that learners' construction of knowledge occurs at mainly two levels. Firstly, knowledge is constructed through the use of a learner's prior mathematical knowledge. Secondly, knowledge is constructed through social interaction that generates contradictions. It is highly unlikely that an individual on his or her own will be dissatisfied with his or her existing knowledge (Hatano, 1996), unless influenced by some external force, such as discussion with other individuals or reading something that does not fit into his or her conceptual structure. From this perspective, contradictions between learners' existing understanding and what the learner experiences gives rise to disequilibrium, which, in turn, leads the learner to question his or her beliefs and to try out new ideas. Disequilibrium forces the subject to go beyond his current state and strike out in new directions. In this vein, Cobb & Bauersfeld (1995) explain that;

...inter-personal interaction triggers socio-cognitive conflict between learners, which, in turn, gives rise to individual cognitive conflict. As learners try to resolve these conflicts, they become aware of their activity and construct increasingly sophisticated systems of thought.

(p.7)

In this way cognitive structures accommodate experience; that is learning.

I believe that the constructivist perspective (Hatano, 1996; Jaworski, 1994; Piaget, 1968; von Glasserfeld, 1990; Siegler, 1995; Smith et al., 1993) is useful to explain and better predict how learners conceive of mathematical ideas including misconceptions. Ultimately,
understanding mathematics begins with learning basic facts, concepts, principles and computational procedures. This understanding is cognitive in nature, so it is fitting to refer to theoretical underpinnings of investigating students' errors and misconceptions in terms of cognitive factors and theories. The researcher argues that errors and misconceptions occur in the cognitive realm because it is the mind that processes information.

The constructivist perspective of learning postulates a learner's mind as the primary unit of learning (Piaget, 1968; Siegler, 1995; von Glasserfeld, 1989). Von Glasserfeld (1989) posited that in constructivism, knowledge cannot be transferred from the teacher to the learner but that knowledge is actively constructed and mediated by a thinking agent and that the chief role of the agent is to adapt and reorganise the experienced world in and with his/her mind. Constructivism negates the belief that learners are *tabula rasa*; or empty slates which are bare of any knowledge (Smith et al., 1993), which is the duty of the teacher to supply the missing knowledge. Learning is viewed as the capability of the individual to change his or her conceptual structures in response to perturbation. Although Piaget is regarded as the father of constructivism, it was scholars like von Glasserfeld (1989), who elevated constructivism to its present pre-eminence in education. Von Glasserfeld postulated radical constructivism which stressed that knowledge can only be actively constructed by an individual and that coming to know is an adaptive process, not about coming to know a pre-existing world. To von Glasserfeld, knowledge is not a commodity residing outside a person. Knowledge cannot be transmitted to the learner even by articulate verbalising and explanation. The teacher can only assist the learner by providing physical or mental models on which the learner can impose and abstract mathematical meaning by him/herself. The constructivist approach urges for empathy on the part of the teacher; to be able to regard ideas in a way that the learner would regard them. This outlook underlined this study, which sought to study errors from the side of the learners.

As referred to earlier, at the heart of the constructivist tenet is the argument that knowledge cannot be transferred *undigested*, from teacher to learner; rather it must be restructured and re-organised by each individual learner to construct meaning for him/herself. In other words, mathematical knowledge cannot be turned over from the teacher to a learner in a manner for example, that audio-visual text is exactly copied from one CD to another. Learners have to
construct their own meanings for ideas communicated to them by negotiating connections between new information and their prior knowledge.

This position competes with the Behaviourist (for example Skinner, 1984); Situated (Lave & Wenger, 1991) and Socio-cultural (Vygotsky, 1978) perspectives of learning. Constructivists argue that to understand is not about being told but about inventing/discovering knowledge for oneself. In contrast Behaviourism postulated that learning occurred in a stimulus-response environment influenced by intrinsic and extrinsic rewards due to learners. This perspective of learning has now been discredited (e.g Piaget, 1968) as too simplistic and mechanistic to account for the complexity of internal processes that occur in learners' minds as they learn. The situated perspective on the other hand argues that learning is not about knowledge acquisition as knowledge is not a thing (Lave & Wenger, 1991). To Lave and Wenger, knowledge is participation in a 'community of practice' (p.56). Learning in this perspective is increasing participation in the community's practice through a process they call Legitimate Peripheral Participation (LPP). Although these theories have important attributes to help in understanding learners' errors and misconceptions, the researcher thinks that constructivism helps researchers a great deal to understand learner errors and misconceptions than the situated perspective of learning. The socio-cultural perspective also is very helpful in that problems in learning and teaching can be understood better if teachers can properly assess the ZPD of their learners in which learners are likely to make errors in the absence of a more capable peer.

Constructivism views learners as actively construct individual ways of knowing and using their prior acquired knowledge via external social negotiation (Cobb & Bauersfeld, 1995; Hatano, 1996). Herein the individual mind is regarded as the primary unit of analysis for learning (Confrey & Kazak, 2006), but which is aided to learn via an external but auxiliary and subordinate social sphere that generates cognitive conflict and provides validation of thought processes. Cognitive conflict occurs when learners become aware that some of their mathematical concepts differ from the expert concepts held by teachers or in mathematics textbooks. This triggers learners to want to close the gap between their current knowledge and targeted concepts by the processes of equilibration and regulation mechanisms. In order to do this learners must struggle to span the gap between the insufficient and inadequate
concepts they have and desired concepts through building understanding of intermediate concepts. The constructivist perspective argues that a teacher cannot cause a learner to learn or transfer knowledge to passive learners; rather a teacher is like a candle that lights up another which then burns with its own fuel.

To sum up, learners acquire mathematical knowledge through construction of more powerful mental structures, concepts or logical structures (Sfard, 1997). Knowledge is restructured after construction through differentiation or integration of old and new ideas (Hatano, 1996). Prior knowledge structures become more logically co-ordinated and organised. Understandings that come with construction are quantitatively and qualitatively richer than before. Learners gain insight that in the final analysis, certain seemingly unrelated mathematical ideas share a fundamentally similar structure; and that seemingly comparable ideas are quite different. This restructuring and conceptual change is often gradual and permanent. This occurs with the conciliation of and growing out of the misconceptions by the learner. As Cobb, Yackel, & Wood (1992) argue, “learning could be viewed as an active, constructive process in which learners attempt to resolve problems that arise as they participate in the mathematical practices of the classroom” (p. 48). Such a view emphasises that the learning-teaching process is interactive in nature and involves the implicit and explicit negotiation of mathematical meanings.

Learners construct knowledge through interpreting, reorganising (differentiating and integrating ideas) and by refining new knowledge in the same grain as their existing knowledge (Confrey, 1990). The resultant knowledge is not only quantitatively more and qualitatively richer and deeper; it is also more subtle and more powerful (van de Walle, 2004). Knowledge-bits that previously appeared disparate and dissimilar become interconnected and integrated. Hatano (1996), for example, notes that the emergence of group theory in mathematics presented a logical structure that integrated and unified many mathematical concepts that had earlier appeared quite distinct. In the same way, as learners’ cogitation on mathematical concepts and mathematics becomes more focused, resolved and distinct differentiation of some concepts occurs. As a consequence, the importance of the constructivist perspective of learning to this study lies in that errors and misconceptions are perceived natural by-products of individual learners’ cognitive efforts of sense making in the
face of mathematical challenge. From this perspective then, misconceptions or misconstructions cannot be avoided when learners do mathematics. As a result, it is important to understand their nature in order to then minimise or repair the barriers and obstructions misconceptions present in the learning of mathematical concepts. Constructivism presents the researcher with a theoretical framework to interpret and examine how and why learners construct errors and misconceptions. This is crucial in mathematics education research in that one cannot predispose learners to lasting solutions to the problem of learning mathematics if one does not know how and why these are occurring at a micro-level in the learner’s mind, concept by concept.

2.3.1.1. Constructivism and misconceptions

Many scholars, for example, Smith et al. (1993) and Nesher (1987) argue that there is a strong link between constructivism and learner mathematical misconceptions. Ernest (1995) writes, ‘Constructivism accounts for the individual idiosyncratic construction of meaning, for systematic errors, misconceptions, and alternative conceptions in the learning of mathematics’ (p. 2).

Confrey & Kazak (2006) argue that teachers do not teach learners the misconceptions they make, rather learners make the misconceptions by themselves. He writes “misconceptions are taken as the strongest pieces of evidence for the constructive nature of knowledge acquisition, because it is highly unlikely that learners have acquired them by being taught” (2006, p. 201). However if the teacher has a misconception, then the teacher can easily pass that misconception to the learner. Constructivism explains errors and misconceptions in that when learners commit them, they occur because of the way learners think about concepts and not because learners are careless (Osei, 2000).

Confrey (1990) has urged that since learners construct their misconceptions, teachers should not attempt to weed out and replace the misconceptions as doing so will disturb the learners in the good concepts they may have that they generalise wrongly to form misconceptions. Rather it is important to help learners to relook and refine their conceptions. Conceptual change is linked to misconceptions because once a learner recognises that a concept be/she
has, has a limitation, the learner is likely to be alerted/poised to correct the misconception in order to resolve it. However, this is not always the case as sometimes learners never resolve their misconceptions in spite of new evidence to the contrary.

According to Piaget (1968), children come to understand mathematical knowledge not through memorising external rules but by internal construction through their own natural thinking. His stage theory explains that some children may not be able to understand certain concepts because of their level of cognitive development. However since all learners in this study are well in their late teens, they are most probably in the formal operations stage and so their misconceptions may not be due to levels of development prior to formal operations. Thus constructivism helps educators to understand that learners' consistency in their formulation of misconceptions is likely to reveal a pattern in their thinking; and that the pattern may not be immediately apparent to the teacher. Analysis of learners' artefacts helps to reveal that erroneous pattern.

2.3.1.2 Concept image and concept definition

Following on the hypothesis that constructivism explains how learners build their mathematical knowledge (including misconceptions). The researcher presumes that Tall and Vinner's (1981) constructs of concept image and concept definition help to give more insight on learner errors and misconceptions particularly in calculus. These scholars referred to this theoretical framework in their own studies (Tall and Vinner, 1981) on students' errors in differentiation and integration. By concept image, Tall and Vinner, refer to a cognitive chunk of ideas the learner has formed in his/her mind regarding all aspects of a specific concept. This is the framework which an individual learner has created and develops as a result of personal experiences with a particular concept. According to Tall and Vinner (1981), the concept image consists of all the mental pictures together with the set of properties which are associated (in an individual's mind) with a given concept.

The concept image is an inner model of reality constructed by the learner as a result of experience with a particular concept. The concept image is likened to Piaget's schema. The concept image is a picture of a concept constructed by the learner to refer to an identified
concept. Concept images then can be correct, partially correct or erroneous, and they are a function of maturity and experience with the concept. So concept images are dynamic and constantly changing as the learner thinks about the concepts.

The perspective of learning as a personal construction of meaning implies that learners build and modify their existing concept images. In the context of this study, a concept image can be considered as modifiable internal structure that is used and evaluated as students interpret and analyse mathematical situations. This means that an individual’s concept images have a fundamental function in learning mathematics.

Tall and Vinner (1981) argue that as individuals are bound to construct varying ideas about a certain concept, a concept image may have competing and contradictory ideas. They argue that “it is not always pure logic which gives us insight, nor is it chance that causes us to make mistakes” (p. 77). By this they mean that unconscious concept images may be responsible for the errors learners commit. The concept images constructed by an individual could differ with formal mathematical knowledge held by the wider mathematical community. The consequences are misconceptions or amateurish ideas that do not help learners to pass if they occur in examinations.

The presence of misconceptions in concept images present obstacles to the construction of new mathematical knowledge in that learners may construct knew mathematical knowledge by interpreting new information with a defective framework. This leads learners to generate a chain of misconceptions that are bound to crash in a chain reaction of cognitive conflicts – mathematics becomes an enormously meaningless body of symbols. However, the concept image might be virtually non-existent in some individuals when they have never met the concepts before; in others it may not be coherent.

Tall and Vinner (1981) noted that the brain sometimes contradicts itself. Thus, stimuli can activate a certain part of the concept image that can be called an evoked concept image. Sometimes, different evoked concept images can occur simultaneously resulting in a learner having unexplained confusion as he/she feels that something is amiss. In such a case, the
student might not realise that the evoked concept images were in conflict with each other. In this case, a student may have a partial understanding of a concept. The concept image can be compared and contrasted to the concept definition.

Tall and Vinner (1981) refer to concept definition as the words specifying the concept as defined by the mathematics experts. The concept definition can be learnt instrumentally or relationally. So personal ‘concept definition’ (concept image) may differ from formal concept definition. The concept definition refers to the outer reality of the learner. Some teachers teach limited concept definitions and learners suffer in that they build incoherent and partially correct concept images. What is required is varying representations or models of the same concept so that learners form a complete picture of the concept in their minds. Partial concept definitions cause learners to form concept images that do not match the concept definition. So to a learner, the concept definition is outer reality that the teacher wants him/her to understand, whereas the concept image is the inner model of that reality, which oftentimes is a misconception of the outer reality. For example a learner might have the concept image that calculus is extended algebra because; he/she notes that calculus is communicated in algebraic terms. Such a learner’s concept image of calculus is quite simplistic and differs from the concept definition that calculus is an analytic method for investigating the rates of change of phenomena.

According to Tall and Vinner (1981), a concept image or concept definition that is likely to conflict with another concept image or definition is a potential conflict factor. These may develop into cognitive conflict factors. Cognitive conflict factors occur when potential conflict factors are brought to the fore simultaneously, yet when considered separately, they are seen as distinct and never seen as conflicting. Potential conflict factors explain the different misconceptions learners have of the same concept. It is an important goal of this thesis that eventually this work will bring to light the potential conflict factors in calculus so that these factors can be resolved and so advance calculus teaching and learning.

Sometimes cognitive conflict factors are stimulated subconsciously so that the conflict only manifests itself in unexplained awkwardness (Tall & Vinner, 1981). If two concept images do not differ between themselves but differ with the concept definition, a learner might not
experience cognitive conflict and the two may exist side by side. This results in a situation of stability where the learner does not feel the need to learn the new material. These factors would be a barrier to the learning of a formal theory, since such students are comfortable in their interpretations; they would regard concept definitions as not needed. These ideas are noted on calculus concepts, where the different concept images that learners have do not conflict with themselves. In this case learners for example might have learnt algebra and regard calculus as unimportant and unnecessary. The awareness of the teacher of the potential conflict factors in calculus – that is stable misconceptions, is important as what is needed is to make them conflict with each other or conflict with concept definition so that learners feel the need to regain equilibrium. Defective concept images can also surface through discussion.

As discussed, misconceptions occur because of variations between the concept images and concept definitions. If learners are aware of the concept but have not constructed the concept, they will only understand the concepts operationally, called ‘weak constructions’ (Jaworski, 1994). Hence, in unfamiliar problems where these concepts will be implicit and required to solve the problem, these learners are likely to be out of their depth.

I have presumed that errors and misconceptions in mathematics occur because of flaws in knowledge acquisition. This may lead to inadequate or defective concept images held by the learner. Mathematics errors and misconceptions occur when learners fail to incorporate or acquire factual or conceptual knowledge into their concept images. The genesis of errors and misconceptions and their perpetuity is due to current inadequate or defective concept images, where the current concept images are not congruent to global and formal expert concept definitions. Mathematical knowledge acquisition is the transformation of knowledge from the forms in which it exists (for example contexts, texts and mathematics teachers’ heads) into forms that the learner can understand and use.

An individual's schema is the totality of knowledge which for her or him is connected (consciously or subconsciously) to a particular mathematical topic. An individual will have a function schema, a derivative schema, a group schema, and so on. Schemas are important to the individual for mathematical functioning say why/ more explicitly. The APOS theory, also closely connected to constructivism is discussed next.
2.3.1.3 Dubinsky’s APOS Theory

According to Dubinsky, Assiala, Schwingendorf & Cotrill (1997), formation of concepts occurs first as an activity, then a process, an object and lastly a schema (APOS). Dubinsky et al. argue that when students learn a concept it at first occurs as an external activity to which the learner may attach little meaning. At the second stage, the learner can view the concept as a process or a procedure semi-external to him/her. At the third stage, the learner now views the concept as an object internalised in his/her mind. Eventually, at the fourth and last stage, the object is incorporated in the learner’s schema, a broader mental picture in terms of Piaget. There is general agreement amongst mathematics educators that activities and processes precede the formation of mathematical concepts (for example Dubinsky, et al., 1997). Gray and Tall (1994) advanced the notion of a ‘procept’ as the hybrid and amalgam of a process and a concept represented by a single symbol.

The APOS loop is crucial in the learning of mathematics and any problems that may occur in this loop do not augur well for the learning of mathematics. Such problems may result in errors and misconceptions. For example, finding the gradient of the tangent to a curve is a process as well as an object. The process of finding the limit and the derivative can be disjointed, although one emanates from the other.

Stein, Smith, Henningsen, & Silver (1993) argue that doing mathematics tasks which involves conceptualising mathematical ideas must be taught in contexts first as this helps learners to develop mathematical meanings before learning procedures. This helps to form the mathematics concepts which are actually mental objects. Doing mathematics tasks provides the child with physical and mental objects upon which mathematical meanings and concepts would be imposed. In essence, once the mathematical concepts have been abstracted from reality through actions and processes, learners can now hold them as generic mental objects which they can apply to other practical and theoretic positions quite different from the concrete situation that bore them. The objects become part of schemas.
2.3.2 Vygotsky's sociocultural theory

Besides cognitive constructivism, the researcher presumed that one of the main theories explaining the learning and therefore the forming of misconceptions in mathematics is sociocultural (Vygotsky, 1978, 1986). The researcher believes that learning mathematics occurs interactively in the social realm as well as in the psychological realm (Sfard, 1997). In this regard the researcher now discusses the features of the socio-cultural perspective and how this informs children's learning.

The socio-cultural perspective postulates the social origin of cognition (Cobb & C. Baueserfeld, 1995; Wertsch, 1979). Vygotsky (1978) suggested that the socio-cultural and historical conditions drive individual learning by providing what is to be learnt and how it will be learnt. He argued that children's learning is the result of their induction to the cultural knowledge that society has accumulated for thousands of years. Essentially, it is suggested that the same thinking occurs at two levels, at the social and at the individual planes. Firstly, learning occurs as a result of "other regulation" on the inter-psychological plane. The inter-psychological plane consists of learners discussing and sharing knowledge during social interaction such as in mathematics lessons. Secondly, learning occurs as a result of "self-regulation" on the intra-psychological plane. The intra-psychological plane occurs in an individual learner's head as she/he integrates and manipulates newly acquired knowledge in order to integrate it with the prior knowledge that the learner already has. Then the learner interprets new knowledge through the lens of what he/she already knows. At the intra-psychological plane, the child re-organises his or her old knowledge in the light of social experience. The individual mental functioning is not a direct copy of the social, as it involves personal meaning. In this Piaget and Vygoysky concur.

Besides the above positions, Vygotsky (1978) contended that learning is assisted or mediated by a More Knowledgeable Other (MKO) through signs, tools and cultural artefacts such as language in a learner's zone of proximal development (ZPD). According to Vygotsky (1978), the ZDP is "the distance between the actual developmental level of the learner as determined by independent problem solving and the level of potential development as determined
through problem solving under adult guidance, or in collaboration with a more capable peer” (p.86). Teaching and learning via the mediation of tools is referred to as semiotics.

As Vygotsky (1978) has argued, the ZPD constitutes concepts and skills that a learner can understand and acquire with the assistance of a more knowledgeable other (MKO) such as teacher or peer, but cannot grasp alone unaided. The process of helping a learner to be able to begin to master concepts and skills within his/her reach is called scaffolding (Vygotsky, 1978). It must be emphasised that the more knowledgeable other is the more competent. If the more knowledgeable other is not competent, he/she cannot appropriately discern the learner’s ZPD and can actually teach the learner mathematical knowledge that is full of misconceptions. The researcher would like to compare such a teacher with an untrained surgeon or untrained pilot given a professional certificate. The MKO must be in a position to assess the level of understanding of the learner as well as his or her misconceptions so that these can be addressed through the intervention of a more competent other. The MKO other then must be able to assess the most suitable means to best scaffold the learner. Semiotics, which is the use of tools and symbols to support teaching and learning makes scaffolding more effective on the inter-psychological, social plane between the teacher and learner in the classroom for example. In this vein language is seen as both a sign and a tool important in semiotics. Later, as described above, the learner reorganizes his/her knowledge in his/her mind as he/she reflects on instruction received. Social conditions and the learner’s mind converge in his/her zone of proximal development. This is because the zone of proximal development is a learner’s psychological space and the teacher or peer mediates to the learner society’s treasured knowledge during social interaction with the aid of language for example. Learning thus takes place in a psycho-social sphere. As such, Wertsch (1979) and Vygotsky (1978) claimed that higher mental processes have social origins, and that mental processes can be understood only if we understand the tools and signs that promote learning through semiotic mediation. As learning occurs and learners master the old ZPD$_n$, a new ZPD$_{n+1}$emerges, so that with progressive sequences of ZPDs, learners become more and more knowledgeable in a discipline.

As Vygotsky’s theory suggests, the interaction with the members of a particular culture gives rise to higher forms of mental development. Thus, this interaction with mathematics teachers
and the mathematics curriculum gives rise to higher understanding of mathematics. Also, as Osei (2000) notes, knowledge cannot be transferred to the learner through linguistic communication; language can be used as a tool in a social sphere in the process of guiding the students' construction of that knowledge.

So the socio-cultural perspectives imply that the role of a teacher in teaching mathematics ought to be that of a knowledgeable guide who helps learners to negotiate their ZPD for their development and progress in mathematics. I argue that it is in learners’ ZPD where misconceptions occur and the teacher needs to be especially alert to the learners’ misconceptions and correct these swiftly. Learners’ ZPDs then must be accurately and appropriately determined to identify the prior knowledge that the learner would use in their intra-psychological plane. It arms the teacher to develop appropriate scaffolding strategy that can be implemented in support of the learner. The analysis of learners’ spoken or written texts provides (as in this study) a situation for teachers to determine the state of a learner’s ZPD. The importance of the socio-cultural perspective of learning to this study is that it will be called upon to infer why learners have errors in introductory differential calculus.

Vygotsky’s concept of the ZPD implies that error analysis should be built into any teaching programme, so as to assess the strengths and weaknesses of a learner on a particular topic or concept. One of the implications of Vygotsky’s work is that socio-cultural contexts facilitate, as well as constrain a child’s learning (Vygotsky, 1978). If the teachers can assess learners’ errors correctly, then the teacher will be able to determine teaching techniques that facilitate learning. However, if the teachers cannot appropriately assess these errors and difficulties, then he/she constrains the learning of the child. The knowledge of teachers about their learners’ thinking thus can make the difference between success and failure. Also if the teacher is knowledgeable he/she can facilitate learning of that knowledge, if however the teacher has little knowledge, then that teacher constrains the learners’ progress. In this situation the researcher refers to content knowledge and pedagogical content knowledge of the teacher.
2.4 Conclusion

This chapter outlined the main theories that framed and guided this research. In particular, the philosophies of mathematics were discussed and it was hypothesised that the Fallibilist philosophy is the most appropriate in studying learner errors and misconceptions in mathematics. This is because Fallibilism sees mathematics as a humanistic activity, a continuous thinking discipline rather than a finished product, which must be drilled into learners’ heads by a knowledgeable teacher. Thus, Fallibilism embraces errors and misconceptions as acceptable phenomena in learning mathematics with the power to push mathematical knowledge forward. I have come to the conclusion that constructivism best describes how learners personally think when forming misconceptions. Constructivists go on to say that the teacher, instead of looking for a simple, short, straightforward path to student success, encourages the exploration of the potential pitfalls and misconceptions with the aim of developing broader, more resilient concepts. The constructs of concept image and concept definition helped to explain that learners form different ideas of the concepts they learn at school. If images that learners form of the concepts learnt do not match reality, they can perpetuate further errors and misconceptions if they are used to interpret and learn new concepts. These errors and misconceptions need to conflict with formal mathematical knowledge if students are to progress otherwise students will be blissfully ignorant and see no need to do away with their misconceptions if they are not unchallenged. Vygotsky’s theory complements constructivism by implying that errors and misconceptions could occur in a learner’s ZPD. For teaching to be effective it is necessary to appropriately assess the learners’ ZPD. This assessment is best done by analysing the errors that learners show in their artefacts such as written answers to mathematics questions.
CHAPTER 3: CONCEPTUAL FRAMEWORK

3.1 Introduction

The conceptual framework of this study concerns the constructs that will inform, first the analysis of the examination items as in the NCS 2008 Paper 1 question paper, and secondly analysis of learners' responses to the introductory calculus examination items. According to Miles & Huberman (2004), the conceptual framework of a research study explains "either graphically or in a narrative form the main factors, constructs or variables – and the presumed relationships among them" (2004, p. 18). The conceptual framework helps to put information in convenient categories for the purpose of analysing the data. In this study, this conceptual framework (see fig. 2) establishes what is to be studied and what data will be collected. This framework is informed by the work of several scholars. For the purpose of analysing the cognitive demand of examination questions, the framework was informed by the Revised Bloom's taxonomy (Anderson & Krathwohl, 2001), the Structural Observed Learning Outcome (SOLO) Taxonomy (Biggs & Collis, 1982; Collis & Biggs, 1983), and Stein, Smith, Henningson, & Silver's (1993) cognitive demand of learning mathematics task descriptions. The reason why I refer to different conceptual frameworks is that I need to draw on the strength of all three, while taking into account their limitations. When analysing the learners' scripts I draw on the work on learner mathematical errors and misconceptions from many scholars. However the main ones are Donaldson (1963), Hirst (2003) and Movshovitz-Hadar, Zaslavsky, & Inbar (1987).
The conceptual framework of the study is discussed in Fig. 2 below.

**Figure 2. Conceptual Framework for the Study**

The different aspects of the framework and their importance is explained and discussed in the following sections.
3.2 Bloom's taxonomy

Bloom's Taxonomy (1956) assumes categories representing an hierarchy of cognitive demands of educational tasks. A revised edition of Bloom's Taxonomy, the Revised Bloom's Taxonomy (RBT) (Anderson & Krathwohl, 2001) recognises the limitations of the original Blooms' taxonomy in that the categories stipulated are not strictly in hierarchy as assumed. In the old Bloom's taxonomy evaluation was the highest level of cognitive operations. In the revised one, 'creating' is the highest having swoped position with evaluation. The revision has changed the hierarchy of the cognitive processes to; 'remember, understand, apply, analyse, evaluate, and create', and also suggests a new axis of knowledge types: 'factual, conceptual, procedural, and meta-cognitive'. This has resulted in six cognitive processes with each having the four dimensions thereby creating a 6 by 4 matrix of cognitive processes and knowledges. In this 24 entry matrix, the revised Bloom taxonomy describes the targeted cognitive process and knowledge levels for educational materials, (see Fig. 3).

**Figure 3.** The Revised Bloom's Taxonomy
According to Anderson & Krathwohl (2001, p. 12), remembering encompasses recognition, recall and naming. Understanding refers to explaining ideas, interpretation of situations and classification of ideas. Applying concerns use of acquired knowledge to solve routine and non-routine problems, while analysis involves differentiating, integrating, constructing and deconstructing of ideas. Still higher, evaluation involves justification of courses of action as well as assessing evidence to make judgments. The highest level is creating new ideas and new products, as well as generating new hypotheses.

The importance of the RBT to the conceptual framework of this study is that it is important for evaluating the level of cognitive demand at which the examinations items are set. This has a bearing on the cognitive level of operation that learners have when answering examination items and the level of misconceptions and errors they make. It is important to note that these levels of operation are not always distinct and that in most cases these are closely intertwined. It is also important to refer to this framework because it is the one that is used by Umalusi, the education quality assurer, and examination standards controlling authority of South Africa (see Umalusi, 2005). However, the shortfall of this framework is that it is generic to all educational disciplines and is not particular to mathematics. Therefore, its use for analysis to mathematical items is limited in that it does not take into account the nature of mathematical knowledge per se.

Another taxonomy complementing the Bloom’s taxonomy is the SOLO taxonomy.

3.3 The SOLO taxonomy

The Structure of Observed Learning Outcomes (SOLO) Taxonomy (Biggs & Collis, 1982; Collis & Biggs, 1983) (see Fig. 4), supplements the Bloom’s Taxonomy by analysing the cognitive demand of the assessment tasks. Biggs & Collis (1982) argue that Bloom’s taxonomy is not helpful enough in describing educational demands on learners as well as their performance against educational demands. The taxonomy consists of two major categories each containing two increasingly complex stages: surface and deep. The surface concerns uni-structural and multi-structural responses. The deep has relational and extended abstract responses. According to Biggs & Collis (1982), the taxonomy makes it possible in
assessing a subject to identify in broad terms the level at which a student is currently operating when responding to educational tasks. In plain language the SOLO taxonomy consists of four levels: one idea, multiple ideas, relating the ideas, and extending the ideas (Biggs & Collis, 1982, p. 39). Biggs and Collis argue that the purposes of the Structured Observed Learning Outcome (SOLO) taxonomy is to balance deep knowledge and surface knowledge when preparing students for high stakes examinations and deep use of knowledge.

Biggs and Collis (1982) based their model on the notion that:

...in any learning episode, both qualitative and quantitative learning outcomes are determined by a complex interaction between teaching procedures and student characteristics. (p. 15)

Each level of the SOLO taxonomy increases the demand on the amount of working memory or attention span. At the surface (uni-structural and multi-structural) levels, a student needs only encode the given information and may use a recall strategy to provide an answer. At the deep (relational or extended abstract) levels, a student needs to think not only about more things at once, but also how those objects inter-relate.

Figure 4. The Solo Taxonomy
The SOLO taxonomy then is useful in analysing at what level learners are operating and what level they are making their errors and misconceptions depending on the level of demand of the tasks. A learner who is thinking uni-structurally is likely to consider a mathematical concept in one dimension and therefore miss the other useful dimensions of a mathematical concept or problem. Similarly he/she may be multi-structured but still fail to relate what he/she is thinking about in relation to other concepts and ideas. These gaps in learners’ thinking levels provide gaps on which mathematical errors and misconceptions occur.

Although the SOLO taxonomy is useful in terms of analysing the cognitive demand of examination items; and learner responses to these, it also tends to be generically applied across many academic disciplines. As a result, it, like the Bloom’s taxonomy cannot appropriately analyse the level of cognitive demand or a learner’s response to mathematics items specifically. A framework more suited to mathematics is presented by Stein, Smith, Henningson, and Silver (1993) and will be discussed next.

3.4 The cognitive demand of mathematics tasks

This research heavily depended on the framework provided by Stein, Smith, Henningson and Silver (1993) to map the cognitive demands of the calculus items in the examination. Stein et al. (1993) argued that in implementing standards based curricula (NCTM, 1989, 2000) teachers must be mindful of the various cognitive demands inherent in mathematics tasks which they present to their learners. They argue that the tasks that a learner engages in while learning mathematics are paramount as they ultimately determine the competency that the learner will eventually have in mathematics. Tasks at the low-level end tend to be superficial and mechanical, whereas those at high-level tend to be linked to broader and deeper mathematical ideas, contexts and problem solving. Stein et al. (1993) argued that if learners are only exposed to low level tasks, there is no way they could learn to do high level tasks. For this reason, it becomes critical for teachers to engage their learners with high level tasks from the start as handling tasks whose cognitive demand is declined would occur naturally. Stein et al. proposed four levels of mathematics tasks. They described two lower level tasks; memorisation tasks and procedures without connection to concepts, to contexts or to
understanding tasks. The two higher level tasks were procedures with connection to concepts, contexts or to understanding and doing mathematical tasks. They argued that doing mathematical tasks are the same as problem solving as they demand complex non-procedural, non-routine and non-linear thinking to do these. The examples of those tasks are spelt out below (see also Fig. 5).

![Diagram](image.png)

**Figure 5. Stein et al.'s levels of cognitive demand of mathematics tasks**

According to Stein et al. (1993), memorisation tasks have the following characteristics:

reproduction of previously learnt facts, rules, formulae, or definitions or committing facts, rules, formulae or definitions to memory. Procedures without connection tasks use procedures specifically and require little or limited cognitive demand for successful completion and there is little ambiguity about what needs to be done. (1996, p. 35)

However, procedures with connection tasks focus students’ attention on the use of procedures for the purpose of developing deeper levels of understanding of maths concepts and ideas. The solution pathways could be direct or indirect and the procedures are broad as to be applicable to different underlying mathematical situations. Doing mathematics tasks can be
compared to problem solving where no clear solution path exists for the learner. These require:

Complex non-algorithmic thinking and there is no fixed predictable, well rehearsed approach or pathway suggested by the task, task instructions, or worked out example. They require students to explore and understand the nature of mathematical concepts, processes, or relationships. Demand self-monitoring or self regulation of one’s own cognitive process. Require students to access relevant knowledge and experiences and make appropriate use of them in working through the task. Requires students to analyse the task and actively examine task constraints that may limit possible strategies and solutions. Require considerable cognitive effort and may involve some level of anxiety for the student due to unpredictable nature of solution process required (1996, p. 36).

Stein, Smith, Henningson, and Silver (1993) and Suh (2007) have stressed the importance of using verbal representations, manipulative, pictorial, procedural and symbolic models to represent the same concept as varied models help learners to understand a concept from different perspectives. The researcher made the assumption that learners experience challenges in dealing with mathematics because learners often have very limited variations or even fixed views of a concept. When learners are required to work with other representations they might not have prior experience with, they tend to have misconceptions or they might have correct concepts in one representation and incorrect concept of the same concept in the other representation (see Erlwanger, 1973). As Stein et al. (1993) have argued, when teachers engage learners in uncomplicated and simple tasks, learners become accustomed to a diet of low level tasks, which does not help them to perceive mathematical concepts in a flexible manner.

Thus, learners who are made to engage in lower level tasks only understand mathematics in terms of memorisation and carrying out mathematical computations with little or no understanding. As a result they develop many errors and misconceptions when they encounter harder mathematics problems to which procedures will form an insignificant component in the solution. In more complex mathematical problems, it is often more important to figure out and try out what to do before an appropriate procedure is chosen.
Sometimes multiple procedures may be required. Blind application of procedures in this case leads to dead end and to errors and misconceptions. This is because procedures address 'what' and 'how' to do in specific situations and not 'why' what is done is done that way.

The framework used by Stein et al. (1993) gives insight on analysing the calculus examination question demands as set out by Umalusi, the examination board. The framework helps in explaining at what cognitive level the learners were being required to perform and at what level they were operating when answering the questions. This framework is thus of crucial importance to this study.

The central issue of classifying learners' errors and misconceptions is discussed next.

3.5 Classification of mathematical errors

Many scholars have attempted to understand the errors and misconceptions of learners in mathematics. This has led them to classify errors and misconceptions in order to understand them more deeply. Some researchers (Heid, 1988; Orton, 1983a, 1983b) have examined student errors as part of their investigation of students' conceptual and procedural understanding in calculus, but neither Heid nor Orton developed a classification system specifically for investigating calculus errors.

I argue that classification of mathematical errors is best done in at least two ways. The first type of classification is 'Domain General Errors' (DGE) which tend to be general error categories in mathematics and other subjects. This typology of errors is not discipline specific. This perspective assumes that there are underlying commonalities that exist across a diverse set of formal and everyday phenomena. On the other hand, there are Domain Specific Error (DSE) classifications which are subject based or topic specific. Both general and discipline specific error classifications were thoroughly investigated for the purposes of fully informing error classification in this research.
In addition, Davis (1984) proposed that errors were generally of two classes. He explained that there were those error patterns common across different learners and that there were also error patterns peculiar to individual learners. Both types of errors are considered in this study.

### 3.6 Donaldson’s error classifications

As early as 1963, Donaldson (1963) proposed three types of generic errors in mathematics that occur while students learn mathematics. Donaldson suggested error categories in mathematics that are very clear even half a century later. These are arbitrary, structural and executive errors. According to Donaldson (1963) these errors are frequent in mathematics teaching and learning. These error classifications are discussed below.

The first type of error is executive. These involve failure to carry out manipulations or procedures even though the required concepts have been understood (Orton, 1983a). Executive errors are related to procedural errors such as failure to properly carry out an algorithm. Errors such as failure to differentiate $3x^4$ by rule are executive errors. These may also be referred to as procedural errors. In this category lie errors such as failure to factorise expressions or expressing a variable as a subject of formulae. In calculus, we may refer to failure to do calculus because of algebra executive errors. Executive errors may occur with or without understanding of underpinning mathematical concepts.

The second type of error is structural. Structural errors are described as those “which arose from some failure to appreciate the relationships involved in the problem or to grasp some principle essential to solution” (Donaldson, 1963, p. 41). Structural errors can be compared to conceptual errors. According to Donaldson (1963), structural errors are due to mistaken perceptions about the nature of mathematical concepts or a fundamental failure to understand the relationship involved in the problem. There is also a failure to grasp some essential rule to a solution (Donaldson, 1963). For example in the functional notation learners required to compute $f(x+h)$ may expand it to $f(x) + f(h)$, thus completely missing the mathematical essence of the symbolism of the function that maps the point $x+h$ in the domain to a corresponding point in the range set. In this case, learners concentrate on the outward form and syntax of the symbolism and are unaware of the substance and semantics of the
mathematical expression. Therefore, such learners completely miss the point. These errors occur due to lack of understanding of target concepts. Structural errors are therefore fundamentally conceptual in nature.

Thirdly, Donaldson (1963) characterised arbitrary errors as occurring when the learner ignores part of the available information while acting on the rest. Arbitrary errors are said to be those in which the subject behave arbitrarily and fail to take account of the constraints laid down in what is given (see Table 3.1). Arbitrary errors could be caused by learners wanting questions to fit what they know or are familiar with. Hirst (2003) proposed more error classifications.

3.7 Hirst’s error classification

Hirst (2003) proposed a classification heuristically different from Donaldson’s classification of executive and structural errors. He also proposed three generic error categories: procedural extrapolation, pseudo-linearity and equation balancing errors. According to Hirst, these errors arise out of learners’ predisposition to generalise what they have learnt to new situations by way of assimilation and accommodation (Piaget, 1968; Siegler, 1995). Such naïve amateur constructions may be robust and resist instructions designed to correct them (Smith et al., 1993).

According to Hirst (2003) in procedural extrapolation, the learner attempts to extend a previously learnt procedure to new situations similar to one learnt in the past. Three examples involving differentiation, and two on integration, are given below.

When asked to find the first five derivatives of \( f(x) = e^{x-x^2} \) common students’ responses are:

\[ f'(x) = e^{x-x^2}; \]

\[ f''(x) = e^{x-x^2} \text{ and } f'''(x) = e^{x-x^2} \] so to speak: they are all the same.
The students use the fact that the derivative of the exponential function is the exponential function;

\[ f(x) = e^x \rightarrow f'(x) = e^x. \]

This, however, can be used as if it were a universal procedure. One can observe this particular extrapolation in many similar contexts. The students appear to be operating on the (exponential) function as an object, having lost sight of its process or action attributes. Or learners could say; since the \[ \int (x^n) \, dx = \frac{x^{n+1}}{n} + c, \]
then

\[ \int x^l \, dx = \frac{x^{-1+1}}{-1+1} = \frac{1}{0} = 0. \]
To evaluate \[ \frac{1}{0} \], some students say it is undetermined, some say it does not exist, some say its zero, some say its 1; all wrong answers, themselves requiring research. Procedural extrapolation then is an over-generalisation of a valid procedure in new situations where it causes errors.

The second characterisation of errors that Hirst explained was erroneous extrapolations; a type of linearity. Fischbein & Barash (1993) identified linearity errors and Hirst (2003) referred to pseudo-linearity errors. Some common examples of pseudo-linearity errors are;

\[ (a+b)^2 = a^2 + b^2, \log(x-y) = \log x - \log y, \sqrt{(9x^2 + 4)} = \sqrt{9x^2} + \sqrt{4} = 3x + 2, \]

\[ \sin(\theta + \beta) = \sin \theta + \sin \beta, \text{ and } (f(x)g(x))' = f(x)' \cdot g(x)' \text{ in differentiation.} \]

Such errors occur before students encounter linearity in an overt, systematic manner as in linear operators in differentiation and integration or linear transformations in linear algebra. One of the underlying possibilities in this type of error is the distributive rule. Norman & Pritchard (1994) label pseudo-linearity errors as ‘misgeneralised distributivity’. These errors may emanate from simplifying expressions like \[ x(y+z) \] to \[ xy + xz. \]

The third characterisation of errors of Hirst is equation balancing. Such errors again emanate from earlier experiences. For instance in elementary algebra the principle; “you do the same thing to both sides of an equation and they are still equal”. This principle can also be a source of errors. What may be significant is that on many occasions in their comments the students replace the phrase “to both sides of the equal sign” with “on both sides”. For example in teaching learners to create equivalent fractions, teachers say that in order to create
equivalent fractions to \( \frac{a}{b} \), the number you multiply the numerator with, you also multiply the denominator with, so \( \frac{3}{4} = \frac{3 \times 5}{4 \times 5} = \frac{15}{20} \). Learners then go on to say

\[
\frac{a}{b} = \frac{a + 2}{b + 2} \quad \text{or} \quad \frac{g'(x)}{h'(x)} = \frac{g(x)}{h(x)}
\]

in calculus following similar balancing laws. Intuitively, this seems perfectly reasonable. But unfortunately such naïve thinking is incorrect.

Hirst’s characterisations of errors demonstrate that errors occurring in calculus and higher mathematics are refinements and extensions of structural errors, which Donaldson (1963) found in elementary mathematics. The errors involve: confusion between action, process and object (Cottrill, Dubinsky, Nichols, Schwingendorf, & Vidakovic, 1996); mis-application of language (Movshovitz-Hadar, Zaslavsky, & Inbar, 1987); confusion between syntax and semantics, and inadequate meta-cognitive control procedures. The types of errors present in elementary mathematics continue into more advanced mathematics. For instance Engelbrecht, Harding, & Potgieter (2005) reported that the calculus errors that South African undergraduate students showed in mathematics originated from their high school. Yet, in teaching mathematics qualities such as flexibility, reversibility, generalisation and intuition are emphasised, and so, paradoxically, it seems that these very qualities can give rise to errors. They give rise to errors as for example when learners generalise the processes that they have learnt. What is important is that learners need to be extremely careful when they generalise earlier results. This lack of care and taking things for granted is the cause of many errors in mathematics. From a constructivist viewpoint, errors occur as learners interpret new situations with the lens of their prior knowledge. Next we consider Movshovitz-Hadar, Zaslavsky, & Inbar’s (1987) error categories.

3.8 Movshovitz-Hadar, Zaslavsky, & Inbar (1987) error categories

Movshovitz-Hadar, Zaslavsky, & Inbar (1987), proposed a generic error analysis of written solutions to test items in Israeli high school graduation examinations in mathematics. They claimed to have found six error categories that cover all mathematics. I do not agree with this finding but all the same, their findings help to clarify my framework. I do not agree because knowledge is not fixed in that new interpretations to the same phenomenon always occur.
The first category they called ‘misused data’. This class of errors relate to variations between the data as given in the item and how the learner refers to them. The error may occur due to a misreading at the beginning of the solution or during solution. The second was ‘misinterpreted language’. According to Movshovitz-Hadar et al. (1987), this category refers to mathematical errors due to poor interpretation of language as well as mathematical symbolism. Such errors could occur during encoding natural language into mathematical expressions such as equation representation. The third error was labelled as ‘logically invalid inference’. These are errors due to false generalisation of old knowledge into new knowledge. The fourth involves ‘unverified solutions’. This occurs when learners work correctly but on a solution that directly does not address a problem given. If the learner checked the solution, there is the strong possibility that the error could have been picked up by the learner him/herself. Some people, for example Corder (1987), call these ‘unsystematic errors or slips’ and they do not regard them as very important because they are not errors due to planning. The fifth error they called ‘technical errors’. According to Movshovitz-Hadar et al. (1987), this category includes errors due to failure to carry out calculations and process computational algorithms, reading data from tables, algebraic errors such as reducing writing $a - 4 - b - 4$ instead of $(a - 4, (b - 4))$ but proceeding as if the parentheses were there as needed and other mistakes in executing algorithms usually mastered in elementary or junior high school mathematics (for example $35,5^0$ written as $35^65'$ instead of $35.50^0$ or $35^030'$).

Although Movshovitz-Hadar et al.’s classification is informative; it was not be used in this research because many of the categories are subsumed into Donaldson (1963) and Hirst’s (2003) work. For example, misused data can be viewed as arbitrary errors and technical errors as executive errors. Some error classifications by Movshovitz-Hadar et al. (1987) seem to be similar, for example, misused data and misinterpreted language. For this lack of clarity of errors made by the learners, Movshovitz-Hadar et al.’s classifications would have made calculus error analysis difficult and inconclusive. However, some categories, such as misinterpreted language and logically invalid inferences, were incorporated in the analysis under a special category.
3.9 Other classifications

Radatz’s (1979) classification was based on the causes of errors, in which, like Davis (1984) there are elements of information-processing theory. He suggested a model that provides a classification of causes of errors by describing five processes that lead to errors in different mathematics topics and concepts.

Radatz concluded that the first cause of errors was due to lack of mastery of language. The second cause of errors in mathematics was due to difficulties in obtaining visual information. The third types of errors were due to a deficient mastery of prerequisite knowledge and skills. According to Radatz, some errors were caused by incorrect associations or inflexible thinking. The last cause of errors was application of irrelevant rules or strategies. This classification of errors by cause attempts to shed light on the cause of the errors explicated by Donaldson and others discussed above. Radatz’s classification on the causes of errors was useful in the discussion section when the causes of errors found in this study were proposed.

Another quite common student error involves incorrectly applying proportional reasoning (Hart, 1984). An example of this occurs as a student attempts to find the number of seats in the 10th row of the Theatre Seats Problem, where the next row has two more seats. The students could use the fact that 19 seats are needed for the fifth row, so they double this amount to find the number of seats in the 10th row, which generates an incorrect number of seats. In this research, there are no instances where proportional reasoning was expected and found even in the examination script analysis. It is mentioned for completeness purposes.

3.9.1 Generalisation

As described by Davydov (1995), “generalization is regarded, as a rule, as inseparably linked to the process of abstracting” (p. 13). Generalization involves the following: deliberately extending the range of reasoning or communication beyond the case or cases considered, explicitly identifying and exposing commonality across cases. Hirst (2003) has proposed errors due to procedural extrapolation, pseudo-linearity, and equation balancing as sub-
categories of structural errors. These errors forms are due to over-generalisation (Fischbein & Barash, 1993).

Cipra (1983) referred to failure of checking one’s solution as one of the errors learners make. This error is related to meta-cognition or “inside critics” (Davis, 1984). The issue is that when learners are aware that they usually make a particular mistake, they should check all the time whether they did the correct thing after working out a problem.

Yet, Chi (2005) argued that misconceptions can be portrayed in one of two ways: as either fragmented or coherent. A fragmented view considers misconceptions as “a set of loosely connected and reinforcing ideas”. In contrast, a coherent view claims that misconceptions are not merely inaccurate or incomplete isolated pieces of knowledge, but rather, they can be portrayed as alternative conceptions that explain how things work. As Anderson and Smith (1987) put it, “there are consistent understandable patterns in the incorrect answers that students give” (p. 90). This “pattern” of coherence was a highly sought after aspect of this research.

3.9.2 Discipline specific errors

Now errors specific to calculus are considered. Orton (1983a, 1983b) observed that most errors that students made in calculus were algebraic in nature. For instance, in solving the equation $3x^2 - 6x = 0$, he indicated that twenty four (24) students out of one hundred and ten (110) lost the 0 root by incorrectly cancelling $x$. A further six (6) incorrectly factorised $3x^2 - 6x$ to $3x(x-6)$, which appeared to be an executive error. Another algebraic error occurred in expanding $3(x+h)^2$. Seventeen (17) students lost the middle term $6ah$ (Orton, 1983b).

On the rotating secant pivoted at point P, Orton (1983b) reported that 43 students out of 110 were unable to state that the secant eventually became a tangent at A, despite encouragement and probing. Students were found to put attention on the chord PQ which did not help them to notice that P and Q eventually collided in the limit to the same point P. There appeared to be considerable confusion in that the secant was ignored by many students; students only
focused their attention on the chord PQ despite the fact that the diagram and explanation were intended to try to ensure that this did not happen. The typical and unsatisfactory responses Orton obtained were: "The line gets shorter"; "It becomes a point"; "The area gets smaller"; "It disappears". It appeared that, in the normal approach to differentiation, students needed considerable help in understanding the tangent as the limit of the set of secants.

Orton also reported failures of interpretation of negative and zero rates of change.

Twelve students out of one hundred could not respond at all when asked to interpret \( \frac{dy}{dx} = -2 \), and a further ten students could only say "decreasing," or "decreasing gradient," or similar, rather than "decreasing function." Twelve students were unable to interpret \( \frac{dy}{dx} = 0 \),

Yet, six of these students were able to put \( \frac{dy}{dx} = 0 \) and obtain values of \( x \) for stationary points, so they must have experienced some confusion in one context but not in the other.

Overall, the many errors of interpretation were structural, but many executive errors were also committed. Two elementary applications of differentiation encountered early in a school study of calculus were tested concerning obtaining the gradient of the tangent to \( y = x^3 - 3x^2 + 4 \) at \( x = 3 \) and was answered quite well by most students (Orton, 1983b).

Numerical errors were rare, but six such executive errors did occur in substituting \( x = 3 \) into \( \frac{dy}{dx} = 3x^2 - 6x \).

Orton (1983b) also indicated concepts in elementary calculus that students did not understand. These are: infinite geometric sequences, limits of geometric sequences, equations, rate of change from straight line graph, rate, average rate and instantaneous rate of change, carrying out differentiation, differentiation as a limit, use of \( \delta \)-symbolism, significance of rates of change from differentiation, gradient of tangent to curve by differentiation and stationary points on a graph.
Orton has reported that a restricted mental image of functions is not always seen as provoking a difficulty in elementary calculus particularly when the subject is seen as focusing on the differentiation and integration of standard functions given as formulae.

Difficulties in translating real-world problems into calculus formulation are part of the folklore of the subject. This area deals with mathematical modelling. There is an item on mathematical modelling in this research.

Orton also reports that although the Leibniz notation \( \frac{dy}{dx} \) is almost indispensable in the calculus, it nevertheless causes considerable conceptual problems. Learners are often confused whether it is a fraction, or a single indivisible symbol. It seems it is both, sometimes taken as indivisible in differentiation and separable for example in facilitating integration and solution of differential equations. Difficulties in selecting and using appropriate representations in calculus are known to be widespread (see for example Shuard & Neill, 1986) given the many notations there are for the derivative; \( \frac{dy}{dx}, Dx, f'(x), y', \dot{y}, \ddot{y} \) and so forth. Hence, difficulties in calculus are sometimes related to unfamiliarity with its rich symbolism and how to use them.

3.9.3 Test taking errors

Nolting (1997) reported of six errors that learners make while writing a test. These errors are general to all subjects and are not specific to mathematics. In misread direction errors learners skip directions or misunderstand directions, but answer the question or do the problem anyway.

In careless errors learners make mistakes which can automatically be caught upon reviewing their work. Nolting (1997) explains conceptual errors as mistakes made when learners do not understand the properties or principles required in an examination item. Application errors are due to failure to apply what one knows to a specific situation where it could help to solve the problem.
Other test-taking errors that Nolting (1997) identified were missing questions in a test and not completing a problem to its last step or not answering a question fully, and study errors, which are mistakes that occur when learners study the wrong type of material or do not spend enough time studying pertinent material. It is clear that some of the error categorisations discussed above were not helpful in this research and so are not be referred to again.
Table 1. Protocol used for analysing mathematics errors in the study

<table>
<thead>
<tr>
<th>Procedural extrapolation errors (Hirst, 2003)</th>
</tr>
</thead>
<tbody>
<tr>
<td>These errors concern over generalizing procedures due to viewing mathematical representations as objects rather than flexible actions or processes. For instance, students use the fact that the derivative of the exponential function is the exponential function: $\frac{d}{dx}e^x = e^x$. Therefore, for $f(x) = e^x$, if $f(0) = e^0$, then $f'(0)$.</td>
</tr>
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</table>

<table>
<thead>
<tr>
<th>Pseudo-linearity errors (Hirst, 2003)</th>
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</thead>
<tbody>
<tr>
<td>The reasoning used is: when you do something to $a + b$ you get the same result by doing it to $a$ and doing it to $b$ for example $\sqrt{5 + 4} = \sqrt{5} + \sqrt{4}$</td>
</tr>
</tbody>
</table>

Equation balancing (Hirst, 2003)

| This emanates from the principle that “you do the same thing to both sides of an equation and they are still equal”; for example $3 = 5 - 2$. |

<table>
<thead>
<tr>
<th>Structural Errors (Donaldson, 1963)</th>
</tr>
</thead>
<tbody>
<tr>
<td>These errors are conceptual in nature emanating from lack of understanding of ideas inherent in a mathematical problem.</td>
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</table>

<table>
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<tr>
<th>Executive Errors (Donaldson, 1963)</th>
</tr>
</thead>
<tbody>
<tr>
<td>These can also be regarded as procedural errors as they are due to failure in executing an algorithm. They can nevertheless sometimes have a structural explanation.</td>
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</table>

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<tr>
<th>Arbitrary errors (Donaldson, 1963)</th>
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</thead>
<tbody>
<tr>
<td>These are due to selective processing of information and ignoring other attributes. For example, $\frac{1}{3}$ is considered as the same as $\frac{1}{2}$. Davis (1962) regarded such errors as due to Pigeon’s law of maximum discrimination.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Random errors</th>
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</thead>
<tbody>
<tr>
<td>These are non-systematic, idiosyncratic errors that do not recur and do not form any perceptible pattern.</td>
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</table>

<table>
<thead>
<tr>
<th>Language Interpretation errors (Movshovitz-Hadar, Zaslavsky, &amp; Inbar, 1987)</th>
</tr>
</thead>
<tbody>
<tr>
<td>These errors are due to failure to interpret language so as to encode it into mathematical symbolism needed to solve the problem.</td>
</tr>
</tbody>
</table>

As discussed in the chapter, the above protocol is a generic error framework that helps to frame my analysis. The researcher searched for literature from across the world in order to find what others have found concerning errors and misconceptions in mathematics and calculus. The researcher believes that the errors and misconception categories that have been discussed in this chapter are some of the most important in mathematics. Having discussed
the background of the research in chapter 1, the theoretical framework in chapter 2, and the conceptual framework here, I now turn to the literature review chapter.

3.10 Conclusion

Given the above review of the mathematical errors and misconceptions obtained from various scholars, I propose to refer to the following conceptual framework when analysing data from scripts. The mathematical error categories were analysed with the help of the following framework in Table 1.
CHAPTER 4: LITERATURE REVIEW

4.1 Introduction

This chapter provides an important basis to the study as it locates it in the realm of comparable previous research. Firstly, it discusses what it means to understand, to know and to be proficient in mathematics in general and calculus in particular. Secondly, previous researches on errors and misconceptions in pre-calculus and calculus topics are discussed. The different research problems, samples and findings of those researches are discussed and critiqued. This literature review particularly focuses on errors and misconceptions in introductory differentiation at high school, although errors and misconceptions at undergraduate level are also discussed. This is because the errors at high school and university are closely related in that students transfer them from high school to university. So in a major way, misconceptions in differentiation at undergraduate level correspond with misconceptions that students have at high school (see study by Engelbrecht, Harding, & Potgieter, 2005). Specifically, research studies by Bezuidenhout (1998); Haripersad & Naidoo (2008); Orton (1983a, 1983b); Tall & Vinner (1981); Vinner & Dreyfus (1989); on errors and misconceptions on major ideas leading to calculus and calculus itself are discussed. These areas include algebra, geometry, co-ordinate geometry, functions, limits, and the derivative itself. Thirdly, this chapter discusses errors and misconceptions learners had in applying calculus to problem solving. Many errors and misconceptions in calculus occur as learners attempt to apply calculus to problem solving (Stewart, 2004). Fourthly, the literature review discusses the nature of errors and misconceptions in calculus and their micro-genesis. These discussions were aimed at providing a solid background on which this research reflected upon. The literature was valuable for comparing, contrasting and validating the findings and recommendations of this study as well as providing insight onto this study.

4.2 The State of Calculus Teaching in Schools

The widespread teaching and learning of calculus in secondary schools is a fairly recent phenomenon. Since the founding of calculus in the late seventeenth century, the study of calculus has been the preserve of elite mathematics and science students in universities (Eves,
The Mathematics Association of America (MAA) introduced calculus as a secondary school mathematics option in 1923. Competitive advanced calculus examinations were offered in secondary schools only after the 1958 Soviet Sputnik-launch which stimulated intense mathematics education reform in the US. Since then, calculus has become an important topic on the school curricula of many countries including South Africa. However, across the world, many scholars (Artigue, 2000; Ferrini-Mundy & Graham, 1994; Orton, 1983a, 1983b) have raised concern that the teaching and learning of calculus in secondary schools and universities has largely been ineffective.

For students intent on careers in mathematics, business, science, engineering and others, calculus lays a strong foundation for studying these fields. As calculus is critical to engendering expertise in these fields and many students experience difficulties to mastering it, many research studies on student learning of calculus have been undertaken (see Confrey, 1980; Dreyfus & Eisenberg, 1986; Engelbrecht, Harding, & Potgieter, 2005; Ferrini-Mundy & Graham, 1994; Orton, 1983a; Vinner, 1991). These studies document generic and specific barriers to the understanding of introductory calculus and integration (together also called elementary analysis). The studies indicate that among other factors, students’ understandings of concepts which build towards introductory calculus, such as function and limit are underdeveloped in most learners (Engelbrecht, Harding, & Potgieter, 2005). Also, these studies report that, in the main, students can successfully differentiate and integrate but often fail to account for the concepts that underpin the differentiation and integration techniques apparently applied (Bezuidenhout, 2001; Ferrini-Mundy & Graham, 1994; Porter & Masingila, 2000). The studies also suggest that many students in university calculus classes possess a superficial and incomplete understanding of basic calculus concepts and that most educators seem not to be aware of this (Bezuidenhout, 2001; Haripersad & Naidoo, 2008; Orton, 1983a; Tall & Vinner, 1981; Vinner & Dreyfus, 1989).

For the study of calculus, a functional understanding of mathematical topics earlier studied at high school is expected (Shuard & Neill, 1986). These topics are namely; algebra (working with variables), equations, exponents, functions, co-ordinate geometry, Euclidean geometry, rates, and mathematal word problems to name a few. The study of calculus demands a good grasp of these topics for learners to begin to construct a sound understanding of it. According
to Shuard & Neill (1986), these pre-calculus ideas fuse with the notion of a limit, to introduce the derivative upon which the whole of calculus is dependent. It is thus crucial to review research on learning difficulties that impact on pre-calculus topics as learners' understanding of these, or lack of it, directly affects their comprehension of calculus ideas.

Cognitive scientists and learning theorists (Bruner, 1966; Davis, 1984; Gagne, 1985; Piaget, 1972; Shulman, 1986; Skemp, 1976; Vygotsky, 1978) have proposed general frameworks and models of knowledge. Models of knowledge are essential for the professional handling of the various facets of mathematical knowledge and the execution of research in the learning of mathematics against those facets. They are important as they attempt to differentiate the different genres of mathematical phenomena (see also Ausubel, 1963, Ausubel, Novak, & Hanesian, 1978; Hiebert & Lefevre, 1986; Sfard, 1992; Tall & Vinner, 1981). According to Grevholm (2008), most of these theories seem to lie on a continuum ranging from lower quality learning to higher quality learning, where higher quality is usually associated with conceptual development.

4.3 Instrumental and relational understanding of mathematics

Ausubel (1963) contrasted meaningful learning and rote learning. He argued that meaningful learning resulted in the creation and assimilation of new knowledge structures whereas rote learning was learning by drill for the purpose of remembering but not rooted in understanding. On the other hand, Davis (1984) argued that mathematics can be viewed either as a routine or a creative activity. It is a routine if it is used for example, for carrying out arithmetic calculations. However, Sfard (1997) argued that mathematics has two notions; operational or structural. Mathematics is operational if it is used for calculations such as in addition, subtraction, multiplication and division. When we study the structure of mathematics we look at the fundamental notions that link superficially disjoint mathematical ideas. For example groups and fields bring out salient features of mathematics that are invisible to the undiscerning practitioner. These structures bring to surface surprising commonalities in different mathematics.
In Skemp's (1976) terms, understanding of mathematics can be either instrumental or relational. Often these two understandings are mutually exclusive in that learners or teachers may think that their understanding of calculus is the only understanding. The type of understanding of what mathematics is often affects the mathematical activities that a person engages in (Boaler, 1998; Lave & Wenger, 1991). As has been discussed, it is not unusual for learners and teachers to believe in only one understanding of mathematics or calculus. In the main, most learners and many teachers have an instrumental view of mathematics (Makonye & Luneta, 2010; van de Walle, 2004). Boaler (1998) and Lave & Wenger (1991) have argued that different learners develop different ideas about the same subject and thus develop different understandings of it depending on the context in which they learn mathematics. The authors argued that the context in which one learns a concept directly affects the activities one engages in and so the type of knowledge one experiences and acquires. It follows that the way learners are taught mathematics (instrumentally or relationally) bears upon what they understand mathematics to be.

Learners and teachers could hold either an instrumental or relational view of mathematics depending on their mathematics education experience and the reflection they might have or might not have about it. Skemp (1976) argues that instrumental understanding involves the skills to carry out mathematical computations and use of formula correctly. He states that this involves rote memorisation of mathematical facts, rules and formulae, without necessarily knowing why and how the rules and formula are derived. The supposed utility of instrumental understanding may be to provide accurate and speedy shortcuts to obtain answers to specific mathematics questions. This is done without any regard to the underlying mathematical structure and validity of those shortcuts. When these algorithms are extrapolated instrumentally to other mathematical contexts, incorrect results are obtained because of the structural mismatch which instrumental understanding has with new problems. Hirst (2003) has identified a mathematical error which emanates from the rigid and inflexible use of procedures as ‘procedural extrapolation’ (see Chapter 3). Then there is also the potential danger of subscribing to instrumental understanding of mathematics at the exclusion of relational understanding as the probabilities of making errors and misconceptions are heightened by instrumental mathematics. The disadvantage of instrumental mathematics is that students often do not have enough time to digest and master the material taught in class. Some students memorize relevant rules and methods in order to deal with examinations, just
as they would recite a cooking recipe. Consequently, most students can, at most, solve only simple problems similar to those appearing in examples and exercises they have studied before, and they are unable to deal with problems from the real world as they cannot extend instrumental understanding further. If they attempt to generalise instrumental understanding, they commit many errors as they do not understand how mathematics works.

On the other hand, relational understanding stresses the understanding of relationships and connections between mathematical concepts and ideas (van de Walle, 2004). It is about how mathematical concepts are related to each other to form a unified logical structure apparently unnoticeable and quite invisible to the untrained eye. Relational understanding concerns how bits of mathematical ideas are differentiated from each other as well as how they connect and integrate with each other (Suh, 2007). It results in developing a flexible, meaningful, organic and deep mathematical knowledge that is general in nature. Van de Walle (2004) envisaged that traditional mathematics teaching emphasised instrumental understanding over and above relational understanding. Learners and teachers who subscribe to instrumental understanding of mathematics may perceive mathematics as a toolkit consisting of rules and formulae, which when correctly recalled can help to get right answers to mathematics questions. It is no wonder that learners who understand mathematics instrumentally regard mathematics as composed of mysterious procedures which when stringently followed lead to mathematics examinations success (van de Walle, 2004). Van de Walle argues that the view that mathematics is about following rules, is computation dominated and answer-oriented distorts what mathematics really is. This approach causes some learners to regard mathematics as illogical, mechanical and meaningless. It causes learners to have many misconceptions in calculus for example, because they cannot establish, link, relate or explain calculus concepts.

Students learn new mathematical concepts and procedures by building onto what they already know (Smith et al., 1993, van de Walle, 2004). What a learner already knows is the window with which new knowledge enters into his/her mind. In other words, learning with understanding can be viewed as making connections or establishing relationships either within already existing knowledge which to the learner appeared unrelated before, or between existing knowledge and new information (Hiebert & Carpenter, 1992). This only occurs when a learner has relational understanding of mathematics.
According to Taylor, Muller, and Vinjevold (2003), traditional mathematics teaching methods were based on the performance model, regarding mathematical ability as a fixed entity. Traditional mathematics teaching thus assessed learners' performance on a norm-referenced assessment base. As soon as a teacher began teaching a class, he/she assessed his/her learners in order to place them into performance groups. A learner was rated as an above average, average or below average performer. Thus if a learner in a ‘below average’ group did well, teachers did not like it as the learner would be performing outside expectation. Teachers did not easily accept that the learner had gained required competency.

Perspectives of reform mathematics however are based on the competence model and assess learners on a criterion referenced base (Taylor et al., 2003). The criterion referenced system regards mathematical learning capability as a variable depending on the support a learner has in learning mathematics. Learning is regarded as a function of the support, effort and time that the learner has (Ausubel, 1978). According to this school of mathematical thought, engaging learners in thinking helps learners to develop conceptual understanding, although the rate of doing this varies from learner to learner. What is required is for teachers to have high expectations and support for all learners (NCTM, 1989), rather than holding the view that, mathematics is complex to understand and only a select few can understand it.

Researchers (Engelbrecht, Harding, & Potgieter, 2005) have argued that over-reliance on procedural knowledge in solving calculus problems results in many errors and misconceptions and ultimately failure in this subject. This is because procedural knowledge addresses how to do a task only. It cannot help to answer the question of what is to be done and why, which is often required in most mathematics questions.

Researchers have pointed out that students generally are mathematically unsophisticated (Vinner & Dreyfus, 1989). To this end, there is a trend amongst students of dealing with calculus at a very superficial level. This leads for some students to regard calculus as a form of advanced algebra (Vinner & Dreyfus, 1989). Such students would learn calculus procedurally and still think that they have adequately learnt it, yet they would have learnt only its techniques. Often students are keen to acquire quick skills that enable them to pass examinations (Engelbrecht, Harding, & Potgieter, 2005). Teachers also find that it is much
quicker and easier to teach procedures than to teach for conceptual understanding (Skemp, 1976). Teachers are tempted to go this way because they are keen for their learners to obtain better grades in their mathematics examinations and so increase their marketability since professionally rewarding careers often demand mathematics qualifications. Also, teachers have found that if they use conceptual questions, learners do not respond to them well. This reinforces them to continue teaching procedurally. Thus, teachers and learners seem to conspire to perpetuate an instrumental but limited understanding of mathematics.

4.4 Procedural and conceptual knowledge of mathematics

Besides Skemp’s elaboration of different mathematical understandings that learners might hold, Hiebert and Lefevre (1986) proposed that the mathematical knowledge that a learner could have can either be procedural or conceptual. Researchers (Byrnes & Wasik, 1991; Haapasalo & Kadijevich, 2000; Hiebert, 1986; Rittle-Johnson & Aiibali, 1999; Star, 1999) have all acknowledged the importance of procedural and conceptual knowledge for mathematics learners and the roles the two knowledge sets play in the development of mathematical proficiency. Hiebert and Lefevre’s (1986) framework of conceptual and procedural knowledge, characterizes the mathematical or calculus knowledge that a learner can have. Since it is the mind that acquires and constructs knowledge (Hatano, 1996; von Glasersfeld, 1989), it is crucial to ascertain the prior knowledge the mind has already to help with the construction of new mathematical knowledge.

According to Hiebert & Lefevre (1986), conceptual knowledge is "knowledge that is rich in relationships" (p.6). On the other hand, procedural knowledge is regarded as "rules or procedures for solving mathematical problems" (Hiebert & Lefevre, 1986, p.7). Conceptual knowledge can also be regarded as knowledge that is rich in relationships and relates to the principles that refine understanding of mathematics and also refers to the interconnections between ideas that explain and give meaning to mathematical procedures. Hiebert & Lefevre (1986) described conceptual knowledge as that knowledge which is part of a network comprised of individual pieces of information and the relationships between these pieces of information. Conceptual knowledge then concerns thinking on the nature of mathematical objects. Hiebert & Lefevre (1986) defined procedural knowledge as knowledge of carrying
out a mathematical task, the *know-how* of mathematics but and not the *know-why*. In addition, Eisenhart, Borko, Underhill, Brown, Jones & Agard (1993) define procedural knowledge as mastery of computational skills and familiarity with procedures, rules and algorithms for solving problems. Procedural knowledge includes both a familiarity with the symbolic representation of mathematics and knowledge of using them. It encompasses knowledge of rules and procedures for carrying out mathematical calculations and how to solve mathematical problems. Procedural knowledge (of calculus) may or may not be learned meaningfully. However, conceptual knowledge by definition must be learned with meaning and understanding. This is because conceptual knowledge is acquired through conceptual understanding. The receiver of knowledge must first understand the concept. In trying to understand the concept the receiver of the concept relates it to concept image that is evoked in the mind.

Figure 6 illustrates that when a mathematics teacher teaches a concept to learners effectively, the learners understand the concept resulting in them forming appropriate concept images which adequately resemble the target concept. In such cases, learners properly align incoming mathematical knowledge with their old mathematical knowledge (through assimilation or accommodation) (Piaget, 1968) thereby forming a more comprehensive concept image. Conversely, if the teacher has weak conceptual knowledge, he/she deploys distorted knowledge to learners. This distorted knowledge may influence learners to form invalid concept images which by definition are misconceptions which influence their mathematical thinking. Similarly, if a learner has a very defective concept image, he/she may not receive incoming knowledge properly, no matter how well articulated by any medium. Rather, the learner might form a distortion/misconception of the new correct knowledge. If the learners are only taught procedures, they only have learnt half-truths because procedures are only useful in specific contexts. Procedures are unlike concepts which are often universal and so can be applied in many different contexts. However, it is true that procedures are also universals but are inflexible to different contexts. So a weak concept image can potentially spawn misconceptions for learners.
Thus, conceptual knowledge provides a higher level of understanding of the principles and relations between pieces of knowledge, while procedural knowledge enables quick and efficient solution to problems. Loosely speaking, one can say that conceptual knowledge is about thinking whereas procedural knowledge is about doing. In calculus, procedural knowledge incorporates strategies of finding limits, sketching graphs of functions, optimising and minimising functions, rules for differentiation and integration, symbolism and so on. In particular the procedural knowledge does not explain why for instance the gradient function $\frac{dy}{dx}$ vanishes at the turning points of a function. Procedural knowledge is also not concerned with the definition of the derivative, or arguments on why a function tends to the limit learners have calculated.
Within the context of teaching and learning calculus, conceptual knowledge is seen as the knowledge of the core concepts and principles and their interrelationship within calculus and outside calculus. Accordingly, it is assumed to be stored in some form of relational representation like schemas, semantic networks, frames or hierarchies. Because of its abstract nature and the fact that it can be consciously accessed, conceptual understanding can be largely verbalized and flexibly transformed through processes of inference and reflection. It is, therefore, not bound up with specific problems but can in principle be generalized for a variety of problem types in a domain.

In contrast, procedural knowledge is seen as the knowledge of operators and the conditions under which these can be used to reach certain goals (Hiebert, 1986). Further, it allows people to solve problems quickly and efficiently because it is to some degree automated. Automisation is accomplished through drill and practice, facilitating a quick activation and execution of procedures, since its application, involves minimal conscious attention and few cognitive resources.

4.5 Notation and symbolism

Calculus concepts are also represented by symbols such as \( \frac{dy}{dx}, \lim_{x \to \infty} f(x) \). The knowledge of these symbols is regarded as procedural rather than conceptual. However, these symbols are representations of concepts. Procedures can be memorised and consolidated through drill and exercises, and may not be connected to the underlying concepts that necessitate them. This is knowledge of processing skills without awareness of the concepts that underlie them. In this regard, learners can make computational errors as a result of a lack of conceptual understanding (Evan & Lappan, 1994).

Hiebert and Lefevre (1986) also make a key distinction between relationships that are constructed at the same level of abstraction as the constituent concepts and therefore do not involve an increase in abstraction, and those "reflective relationships" that are constructed at a higher level. They use the term "abstract" to refer to the degree to which a relationship is
freed from specific contexts. This is similar to what Skemp (1976) had referred to as primary or secondary mathematical concepts. According to Skemp, primary concepts are those formed from our senses and from operations on physical objects existing outside our minds. The mind imposes concepts on relations between physical objects, which relations do not exist in the objects themselves but are discerned and abstracted by the mind. These primary concepts are formed through the use of our five senses (van de Walle, 2004). Secondary mathematics concepts are purely mental; they are not directly connected to the physical contexts which induce them in the first place. Thus while we can touch a set of 3 counters and 2 counters in the physical space and combine them to get 5 counters (primary concepts); 3x and 2x are secondary concepts in the mind, we cannot touch 3x, it only has a mental existence. Because advanced mathematical concepts are the result of several abstraction sequences, the network of relationships among concepts can be extremely complex secondary concepts (van de Walle, 2004).

Hiebert & Lefevre (1986, p. 8), propose that the key word for procedures is "after" in the sense of "after this step comes the next step". Procedural knowledge may or may not be supported by conceptual knowledge. Unsupported procedural knowledge has connotations of rote, manipulative learning. Rote learning is defined as learning that is habitual repetition and devoid of conceptual understanding. This has implications as far as this study is concerned, as rote learning is limited to specific knowledge and not very helpful in problem solving situations.

There is debate amongst researchers on which knowledge must come first during teaching and learning of mathematics. Some, for example Rittle-Johnson, Siegler, & Alibali (2001), believe that the order of acquisition of mathematical concepts (conceptual knowledge) and the acquisition of mathematical skills (procedural knowledge) is of no consequence. But others, for example: Cipra (1989), Orton (1983a), Tall & Vinner (1981) and Vinner (1989) strongly argue that the main problem with traditional calculus teaching is that it teaches procedural knowledge at the expense of or before conceptual knowledge is established. To them, that is why learners have problems with calculus.
In answering the above question on which knowledge must come first in mathematics teaching, Star (2005) introduces the term teleological semantics. Teleological semantics tries to link conceptual knowledge and procedural knowledge. The teleological semantics of a procedure is "knowledge about [the] purposes of each of its parts and how they fit together. ... teleological semantics is the meaning possessed by one who knows not only the surface structure of a procedure but also the details of its design" (Star, 2005, p. 95). Van Lehn and Brown (1980) note that a procedure can be cognitively represented on a very superficial level (as a chronological list of actions or steps) or on a more abstract level (incorporating planning knowledge in its representation). According to Van Lehn and Brown (1980), planning knowledge includes not only the surface structure (the sequential series of steps) but also "the reasoning that was used to transform the goals and constraints that define the intent of the procedure into its actual surface structure" (p. 107). In other words, planning knowledge of a procedure takes into account the order of steps, the goals and sub-goals of steps, the environment or type of situation in which the procedure is used, constraints imposed upon the procedure by the environment or situation, and any heuristics or common sense knowledge which are inherent in the environment or situation. Within Hiebert's framework, this type of knowledge does not fall neatly into either conceptual knowledge or procedural knowledge. In essence, teleological semantics is conceptual knowledge about a procedure - it is both procedural and conceptual knowledge. From the above, it would appear that learners with teleological semantics can negotiate their errors and make progress in learning mathematics.

As has been stated, memorised rules and facts cannot form a schema on which new mathematical knowledge can be properly accommodated and assimilated. If this is done, there is always the danger of oversimplifying or over-generalisation (Coombe, 1981). Some scholars (for instance Kilpatrick, Swafford, & Findell, 2001) argue that there is a symbiotic relationship between procedural and conceptual knowledge in that they are interwoven. They argue that acquisition of one reinforces the other. But many disagree and argue that conceptual knowledge must always be taught before procedural knowledge. In my opinion, there ought to be a balance between procedural and conceptual calculus knowledge that learners learn at school. Conceptual and procedural knowledge are iterative and mutually depend on each other for balanced mathematics learning. Both are very important as one cannot exist without the other in the real world.
4.6 Notion of mathematical proficiency

In addition, the work of Hiebert and Lefevre (1986) and Skemp (1987), Kilpatrick, Swafford, and Findell (2001) proposed the notion of mathematical proficiency to which every learner must strive to have a balanced mathematics education. Kilpatrick et al. (2001) describe mathematical proficiency as “expertise, competence, knowledge, and facility in mathematics” (p. 116). They further elaborated that mathematical proficiency has five interwoven strands: conceptual understanding; procedural fluency; strategic competency; adaptive reasoning; and productive disposition. The strands of conceptual understanding and procedural fluency are in essence similar to the terms used by Hiebert and Lefevre (1986) of conceptual knowledge and procedural knowledge.

According to Kilpatrick et al. (2001), “conceptual understanding refers to an integrated and functional grasp of mathematical ideas” (p.118). These ideas are mathematical concepts, operations and relationships. Without conceptual understanding, mathematics becomes a subject of unconnected facts that have to be learnt by rote. Here, Skemp’s (1987) relational understanding is equivalent also to the conceptual understanding strand. Procedural fluency is the skill in carrying out procedures flexibly, accurately, efficiently, and appropriately (Hiebert et al., 1986). Strategic competence is about problem solving. It is the ability to formulate, represent, and solve mathematical problems. Strategic competence is extremely important because it calls upon the input of learners’ repertoire of mathematical competencies and practices to bear upon the problem in order to solve it. Adaptive reasoning refers to the capacity to think logically about the relationships among concepts. The ability to reason is such a vital part of mathematical behaviour that it is safe to assert that mathematics cannot be done without it. Adaptive reasoning is useful because it is used in interpreting and making sense of mathematics; of linking new ideas and old ones; of reflecting on old ones and reorganising and refining them to make them clearer and more powerful than before. The productive disposition strand occurs when learners begin to see mathematics as an interesting subject that is worthwhile for its own sake. People in the history of mathematics such as Isaac Newton and Wilheim Leibniz had a superb productive disposition because they produced new mathematics, in this case calculus. Thus procedural knowledge comes second to the enjoyment and wonder of doing mathematics seen in these two prodigies.
Consequently, it can be seen that different scholars have various descriptions of what it means to understand mathematics (Skemp, 1976), know mathematics (Hiebert & Lefevre, 1986) and to have proficiency in mathematics (Kilpatrick, et al., 2001). The connecting argument in all these variable positions is that learners should not only know how to carry out mathematical computations to the exclusion of other competencies as there is much more in mathematics than that. Learners should also be able to explain the mathematical relationships which necessitate those mathematical computations. Researchers have found that students do differentiate, find limits and integrate among other techniques (Engelbrecht, Harding, & Potgieter, M. 2005; Orton, 1983a) yet the students seemed unaware of the conceptual nuances that underly these procedures.

4.7 Research on the teaching and learning of Algebra

Bednarz, Kieran, & Lee (1996), Kilpatrick, Swafford, & Findell, (2001) and Kieran (1992) have reported on the challenges learners face in the transition from arithmetic to algebraic reasoning. A number of different characterizations of algebra can be found in mathematics education literature. Usiskin (1988) categorised four algebraic conceptions of generalized arithmetic, the set of procedures used for solving certain problems; the study of relationships among quantities, and the study of structures. At the same time, Kaput (1995) identified five characterisations of algebra known as: “generalization and formalization; syntactically guided manipulations; the study of structure; the study of functions, relations, joint variation; and a modelling language” (p. 73). Makonye & Luneta (2010) and Tall (1992) have identified several errors and misconceptions that students encountered in answering calculus questions. These included problems of algebraic manipulation. Makonye & Luneta (2010) noted that students encountered many difficulties in simplifying expressions where laws of indices were required to simplify algebraic expression before differentiating them. Successful teaching and learning of calculus depend on learners’ competency in many areas of mathematics including geometry, co-ordinate geometry, functions and algebra. Therefore, this section describes some of the difficulties learners face in learning algebra which has a strong effect on whether learners would understand calculus or not.
According to Kieran (1992), algebra is the science of solving equations and provides important tools for representing and solving problems. Notwithstanding the potential empowerment that algebra would give to learners of mathematics, this branch of mathematics remains very difficult to teach and learn because it is abstract (Gough, 2004; Travis, 1985). Algebra, unlike arithmetic, is not derived from the logico-mathematical relationships (Piaget, 1968) that we impose on physical objects. Arithmetic affords us the primary concepts (Skemp, 1976) of addition of two numbers for example when we join together a set with two objects and a set with three objects. We can touch, see, and physically count the objects in the new set to have five objects. However algebra relies on secondary concepts (Skemp, 1976). For instance, how can we touch one $2x$ or $3x$? We have to depend on primary concepts in order to add $2x$ and $3x$ for $2x$ and $3x$ exist only in our minds and not in the physical space. This is one of the reasons why algebra is hard for students to learn. Kieran (1992) argues that in the transition for learners from doing arithmetic (at primary school) to learning algebra (at secondary school), students often meet academic challenges which they must negotiate in order to learn algebra successfully. According to Kieran, this challenge affects all learners: those who are proficient in arithmetic and those who are not. She argues that mostly, learners tend to focus on operations on mathematical quantities in order to obtain answers through manipulating them. These manipulations rarely focus on representations. Kieran explains that, in arithmetic, a sum of two numbers such as $3 + 5$ is a signal to compute its answer. That is why students will write $13$ in the space for the equation $8 + 5 = - + 9$ instead of the correct value of $4$. Such students do not yet regard the equal sign as a relation symbol for quantities on either side of it, but as a command to put an answer. Such computation-oriented students are perplexed by an expression such as $2x + 3$. Many such students would not want to leave the expression as it is but are compelled to act on the $2$ and $3$ and the $+$ sign they are seeing. In this case it is very common for the learners to offer an answer of $5x$, though they may doubt that answer to a certain extent. Students are very uncomfortable to leave $2x + 3$ as it is because they are not aware that it is an important general representation by itself of adding $3$ to twice an unknown quantity (see Travis, 1985, Fig. 7).

Thus, students operating in an arithmetic frame of reference tend to overlook the relational aspects in algebraic expressions as they focus mainly on answer-oriented computations (Kieran, 1992). Kieran argues that for the improvement of algebra learning, instruction must focus firstly on relations in mathematical quantities and not merely on the calculations.
Secondly, it must focus on operations as well as their inverses; on the related idea of doing and undoing. Thirdly, teaching and learning must focus both on representing and solving problems rather than merely on computation or on solving. Fourthly, there must be use of both numbers and letters, rather than on numbers alone, so that learners see that algebra makes sense. This includes, working with letters that may at times be unknowns, variables, or parameters; accepting unclosed literal expressions as responses. Lastly it is also important to refocus on the meaning of the equal sign (Kieran, 1992).

Stacey (1997) asked what it means for a student to ‘understand’ algebra. She proposed that we could accept that a student understands what he or she is doing algebraically if the student has some capacity to detect errors in the algebraic answers. Stacey urges that learners develop an ‘algebra sense’, much like the obvious counterpart to the arithmetic-related idea of ‘number sense’. Meanwhile, Swedosh and Clark’s (1997) ‘easy and effective method’ (p. 193) consists of substituting in simple expressions evaluated by the learners to lead to ridiculous conclusions. This establishes a cognitive conflict for the student. Here the student’s erratic algebraic conclusions are made to clash with correct results obtained by substituting numbers to their results.

Such a confrontational approach to misconceptions (Davis, 1997) is called ‘torpedoing’. It presents an immediate counter-example refuting a student’s naive conjecture. Misconceptions in algebra also occur to students at undergraduate level. To research student errors in algebra, Travis (1985) asked students to answer the question;

**Three more than twice a certain number is 57. Find the number.**

Travis (1985) reported that 19 out of 31 undergraduate students made the following erratic responses in Fig. 7:
Error pattern #1: \( x + (3 + 2x) = 57 \)
Error pattern #2: \( x^3(x2) = 57 \)
Error pattern #3: \( 3x + 2 = 57 \)
Error pattern #4: \( 3x + 2x = 57 \)
Error pattern #5: \( x + 3 = 57 \)
Error pattern #6: \( 3 + 2x^2 = 57 \)
Error pattern #7: \( x^2 + 3 = 57 \)
Error pattern #8: \( 2(x + 3) = 57 \)

**Figure 7.** Different error patterns in response to an algebra question

This showed that even undergraduate students make many cognitive errors in basic algebra.

On another question (Fig. 8), students showed even more surprising incompetency in algebra:

*The length of a rectangle is 7 inches greater than its width. If its length is increased by 2 inches and its width is decreased by 3 inches, its area is decreased by 37 square inches. Find its dimensions.*

**Figure 8.** Word mathematics question given to students

Travis (1985) reported that only 4 out of 31 students successfully solved this problem (Fig. 8). Travis reported that only two students used a visual to formulate the required equations. Such incompetency within undergraduate mathematics students shows that the teaching of calculus can be very problematic. This is because mathematical arguments in calculus are often expressed and communicated in algebraic terms. It therefore means that with no competency in algebra, epistemic access to calculus is doubly difficult. This is why algebra is such an important pre-calculus topic.

### 4.8 Research on the teaching and learning of Geometry and Co-ordinate geometry

Besides competency in algebra, pre-calculus students are expected to have competency in geometry and co-ordinate geometry. This is because students also exhibit many errors and misconceptions in calculus related to its graphical and visual representation in the Cartesian plane (Makonye & Luneta, 2010; Orton, 1983a). In geometry it is important for students to
understand the notion of a curve. It is important to study what occurs when curves, lines and points intersect. In this way students are expected to learn about chords and tangents to curves. They also learn that whenever a straight line meets a curve, the straight line either cuts (crosses) the curve or it just touches the curve whereupon it is called a tangent.

This prior geometric knowledge of curves, lines, angles and points ought to be followed by the study of co-ordinate geometry on the \( x-y \) plane. Concepts to be studied include graphical representations of mathematical objects on the \( x-y \) plane; points, lines and curves. The concept of the slope of a straight line must be studied as well. In particular, learners must estimate the gradient of a curve at a point through scale drawing. How a function is graphically represented on the Cartesian plane, dependent and independent variables and its table of values are important pre-calculus notions.

### 4.9 Research on teaching and learning functions

Tall (1996), reports that many researchers (for example Dubinsky & Harel, 1992; Leinhardt, Zaslavsky & Stein, 1990) have found that students who study calculus have a limited view of a function. They found that students generally rely predominantly on the use of and the need for algebraic formulae when dealing with the function concept (see Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Dreyfus & Eisenberg, 1982; Eisenberg, 1992; Vinner, 1989; Vinner & Dreyfus, 1989). Earlier, Tall and Vinner (1981) had suggested that student difficulties in dealing with a function given in graphical form may be a result of traditional instructional methods. Tall (1992) reported that students had restricted mental images of functions.

The concept of function, also called the Dirichlet-Bourbaki concept, is that of a relation between two sets assigning each element or point in the first set (the domain) to exactly one element in the second set (the range). One may perceive a function as a rule that runs on the whole set of the independent variable \( x \), mapping each value of the independent variable \( x \) to one and only one value in the set of the dependent variable, \( y \). This creates a set of ordered pairs \((x, y)\). The Dirichlet-Bourbaki definition though completely capturing the essence of a function is of little help to calculus learners unless the function is expressed in the form of polynomial or other formula. Moreover the Dirichlet-Bourbaki concept is of no value if the
values in the domain and range are not numerical. The reason why the Dirichlet-Bourbaki function may not be always helpful is that it also encapsulates non-continuous functions. By definition, non-continuous functions are not differentiable at points where they are not continuous. Even at points of continuity, the derivative from the above must be equal to the derivative from below; otherwise the function would not be differentiable at those points despite that the function may be continuous at those points. Some functions are continuous over their entire domain, some are not. Even though some functions could be continuous over their entire domain, but may not be differentiable at all points in their entire domain as continuity in the domain is not a sufficient or enough condition for differentiability at all points.

Several authors attribute the errors that the learners have in calculus to their teachers. Tall & Vinner (1981) illustrated how teaching may contribute to learner misconceptions. For instance, the concept definition of a mathematical function might be taken to be “a relation between two sets \( A \) and \( B \) in which each element of \( A \) is related to precisely one element in \( B \).” But individuals who have studied functions may or may not remember the concept definition and the concept image may include many other aspects, such as the idea that a function is given by a rule or a formula, or perhaps that several different formulae may be used on different parts of the domain \( A \). There may be other notions, for instance the function may be thought of as an action which maps \( a \) in \( A \) to \( f(a) \) in \( B \), or as a graph, or a table of values. All or none of these aspects may be in an individual’s concept image. But a teacher may give the concept definition and work with the general notion for a short while before spending long periods in which all examples are given by formulae. In such a case, the concept image may develop into a more restricted notion, only involving formulae, whilst the concept definition is largely inactive in the cognitive structure. Initially, the student in this position can operate quite happily as his/her restricted notion is adequate in its restricted context. He may even have been taught to respond with the correct formal definition whilst having an inappropriate concept image. Later, when he meets functions defined in a broader context he may be unable to cope. Yet the teaching programme itself has been responsible for this misconception.

Some functions are defined on split domains (i.e., by different rules on different subdomains). Although the Dirichlet-Bourbaki approach is frequently presented in textbooks and curricula
materials, the examples used to illustrate and work with the concept are usually functions whose rule of correspondence is given by a formula \( y = f(x) \). This practice may lead to students' images of a function based on the appearance of a formula, even though their definition may well be of the Dirichlet-Bourbaki type. Thus, when asked about the function definition, a student may well come up with the Dirichlet-Bourbaki formulation, but when working on identification or construction tasks, his or her behaviour might be based on the formula conception. This inconsistent behaviour is a specific case of the compartmentalization phenomenon mentioned in Vinner, Hershkowitz, and Bruckheimer (1981). This phenomenon occurs when a person has two different, potentially conflicting schemes in his or her cognitive structure about a mathematical concept. This failure to reconcile different representations of a function concept has potential for generating errors and misconceptions on the use of the function concept. One purpose of the function is to represent how things change. That is how one quantity changes as the other changes.

Hence, on the function concept, certain situations stimulate one scheme, and another situation stimulates the other. This inconsistent behaviour is an indication of compartmentalization (Dubinsky, 1990). The unfortunate outcome is that sometimes, a given situation does not stimulate the scheme that is the most relevant to the situation. Instead, a less relevant scheme is activated. For instance, respondents can give the Dirichlet-Bourbaki definition and even accept that a certain discontinuous correspondence is a function; when asked to justify this, however, they do not use the definition but rather say that it is a discontinuous function. Notwithstanding the discussion, the much more important type of a function in mathematics in general and calculus in particular is the formula type given in terms of variables.

Tall & Vinner (1981) suggest that although the general notion of function, which includes functions defined by graphs, may be utilized for a short time, the predominant use of functions given by algebraic formulas may contribute to the development of a restricted concept image of function. In this study, this observation is not important, because in calculus it is important to integrate the formula image of a function with the graphical image of the function, as analysis and examination of the two helps to bring a strong conceptual basis on which learners can understand the derivative.
The limited conceptual view of the function concept held by students in calculus has also been reported (see Sfard, 1992; Orton, 1983a). In particular, Sfard (1992) reports that even students who performed very well on routine calculus problems found great difficulty and had little or no success in dealing with problems that were non-routine involving functions. Sfard (1992) argued that it was of prime importance to develop a strong process conception of function prior to an object conception in the development of the students' understanding of the function concept. She believes that a lack of, or weakly developed, process conception of function contributes to student difficulties in dealing with the function concept particularly where applications of it are required as in differentiation.

One example of the need for a strong process conception of function is required to obtain the limit of a difference quotient,

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Despite the many difficulties that learners may exhibit by having restricted mental images of functions, these seem not to hinder them in elementary calculus, at least not at the beginning (Orton, 1983a).

Research into the understanding of calculus has shown a whole spectrum of concepts that cause problems for students. In particular, students have difficulties with the abstract concepts of rate of change (Orton, 1983a), limit (Cornu, 1991; Tall & Vinner, 1981; Wu, 2004; Zandieh, 2000), tangent (Vinner, 1989; Tall, 1987), and function (Dreyfus & Eisenberg, 1982; Even, 1993; Vinner, 1989; Vinner & Dreyfus, 1989).

These concepts involve mathematical objects or processes specific to calculus. Another aspect that needs to be considered is the question of what other concepts are involved in applying calculus knowledge. Much of the research on calculus learning has shown that students are able to successfully carry out methods of differentiation and integration but sometimes lack the conceptual underpinnings necessary to explain procedures, to work
through problems using multiple strategies, and to make connections between concepts (Ferrini-Mundy & Graham, 1994; Orton, 1983a; Vinner, 1989).

Student difficulties with the abstract concepts of rate of change (Orton, 1983a), limit (Cornu, 1991; Tall & Vinner, 1981), tangent (Porter & Masingila, 2000; Tall, 1987; Vinner, 1989), and function (Dreyfus & Eisenberg, 1982; Even, 1993; Vinner, 1991; Vinner & Dreyfus, 1989) are well documented. These concepts involve mathematical objects or processes specific to calculus. These are discussed in the following sections.

4.10 Research on the teaching and learning of the limit concept

To calculus, the idea of a limit is extremely important because the limit concept is the glue that holds calculus together. According to Artigue (1996) limits are important to a real understanding of analysis (i.e. integration and differentiation), but generally not much school time is devoted to understanding of the limit concept. Introductory calculus institutes in learners the sudden transition from the study of the discrete and the finite, to the study of the continuous and infinite (Tall, 1987). This transition is one hardest for learners to master (Orton, 1983a). Learners now encounter concepts difficult to conceive and handle such as limit, infinity, division by zero, hyperbolas, and infinitesimals and so on. Some quantities are regarded as infinitesimally small, yet others are infinitely large. These notions are often regarded as processes or objects (Dubinsky, 1990). Learners require a good understanding of these notions if they are to make headway in understanding the derivative. The limit is one of the new concepts to confound the learners at this level.

Davis and Vinner (1986) argue that students fail to explain why the concept of 'limit' is fundamental to calculus. The limit; the least upper bound or the greatest lower bound of a function is encountered unexpectedly by students. So the calculus represents a time in which the student confronts the limit concept head on. It can no longer be postponed. This concept at first appears odd/weird/queer to many learners. The limit involves calculations no longer performed just through simple arithmetic and algebra, but also requires intuitively careful arguments on infinite processes. Now logical mathematical arguments become more and more important than just procedures.
Thus Orton (1983a, 1983b) has commented that errors made when dealing with limits tend to be structural, revealing an absence of real understanding, though it is possible that some of the incorrect numerical limits were the result of executive errors. Davis and Vinner (1986) argued that students' conceptual understanding of limits and calculus concepts go hand in hand with the development of their manipulative skills. This is because concepts and skills are inter-related and cannot exist independently of the other.

Artigue (1999) reported that students often had difficulties whether the limit can actually be reached. Also students had confusions over the passage from finite to infinite, and in understanding "what happens at infinity" (Cornu, 1991; Orton, 1983a, 1983b; Schwarzenberger & Tall, 1978). Cottrill, Dubinsky, Nichols, Schwingendorf, Thomas, & Vidakovic (1996) using the APOS theory, studied students' understanding of the concept of a limit. They concluded that even at its most formal conception, the limit is a schema involving dynamic aspects rather than solely a static concept, which makes for great difficulty for learners.

Bezuidenhout's (2001) research shed light onto ways undergraduate students think about the idea of a limit. His study showed that many students' knowledge and understanding of the 'limit', was largely based on fragmented facts and procedures, and that they could not link these concepts with each other. Bezuidenhout (2001) argued that such a situation may be the result of teaching approaches that emphasize procedural aspects of the calculus, while sideling the foundational ideas that formulate the calculus. Bezuidenhout (2001) postulated that the oversimplified calculus exercises found in many textbooks often encouraged procedural thinking at the expense of the conceptual understanding of calculus. He argued that this played a crucial role in learners' difficulties with the calculus.

4.11 Research on teaching and learning of the derivative

Literature on understanding the derivative is discussed next. Many authors have discussed problems students have with interpretation of the derivative (see for example Bezuidenhout, 1998; Brown & Burton, 1978; Cipra, 1989; Hart, 1981; Maurer, 1987). This research on the
understanding of calculus showed that grasping the notion of the derivative causes problems for students as there are many different concepts associated with it (Fig. 9).

![Concept Map of the Derivative](image)

**Figure 9.** Concept Map of the Derivative

(Adapted from Grevholm, 2008)

Students in Orton’s (1983a) study on errors on differentiation produced many unsatisfactory responses which appeared to suggest widespread misconceptions. For example, the idea of the rotating secant was intended to relate to the limit and to differentiation. The students appeared only to focus their attention on the chord $PQ$, despite the fact that the diagram and explanation were intended to try to ensure that this did not happen. Typical unsatisfactory responses included: "The line gets shorter"; "It becomes a point"; "The area gets smaller"; "It disappears". Orton reported that in introducing differentiation, students may need considerable support and time to understand the tangent as the limit of the set of secants.

Vinner & Amit (1990) reported that some students equate the derivative of a function with the equation for the line tangent to the graph of the function at a given point. Also, Makonye and Luneta (2010) found that students had problems in understanding the meaning of the derivative when it appeared as a fraction or the sum of two parts. Students also showed difficulties in applying the chain rule for differentiation and use of the parameters in partial
differentiation. In addition to these, Vinner & Amit (1990) found that students showed errors and misconceptions in executing procedures for maxima and minima problems. Tall (1992) reported that in doing calculus, students showed preference for procedural methods rather than conceptual understanding. He also found that students had difficulty with translating real-world problems into calculus formulations. Students in Tall’s study also exhibited difficulty with calculus notation.

On the other hand, Orton (1983a) noted that students’ routine performance on differentiation items was adequate, but they had little intuitive or conceptual understanding of the derivative concept itself. Also Vinner & Amit (1990) reported that students failed to distinguish the derived function and the derivative. Orton also reported that student difficulties with graphical interpretations of the derivative can occur in the case of straight lines, not only with more complicated curves. He also found that many students (about 20% in his study) confused the derivative at a point with the ordinate, or the y-coordinate of the point of tangency. Meanwhile Leinhardt, Zaslavsky and Stein (1990), reported that students had difficulty in computing slopes from graphs. All researchers recommend a greater emphasis of teaching calculus from graphs to help students deal with difficulties they encounter in learning about the derivative.

In the expansion of $3(a + h)^2$ a term in the definition of the derivative, Orton (1983a) found that 17 students out of 50 had the error of losing the middle term, $6ah$. This showed lack of understanding of the algebraic manipulation of the quadratic expansion. Orton argued that such errors were more structural than executive. He argued that the main point which students must grasp in a graphical study of rate of change concerns the difference between straight lines and curves. He explained that for a curve an average rate of change can be calculated in the same way as for a straight line, but the idea of rate of change at a point on the curve and every point on the curve lead to a different value for the rate of change at different points on the curve. In the case of a straight line one can also obtain the rate of change at a point, but its value is the same for every point. Indeed its value is the same as the average rate of change over an x-interval. Orton (1983a) reported that students found it difficult to understand the rate of change of gradient at a point for the reason that on a curve the gradient was variable whereas in a straight-line it is constant. Orton argued that as the
most elementary graph is the straight line which has a constant rate of change, students found
the distinction between average rate of change and rate of change at a point as having very
little meaning to them. A very prominent error was to state that the formula measured the rate
of change "from P to Q" that is, over the whole section of curve between P and Q. Orton
thought that such a response may have been a guess in response to an unfamiliar situation.

In his study, Orton also noted that students also had difficulty in dealing with differentiation
symbols. Some students could not interpret the meaning of $\frac{dy}{dx}$, $\delta y$ or $\frac{dy}{dx}$ notations.
Orton reported that the symbols which caused the greatest problems to students were $dx$ and
d$y$. This was expected in the sense that the symbols are not really meaningful except when
$\frac{dy}{dx}$ or when used in integration, for example, $\int f(x) \ dx$. Orton reported three
main types of incorrect responses apparent in addition to nil responses. For example, twenty­nine
students explained $dx$ as "the differential of $x$" or "the rate of change of $x$". A further
twenty-five students explained $dx$ as "the limit of $f(x)$ as $x \to 0$". Another twenty students
thought that $dx$ was an "amount of $x$" or "$x$-increment," in other words, was more or less the
same as $\delta x$.

It has been reported that the Leibniz’s $\frac{dy}{dx}$ notation proves to be almost indispensable in the
calculus, yet it can cause serious conceptual problems. Is it a fraction, or a single indivisible
symbol?

What is the relationship between the $dx$ in $\frac{dy}{dx}$ and the $dx$ in $\int f(x) \ dx$

$\frac{dy}{dx} \neq \frac{\delta y}{\delta x} \cdot \frac{du}{dx}$

Can the $du$ be cancelled in the equation $\frac{dy}{dx} = \frac{\delta y}{\delta u} \cdot \frac{du}{dx}$?

On the other hand, failing to give a satisfactory coherent meaning leads to cognitive conflict
which is usually resolved by keeping the various meanings of the differential in separate
compartments.
Orton noted that eleven students responded that \( \frac{dy}{dx} \) and \( \frac{\partial y}{\partial x} \) were the same as each other. Eleven students said something like "\( \frac{\partial y}{\partial x} \) gets smaller until it is so small it is called \( \frac{dy}{dx} \)." Six students said that \( \frac{dy}{dx} \) and \( \frac{\partial y}{\partial x} \) were approximately equal. On the routine aspect of differentiation there was failure to interpret negative and zero rates of change. Twelve students could not respond at all when asked to interpret \( \frac{dy}{dx} = -2 \), and a further ten students could only say "decreasing," or "decreasing gradient," or similar, rather than it implied "decreasing function." Twelve students were unable to interpret \( \frac{dy}{dx} = 0 \), yet in an application task, six of these students were able to put \( \frac{dy}{dx} = 0 \) and obtain values of \( x \) for stationary points. This showed that students experienced confusion in one context but not in the other. Particularly they showed that they did not understand the conceptual basis of the procedures they were so good at.

Difficulties in selecting and using appropriate representations are known to be widespread. Dreyfus & Eisenberg (1986, 1991) reported that the students who were the most successful were invariably those who could flexibly use a variety of approaches: symbolic, numeric, visual. Dreyfus & Eisenberg also reported on students' reluctance to use visual reasoning in calculus. They gave examples where visual representations would solve certain problems almost trivially, yet students refrained from using them because the preference developed over the years was for a numerical and symbolic mode of approach (Eisenberg, 1992; Vinner, 1989). Yet research shows that visual images can provide vital insights to solving mathematics problems.

Thus students experience many difficulties when learning about the derivative; from terminology, notation, and ideas constituting the derivative itself, besides difficulties with pre-calculus topics. The next section discusses the difficulties learners encounter in applying the derivative in solving problems.
4.12 Research on applying calculus to solving problems

The application of calculus in solving multitudes of theoretical and practical problems in diverse areas of human expertise represents its immense utility. Its immense utility is the reason why researchers have always found it imperative to research barriers learners have in learning and applying calculus. White & Mitchelmore (1996) researched on learners' problems in the application of calculus. They reported that in calculus, the context of an application problem may be a realistic or artificial "real-world" situation, or it may be a theoretical mathematical context.

According to Stewart (2004), a mathematical model is a mathematical description (often a function or equation) of a real world phenomenon such as the size of a population, demand for a product, the speed of a falling object, the life expectancy of a person at birth or the cost of emission reductions. The purpose of the model is to understand the phenomenon and perhaps to make a prediction (see Fig. 10).

![Figure 10. Mathematics modelling process](Adapted from Stewart (2004))
According to Stewart (2004), mathematical modelling begins by registering the real world situation as a problem. Then a mathematical model is created for it. This occurs through the identification and naming of independent and dependent variables in the real-world problem. It involves making assumptions that simplify the problem so that it is mathematically solvable. This mathematical model must capture the most essential and key aspects of the problem. Then knowledge of the physical situations and that of mathematics help to formulate functions or equations that describe the real-world problem mathematically. Where no clear mathematical relation is discernible, data is collected, tabulated or graphed in order to investigate patterns that it may form. The equations are solved using the tools of mathematics such as calculus to derive mathematical conclusions. Then the mathematical conclusions are used to interpret the real-world problem by offering explanations and predictions. Finally, the explanations and predictions are checked against new real data. If the predictions do not compare well with data, the mathematics model is refined or another may be formulated to start the cycle again. If the mathematical model was good then it can be preserved to solve quickly the real-world problems of the same class should they occur again. Given a mathematical model may not be a completely accurate representation of the physical situation, as it is idealised. However a good model provides valuable conclusions about the real-world problem.

### 4.13 Misconceptions

What are mathematics misconceptions? Chi (2005) showed that misconceptions can be portrayed in one of two ways: as either fragmented or coherent. A fragmented view considers misconceptions as “a set of loosely connected and reinforcing ideas” (Smith, diSessa & Roschelle, 1993). For example, a misconception such as “multiplication makes bigger” (Olivier, 1992), is a simple abstraction of a common early experience with multiplication. In this fragmented view, students have a naïve explanation of a mathematical phenomenon. In contrast, a coherent view claims that misconceptions are not merely inaccurate or incomplete isolated pieces of knowledge, but rather, they can be portrayed as alternative justifiable conceptions. As Anderson and Smith (1987) put it, “there are consistent understandable patterns in the incorrect answers that students give.” (p. 90). An example of a theory like
meta-misconception could be that mathematics is essentially a set of accumulated calculation procedures.

The inability of many teachers to address their pupils’ misconceptions in mathematics accounts more than anything else for their failure to promote students’ conceptual understanding in this subject. Students tend to memorise the action schemes of their teachers without understanding the concepts being developed (Hoffer, 1981; Clements & Battista, 1990).

A key feature of this research is on children’s misconceptions (Nesher, 1987; Smith et al., 1993) or rational errors (Ben-Zeev, 1996, 1998). Similarly, Confrey (1981) argues that “students’ errors are reasoned and not capricious” (p.11). While mathematically incorrect, misconceptions make sense when viewed from a more limited perspective, which is likely to be the perspective of the child. Nesher (1987) and Erlwanger (1973) explicated that the learner’s knowledge generating misconceptions have contexts of successful use but fail when applied to more advanced mathematical concepts.

Misconceptions can occur because learners cannot reconcile different representations of the same concept. For example, Ball & Wilson (1996) reported that third graders would use manipulatives to correctly find the sum of $\frac{1}{6} + \frac{1}{6}$ to get $\frac{2}{6}$ but when using symbols would get $\frac{2}{12}$. They observed that the children were comfortable with these results and did not try to resolve the situation as they felt that the context they worked with affected the answer. So teachers must work with learners’ current knowledge; that is learners’ correct concepts as well as learners’ incorrect concepts as shown in the above example. We have seen in the several cases considered earlier that learners have misconceptions because of different representations of the same mathematical concept (see Hiebert & Wearner, 1986; Nesher, 1987).

Nesher (1987) reported that in a learning situation the expertise of the learner is to make errors. She argued that a good instructional design seeks errors rather than condemn them. She recommended that the teacher should use errors as resources for teaching for real
learning to occur. Such teaching is constructivist in that it takes into account the thinking of learners. It is also socio-cultural in that it is teaching which engages the learner’s ZPD and seeks to scaffold the learner in dealing with the concepts they are still not able to deal with without expert help. It is crucial to know specifically how the already-known knowledge may interfere with new experiences.

Analysing errors made by students provides teachers insight regarding their students’ "procedural and conceptual misunderstandings" (Ginsburg, 1977; Mercer & Mercer, 2005). The errors that students make can sometimes be even more informative to teachers than correct responses. Student errors often provide insight into students' misunderstandings about a particular mathematics concept or skill.

Errors are mistakes in that they are incorrect responses to questions. These incorrect responses could be due to inattention or they could be a result of planning. Some errors are trivial but some represent a profound misunderstanding of the situation. When students produce errors in the process of engaging with mathematics, it can be a moment of exploration and learning if the error sets up an occasion of serious dialogue, exchange and consideration of the ideas involved in making the mistake. Carelessness and lack of understanding of basic facts remain the major causes of errors in mathematics. However, it is interesting to note that even careless and uninformed students still tend to follow the same pattern of errors each time they attempt to solve similar problems because this procedure makes sense to them.

4.14 Some approaches to teaching calculus that cause misconceptions

This view of errors as ‘sites for learning’ (Borasi, 1994) is critical in education. With such a philosophy, mistakes are accepted as a precondition for learning. As Smith, diSessa, and Roschelle (1993) state, “it is time to move beyond simple models of knowledge and learning in which novice misconceptions are replaced by appropriate expert concepts” (p. 117). Now researchers need to redouble their efforts to understand the nature of student misconceptions and document how students move from amateur conceptions toward scientific concepts.
Research on student errors has an early history. For instance, research in errors in mathematics education is recorded in 1925 by Buswell and Judd. Such scholars have diagnosed a variety of errors that are resistant to instruction. Many theoretical perspectives have been offered of how teachers may view student errors, but the Behaviourist perspective of uprooting and eliminating errors is now discouraged (Smith et al., 1993). This included giving negative reinforcement such as punishing those who make errors and giving rewards to those who avoided making errors in mathematics.

Teachers know that conceptual questions are rarely answered correctly, by their learners, so they compromise by giving learners procedural questions which they know are easy to answer. Often those students who start a calculus course with mainly a diet of technical procedures, often encounter insurmountable problems when they are required to apply calculus more rigorously. At that stage, they commit many errors and misconceptions. This is because technical approaches lack the deeper thinking required in extended abstracts (Biggs & Collis, 1982) needed in solving non-routine problems in calculus.

Dali (2008) has advised that it is not wise for teachers and parents to openly criticise a learner's mistake. He argued that it is wiser to first understand the learners' mistakes and then find the wise ways to sublimate them so that the learner could correct them for him/herself. Thus, errors and misconceptions are viewed as opportunities for deepening one's understanding of learners' difficulties and are of critical importance in furthering learning. The view of errors as a vehicle for learning, to be understood rather than to be replaced and eliminated, is gaining support in mathematics education (Smith, et al., 1993).

As argued earlier, one of the main reasons responsible for errors is the misconception of over-generalising. As Davydov (1995) explains, "generalization is regarded, as a rule, as inseparably linked to the process of abstracting" (p. 13). Generalising involves abstracting particular processes in an action. It involves deliberately extrapolating the range of reasoning beyond the specific examples considered, attempting to expose commonality across situations. Generalisation involves extending the reasoning to the focus not on the cases or specific situations themselves but more on the patterns and structures across situations (Kaput, 1995, p. 136). Thus when learners make errors, they will be thinking, attempting to
generalise across cases. Mason (1996) has commented that generalizing involves, “seeing a generality through a particular and seeing the particular in the general” (p. 65).

Students’ erratic thinking can be categorised into different levels (see Fig. 11) below.

![Figure 11. Level of errors in mathematics](image)

The level, ‘Not an Error’ occurs if a student’s response is correct and does not have an error. In an ‘Instance-Level Error’, the student makes an error only in a particular instance or a few isolated instances of the situation. With ‘Problem-Level Errors’, errors are viewed as incorrect for any application in a particular problem context, while in ‘Cross-Problem Level Errors’, they are recognised to occur in different classes of problems.

If a misconception is the result of a mis-application of a rule, as Drews (2006) points out, then the learner’s understanding of a rule is procedural or instrumental, otherwise the learner can judge why the rule does not apply and be able to modify it if necessary. If it is an error of over-generalisation or under-generalisation, it means that the learner’s conceptual grasp is weak. The study of errors and misconceptions is important in problem solving due to the Einstellung effect. Learners continue to resort to previously successful algorithms even though fundamental conditions of the mathematical task have changed. According to Haylock (1987), in the problem solving set, Einstellung or learned helplessness denotes a learner’s
tendency to dogmatically resort to an inappropriate but once successful strategy to solve a
new problem to the exclusion of new and more effective ways of solving the problem. The
Eistellung effect, amply demonstrates that within previous success therein lie the seeds of
failure when learners exhibit inflexible thinking. Often learners are confounded that a method
that had worked perfectly in the past is found to be lacking in a new task. Hence, the
Einstellung effect has a negative effect to the solution of new problems as learners stick to
old procedures. It results in negative transfer, culminating in confused and unsuccessful
problem solving. In calculus classes this results in inhibited progress as the learner becomes
resistant and impervious to new knowledge and techniques.

4.15 A layman shakes the foundations of calculus: Bishop Berkeley

Often, in science, the exposure of its theory to criticism can lead to questions which have
profound effects (Pierce, 1958/1887). This is the case with George Berkeley who was an Irish
bishop and philosopher in the 17th century, when calculus had just been invented. His work
provides a fine example of how identification of an error or misunderstanding can fuel the
advancement of knowledge. His contribution to mathematics was his attack on the logical
foundation of the calculus as developed by Isaac Newton (Fig. 12) and Gottfried Wilhelm
Leibnitz (Fig. 13).
In his 1734, publication; ‘The Analyst: A Discourse Addressed to an Infidel Mathematician’, he attacked the principle of infinitesimal change leading to the limit which is the genesis of
calculus. Berkeley coined the phrase: ‘Ghosts of Departed Quantities’, which captures the gist of his criticism in that he saw quantities tending to zero as ceasing to exist once there was nobody around to perceive them. To him, a quantity was there then suddenly it was not there, metaphorically this was the very idea used to denote the derivative. However, Berkeley regarded this as intellectual dishonesty. This Bishop Berkeley story is re-played all the time in mathematics classes when learners and teachers do not understand each other when they do mathematics.

Berkeley's criticisms or misconceptions about the limit concept were important in that they urged mathematicians to base calculus on a more solid and logical foundation. This was done 100 years later when a group of mathematicians; Augustine Louis Cauchy (Fig. 14), Karl Weierstrass, and Benhard Riemann reformulated Calculus in terms of limits rather than on infinitesimals. Finally, Calculus was put on a logical footing and was better able to extend its results to other areas of mathematics. It also came to terms with some of the more subtle aspects of its theory.

Figure 14. Augustine-Louis Cauchy (1789-1857)

(Adapted from Eves, 1990)
This uncertainty in mathematics is carried on to this day, everyday in mathematics classrooms and it is the duty of research to come to understand this interplay to further mathematical teaching and learning for the benefit of society.

4.16 Conclusion

Developing students' understanding in mathematics is an important but difficult goal (van de Waalle, 2004). Being aware of student difficulties and the sources of the difficulties, and designing instruction to diminish them, are important steps in achieving this goal. Student difficulties in learning written symbols, concepts and procedures can be reduced by creating learning environments that help students build connections between their formal and informal mathematical knowledge; using appropriate representations depending on the given problem context; and helping them connect procedural and conceptual knowledge. This process is assisted by error diagnoses that inform teaching and learning to focus on learners' needs. The literature review on these items goes a long way to help this research by illuminating what has been done and found and what has not to foreground this research.

To sum up, as Porter & Masingila (2000) have reported, it is not unusual to find students who use calculus procedures with little or no understanding of the concepts behind those procedures (Hiebert & Lefèvre, 1986; Schoenfeld, 1985). In fact, some students are not even aware that there are concepts underlying the procedures they use (Oaks, 1988, 1990). Such students may not realize that there is meaning in the calculus. They believe that doing mathematics means performing (pointless, sic!) operations on meaningless symbols, and that everyone, including the teacher, learns mathematics un-meaningfully (Oaks, 1988). Oaks suggested that students’ difficulties in doing mathematics can be related to their conceptions of mathematics (i.e., the ways in which the students view mathematics). This view had been echoed by Skemp (1976) when he discussed notions of understanding mathematics. A rote conception of mathematics can interfere with students’ procedural ability. It can also prevent students from gaining an understanding of mathematical concepts. Both procedural ability and conceptual understanding are necessary for success in mathematics (Hiebert and Carpenter, 1992). Without them mathematics learning is fraught with errors and misconceptions.
CHAPTER 5: RESEARCH METHODOLOGY

5.1 Introduction

This chapter explains the theory and practice of the methodology, design and data analysis (Brown & Dowling, 1998; Miller, 1983) underpinning this study. This study focuses on learner errors in calculus, specifically in the introductory differentiation assessment standards stipulated in the National Curriculum Statement (NCS) mathematics syllabus at Further Education and Training (FET) level in South Africa (see appendix A). Philosophically, the methodology, research design and data analysis ought to align to one or a mixture of research paradigms befitting the study. Merriam (1992, p. 3) explained that ‘linking research and philosophical traditions helps to situate the special characteristics of different research orientations or paradigms’. Therefore, undertaking the study within a particular philosophical research tradition is professional practice that clarifies. The research paradigm chosen determines the research methodology and anchors the study in the appropriate ontological and epistemological perspective. This is because different research paradigms hold different views on the nature of knowledge and how we come to know of that knowledge. It is crucial to weigh and discuss competing research paradigms as they relate to research in education in general and mathematics education in particular. Then it is necessary to justify why the chosen paradigm befits the study on learner calculus errors in a research design that obtains data from learners’ examination scripts. That task is done next followed by other sections of research design, sampling, content analysis, pilot study, reliability and validity of the study as well as research ethics.

According to Hatch (2003), the main research paradigms are positivism, post-positivism, interpretivism, critical theory and post-modernism. Each ascribes to different epistemological and ontological notions.

5.2 What is a research paradigm?

The term paradigm was introduced by Thomas Kuhn. It originates from the Greek word paradeiknyai which means to show side by side. A paradigm is a presumed world view by
which a researcher regards phenomena. Kuhn (1970) introduced the term paradigm as "universally recognized scientific achievements that for a time provide model problems and solutions to a community of practitioners" (p.viii). Kuhn (1970) wrote; "it stands for the entire constellation of beliefs, values and techniques, and so on shared by the members of a community." (p. 175). The word paradigm connotes the idea of a mental picture or pattern of thought. In general, a paradigm may be viewed as a set of basic beliefs that deal with ultimates or first principles. According to Guba & Lincoln (1994), it represents a worldview that defines for, its holder, the nature of the 'world' the individual’s place in it, and the range of possible relationships to that world and its parts. “The beliefs are basic in the sense that they must be accepted simply on faith (however well argued); there is no way to establish their ultimate truthfulness” (1994, p. 107).

Also Henning, Gravett and van Rensburg (2003) view a paradigm as a framework within which theories are built and fundamentally influence how one sees the world. It is viewed as determining one’s perspective and understanding of how things are connected and way that things are done. Holding a particular world view influences personal behaviour, professional practice, and ultimately the position taken with regard to the subject of research. For example, a researcher who holds strong positivist views believes that research must produce testable and duplicable results if not then that research is wishy washy and of no particular use.

Furthermore, Guba and Lincoln (1994) argue that research paradigms define for the researcher “what it is they care about, and what falls within and outside the limits of legitimate research” (p. 108). Every research paradigm is constituted of four closely linked elements; an epistemology, an ontology, an axiology and a methodology.

The term epistemology itself originates from the Greek term epistēmē which means knowledge, and from epistanaí; that is to understand, know, and therefore epi- + histanaí; to cause to stand (Gettier, 1963). Epistemology then is the study or a theory of the nature and grounds of knowledge especially with reference to its limits and validity (Burrell & Gareth, 1979). It is concerned with the nature, sources and validity of knowledge and how we come to know. Epistemology has been primarily concerned with propositional knowledge, which is
knowledge that such-and-such is true, rather than other forms of knowledge, for example, procedural knowledge; knowledge of how to do something or declarative knowledge. There is a vast array of views about propositional knowledge, but one virtually universal presupposition is that knowledge is true belief (Klein, 1998, 2005). The notion of true belief is explained below. The historically dominant epistemological tradition asserts that it is the quality of the reasons and basis for our beliefs that converts true beliefs into knowledge. When the reasons are sufficiently cogent, we have knowledge. This sums up the Justified True Beliefs (JTB) (Gettier, 1963) as the basis for knowledge. Since epistemology is the theory of knowledge, a central question of the discipline is: under what conditions does a subject know something to be the case? Justification is the process of proving that a case is true and plays a central part in epistemology.

The question of epistemology in this research was important in that the researcher wished to obtain and explore the justified true beliefs (JTB) about the nature of students' calculus errors. Otherwise the findings would not be true knowledge but only subjective views, not validated by data on how students think. Mudavanhu (2010), argue that epistemologically, there are three inherent issues beginning with the relationship between the researcher and the researched. The relationship assumed in the current study is "subjective, value-mediated, negotiated as in the social world where the relationship between the researcher and social phenomena is interactive", (Rotchie & Lewis, 2003, p. 13). The assumption is "that knowledge is essentially a relation between the learner and the phenomena being learned – between the knower and the known, the learner and the learned", (Booth, 2008, p.451). The second issue addresses theories about 'truth'. The stance adopted in the current study is multiple realities constructed and altered by researcher and participants, and that knowledge is the best understanding of phenomenon achieved so far (Laverty, 2003). The third issue looks at the way in which knowledge is acquired; its methodology. Rotchie & Lewis (2003), argue that "induction or looking for patterns and associations derived from observations of the world; and deduction whereby propositions or hypotheses are reached theoretically, through a logically derived process" (2003, p. 14). Rotchie & Lewis state that induction and deduction are the most used analytical methods that help to create knowledge. Also rationalistic thinking is crucial to draw logical conclusions on the themes emerging from the top-down as well as the bottom-up analysis. The ontological dimensions of a paradigm are discussed next.
The word ‘ontology’ is used to refer to philosophical investigation of existence, or being (Klein, 1998, 2005). Such investigation may be directed towards the concept of being, asking what ‘being’ means, or what it is for something to exist; it may also be concerned with the question ‘what exists?’, or ‘what general sorts of thing are there?’ The philosopher’s ontology generally refers to the kinds of things taken to exist by that thinker. The ontology of a theory then refers to things that would have to exist for that theory to be true. Ontology then is concerned with the nature of reality; the beliefs about ‘what there is to know’ about the world. Realism, materialism, idealism and metaphysics discussed below are competing ontological perspectives that lie in a continuum; on one extreme sits realism and on the other metaphysics.

The first ontological perspective discussed is realism. Rotchie & Lewis (2003, p. 11) argue that, realism posits an external reality existing independently of people's beliefs or understanding about it. This means that there exists a disparity between what the world really is and the understanding of the world held by individuals, which duty research is to bridge that disparity. The second ontological perspective is materialism. According to Rotchie & Lewis (2003, p. 11), "Materialism claims that there is a real world but that only material features, such as economic relations, or physical features of that world are reality. Values, beliefs or experiences are 'epiphenomena' - that is features that arise from, but do not shape, the material world". Materialism is closely related to Marxist philosophical thought. This stance is difficult to sustain within an interpretive research as this one. The third ontological perspective is idealism. According to Rotchie & Lewis (2003, p. 11), "idealism asserts that reality is only knowable through the human mind and through socially constructed meanings". The last ontological perspective considered here is metaphysics. Metaphysics concerns the beliefs in a spiritual and immaterial world that humans hold of the supernatural powers of the world. With regard to the aforementioned ontologies, most contemporary researchers use qualitative methods which maintain that the social world is regulated by shared understandings and hence the laws that govern it are relativistic, ever-changing and never fixed which is idealistic (Merriam, 1992; Brown & Dowling, 1998). Therefore, idealist ontology is adopted in this thesis.
Realist ontology links comfortably with the positivist paradigm while idealism compliments the constructivist-fallibilist research paradigm. Realism links with positivism in that both assume knowledge in the world which has objective existence whereas idealism like the constructivist-fallibilist paradigm holds that knowledge is essentially subjective to the knower. Husserlian phenomenology investigating subjective phenomena argues that essential truths about reality are grounded in people’s lived experiences (Rolfe, 2006). This implies that it is the research participant who has direct access to “essential truths about reality”, which a researcher can uncover by bracketing all his pre-conceived ideas (Mudavanhu, 2010). Thus phenomenology is an interpretivist methodology (Rolfe, 2006). This methodology is suitable to this study as it studies the phenomena of students’ mathematical misconceptions.

5.3 Classifying research paradigms

Guba and Lincoln (1994) state that the basic beliefs that define a particular research paradigm may be summarised by the responses given to three fundamental questions. The first question is that of ontology and its central question is; what is the form and nature of reality? The second question is epistemological; what is the basic belief about knowledge and what can be known? The third question is methodological and is intimately related to the above two; how can the researcher go about finding out whatever s/he believes can be known and is worth knowing?

Haack (2003) describes in detail the features of the three main research paradigms with respect to their ontology, epistemology, methodology as well as the roles of theory and researchers. Ontologically, positivism asserts that objective and true reality exists which is governed by unchangeable natural cause-effect laws (Voce, 2004). Positivism regards reality as consisting of stable pre-existing patterns or order that can be discovered. Reality is regarded as neither time nor context-bound and that it can be generalised. According to Voce (2004), positivist research findings are true if they can be measured, hence the scientific method is prominent here. Here, the role of research is to uncover reality which is in essence is objective and apersonal. Positivist researchers often control the investigation, bracketing their feelings and emotions outside the phenomena being investigated. Axiologically,
positivism assumes scientism (Haack, 2003; Hayek, 1980), the ideology that science is objective and value-free and the best way to acquire knowledge.

The second main school of thought is the interpretive paradigm. The interpretive paradigm arose when many researchers were dissatisfied with positivism which they felt had many shortfalls. According to Voce (2004), Interpretivism regards the world as complex and dynamic and constructed, interpreted and experienced by people in their interactions with each other and with wider social systems. Interpretivism assumes that reality is subjective and people experience reality in different ways. Subjective reality is important and what people think, feel and see is crucial. Interpretivist philosophy assumes that reality can only be imperfectly grasped. People are seen as social beings who create meaning and who constantly make sense of their worlds. Epistemologically, knowledge is constructed and knowledge is about the way in which people make meaning in their lives, and what meaning they make. Theories constructed from multiple realities are revisable and sensitive to context. Ontologically, interpretivism embraces Fallibilism (Lakatos, 1976; Powell, 2001) in that absolute knowledge may not be possible. Research is viewed as a communal process, informed by participants, and scrutinised and endorsed by others. The researcher is co-creator of meaning and brings own subjective experience to the research. The researcher tries to develop an understanding of the whole and a deep understanding of how each part relates and is connected to the whole. The role of research is to study mental, social, cultural phenomena — in an endeavour to understand why people, and natural and unnatural events behave the way they do. Research aims to grasp the ‘meaning’ of phenomena and describe multiple realities. Axiologically, interpretivism holds that values are an integral part of social life and that no values are wrong; only different (Voce, 2004). The interpretivist paradigm suits this study which focuses on how different learners think as they respond to mathematical tasks. It helps to study how different learners construct meaning, even if that meaning contains misconceptions.

The shortfalls of positivism raised by the interpretive paradigm are that human beings cannot be treated like inanimate materials as humans create reality based on their subjective perceptions. It is argued that humans have free will and so people should be studied as active agents in order to get meaningful interpretation of their condition.
According to Merriam (1992), the methods of interpretive research are; unstructured observation, open interviewing, discourse and content analysis. Interpretive research ultimately strives to capture "insider" knowledge. Knowledge from interpretive research emanates from Grounded Theory (Glaser & Strauss, 1973). This research uses content analysis as the main methodology for analysis of examination scripts.

The third main research paradigm is critical theory. This paradigm originated from the Frankfurt School scholars (see Habermas, 1990). This paradigm focuses on oppression. For Freire (1970) the way students are taught and what they are taught represents a political agenda which can serve the interests of oppressors. He argued for critical education where learners have to liberate themselves from the banking concept of education which serves the interest of the oppressors. The banking concept of education assumed that learners are empty vessels to be filled by the teacher. The researcher critically agrees with Freire in that learners are not empty vessels, to teach them well, we have to listen to them to have insight into their difficulties. For Apple (1997), critical theory postulates a hierarchical society structured to serve the interests of a tiny but dominant class. In critical theory, the main purpose of research is to uncover and lay bare social injustice and inequality with the goal of liberating the majority from oppression in order to set up an egalitarian society where all people are treated equally and enjoy economic equity. Ontologically, critical theory is concerned with conflicting, underlying social, political, cultural, economic, and ethnic or gender structures. It is argued that people can reconstruct a better world through critical reflection and action. Epistemologically, critical theory perceives knowledge as a source of power. Critical theory is constituted by the lived experience and the social relations that structure these experiences. Events are understood within the social, political and economic contexts of the agents. Within this paradigm, theories are built from deconstructing the world and from analysing the inherent power relationships. The role of research is to promote critical consciousness as well as breaking down institutional structures that produce oppressive ideologies and social inequalities. Critical research aims to shift the balance of power so that power may be more equitably distributed, to bring political and economic emancipation (Geuss, 1981; Habermas, 1990). Critical research aims to unveil illusions and false beliefs that hide power and 'objective' conditions. The methodology of the research is participatory action research and dialogical methods which encourage dialogue between researchers and researched.
The foregoing discussions on the nature of research paradigms illustrate the importance for researchers to properly position their research in a particular philosophical orientation which helps to place the research in a scholarship tradition. This placing is critical since a paradigmic mismatch between research questions, methodology, research design and data analysis creates a confused and contradictory matrix for the research and its readers.

According to the interpretive paradigm under which this research mainly falls, “education is a process and school is a lived experience, understanding and meaning of the process or experience constitutes the knowledge to be gained from an inductive, theory generating mode of inquiry” (Merriam, 1992, p. 4). The researcher used approaches which seek to understand how students think in committing errors. Firstly, this analysis helped me to gain insights into students’ meanings as required in the interpretive paradigm. In interpretive research, new knowledge or findings emerge from the interactions between the researcher and data (Creswell, 2008). Secondly, in interpretive research the element of subjectivity is valued, and the research is shaped by questions asked and responses obtained, and the ways meanings are generated, negotiated and interpreted (Ajjawi & Higgs, 2007). I argue that the interpretive paradigm was the most suited for the research problem and research questions in this study because of the dialectical nature of this paradigm.

Qualitative methods dovetail with the interpretive paradigm. Creswell (2008) defines qualitative research as: “an inquiry process of understanding a social or human problem, based on building a complex, holistic picture, formed from words, reporting detailed views of informants, and conducted in a natural setting” (p.1). Qualitative research is “any kind of research that produces findings not arrived at by means of statistical procedures or other means of quantification” (Strauss & Corbin, 1990, p. 17). In order to understand students’ perceptions and what went on in answer settings; the synergies, contradictions and tensions in and between these settings, it is necessary to use a qualitative method.

This research aimed to produce thick and rich descriptions of students’ misconceptions in calculus. This can be aided by phenomenography. Booth (2008) argues that phenomenography is an important approach for understanding people's ways of experiencing
the world. The aim of phenomenography is to describe qualitatively different ways of experiencing phenomena, in this study to illuminate the variations of misconceptions in ways in answering mathematics questions.

In accordance with the extended discussions above, the selected research paradigms suitable for this research is the interpretive, subordinated by the positivist. The qualitative takes precedence over the quantitative. This selection of research methodology is argued below. In this study, interpretive inquiry aims to understand how people experience the world and construct subjective meanings of their experienced world resulting in multi-knowledges (Booth, 2008). Such multiple knowledges are evident in how learners think differently to solve mathematics problems. Each learner develops a distinct way of reasoning and thinking that may not be similar to the way others think, yet that thinking is important in its own right. However, the positivist paradigm which centres on quantification and measurement of research variables helps to bring understand the research results more fully. Quantification is important so that the most important and prevalent errors are highlighted. The frequency of different error categories in the scripts is better reported quantitatively rather than qualitatively.

Positivism linked to quantitative methodology plays a minor role in measuring the prevalence of specific errors and misconceptions as well as correlations of these among different learner performance groups. In the next section, I describe the qualitative and quantitative research methodology used in this research.

5.4 Qualitative and quantitative methods

A qualitative study methodology suited the proposed research as it concerns description of thinking processes exhibited in learners’ answers. From an epistemological perspective qualitative research can be described as “interpretive” or ‘constructivist’ with the emphasis on understanding the world through the perceptions of its participants (Bryman, 2004; Hatch, 2003; Yin, 1994). Merriam (1992) highlights the characteristics of qualitative research. Firstly, it is concerned with the meanings that people have constructed in their contexts. These contexts occur in an empirical field (Brown & Dowling, 1998), which subsumes the
space in which data is collected and findings arise. Merriam (1992) argues that, achieving a deep understanding of a specific phenomenon and to probe beneath the surface of a situation and to provide a rich context for understanding the phenomena under study is the aim of qualitative research. Secondly, in qualitative research, the researcher is the primary ‘instrument’ for data collection and analysis. In general, this implies that qualitative research depends heavily on the “integrative powers of the researcher” (Benbasat, Goldstein, & Mead, 1987). The researcher was meticulous in devising a research design, selecting an appropriate sample, and being most aware of how data was collected and analysed. The researcher was aware of personal biases. In this regard, he devised ways of minimising these so that these personal limitations did not affect the validity of research findings. I did this by ensuring that I analysed the data over and over again so that each time I analysed the data I scrutinised my earlier analysis. Thirdly, qualitative research usually involves fieldwork with emphasis on studying phenomena in their natural setting. In this study, fieldwork was substituted with the actual 2008 examination scripts. Fourthly, in analysing qualitative research, an inductive approach is used together with a typology. According to Hatch (2003), inductive analysis seeks to uncover patterns and regularities in data that are hitherto concealed from consciousness whereas in typological analysis, the researcher approaches data with the intention of assigning it to pre-conceived categories. Consequently, qualitative analysis is theory/hypothesis building, rather than hypothesis testing. The product of qualitative research is richly descriptive. Thus qualitative research is exploratory seeking to explain and understand processes in depth rather than focusing on outcomes or outside appearances only. Qualitative research focuses on process meaning and understanding. In order to arrive at judgements regarding the nature of learner misconceptions, I would need “rich descriptive data” (Cohen & Manion 1994, p. 29). Words and graphics are therefore used to convey the outcomes of the research findings. Data are also in the form of the participants’ own texts, such as vignettes are used to explain and support findings.

According to Yin (1994), qualitative studies aim to gain an understanding of underlying reasons and motivations and to provide insights into the setting of a problem, generating ideas and/or hypotheses for later quantitative research. The aim is to obtain a complete, detailed description. Qualitative studies aim to uncover prevalent trends in thought and opinion. Researchers may only know roughly in advance what they are looking for.
In a qualitative study, the sample is usually purposefully selected. It is usually a small non-probability sample that is non-representative. In a non-probability sample, the sample chosen is usually non-representative of the whole population. The non-representativeness is not important because the findings of the study are not generalisable to the population but are important in generating theory that may be later tested in a generalised population. Often, the data is in the form of words, pictures or objects. Qualitative data is more 'rich', but it is also time consuming to analyse. The findings in qualitative research are not necessarily conclusive and cannot be used to make generalizations about the population of interest. As Yin has argued, qualitative research is about theory generating rather than theory testing. Qualitative research develops an initial understanding and sound base for further decision making. Qualitative research may be necessary in situations where it is unclear what exactly is being looked for in a study, so that the researcher needs to be able to assess what data is important and what is not. While the quantitative researcher generally knows exactly what to look for before the research begins, in qualitative research the focus of the study becomes more apparent as time progresses.

Often data presented from qualitative research is much less concrete than pure numbers. Instead, qualitative research may yield stories, or pictures, or descriptions of feelings and emotions and ways of thinking. The interpretations given by research subjects are given weight in qualitative research, so there is no seeking to limit their bias. At the same time, researchers tend to become more emotionally attached to qualitative research, and so their own bias may heavily influence the results. Many scholars (see Yin, 1994) argue that the primary benefit derived from qualitative research is that it allows for a richer study of a subject, and allows for information to be gathered that would otherwise be entirely missed by a quantitative approach. In the next section I discuss features of quantitative research which underpins positivism.

Quantitative data is numerical in form. Quantitative research is more closely aligned with what is viewed as the classical scientific paradigm of positivism. It involves gathering data that is absolute, such as numerical data, so that it can be examined in as unbiased a manner as possible. Quantitative research generally comes later in a research project, once the scope of the project is well understood through qualitative methods (Hatch, 2003). However in certain
circumstances it does not matter whether one starts data collection and analysis with qualitative methods or starts with quantitative methods.

The main idea behind quantitative research is to be able to separate things easily so that they can be counted and statistically modelled to remove intervening variables likely to distract from the intent of the research. At the outset, a researcher generally has a very clear idea what is being measured before they start measuring it, and the study is set up with controls and a very clear blueprint. The result of quantitative research is a collection of numbers, which can be subjected to statistical analysis to corroborate or negate a prior set hypothesis.

Being objective, emotionally detached and separate from the research data is a key aspect of quantitative research to eliminate researcher bias. The purpose of quantitative research is to quantify data and generalise results from a sample to the population of interest. The aim is to classify features, count them, and construct statistical models in an attempt to explain what is observed. As a result, the sample for a quantitative study must be carefully chosen so that it is representative of the population of interest to eliminate sample bias. Quantitative data is more compact than qualitative data and capable of having hypotheses tested. However, its weakness is that it often misses contextual detail. Because of this, sometimes quantitative studies are followed by qualitative research which explores the quantitative findings further. The data is usually in the form of numbers and statistics. However, even qualitative data can be put into categories or measured with a Likert scale on which statistical techniques can then be applied. This technique was used in this study as categories of learners' errors and misconceptions were also statistically quantified. The outcome of quantitative research is to make a judgement and recommend a course of action.
I chose a qualitative methodology, because research into understanding the thinking process requires a detailed description of what learners think in order to capture the essence of what they have written in response to mathematics task demands. Moreover, a qualitative methodology seeks patterns and relationships. In this context, a qualitative study reveals the processes that affect learner responses to calculus questions. A qualitative study was more suitable for this study rather than a quantitative one because the intention was to comprehensively understand learners' errors and misconceptions. The quantitative method complemented the qualitative study by exploring the prevalence learner errors in the scripts. This occurred when quantitative methods were used to analyse performance and error prevalence before qualitative methods were used to deepen the understanding of the errors.

5.5 Primary data and secondary data

Data collected in research can be classified as primary or secondary data (Opie, 2004). Primary data is first-hand data collected by a researcher solely for the purpose of the research. The popular ways to collect primary data consist of surveys, interviews, observations and focus groups. Primary data is more accommodating to the research as it shows latest information. Primary data is accumulated directly by the researcher particularly to meet the
research objective of the project. Primary data is completely tailor-made and there is no problem of adjustments often necessary in secondary data. Primary data takes a lot more time to collect than secondary data and the unit cost of such data is relatively high. Often expensive and sophisticated equipment such as video cameras and DVDs are needed to collect primary data effectively, and usually the data has to be transcribed as well. This makes collection of primary data much more complicated and involved although this advantage is cancelled out by the quality data collected.

Secondary data, however, is obtained from already collected information but which was collected for some other purpose. In secondary data, information relates to a past period. Hence, it lacks aptness and therefore, it may have unsatisfactory value. Secondary data is obtained from some other organisation than the one instantaneously interested with a current research project. Although secondary data may be old, it may be the only possible source of desired data as recreating the past is impossible. Secondary data is available rapidly and inexpensively.

The problem of secondary data is that often the reliability, accuracy and integrity of the data is uncertain. In short, primary data is expensive and difficult to acquire, but it's trustworthy. Secondary data may be easily accessible, but it must be treated with caution as the purpose for its collection and how it was collected must be fully understood.

In this research, primary-secondary data was used. Although students’ answered the examination questions for the purpose of passing the mathematics examination, the data to this study is primary as it directly attached to the concern of this research. The researcher found that the data was appropriate because many scripts had been randomly selected from about sixty different schools in Gauteng Province. It would have been financially and logistically difficult for the researcher to get data as much and varied from sixty schools, scattered across the province by himself. Further this is real and useful data as learners were quite serious as they wrote the high-stakes examinations set to determine their future. Learners could be assumed to have been diligent and earnest in their efforts to get the best grade possible. I assume that students may not have been so serious in their responses had I given them questions myself. I assume that they would not have taken the questions as
seriously as it would have been just an uneventful exercise. For the above reasons, the data used in this research was very reliable and valid.

5.6 Research Design

The research design takes into account the research questions, theoretical and conceptual framework as well as the data collection methods and analysis (Bergman, 2008). It attempts to bring these together for the purpose of ultimately realising research goals in a systematic manner. In writing about research design, Yin (1994) explicates that, “a research design is the logic that links the data to be collected to the initial questions of the study” (p. 18). Thus, a research design is a mapping strategy, essentially a statement of the object of the inquiry and the strategies for collecting the evidence and reporting the finding. A research design guides the researcher on how to answer the research questions, what data to collect and how to analyse and interpret observations (Yin, 1994). In this study, the research design concerned analysing scripts using the conceptual framework discussed in Chapter 3. The content analysis technique was used to analyse the scripts. The explanations of the errors were guided with the lens of the theoretical framework of Chapter 2.

5.7 Sampling

On sampling, Merriam (1992), explicates that, “once the general problem has been identified, the task becomes to select the unit of analysis” (p. 60). The unit of analysis is the object which is to be studied in terms of research variables that constitute the construct of interest (Brown & Dowling, 1998; McMillan & Schumacher, 2001; Yin, 1994). In this study, learner errors and misconceptions on calculus examination items are the units of analysis. The unit of analysis, a construct, is located in the sample. Participants’ written work was composed of written symbols that examinees consciously made. The written work provided evidence of the thinking processes of the examinees. Qualitative sampling is theory-driven because selection of participants, settings, and interactions are determined by a conceptual question not a concern for representativeness (Miles & Huberman, 2004; Yin, 1994).
A sample is a finite part of a statistical population whose properties are studied to gain information about the whole (Crawshaw & Chambers, 2008). When dealing with people, it can be defined as a set of respondents selected from a larger population for the purpose of a survey. A population is the group of individuals, objects, or items from which samples are taken for measurement. Sampling is the act, process, or technique of selecting a suitable sample, or a representative part of a population for the purpose of determining parameters or characteristics of the whole population. To draw conclusions about populations from samples, we must use inferential statistics which enables us to determine a population's characteristics by directly observing only a portion (or sample) of the population. We obtain a sample rather than a complete enumeration (a census) of the population for many reasons. It is cheaper to observe a part rather than the whole, but one ought to be wary of dangers of using samples. A sample may also provide needed information quickly.

Miles & Huberman (2004) stressed that meticulous sampling is crucial for later analysis. Fraenkel and Wallen (1993) write that the 'ecological generalisability' of the sample refers to the level to which the findings of a study can be extended to other settings. So researchers must describe fully the characteristics and selection of the sample. Although this research was mainly qualitative, ecological generalisability was important as the findings of this study were to be used by many educators to improve teaching and learning in many different school contexts.

The methods of sampling in a qualitative study and in a quantitative study are different. In a qualitative study, non-probability sampling is used but in a quantitative study, probability sampling is necessary. In a quantitative study, by studying the sample we may fairly generalize results back to the population from which the sample was chosen. But in a qualitative study results may not be generalised, but are useful to generate theory.

In a qualitative study, a purposive sample is often subjectively selected by a researcher. The researcher attempts to obtain a sample that saturates the data to answer his/her research questions. In a study like this one, it was prudent to select scripts from a spectrum, from very highly performing learners with few misconceptions and very low performing learners with many errors and misconceptions.
In this research a compromise between quantitative and qualitative sampling was achieved. My sample had 1000 scripts and this subscribes to the law of the inertia of large numbers (LLN). This law states that as the sample becomes bigger and bigger the sampling error becomes smaller and smaller (Crawshaw & Chambers, 2008). This makes the sample more representative of the population. Also, large groups of data show a higher degree of stability than smaller ones; there is a tendency for variations in the data to be cancelled out. I consider the random sample of 1000 scripts to be large enough to provide reasonable arguments for the study.

The researcher would have wanted statistically representative samples, characteristic to quantitative research studies. But this was not possible for logistical reasons. Gauteng Department of Education would not allow me to select my sample from all examination scripts. This is because by nature examinations materials are private and confidential whose access is highly restricted. As a result I accepted working with the 1000 scripts obtained from the Script Analysis Project. In this study then, convenience sampling was used, but was also linked to purposive sampling.

According to Merriam (1992), purposive sampling is based on the hypothesis that the researcher wants to learn, comprehend, and gain knowledge about a research problem and therefore must select a sample with phenomena that instantiate the issues being researched. Purposive sampling is the method of choice for most educational qualitative research (Merriam, 1992). In purposive sampling, the researcher carefully chooses the cases to be included in the sample on the basis of the judgement of their suitability for a specific purpose (Cohen & Manion, 1994; McMillan & Schumacher, 2001). Purposive sampling permits exploration and the size is often determined by resources, time available and theoretical saturation (Fraenkel&Wallen, 1993). Theoretical saturation is the point in data collection when new data no longer brings additional insights to the research questions. Patton (1990) argues that the rationale and thrust of purposeful sampling lies in selecting cases that generate pertinent information for study in depth.
I analysed the calculus examinations items for their academic and mathematical demand using three frameworks; the Revised Bloom’s Taxonomy (Anderson & Krathwohl, 2001), the Structured Observed Learning Outcomes (SOLO) taxonomy (Biggs & Collis, 1982, 1983) and Stein et al.’s (1993) Mathematical Task Demand Levels. All three were used to determine the cognitive level expected of examinees. It was necessary to analyse the questions using these three different frameworks because each has its advantages and disadvantages. In particular the Revised Bloom’s taxonomy attempts to address the limitations of the first Bloom’s 1956 taxonomy. The SOLO taxonomy discounts the Bloom’s taxonomy as impractical and confusing (Biggs & Collis, 1982). The SOLO taxonomy argues that academic tasks or performance on them can be either surface or deep. Stein et al.’s framework however seems to be more pertinent to this study than the other two because it refers to mathematics per se whereas the other two are generic. I decided to use all the taxonomies for the sake of strengthening my analysis and for completeness. These conceptual frames on the examinations items provided a comparative domain on which to refer back the errors and misconceptions of the examinees.

Like most taxonomies, the SOLO taxonomy describes the cognitive depth inherent in a question on a scale of increasing difficulty or complexity. The Revised Bloom’s Taxonomy (RBT) refers to the type of thinking or processing required in completing educational tasks or answering questions; that is, know, comprehend, apply, analyse, synthesise, and evaluate (Bloom, Engelhart, Furst, Hill, & Krathwohl, 1956). Stein et al. (1993) have argued that in implementing standards based curricula (NCTM, 1989, 2000) teachers must be mindful of the various cognitive demands inherent in mathematics tasks which they expect their learners to learn. Tasks at low level tend to be superficial and mechanical, whereas those at higher level tend to be linked to broader mathematical ideas and problem solving.

As earlier indicated, the purpose of this study was to examine the nature of calculus errors and misconceptions held by National Senior Certificate candidates as displayed in their actual examination scripts. For this purpose, there were various possible analysis frameworks proposed, for example Hirst (2003), Movshovitz-Hadar, Zaslavsky, & Inbar (1987) and
others (see Chapter 3, Table 1). A discussion of these typologies was made in the conceptual framework chapter. The main method for data collection in this study revolved around content analysis.

5.9 Data Analysis

Content analysis is an instrument of data analysis used in the social sciences. Holsti (1999) offers a broad definition of content analysis as "any technique for making inferences by objectively and systematically identifying specified characteristics of messages" (p. 53). Content analysis is regarded as the study of recorded human communications such as books, websites, paintings and so on. As a research tool, it is focused on the actual content and internal features of medium of communication.

The content analysis technique is regarded as the, “The Constant Comparative Method of Qualitative Analysis” (Glaser & Strauss, 1973). This leads to “Grounded Theory” (Glaser & Strauss, 1973). Amounts of textual information are analytically categorised to finally provide a meaningful interpretation and understanding of content under scrutiny. The creation of coding frames (Miles & Huberman, 2004) is intrinsically related to a creative approach to variables that exert an influence over textual content. Finally, in content analysis judgments need not be based on value statements if the research objective is aimed at presenting subjective experiences.

5.10 Uses of content analysis

Holsti (1999) explains the uses of content analysis. These are to make inferences about a communication, to describe and make inferences about characteristics of a communication, and to identify the intentions, focus or communication trends of an individual, group or institution.

There are several uses of content analysis. These depend on the purpose, question and use of the analysis. If the purpose is to describe and make inferences about the characteristics of the
communication, the analysis occurs through encoding. The researcher seeks to answer the questions, what, how, and why? In relation to mathematical errors, the researcher analyses trends in thinking and relate known characteristics of learners to the errors produced.

Krippendorff (2004) argues that six questions must be addressed in every content analysis. These are; which data are analysed? How are they defined? What is the context relative to which the data are analyzed? What are the boundaries of the analysis? And what is the target of the inferences? Qualitatively, content analysis thus involves analysis where written text is categorised and classified. There are two general categories of content analysis: conceptual analysis and relational analysis (Holsti, 1999). Conceptual analysis can be thought of as establishing the existence and frequency of concepts in a text whereas relational analysis builds on conceptual analysis by examining the relationships among concepts in a text. This research used both conceptual analysis and relational analysis.

Content analysis has most often been thought of in terms of conceptual analysis. In conceptual analysis, a concept is chosen for examination and the number of its occurrences within the text recorded. Because terms may be implicit as well as explicit, it is important to clearly define implicit terms before the beginning of the counting process. To limit the subjectivity in the definitions of concepts, specialized registers are used.

As with most other research methods, conceptual analysis begins with identifying research questions and choosing a sample or samples. Once chosen, the text must be coded into manageable content categories. The process of coding is basically one of selective reduction, which is the central idea in content analysis. By breaking down the contents of materials into meaningful and pertinent units of information, certain characteristics of the message may be analyzed and interpreted.

An example of a conceptual analysis would be to examine a text and to code it for the existence of certain words. In looking at this text, the research question might involve examining the number of positive words used to describe an argument, as opposed to the number of negative words used to describe a current status or opposing argument. The
researcher would be interested only in quantifying these words, not in examining how they are related, which is a function of relational analysis. In conceptual analysis, the researcher simply wants to examine presence with respect to his/her research question, i.e. whether there is a stronger presence of positive or negative words used with respect to a specific argument or respective arguments.

As stated above, relational analysis builds on conceptual analysis by examining the relationships among concepts in a text. And as with other sorts of inquiry, initial choices with regard to what is being studied and/or coded often determine the possibilities of that particular study. For relational analysis, it was important to first decide which concept type(s) were to be explored in the analysis.

There are many techniques of relational analysis available and this flexibility makes for its popularity. Researchers can devise their own procedures according to the nature of their project. My work differs from the analyses of the authors mentioned above in that I was dealing with summative examinations for formative purposes as I wish to find in detail the corpus of learner calculus responses. Content analysis is important in this study as scripts were conceptually analysed with the help of a conceptual framework discussed in Chapter 3. Relational analysis is important to compare errors and misconceptions among learners to perceive the common threads among them.

5.11 Documents as sources of data: Examination scripts

In this research, students' examination scripts were the documents on which content analysis was done. According to Prior (2003), documents are “fields, frames and networks of action” (p. 2). Documents have creators, and in this case, the students who wrote the end of school mathematics examinations were the creators. The settings for the examination documents were that they were written in a highly controlled environment in which the students were expected to write the examination in a period of three hours. Students were also forbidden to use any other resources to answer the examination questions. They were only provided with the question paper, writing paper, calculator, ruler, pencil and pen. Also, formulae sheets for helping to answer some questions were provided. No communication was ever allowed
between the candidates and copying someone else’s work was completely forbidden. Students were expected to depend on whatever prior knowledge and skills they would have acquired in mathematics in their school life. Students were expected show competence on answering questions on NCS calculus assessment standards (see Appendix A).

After the examination, the answer scripts or documents were intended to provide evidence for the owner’s competency or lack of it in all school mathematics. According to Prior (2003), “documents are essentially social products, consumed in accordance with rules, they express a structure, they are nestled within a specific discourse, and their presence in the world depends on collective, organized action” (p. 12). The examination scripts were the essential documents in this study to understand learner perceptions on calculus concepts.

In this study, documents as a data collection tool were involved in identifying the creators’ motive; the extent to which the scripts reflected solution processes to calculus items. The researcher allowed the scripts to speak for themselves on the difficulties that their creators met or did not meet when answering calculus questions to eliminate the potential bias of the researcher imposing preconceived meanings/expectations on the data and the analysis of the data.

This work was in accordance to the description of the authors mentioned above. The study analysed examination scripts to evaluate learner cognitive processes for future formative purposes. I wished to find in detail learner calculus errors and misconceptions in their scripts. This involved collecting and analysing data on errors and misconceptions in elementary analysis. The data from this analysis was compared to data obtained from literature in the form of discourse analysis.

5.12 Analysis strategies employed: Deductive and inductive analysis

In general, research data can be analysed inductively or deductively or through a combination of both (see Hatch, 2003; Yin, 1994; Merriam, 1992). Thorough analysis occurs when rationality is employed in a forwards and backwards movement of deductive and inductive logic, until patterns in the data begin to emerge in a way that research questions could be answered. According to Miles & Huberman (2004), deductive analysis is based on the
creation or design of a theory and pre-determined assumptions in relation to that theory in the face of empirical data. The assumptions are inferred from a theory and examined in order to prove or disprove a theory. In this research, the assumed theory was the conceptual framework (see Chapter 3) given in the form of error categories explained and described in earlier researches. Inductive analysis on the other hand starts with observation and examination of events or processes with the aim of reaching more general explanations induced by those events or processes (alphabetical order). The observations from the research results create a grounded theory. This way inductive qualitative data analysis generates theory.

This research used both inductive and deductive analysis. It was more practical to firstly use the deductive approach because inter-textual references availed the errors learners commit in a first course in calculus in different countries (for example Artigue, 1996; Hirst, 2003; Orton, 1983a; Tall and Vinner, 1981). Literature used suggested analytical frames for error categories in mathematics and calculus. This pre-constructed template was made to bear in scrutinizing the learners' responses to the 2008 NSC mathematics examination calculus items.

The foregoing data analysis techniques are summarised by Miles and Huberman (2004). According to Miles and Huberman (2004), once the research problem and research questions have been determined and data collected, analysis begins with the construction of a conceptual framework that describes, explains and determines the variables and constructs to be analysed. During analysis, empirical data obtaining is mapped with the lens of that conceptual framework. According to Miles and Huberman (2004), deductive analysis is done through coding and memoing. The first level of analysis occurs when data is coded according to categories explicated in the conceptual framework. This deductive analysis attempts to fit chunks of data into those pre-determined codes. As data is coded, the researcher begins to see patterns and regularities in the data which is then memoed. The memos are theory-like and help to pull out important interpretations of the data which at first might not be apparent to the researcher. Miles and Huberman argue that good data analysis is often serendipitous, in that valuable unexpected results can be found.
Miles and Huberman argued that memos assist to label and retrieve data, thereby speeding up analysis. The clustering of codes inductively produce memos as data analysis moves from specific data bits to general patterns in the data. According to Glaser (1978), the aim of data collection and analysis is to saturate the data, so that no important information is omitted in data collection and no information of importance is left out in data analysis. In my analysis, memos were on big ideas, often unifying different error codes by singling out a common feature among them. Thus error codes and memos provided a methodical way to capture the thinking that learners have constructed. They attempted to interpret the concept images that learners had on certain mathematical concepts.

As had been referred to, data analysis in this thesis occurred in two ways, forwards; through deductive typologies and backwards; through inducted theory drawn from specifics. During deductive analysis students' responses were coded through comparing responses with the categories suggested in the conceptual framework. The suggested conceptual framework describes various types of errors in mathematics raised in literature (for example Donaldson, 1963; Hirst, 2003; Chi, 2008; Clement, 1993). My assumption was that this conceptual framework with the errors identified by these scholars, though important, was not adequate to completely explain and account for all learners' calculus errors in the South African educational landscape. This is because the South African educational system is different from other countries in that South African learners' performance in international comparison tests in mathematics (and science) is markedly well below the benchmark. I presumed that the errors learners had in calculus were different to the rest of the world so an analytical framework drawn from the wider world would not be appropriate to the South African landscape. This conjecture of unexplored learner calculus errors in the South African empirical field provided me space to explain these from the script analysis as grounded theory.

As Booth (2008) explains, grounded theory analysis “can be seen as an issue of working with wholes and parts of wholes, de-contextualising and re-contextualising parts to form new wholes that tell a different story from the original work” (p. 453). Grounded theory is a repetitive process, of “constant comparison” (Glaser and Strauss, 1973; p.36), in which data is compared to other data again and again until unforeseen and concealed patterns in the data
begin to emerge. Grounded theory revealed current errors and misconceptions in learners’ answers. Through constant comparison theory emerged, and the latter became points of reference, to compare new data with. Although Glaser argued that data must be analysed without forcing it into pre-existing typologies, I agree with him only to a certain extent on this view of analysing data.

As reported, this analysis was not purely based on grounded theory, because the researcher reasoned that such analysis would be not pragmatic (see Fig. 16). Miles and Huberman (2004) have warned that while it would be commendable to do analysis without any pre-existing hypothesis impinging on the data, such an approach is suitable only to extremely experienced researchers. In this research, I appropriate Miles and Huberman’s stance and approach analysis not with empty hands but with errors and misconceptions that other researchers have found before me. This approach with prior categories was good because it sharpened my perception of the error categories potentially appearing in the scripts which was catered or not catered for in my analytical framework. I argue that deductive analysis is verificational of prior hypothesis and is influenced by my conceptual framework. Hence I used mixed methods of analysis; theory proving and theory generating. I believe this combination, offered the most pragmatic and compact analysis for this study. This combination of approaches involving rigorous logic and painstaking rationality triangulated analytical strategies so that results and recommendations from this study are reliable and valid. This multiple movement in analysis was dialectical and aimed at producing a cogent and well-argued calculus error analysis protocol.
Primary analysis occurred with the help of the first hand empirical data found in examinee scripts. This analysis was done with the aim of ultimately constructing a protocol for analysing learner errors and misconceptions on the topic of Introduction to Differentiation at FET level. This typology is an important tool that helps practitioners to better handle the teaching and remediation on this topic as well as its enrichment.

5.13 Pilot Study

In preparation of the full-scale study, a pilot study was undertaken from April to August, 2009. According to Polit, (2001), the term pilot study refers to so-called feasibility studies or baseline studies which are "small scale version[s], or trial run[s], done in preparation for the major study" (p. 467). However, a pilot study can also provide a springboard to try out a particular research instrument. The main benefit of conducting a pilot study is that it could give advance warning about where the main research project could meet future problems. It also helps to show whether the research methods or instruments are adequate or not to collect the required data, whether they are realistic or workable as well as assessing whether the proposed data analysis is good enough. In other words, the pilot study is useful in uncovering potential problems.
A pilot being a feasibility study helped to improve the quality of the oncoming study (Mason and Zuercher, 1995). Although the pilot study was time-consuming, it exposed several unforeseen shortcomings in the design of a study. These shortcomings were addressed in time before financial and human resources were expended on the bigger study. A pilot study was important because good research requires careful planning and a pilot study is good preparation for the study. Despite being a small scale undertaking compared to the main study, a pilot study could have the weakness of providing only limited information on the actual study. Some of these weaknesses only become apparent when the full-scale study was carried out. Nevertheless, the pilot study gave vital information on the efficiency and effectiveness of the data collection methods and analysis.

Another limitation of a pilot study arises from contamination. Contamination occurs when data is collected from the same participants twice; in the pilot and in the main study. Contaminated data is biased and therefore unreliable. Pilot studies also do not have statistical robustness because the numbers on which it is done are nearly always small. Holloway (1997) argued that contamination is less of a concern in qualitative research, where researchers often use some or all of their pilot data as part of the main study. In this study contamination was not an issue as all scripts were analysed using the reworked analytical framework induced by the pilot study.

As is argued above, a pilot study is a “foretaste” of the actual study. The pilot study was critical to fine-tune data collection methods and analysis well before time. In the pilot study, 100 scripts were analysed under the Script Analysis Project run by the University of Johannesburg. Six subjects among them mathematics were analysed for errors and misconceptions in learners’ scripts.
Table 2. Error categories used in the pilot study

<table>
<thead>
<tr>
<th>Category</th>
<th>Type of response</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Correct</td>
</tr>
<tr>
<td>2</td>
<td>Partially correct</td>
</tr>
<tr>
<td>3</td>
<td>Incorrect</td>
</tr>
<tr>
<td>4</td>
<td>Un-attempted</td>
</tr>
</tbody>
</table>

The pilot study was quite different from the main study in that it analysed errors and misconceptions in all questions in Mathematics Paper 1. In this paper, the calculus questions consisted only 15% of the total marks. This analysis however was very critical in influencing the main study in a positive manner. At first the examination questions were analysed for their mathematical demand using Stein et al.s' (1993) mathematics task demand framework. In the analysis of the pilot scripts, a protocol was used to analyse examinee errors according to pre-set categories of careless errors, procedural errors, conceptual errors and application errors. Also the mathematics examinations items were initially analysed quantitatively to determine the frequency of fully correct, partially correct, incorrect and blank responses (see Table 2). These frequencies were recorded on a spreadsheet. On the spreadsheet were also columns for misconception and misinterpretation. The frequencies of these were noted and recorded as percentages.

Then the scripts were analysed for learners’ errors. Letters were used alongside errors in the scripts to code the type of errors made by the learners. These were; P for procedural error, C for conceptual error, A for Application error, Ca for careless error and M for misinterpretation error. The symbol ME was also used to indicate markers’ error.

An item was scored fully correct if the learner gave the correct answer following a correct method. In the case where the answer was wrong, the researcher looked for a misconception or error made. Where the student got the question wrong, say through a misread, I looked at the process of getting the answer to see whether there was a misconception or
misinterpretation. This analysis was by no means linear as different lines of thinking might have been used by the learner to arrive at a response. Considerable time was taken before I could figure out what could have been the best interpretation of the error.

The pilot was also conducted to test the analytical framework for analysing errors. The results of the pilot showed that the analytical framework which categorised errors into careless errors, procedural errors, conceptual errors and application errors was not sufficient or robust enough for the study. It occurred that many of the errors that occurred in the scripts could not be simply categorised into only those four categories. Also, it became clear that when learners made errors on examination items, they often made more than one type of error, contrary to the thinking prior to the pilot, that a learner made one error in response to a mathematics item. It became clear that the analytical framework had to be overhauled and reworked for it to be practical and usable in the main study. The pilot study's results led the researcher to review literature more critically to find more about error types other researchers had found in mathematics and calculus. I report that this search was fruitful hence the improved framework that was used to analyse data in this study.

The researcher learnt a great deal from the pilot study. The pilot also showed that there was such a variation of errors and misconceptions that it would be impossible to categorise them all in a distinct manner. It became clear that some errors and misconceptions are inter-related and inter-dependent, rather than distinct and mutually exclusive. Yet some errors were easily differentiated from each other. This evoked the researcher to come to understand that there were varied and general errors and misconceptions in mathematics. It also became clear that there were errors and misconceptions peculiar to calculus. Thus the pilot helped me to become aware of the major weaknesses and pitfalls in my analysis. It helped me to rationalise my data collection methods as well as redesigning my analysis so that it would capture a greater variation of learners’ thinking. In short, the pilot study helped to deepen my analysis technique. It increased the validity and reliability of my research by sharpening my analysis. The pilot study also helped me to craft a research proposal that was accepted by our university’s Higher Degrees Committee the first time it was presented for examination. In short, the pilot study enabled me to reflect and review on my research in a way that helped
me to improve my methods and analysis. This helped me to push forward with my research much more quickly than I imagined.

5.14 Academic Background of Students

According to the National Curriculum Statement (NCS, 2003), all learners in South African schools must either study mathematics or mathematics literacy up to grade 12. (Pure) Mathematics rather than Mathematical Literacy is studied by the more capable students from grade 10 to 12. Mathematics is studied by learners who wish to pursue mathematics oriented professions such as science, engineering and commerce when they leave school. However, realising that mathematics is vital to the civilised function of every modern citizen, the NCS expects that any learner not doing mathematics proper must study mathematical literacy instead. The philosophy behind this policy is for each student to have an opportunity to learn of the critical importance mathematical applications play in solving problems learners would encounter in their daily life; at present and in the future.

Generally, mathematics is more demanding than mathematical literacy. However, it is not always the case that students who study mathematics are the better able students. In some cases, students study mathematics rather than mathematical literacy. This is because learners or their parents value the potential mathematics has in opening windows of opportunity to those learners who pass it at grade 12. Past experience shows that a mathematics pass at Grade 12 is the passport to valued employment or career opportunities. For this reason it would be simplistic to presume that students in the sample of this study were better in mathematics than those who studied mathematical literacy.

In addition, South Africa is famous for its shortage of mathematics teachers, so some schools do not have enough well qualified mathematics teachers (Jansen, 1999b). As a result, the students at such schools may study mathematical literacy not by their own volition, but because of a dearth of teachers qualified to teach mathematics. There are many cases of mathematics classes being taught by under-qualified teachers (Taylor, Muller, & Vinjevold, 2003). Yet other South African schools are endowed with a wealth of resources and highly
qualified mathematics teachers. This situation underlines the challenges that different students face in learning mathematics in South African schools.

5.15 Reliability and validity

According to Cohen & Manion (1994), validity concerns what a study researches, in that it researches what it sets out to research and how well it does so. There are different types of validity. For example, face validity and internal validity. Cohen and Manion (1994) note that face validity pertains to whether the research "looks valid" to the stakeholders who read it, and other technically interested observers. Internal validity is concerned with whether the processes of the actual study are appropriate to the research. Reliability concerns getting consistent results if the research is carried out by some other researcher again.

Qualitative research, with its distinctive approach to harnessing the analytical potential of exceptions, allows a research question to be examined from various angles. As Mays and Pope (1993) conclude, comprehensiveness may be a more realistic goal for qualitative research than is internal validity. According to this approach, apparent contradictions (or exceptions) do not pose a threat to researchers' explanations; they merely provide further scope for refining theories.

The errors that learners made were analysed from many angles. This analysis involving constant comparison of learner errors helped to highlight the enduring nature of the learners' errors and misconceptions. Also, these errors were compared to those that were found in literature. This helped to balance my findings with what other researchers have earlier found.

5.16 Involvement of co-researchers in data analysis

The researcher realised that for the analysis to be more reliable, he needed to be helped to do that by others; a sort of analysis triangulation. In this way, the analysis was done with the help of practising teachers. At first error analysis was done under the aegis University of Johannesburg Script Analysis Project. In this project there were six mathematics team
members including three teachers who were teaching grade 12. These teachers helped the researcher to analyse some of the differentiation errors in learner’s scripts. After the researcher was writing up the analysis of his study he visited the teachers and discussed with them the error analysis protocol on differentiation he had compiled. The teachers examined it against the scripts and gave more suggestions on improving the analysis. The researcher re-analysed the scripts and incorporated the views of the teachers in the protocol. In this manner the researcher attests that the error analysis protocol was a product of more than his solo efforts.

5.17 Research Ethics

By its nature, protracted research such as this one is competitive and at the cutting edge as it seeks to create new knowledge. In pursuit of its goal to push forward the frontiers of knowledge, doctoral research aims to contribute to new knowledge, yet in doing so there is danger in violating ethical codes of conduct through assault on the privacy of participants or by harming them either physically or psychologically (Opie, 2004).

The ethical principles for research are; voluntary participation informed and understood consent, as well as keeping the confidentiality and anonymity of research participants (Graue, & Walsh, 1998). Voluntary participation stipulates that participants are not coerced to take part in research. Informed consent stipulates that voluntary participation occurs when participants have been informed about the nature of the research, its aims and methods, its effects on participants (possibly physical or psychological harm), and how the results of the research are distributed. It underlines the fact that participants must be fully informed of the harm, if any, they could suffer through participating in the research. The protection of participants against invasion of privacy and psychological harm is often present in social research such as in education. Participants must be made fully aware of these if any, before they sign their consent. Most of these factors do not hold for my study but I raise them to inform readers of my awareness of them. An important aid to anonymity is that the examination scripts that were used in the study only had centre numbers and candidate numbers and the names of the candidates and their schools were absolutely absent from their scripts.
This study was done in relation to the Examinations and Assessment Board, University of Johannesburg Script Analysis Project (EAB/UJ/SAP). It therefore used the Ethics Clearance obtained by the University for their Project. This is because the aim of this research project was related to a certain extent to the aims of SAP; namely to find out what is of importance that can be learnt from the examination scripts that can be communicated to stakeholders in mathematics education to improve the teaching and learning of mathematics in South Africa. As such, I did not need to directly seek consent and permission for the use of examination scripts in my research. However, I wrote a letter to the SAP Lead Researcher at the University of Johannesburg seeking permission to use scripts in SAP and was granted permission to research within this project. In particular I was given open and free access to the mathematics examination scripts that SAP had obtained from the Gauteng Department of Education (GDE).

As the research was done on marked scripts of a previous year, there was the danger in exposing the results of the students or commenting about them to the general public in a way that the students and their schools are identifiable. The researcher undertook that as the purpose of this research was to pursue doctoral research, availing the results to the public or private individuals would not be done. Also, it was not necessary to seek permission and consent from the schools and learners or their guardians as the scripts were now in the custody of GDE. Legally, the examination scripts were the property of GDE. Regulations stipulate that GDE can keep the examination scripts for a number of years before they can destroy them. Another important issue is that since the study was carried out under the aegis of the University of Johannesburg, Faculty of Education Script Analysis Project, which sought ethical clearance for the project; this project being minor to SAP was automatically covered by the ethical clearance SAP obtained in its study. The results of this research may be presented at educational conferences and seminars or published in journals. GDE as an educational entity is often pleased with dissemination of research findings that it can use to improve teaching in its schools. In addition, I obtained ethical clearance from the University of Johannesburg Faculty Of Education Higher Degrees Ethical Clearance Committee (UJFEHDECC).
5.18 Conclusion

This chapter explicated the methodology of the study as well as how it was married to the theoretical and conceptual framework. The chapter also discussed the sampling of the scripts used in the research. Having done that, the chapter described the analysis of the data. The content analysis methodology was described as was the pilot study, the validity and reliability of the study and the ethical clearance.
CHAPTER 6: DATA ANALYSIS

6.1 Introduction

This chapter is the culmination of an extensive study which sought to determine and understand the errors learners hold in answering examination questions from a first course in calculus. This analysis sought to obtain answers to the following research questions:

What is the nature of common errors that students displayed in answering grade 12 mathematics examination questions on introductory differential calculus?

The sub-questions were:

(a) What are the types of errors made by learners in response to grade 12 differential calculus questions?
(b) Why do learners make those errors and misconceptions?
(c) How can learner errors in introductory differentiation be explained in an analytical protocol?

This chapter is structured in the following manner. Firstly, the two main data analysis approaches; namely deduction and induction, as well as a Newman’s (1983) procedure for mathematical error analysis are discussed and compared in relation to this study. It is argued that these three data analysis approaches were from certain angles appropriate for this study. Secondly, the preliminary quantitative analysis is done for the performance of the learners on the whole of Paper 1 followed by a more detailed statistical analysis of performance on the calculus items. This quantitative analysis helps to foreground the qualitative analysis by highlighting the general and specific weaknesses of the learners on particular items. Included also in this analysis is the number of blank answers. The blank answers to the items imply that the learners found the items so difficult that they could not even figure out how to begin answering them, although at times they imply that the learners might have run out of time to do them. Thirdly, the mapping of the calculus items against their cognitive and mathematical
demand is done. This mapping also provides a foreground for the analysis of errors. Also discussed at this stage is the suitability of the items for the examination. In this regard, the items are compared to NCS assessment standards; 'the knowledge, skills and values that learners need to show to achieve the Learning Outcomes in each grade' (Department of Education, 2002, p.125). The fourth stage concerns the actual error analysis of learners' work. In this regard, scanned vignettes from actual learners' scripts are presented and analysed in vivo. This analysis aimed to describe, discuss and identify the errors in those learner micro-worlds (Davis, 1984). I used thick descriptions to analyse the errors. The fifth stage summarises this analysis in preparation of the concluding chapter of this thesis.

6.2 Newman's procedure for error analysis in mathematics

Content analysis was done through reading the mathematics items; and reading learners' responses to mathematics items. In reading responses to mathematical items it was critical to comprehend what learners had written to determine how learners were thinking in coming out with their responses. It was necessary to infer the most appropriate/effective mathematical strategy that learners were using to formulate their answers. In doing so, the researcher was aided by the conceptual framework which provided codes to categorise errors and misconceptions in the scripts. The method of analysis was not uniform but the researcher moved back and forth trying to put responses in one category and then another depending on what insight he had.

Newman used the word "hierarchy" because she argued any failure to follow the above sequence prevents problem solvers from obtaining satisfactory solutions. However one often revisits lower levels of the hierarchy in a meta-cognitive manner to assess the progress of the solution. This procedure provides an additional scope for analysis. In addition to the aforesaid analytical approach, the researcher also used Clements (1980), error analysis technique. Clements (1983) illustrated the Newman technique with the diagram shown in Figure 17 below. According to Clements (1983), errors due to the form of the question are essentially different from those in the other categories shown in Figure 17, because the source of difficulty resides fundamentally in the question itself rather than in the interaction between the problem solver and the question. Two other categories, "Carelessness" and "Motivation,"
have also been shown as separate from the hierarchy although these types of errors can occur at any stage of the problem-solving process. A careless error, for example, could be a reading error, a comprehension error, and so on. Similarly, someone who had read, comprehended and worked out an appropriate strategy for solving a problem might decline to proceed further in the hierarchy because of a lack of motivation.

Figure 17. The Newman hierarchy of error causes

(Adapted from Clements, 1980)

Newman (1983) suggested a new error called 'errors due to form of a question'. This notion from Newman implies that this study needed to analyse how the questions were asked; were the items valid for the examination? Were these worded clearly and unambiguously? These questions were addressed later in the chapter.

6.3 Quantitative analysis on whole paper performance

This section reports on the quantitative variation in examination performance on the whole Mathematics Paper 1 examination from amongst the 1000 sample scripts (see Table 3, Fig. 18 & 19). Descriptive statistics including simple measures of central tendency and measures of dispersion were used. This foregrounding was important to get the general level of performance of the students before the detailed qualitative analysis on the actual items.
Table 3. Summary statistics of marks scored on the whole paper

<table>
<thead>
<tr>
<th>Possible Mark</th>
<th>150</th>
</tr>
</thead>
<tbody>
<tr>
<td>Range</td>
<td>146</td>
</tr>
<tr>
<td>Mean</td>
<td>46</td>
</tr>
<tr>
<td>Mean</td>
<td>28%</td>
</tr>
<tr>
<td>Percentage scoring less than 50%</td>
<td>79</td>
</tr>
<tr>
<td>Percentage scoring more than 50%</td>
<td>21</td>
</tr>
<tr>
<td>Percentage scoring more than 20%</td>
<td>40</td>
</tr>
<tr>
<td>Mode</td>
<td>28%</td>
</tr>
<tr>
<td>Median</td>
<td>28%</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>24.6%</td>
</tr>
<tr>
<td>Lower quartile</td>
<td>8%</td>
</tr>
<tr>
<td>Upper quartile</td>
<td>44%</td>
</tr>
<tr>
<td>Inter quartile range</td>
<td>32%</td>
</tr>
</tbody>
</table>

These statistics show that in any group of students, the likelihood of a learner failing was greater than 0.8. In every 100 students, the number of students likely to fail the paper was greater than 80. The lowest performers scored 0 out of 150 (there were seven of these) while the highest performer scored 146 out of 150. The modal class on learners' performance in mathematics was 20-29 out of a possible 150.
Figure 18. Learners’ performance in Mathematics Paper 1 Examination (2008)
The frequency polygon shows that there was a steep positive skew in learners’ performance in Mathematics Paper 1 examination of 2008.

6.4 Statistical analysis of performance on calculus items

The calculus examination questions were renamed to use item numbers instead of the original question numbers used in the question paper. This has been done to adhere to common practice of calling the examination questions items. The term question was reserved for the
thesis research questions. Also, it was much easier to refer to items numbered from 1.1 to 3.3 instead of questions 8.1 to 10.3 since the calculus questions only start from question 8 until question 10, and in between them there are some non-calculus questions as well. The last question in the paper, question 11 was not calculus. The item numbering directly shows the appropriate number of questions under analysis.
### Table 4. Frequency and marks scored per item

<table>
<thead>
<tr>
<th>ITEM (POSSIBLE:5)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>NO ATTEMPT</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>232</td>
<td>15</td>
<td>137</td>
<td>111</td>
<td>120</td>
<td>333</td>
<td>52</td>
</tr>
<tr>
<td>1.2</td>
<td>518</td>
<td>74</td>
<td>75</td>
<td>102</td>
<td>N/A</td>
<td>N/A</td>
<td>231</td>
</tr>
<tr>
<td>2.1 (POSSIBLE:4)</td>
<td>411</td>
<td>32</td>
<td>33</td>
<td>20</td>
<td>172</td>
<td>N/A</td>
<td>352</td>
</tr>
<tr>
<td>2.2 (POSSIBLE:5)</td>
<td>196</td>
<td>94</td>
<td>94</td>
<td>101</td>
<td>44</td>
<td>91</td>
<td>380</td>
</tr>
<tr>
<td>2.3 (POSSIBLE:4)</td>
<td>325</td>
<td>15</td>
<td>32</td>
<td>41</td>
<td>178</td>
<td>N/A</td>
<td>409</td>
</tr>
<tr>
<td>3.1 (POSSIBLE:2)</td>
<td>409</td>
<td>8</td>
<td>141</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
<td>442</td>
</tr>
<tr>
<td>3.2 (POSSIBLE:2)</td>
<td>400</td>
<td>0</td>
<td>79</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
<td>521</td>
</tr>
<tr>
<td>3.3 (POSSIBLE:5)</td>
<td>380</td>
<td>8</td>
<td>16</td>
<td>0</td>
<td>16</td>
<td>33</td>
<td>547</td>
</tr>
</tbody>
</table>
The marks scored per item and the frequencies are shown in Fig. 19 – 27 below.

<table>
<thead>
<tr>
<th>ITEM 1.1</th>
<th>MARK SCORED</th>
<th>FREQUENCY</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>0</td>
<td>50</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>60</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>100</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>150</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>200</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>250</td>
</tr>
</tbody>
</table>

**Figure 19.** Marks scored on item 1.1
Figure 20. Frequency and marks scored on Item 1.2.
Figure 21. Frequency of marks scored for Items 1.2 and 1.2
Figure 22. Frequency of marks scored for Item 2.1

Figure 23. Frequency of marks scored for Item 2.2
Figure 24. Frequency of marks scored for Item 3.1
Figure 25. Frequency of marks scored for Item 3.1
Figure 26. Frequency of marks scored for Item 3.2

Figure 27. Frequency of marks scored for Item 3.3
6.4.1 Summary on learners' scoring on calculus items

The statistical overview shows that by far, the majority of the learners had lots of problems dealing with calculus items, hence the high frequency of zero scores and no attempts in the answers (see Figs. 19-27). This implied that learners had many errors and misconceptions preventing them from scoring. Just as in the general paper, students' performance on the calculus items was also quite bad. This is demonstrated by the descriptive statistics in tabular and graph form that have been shown. This performance reflects the general low level of mathematical proficiency in South African learners.

The descriptive statistics presented above indicate that by far, the modal mark in all but one calculus item was zero. Only item 1.1 had the modal mark of 5. More than half of the zero scores were obtained from non-attempts. Thus in general, learners' results in calculus were much poorer than in the in whole paper. Discussion of the academic demands of the questions now follows.

Before mapping the items to the above frames, the concepts and procedures needed to successfully perform on the items were enumerated in Table 5 below.
Table 5. Concepts needed to answer calculus items

<table>
<thead>
<tr>
<th>ITEM</th>
<th>CONCEPTS AND PROCEDURES NEEDED TO ANSWER THE ITEM</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td><strong>Determine $f'(x)$ from first principles if $f(x) = -3x^3$</strong></td>
</tr>
<tr>
<td></td>
<td>Remembering or re-constructing the difference quotient, function notation, evaluating (or substitution) the value of a function at a point, squaring a binomial (optional), addition and subtraction of algebraic terms including directed numbers, gradient of a straight line, the idea of a limit, calculating a limit algebraically, graphical meaning of a derivative, tangent to a curve at a point, tangent as the limit of a chord or secant, co-ordinate geometry</td>
</tr>
<tr>
<td>1.2</td>
<td><strong>Determine, using the rules of differentiation:</strong></td>
</tr>
<tr>
<td></td>
<td>$\frac{dy}{dx} = \frac{\sqrt{2}}{2} - \frac{3}{5x^2}$</td>
</tr>
<tr>
<td></td>
<td>Show ALL calculations.</td>
</tr>
<tr>
<td></td>
<td>Simplifying terms with integer and non-integer indices, surds, reciprocals, power rule of differentiation for integer and non-integer indices, operations on directed numbers including fractions</td>
</tr>
<tr>
<td>2.0</td>
<td><strong>QUESTION:</strong></td>
</tr>
<tr>
<td></td>
<td>Sketched below is the graph of $g(x) = -2x^2 - x^3 + 12x + 30 = -(2x + 3)(x + 2)^2$</td>
</tr>
<tr>
<td></td>
<td>A and T are turning points of $g$. A and B are the x-intercepts of $g$.</td>
</tr>
<tr>
<td></td>
<td>P(1, 11) is a point on the graph.</td>
</tr>
<tr>
<td>2.1</td>
<td><strong>Determine the x coordinate of T.</strong></td>
</tr>
</tbody>
</table>
|      | Differentiating polynomials, factorising a quadratic polynomial, solving a quadratic equation, finding the value of $y$ or $f(x)$ if the value of $x$ is known, the notion of the turning point of a function, co-ordinate geometry, function concept, differential conditions for turning points, meaning of the zero derivative, testing the nature of a turning point by determining the gradients of the function at the left and at the right.
of a point, second derivative test for a turning point (optional), requisite terminology: turning point, extreme values of a function, minimum point, maximum point, global/local minimum, global/local maximum, derivative, derived function, sketch graph of a function, co-ordinate geometry.

2.2

2.3

Determine the equation of the tangent to \( y = x^2 + 1 \) at \( P(3, 11) \) in the form \( y = \ldots \)

Curve, (coordinate) point on a curve, co-ordinate geometry, tangent to a curve at a point, differentiating polynomials, derived function, determining the gradient of a point on a curve, gradient of tangent as gradient of a point on a curve, substitution, finding the equation of a straight line when its gradient is known as well as a point it passes on.

2.3

Determine the x-coordinate of the point of inflexion

As in 2.1 and also conditions of a point of inflexion; sign of the derived function changes.

3.0

**QUESTION 10**

A drinking glass, in the shape of a cylinder, height \( h \) and base radius \( r \), must hold 200 ml of liquid when full.

3.1

10.1 Show that the height of the glass, \( h \), can be expressed as

\[
h = \frac{200}{\pi r^2}
\]

Properties of a cylinder, reducing a whole to parts, area of net of a cylinder (drinking glass) is the sum of the circular base and a rectangle which is curved surface of cylinder that is a rectangle whose dimensions are the circumference of the cylinder and its height; area of a circle, area of a rectangle, volume of a cylinder, the concept of a uniform prism; volume of a prism as area of circular base times height, circumference and radius of a circle, height of a cylinder, algebraic elimination and substitution, literal equations, subject of formula, mathematical modelling, forming mathematical concepts from visuals, visual reasoning.

3.2

10.2 Show that the total surface area of the glass can be expressed as

\[
S(r) = \pi r^2 + \frac{400}{r}.
\]

Area of a circle, area of a rectangle, area of a curved surface of a cylinder, curved surface of cylinder equal to area of rectangle, subject of formulae, literal equations, function concept and notation.

3.3

Hence determine the value of \( r \) for which the total surface area of the glass is a minimum.

Differentiation variables other than \( x \), differentiating reciprocal functions, addition and subtraction of algebraic expressions, derivative is zero at maximum point, calculations using fractional powers, algebraic manipulation, calculator skills.
In South Africa, the management of examinations is the mandate of Umalusi. Umalusi, which means shepherd in isiZulu, is a South African statutory body, with headquarters in Pretoria, the administrative capital of South Africa. Umalusi is the standard bearing body of all school examinations in South Africa. Among other duties, it monitors item writing for school leaving examinations and controls examination standards. Umalusi also administers the marking of the school leaving examinations as well as grade the results and issues the valuable school leaving certificates.

In spite of the checking of the standards of examinations done by Umalusi, I did my own analysis of the cognitive demand of the examination items against the NCS assessment standards (see Appendix A).

Generally in designing examination, it is necessary to write the items in the light of the intended curriculum as reflected in the curriculum policy documents. There ought to be agreement between the intended curriculum, implemented curriculum and examined curriculum (Reddy, 2006). This link is critical for the education system to be both valid and reliable. So the items must be carefully worded and stipulate the appropriate content at the appropriate level. Another important principle is that examination items must be short and simple for clarity.

In analysing the scripts, the researcher also sought to discover factors affecting the examinees’ responses. One factor could be that in real life, a drinking class is formed with solid material, called glass. The question itself could then be misleading. It was supposed to specify that, what were given were the internal dimensions of the drinking class. This item was not explicit. With that assumption not given, the item remained insoluble and all solutions become probables. This is because low income learners require teaching to be explicit (Makonye & Luneta, 2010). Low income learners need language to be explicit because schooling often implicitly mirrors middle class culture and ways of life that low income learners are not familiar with.

However, I analysed the question with the assumption that it was the internal dimensions of the drinking class that were given or that it was a theoretical drinking not made of any solid material. This question illustrates the danger of using contexts in the teaching of mathematics.
which are having a veneer of reality (Boaler, 1998). It is not clear how many students had these contextual misconceptions but they were real possibilities of these occurring. Such contextual issues further affected the weaker students’ responses to this item.

In consideration of the above, I mapped the items into the cognitive demands as indicated in Figs. 28-35 below. On each pyramid, representing each item there were three categories. The apex represents Stein et al.’s (1993) classification of the item. The bottom left is from the SOLO taxonomy and the bottom right is from the RBT. In my opinion, the questions could be mapped into the following categories.

![Figure 28. Mapping of the mathematical demand of Item 1.1](image-url)
Figure 29. Mapping of the mathematical demand of Item 1.2

Figure 30. Mapping of the mathematical demand of Item 2.1
Figure 31. Mapping of the mathematical demand of Item 2.3

Figure 32. Mapping of the mathematical demand of Item 2.4
Figure 33. Mapping of the mathematical demand of Item 3.1

Figure 34. Mapping of the mathematical demand of Item 3.2
The researcher discovered that mapping the cognitive demand of the questions could not be easily compared between the three frameworks of Biggs and Collis (1982), Anderson and Krathwohl (2001), and Stein et al. (1993). This is because the frameworks categorise cognitive levels differently and so their categories cannot be directly compared on a one to one basis. Although Stein et al. (1993) and Biggs and Collis (1982) have four levels each in their frameworks, it is too simplistic to compare them on a one to one basis. Overally, the researcher felt that the calculus items were generally of upper medium difficulty and that the more challenging item was 2.3 which had an extended abstract demand or doing mathematics. Item 1.1 seemed to be the simplest as it was a multi-structural (Biggs & Collis, 1982), understanding and procedure with connection task (Stein et al., 1993). When compared to the NCS policy documents, the items reasonably reflected the required standard of the syllabus.

6.5 Coding of learner errors and misconceptions: Vignettes of learners' responses

In consideration of the above analysis, I found that the items were suitable for the grade 12 examination, but they were not very challenging for an examination at this level. However they still reasonably reflected the desires of the NCS assessment standards. The analysis of learner errors and misconceptions occurred under this background. I became very interested to know how those students who attempted these medium difficulty questions committed
errors in answering them. That analysis began in earnest with examining vignettes in the following section. In doing so, I remembered to refer to my conceptual framework (see Table 1, Chapter 3).

In the following section, learners' work is presented to analyse firsthand how their errors and misconceptions were expressed. The 26 vignettes covering the three items and their sub-items are analysed. Item 1 corresponds to question 8, item 2 to question 9 and item 3 to question 10.

Vignettes are rich pockets of especially representative data (Miles & Huberman, 2004). These are summaries that can be pulled together in a focused way for interim understanding. Vignettes are a focussed description of a series of events taken to be representative, typical or emblematic. They are vivid, but one must be careful not to exaggerate when reporting about them. Also they are representative caricatures. Many vignettes of learners' written work have been provided in this analysis. The analysis starts with Fig. 36, Vignette 1, through Fig. 61, Vignette 26.

6.5.1 Item by Item Analysis of selected vignettes

Item 1.1 Task 1

8.1. Determine $f'(x)$ from first principles if $f(x) = -3x^2$
Figure 36. Vignette 1 for Item 1.1 (QUE: 9)

The errors shown in this vignette were:

- Structural error - the learner fails to substitute for $f(x+h)$. Learner has a concept image of a function that is very inadequate.

- Executive error - in simplifying terms in $x$ for $f(x+h)$ in line 4, but learner has some correct knowledge, $-x-3x = -4x$. (Line 6)

- There is also the problem of signs for $-f(x)$ in line 4. This could be due to Feigenbaum law of minimal-discrimination (FLMD) in regarding $-(-3x)$ as the same $-3x$. 
Figure 37. Vignette 2. for Item 1.1 (QUE: 9)

The errors shown in this vignette were:

- Structural error in lack of knowledge of function concept
- Semantic error – resorting to prior understanding for multiplying algebraic terms in a totally different context, $f(x+h) = fx + fh$ (Line 1 to 2). The learner is devoid of the meaning of the functional notation.
- There is also the primary grade multiplication error in that $f$ must not be multiplied twice to $x$ and $h$.
- Also $3x + h$ is simplified to $3xh$. There is the idea that addition is putting things together from primary school.
- Fragmented and executive errors in ‘adding’ terms in $x$, $h$ and numbers as well as dividing by $h$ (Line 5 to 6). These errors are algebraic in nature.
As seen the learner also differentiates coefficients and not just the powers of the variable x. Moreover learner assumes that the derivative of products is found by multiplying the derivatives. This shows lack of procedural understanding of the power rule. The learner has theory like error and pseudo-linearity errors.

\[
\frac{d(fg)}{dx} = \frac{df}{dx} \cdot \frac{dg}{dx}
\]

He/she shows that \( \frac{df}{dx} \cdot \frac{dg}{dx} \) which is pseudo-linearity of the differential operator.
• The learners illogically infer that the 6 can continue to be attached to the term in x. Hence it moves into the numerator without any changing. The learner fails to as a product of the 2 which can and should be easily separated and decomposed to facilitate answering the question. The learner has parenthesis, logically invalid inference and executive errors.

The learner also does not simplify $\frac{1}{2}$, which is a structural error.
This learner has many good procedures. His only error is that of over-generalisation. The learner has the concept image of equation balancing in that he/she thinks if the “same thing” (differentiation) is done to the numerator and the “same thing”; (differentiation) is done to the denominator; that does not change the ‘balance’ of the quantities, so a correct derivative results. This rule often applies to reducing fractions to the lowest terms and extended here incorrectly. The learner seems to operate at a stage were he/she is ready to be confronted through cognitive conflict, to challenge the current concept image.
The learner shows arbitrary error and equation balancing.

The learner invents an own law for simplifying fractions

\[
\frac{a}{c} = (a+b) + (c+d).
\]

Namely, \( \overline{b} \pm \overline{c} \). To him/her it is also a derivative which is equated to zero. This suggests a link to finding turning points. This shows a structural error. Also the learner does not show any evidence of knowledge of differentiation except copying the sign \( \frac{dy}{dx} \) and equating this to 0.
Figure 42. Vignette 7. for Item 1.2 (QUE: 8)
Figure 43. Vignette 8, for Item 2.1 (QUE: 9.2)
Here the learner shows structural errors as well as hybridisation errors. Learner links the derivative and quadratic formulae where that does not apply.

9.2 A and T are turning points of \( g \). Determine the \( x \)-coordinate of T.

**Figure 44.** Vignette 9 for Item 2.1 (QUE: 9.2)

- Hybridising error. In this case the learner uses procedural techniques for finding turning points for quadratic functions in a non-quadratic function. The word turning point and a look at the sketch of the graph act as a cue and link to turning points of quadratic functions.
9.2 A and T are turning points of \( g \). Determine the \( x \)-coordinate of T.

Figure 45. Vignette 10 for Item 2.1 (QUE: 9.2)

- Terminology and structural errors: The learner associates the midpoint (which is written coordinate) with turning points, so there is a terminology error. The error is also structural in that the learner fails to use deeper methods that could be afforded by calculus to do this task.
9.2 $A$ and $T$ are turning points of $g$. Determine the $x$-coordinate of $T$.

Figure 46. Vignette 11 for Item 2.2 (QUE: 9.3)

- This is an error due to language or terminology. The learner confuses the midpoint and a turning point.

- It is also a hybridising or fragmented error because the learner’s concept image has conflicting factors about an equation and the “midpoint”.

- It could also be an arbitrary error in that the learner answers the question she/he chooses.
9.3 Determine the equation of the tangent to \( g \) at \( P(-3; 11) \), in the form \( y = ... 

Figure 47. Vignette 12 for Item 2.2 (QUE: 9.3)

The learner has knowledge of finding the equation of a straight line once its gradient and the point it passes is known. The learner actually finds the gradient of AP which is a chord to the curve, not a tangent. This could be an arbitrary error. It could also be an error due to
terminology in failing to differentiate a chord and a tangent. It is a structural error of not knowing that differentiation can help to find the gradient of a tangent of a point on a curve.

9.3 Determine the equation of the tangent to \( g \) at \( P (-3; 11) \), in the form \( y = \ldots \)

\[ m = \frac{3}{11} \]

\[ y - y_1 = m(x - x_1) \]
\[ y - 11 = \frac{3}{11}(x + 3) \]
\[ y = \frac{3}{11}x + \frac{36}{11} + 11 \]
\[ y = \frac{3}{11}x + \frac{183}{11} \]

**Figure 48.** Vignette 13 for Item 2.2 (QUE: 9.3)

This learner has a structural error of failing to define the gradient of a tangent to a curve. It can also be a terminology error as the learner seemed not to understand the term tangent. In the end the learner finds the equation of the line \( OP \) which is an escalation of the terminology error. There is also error of notation because the \( \partial \) notation is inappropriate in this case.

Actually the gradient of a tangent is not \( \partial y \) rather it is its limit as \( \partial x \to 0 \).
9.3 Determine the equation of the tangent to \( g \) at \( P (-3; 11) \), in the form \( y = \)

Figure 49. Vignette 14 for Item 2.2 (Q: 9.3)

- **Structural and arbitrary errors:** The learner cannot figure out how to find the gradient of the tangent using calculus or otherwise, which is a structural deficit. The learner seems not to have that concept image. It appears the gradient of 11 was chosen arbitrarily.
Terminology, structural, procedural extrapolation: The learner seems not to understand the term tangent. The learner calculated the gradient of the line OP instead. Perhaps, having learnt before that the equation has a gradient; the learner extrapolates this procedure here. The error is also structural as the learner fails or cannot proceed to form an equation having established the gradient and a point it passes through.

9.5 Determine the $x$-coordinate of the point of inflection.

Application, structural and meta-cognition errors: The learner fails just to finish the calculations. It seems that this is because he/she is not thinking about what she has written in relation to the question that is why she/he leaves the solution hanging.
Figure 52. Vignette 17 for Item 2.3 (QUE: 9.5)

- Careless error: Here the learner copies -12 as -2 in the second line to produce a careless error.

Item 3 Tasks

A drinking glass, in the shape of a cylinder, must hold 200 ml of liquid when full.

10.1 Show that the height of the glass, $h$, can be expressed as $h = \frac{200}{\pi} r^2$
• Procedural extrapolation and visualisation: The word area was a cue to the LxB fixation of area learnt at primary school. This knowledge though useful was being applied blindly as the learner could not visualise how to apply it.

Figure 53. Vignette 18 for Item 3.1 (QUE: 10.1)

In this one the learner just adds the 200 and 400, the numbers appearing in the question and arbitrarily takes them as the length and the breath. The learner declines the x symbol to the + symbol due to primary grade operation frame, where the learner first learnt the + symbol and so regards all operations as addition.

Figure 54. Vignette 19 for Item 3.1 (QUE: 10.1)
10.2 Show that the total surface area of the glass can be expressed as

\[ S(r) = \pi r^2 + 400/r. \]

**Figure 55.** Vignette 20 for Item 3.2 (QUE: 10.2)

Arbitrary error or visualising error or logically invalid inference through considering the glass as closed on both ends

**Figure 56.** Vignette 21 for Item 3.2 (QUE: 10.2)

- Hybridising and structural: the \( r \) cues the ideas of the equation of a circle centre \((a,b)\), which is conceptually off the mark.
10.3 Hence determine the value of $r$ for which the total surface area of the glass is a minimum.

- Hybridising error due to Feigenbaum minimal-discrimination rule. The $r$ acts as a cue for formulae for the sum of a convergent geometric series.
Syntactical, structural and executive errors: Learner did not understand the difference between $S(r)$ and $S'(r)$. The learner showed no calculus conceptual knowledge needed to work out the question. The learner also had an executive error in line 3 in failure to clear the fraction. In line 5 the learner had some executive errors on differentiation. Only the first term was differentiated but the second was not. This could be due to Feigenbaum law of minimum discrimination, in that when a student differentiates the first term he/she thinks that is sufficient. The learner also makes an executive error in failing to assign the proper sign for the final value of $r$. This can also be structural error on dealing with signs.

Hybridising error on item 3.3, in quoting and using the area of a triangle where it clearly does not apply. The word area was a cue to this. This is also a structural error caused by failure to comprehend the differentiation concepts required to work out the problem.
This is an equation balancing in that the learner multiplies the numerator and denominator by \( r \); line 2 to 3. This can also be taken as an arbitrary error.

The learner differentiates \( \frac{400}{r} \) to obtain 400. This is an executive differentiation error which is also arbitrary. Learner considers \( \frac{1}{r} \) just the same as \( r \).
6.6 Further general item by item qualitative analysis for errors and misconceptions

Item 1.1: Determine \( f'(x) \) from first principles if \( f(x) = -3x^2 \)

This item required the use of differentiation from first principles to establish the derivative of \(-3x^2\). In general, most learners remembered the difference quotient:

\[
\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]

required for doing this task. However, examinees encountered a number of challenges in executing and applying it in order to get the result. The main difficulties that learners displayed on this question were failure to substitute properly in the functional format inherent in the difference quotient. Examinees found it quite challenging to substitute properly the value of \( f(x+h) \) given the formula of \( f(x) \). Correct substitution was first critical technique in this procedure. As such for many learners who could not forge ahead, procedural fluency hampered them if they did not understand functional symbolism. The failure to substitute properly seems to have emanated from their lack of understanding the deeper meaning and semantics of the processes represented syntactically by the functional notation and the difference quotient. So errors in this item were structural, executive and due to semantics.

30% of the students correctly recalled the definition of the derivative at the point \((x, f(x))\) as \( \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \), thus remembering the required definition to begin to respond to the question correctly. A further (give number of participants 400) 40% had some inaccuracies in recalling this definition; some learners for example put a plus instead of a minus required to denote the infinitesimal change in the mantissa, thereby committing a structural error. This is similar to Davis' (1984) Primary Grade Sign Error (p. 123) where it was argued that learners are fixated to the addition operation because that was the first sign they learnt at school. About 30% of the students seemed to be unaware that this definition or
its alternatives; \( \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \) or \( \lim_{\delta x \to 0} \frac{\delta y}{\delta x} \), were the solution path to the question. The ignorance of these alternatives may be due to the fact that they were not exposed to these notations at school. Or they just could not recall or re-construct them in the heat of the exam.

The word tangent is derived from the Latin word tangentes which means touching. Thus the tangent to a curve is a line that touches the curve. So the tangent has to have the same direction as the curve at the point of contact. The essence of calculus is the limit and the derivative. (Fig. 62) The derivative is the instantaneous rate of change of a function with respect to one of its variables. This is equivalent to finding the slope of the tangent line to the function at a point.

As the derivative can be interpreted in two ways.

As the slope of a tangent

\[
\frac{\text{dy}}{\text{dx}} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}
\]
Secondly, the derivative at $a$ can be written as

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Thirdly, the derivative can be thought in terms of infinitesimals, small increments in $x$ resulting in a small increment in $y$ (see Fig. 63).
This is usually written as:

\[
\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}
\]

\[
= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]

\[
= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.
\]

Student errors and challenges on this item were analysed at a micro-level below.

Once the wrong substitutions (and correct ones) were made in the difference quotient, fragmented and random errors of all sorts emerged from algebraic simplifications of those substitutions. For example, simplifying \((a+b)^2\) represented in \((x+h)^2\) is supposed to be well-

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known and standard result at this level, but it inherently brought countless difficulties; if the
students recognised the need to apply that standard formula at all.

For the 30% students who could recall the formula correctly, the main difficulty displayed
was failure to operationalise the functional notation through proper substitution to map \((x+h)\)
on to \(f(x+h)\) in the range set. This failure could be borne out of the formula function concept
image that students may have had on the function. This understanding of a function is
superficial and partially correct concept image of a function concept. This showed a surface
understanding of the concept of a function bereft of its deeper meaning. This understanding
was mechanical and instrumental as structural. What is crucial was for students to be
extremely clear that the point in the domain set was to be mapped onto its proper point in the
image set. Once a point, in this case \(x+h\), is NOT mapped to its proper image then the
mathematics immediately suffers; structurally and executively. The level of violation of the
mathematics is analogous at kindergarten level to writing that \(2 + 3 = 7\) for example. Then
the plus sign represents an operation; a function mapping points in a plane \(\mathbb{R}^2\) onto points in
the real line \(\mathbb{R}\). This can be generalized as \(f(x,y) = x+y, (= f(y,x))\). In particular, points in a
plane are mapped to a single point in the real line by the addition operation/function. If we
have \(f(3,2)=7\), even elementary mathematics learners will remark that this is incorrect, if our
function \(f\) represents addition. But at grade 12, learners do not realise this very same error
when they commit it in evaluating functions at particular points. In calculus when the
functional notation is used it is assumed that \(f(x)\) is a unique point.

Some students were confused by notation, failing to note the difference between, \(f(x)\), from
\(f'(x)\), for example. So they had errors of interpretation of symbols, which is quite serious
in mathematics as symbols communicate mathematics concepts. At the same time students
failed to differentiate between \(\delta y\) and \(\frac{dy}{dx}\). Students treated these as though they were
the same. These were also syntax errors. Such errors are important because much of calculus is
communicated through symbols, one cannot understand and do mathematics without
understanding its symbolism since mathematical knowledge is abstractly represented via
written symbols. In Vygotskian terms, symbols are also tools that signify concepts as well as
the oil that lubricates mathematical discourses within and across persons. In Vygotskian
terms, symbols are both signs and tools. They communicate mathematics concepts and help learners to think about mathematical ideas.

\[
\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \left(\frac{-(x+h)^2 - 3}{h}\right)
\]

One student wrote, \(\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}\) to represent the formula, \(\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}\) that he/she had presented. Such substitution showed that some students had superficial conceptual knowledge of a function and its notation. Such errors seemed to be also random as they showed no pattern of recurrence.

Yet another wrote \(-3(x+h)^2 - (-3x^2)\) which was then reduced to \(-3x - 3h - x^2 - xh - xh - h^2 - 9x\). This student apparently had great difficulty in removing the parenthesis. Also \((-3x^2)\) was reduced to \((-3x)(-3x) = -9x\). This student exhibited primitive knowledge gaps in algebraic conceptual and procedural knowledge involving at least manipulation of directed numbers, binomials and indices. Such difficulties were faced by the majority of the students. To them, algebra remains an untraceable maze that negatively affects their mathematical performance.

Notations were sometimes used loosely for instance \(f'(x+h) = -3(x+h)\) but subsequent writing showed that that learner meant \(-3(x+h)^2\).

One had correct formula but wrote \(\frac{h}{3x+x}\). Another wrote \(f(x) - f(-3x^2)\). Which was reduced to \(f/9x\). Another had \((x+h)(-6x)\). One actually had a wrong formula:

\[
\frac{(fx-h)-3x^2}{h}
\]

Another used inverse functions \(f(x) = -3x^2\).

Other students were more methodical and inventive in their errors. For instance, one invented an interesting and logical way (though it was incorrect) of simplifying \(-3(x+h)^2\). The student wrote that \(-3(x+h)^2 = -3x^2 - 3h^2 - 9xh\). This example shows clearly that students can be methodical and can be cunning in the mistakes they make.

One student wrote the required formula but failed to figure out how to proceed. I would assume that this student could not figure out how to substitute in the generic formula.
Another student wrote \( f'(x) = \log \frac{3}{x} \). This could have been an error due to procedural extrapolation on concepts of logarithmic differentiation. Another used inverse functions and second derivatives writing \( f(x) = \frac{1}{3x}; f''(x) = \frac{x}{9} \), yet others referred to second derivatives such as \( f''(x) = -9x \). Such learners had structural misunderstandings. They could not demonstrate how mathematical topics link and do not link, otherwise how could the learners link differentiation of polynomials with logarithmic differentiation? The researcher thinks that these examinees had been exposed to logarithmic differentiation but failed to understand how it linked with the current item. I suspect that mixing unrelated concepts occurs as students try to interpret some mathematics concepts with the lens of very defective and limited concept images. The error of mixing up procedures or concepts can be referred to chunking and hybridization.

It is important to indicate that in the students’ scripts, there was very little evidence that they struggled with the limit concept. Students just mechanically substituted \( h \) with \( 0 \) in the last line when differentiating from first principles. It could be that students might not have reached the level of analysing the limit concept critically. I believe that students were regarding and handling the limit concept procedurally and instrumentally, rather than conceptually. As a result there were more procedural errors and fewer conceptual ones.

Any manipulations the examinees showed were strictly mechanical and not based on any conceptual understanding.

Besides the epistemic gates that were locked by their lack of understanding calculus, these findings were important because calculus questions constitute 35 marks out of 140 allocated to paper 1. The learners exhibited serious gaps in algebraic knowledge of multiplying binomials.

**Item 1.2**

This question was very badly done by the learners. However, the pattern of their thinking could be deciphered.
This item required the use of the power rule of differentiation that if

\[ f(x) = x^n \text{ then } f'(x) = nx^{n-1} \]

or

\[ \frac{d}{dx}(x^n) = nx^{n-1}, \] where \( n \) is any real number.

Student difficulties with this item proved to have three parts. The first part was simplifying the expression

\[ y = \frac{\sqrt{x}}{2} - \frac{1}{6x^3} \]

so that its terms were in a format on which the power rule of differentiation could be easily applied; that is the terms of \( x \) were first to be re-expressed free from radicals and reciprocals. Students committed structural, executive and arbitrary errors at this level. The second part of the solution concerned applying the power rule to the simplified version of \( y \). At this level the errors committed were mainly executive in failure to deal with non-integral powers for example. The third and unimportant stage was re-writing the answer in standard, surd and index form.

About 80% of the students' failure was due to errors in the algebraic procedure to convert

\[ \frac{\sqrt{x}}{2} - \frac{1}{6x^3} \]

to

\[ \frac{x^{1/2}}{2} - \frac{x^{-3}}{6}, \]

the format needed to avoid unintended errors when applying the power rule. This difficulty emanated from student problems with algebraic competency.

It was surprising that 85% of the students in the sample simplified \( \frac{1}{6x^3} \) to \( -6x^3 \) or \( -6x^{-3} \), simply transferring 6 to the numerator. This was one of the most common errors shared by different learners. Apparently, students failed to detach 6 and \( x^3 \) in \( 6x^3 \). The first group had an arbitrary error of regarding the fraction \( \frac{1}{6x^3} \) as equal to \( 6x^3 \). Indeed these students disregarded the fraction concept. They changed the question to a form that was accessible and sensible to them thereby committing an arbitrary error. The second group of students reduced the fraction to \( -6x^{-3} \). These students acknowledged the fraction concept but treated it as though \( 6x^3 \) was in parenthesis. This was an error of mishandled parenthesis; it was a failure to understand mathematical syntax. By far this was one of the most prominent errors of the students on this item. Students also had problems restating \( \sqrt{x} \) from surd to index form, some wrote it as \( x^2 \). Others wrote \( x^{1/2} \), but had problems of applying the power rule on
the non-integral power. One had \(2x\), suggesting that the power was of \(\frac{1}{2}\) was taken as just a 2. These errors were due to dropped exponents or were arbitrary errors.

In simplifying \(\frac{1}{2}x^2 = 6x^3\), had an error of extended attachment, of delayed decomposition which can also be regarded as an arbitrary error due to FLMD.

Some students who had the procedural knowledge of differentiating terms that had integral powers, showed that they failed to generalise that rule to non-integral powers. Failing to apply the rule properly, they ended up with a myriad of wrong numbers. This question amply demonstrates that learners cannot hope to perform well in problem situations if they by themselves cannot diligently think forwards and backwards, to figure out the case for themselves. Although they did possess the concept image of the power rule, those who tended to apply the rule blindly produced many errors.

Some learners did their work which showed that they had no idea of a derivative at all. For instance one examinee had an answer of \(\frac{x - 4.63}{2}\) in response to Item 1.2.

**Item 2.1** Determine the x-coordinate of T

In terms of Stein et al. (1993), this task fell under the procedures with connections task. This is because it required learners to apply their known calculus procedures and interpret their value in order to solve the calculus problem. This is a classical application of calculus problem type often taught in introductory calculus lessons.

**Item 2.2** Determine the equation of the tangent to \(g\) at \(P\ (-3,11)\), in the form \(y=\ldots\)

This is another fairly familiar notion that students/learners put to use to find an equation of a line for which only one point on a curve it passes through is known. It is a procedure with a connection task. Initially it requires learners to determine the gradient of the tangent at the known point. When that gradient is found, it is then used to find the equation of the tangent,
as once the gradient of a line and a point on which it passes are known then the equation of
the line can easily be found. The procedure of finding the gradient of a curve at a point is
differential and requires the limiting process. However, it can also be found through the
‗power rule‘ that is deduced from first principles of differentiation. Suffice to say that the
power rule itself can be proven at a deeper level through mathematical induction.

Some learners knew that substituting the value of x in the derivative of
\[ g(x) = -(2x-5)(x+2)^2 \] gives the gradient.

However, having correctly found the derivative, \(-6x^2 - 6x + 12\) they reduced it to \(-x^3 - x + 2\), in
the previous turning point question. This was changed to \(x^2 + x - 2\) to avoid working with
negative xs. This error emanated from students‘ procedures in finding x values of turning
points where the derivative is zero and changing signs like this is helpful to solve for x.
Students just picked this expression from the previous question and presumed it to be the
derivative. This error is very much a conceptual error. It is due to lack of meta-cognition.
Students then substituted -3 into \(x^2 + x - 2\). Because of this error, the answer 10 was very
common. In the end, the sign of the gradient found was opposite and still needed to be
multiplied by 6. This error of changing the signs of the derivative was also very common
even though the derivative was not factorised. This means that students did not know that the
derivative follows the shape of the curve.

\[
\frac{dy}{dx} > 0, \text{ and if it is decreasing, } \frac{dy}{dx} < 0. \text{ It is caused by}

students failing to monitor and check their work. So here there was an arbitrary error and
meta-cognition error.

As expected, it was common for students to differentiate in the brackets separately and then
multiply these. This was a pseudo-linearity error. Many weak students went on to use the
formula of the midpoint to find the gradient. Other weak students just chose the co-ordinates
of the given point \(P(-3,11)\) and another point mostly \(A(-2,0)\) and found the gradient of \(PA, -11\). The answer -11 then, was very widespread. They then used this gradient to correctly use
the procedure of finding the equation of the line, which was in fact not the tangent. The
answer \(y = -11x - 22\) was very common and represented this error. Some students used this
approach on points $PT$. In this students showed enough knowledge of the equation of a straight line but failed to use calculus to generalise their knowledge to tangents. Clearly, students seem not to understand the meaning of the term: 'tangent'. Here, students answered a question familiar to them, of finding the equation of a straight line where two points it passes were given. I doubt that students were convinced that this was a tangent. They could have wanted the examiner to see what they knew hoping that they will get some credit for that. They did this as they were awarded marks for the procedure of finding the equation of a line even though their gradient was wrong. Students' answers show their unwillingness to conceptualise the idea of a tangent to a curve.

Other students used the x intercepts points $A(-2, 0)$ and $B(2, 0)$ to find the gradient of the tangent. They used the gradient in a correct procedure to find the equation of a straight line.

There were students who regarded the tangent as having gradient $m = \tan \theta = -0.27$. Where $-0.27$ came from $\frac{-11}{3}$, the coordinates of $P$. Other students, having correctly found the expression of the derived function, went on to express it as the equation of tangent, rather than the gradient of the tangent. Other students wrote $y^2 = (-3)^2 + (11)^2$. In this case they were finding the length of the line $OP$. Again this was an interpretation or arbitrary error.

It was pleasantly surprising to discover that students had grasped well the procedure of finding the equation of a straight line. But they were affixed to this procedure and seemed reluctant to acknowledge that this procedure was insufficient to determine the equation of a tangent of a curve.

Other students regarded the gradient as $\frac{-11}{3}$ and others took $\frac{-11}{3}$, making use of the $x$ and $y$ coordinates of the point of tangency $P(-3, 11)$. Inevitably, they went on to correctly use the procedure for the equation of a straight line. It seems that the students had a misconception that any line passing through $P(-3, 11)$ was a tangent to the curve. These were arbitrary errors. Other students regarded the $x$ or $y$ co-ordinate of $P(11, -3)$ as the gradient of the tangent, and the expression $11 = -3x + c$ was very common. This was a slavish and uncritical application of the generic formula $y = mx + c$. These were errors due to comprehension that.
Newman (1977a, 1977b) and Clement (1980) had alluded to. Other students had errors of simplifying the derivative having substituted \( x = -3 \) in the correct expression.

One student just wrote \( y - 3 - 11 = 14 \), just using the coordinates of P. Another student just wrote \( y = \tan x \). Other students took the \( x \)-coordinate of T as the gradient of the tangent. It would appear that such students seemed to think that a value found in the previous question must be used as an input to the next.

Other students had procedural extrapolation errors. Having correctly found the derivative they went on to equate it to zero and then solved the quadratic equation for \( x \). Clearly, they were using the procedure for finding the turning points here, which was not asked for in the question. Other students confused this question with transformations of graphs. The expression

\[
y = x^3 + q = x + 3 + 11
\]

occurred several times, suggesting transformations of hyperbolic functions; a topic in the mathematics curriculum cued this idea. This then was any error of hybridisation.

Thus students had many terminology errors. The issue of mathematical terminology raises the important aspect of language (for example Vygotsky, 1978) which could be a main cause of students' failure to learn and understand mathematics better. When these students saw the graph of \( g \) they immediately thought of transformations, irrespective of what the question was asking. The symbol \( g \) then put into consciousness conflict factors (Tall & Vinner, 1981) that triggered linking \( g \) to hyperbolic functions where this symbol had been used together with \( f \). Other students wrote \( y = \log_4 x \). These were conceptual errors of failing to differentiate between different but related mathematical notions, which could be seen as hybridisation. Other students took the second derivative and equated it to zero. Then they took the holding value of \( x \) as the gradient of the curve.
Item 2.3 Determine the x co-ordinate of the point of inflexion

This question was most unpopular. About 70% of the examinees never attempted it. Some found the derivative but could not proceed. The question required meta-cognition that seemed out of reach to students at this level.

Item 3.1 Show that the height of the glass, h, can be expressed as $h = \frac{200}{\pi r^2}$

This was a doing mathematics task as it involved mathematising and making links between the volume of a cylinder and its dimensions. Students were expected to express the height $h$, in terms of $r$ using the volume given. From this, they were expected to substitute this value of $h$ in the area so that a function $S$ of $r$ could be written down for calculus techniques to be used. This question required students to have strong visual representation skills of three dimensional figures. This could have provided learners with an extra cognitive load. A cognitive load occurs because of redundant representations in a question that may make its solution difficult.

Essentially, the examinees were required to visualise the cylinder and transform the 3D shape to a 2D shape. As the formulae of the surface area and volume of the cylinder were provided in the examination, the task was essentially reduced to that of manipulating the variables Volume ($V$), Surface Area ($A$), height ($h$) and radius ($r$), so that from the given volume of $200\text{cm}^3$, the variable $h$ could be substituted in the area in terms of $r$.

Some of the students’ responses were:

\[
\frac{h}{r} = \frac{200}{r^2} \quad \text{implying that } h \text{ and not volume was } 200\text{ml}
\]

\[ h = \frac{200}{\pi r^2} \quad \text{or just copying } h = \frac{200}{\pi r^2} \] was very widespread but it was never shown how it came to be. This misconception could be from language interpretation.

\[ \text{Height} = \text{length} \times \text{breadth} = \frac{200}{\pi r^2} \] This is just similar to the area of a rectangle long learnt in primary school. This was a primary grade fixation with the area of a rectangle.
200= πr^2 just seeing these formulae triggered the idea of the area of a circle, perhaps anything with pi was related to a circle which however was not very wrong. So this was a problem of application.

H=200ml of liquid x r^2

L.B.H =volume, students could not free themselves from the fixation of volume of a cuboid when they heard it for the first time at primary school. They thought the formula for the volume was always the same.

-1<r<1, -1<200<1. This is hybridizing with the geometric series for conditions of convergence of a geometric series. The symbol S(r) triggered the formula of the sum to infinity of geometric progression with common ratio r. The -1<r<1 indicated that the learner was investigating condition of convergence of the geometric progression.

\[ H = \frac{200}{\pi r^2} = 31.83 \text{, that is } \frac{200}{2\pi} \] This showed the rush to find h, disregarding other conditions.

H=63.66 came from \[ \frac{200}{\pi} \], then h = 7.98; that is the square root of 63.66

H=200/9.869 = 20.26 coming from r = 3.14, where H = \[ \frac{200}{\pi r^2} \]

Arbitrary took r=5

\[ A = \pi r^2 \cdot h \cdot b \]

Thus the learner errors were varied and most were arbitrary errors in that students put in the questions they wanted to answer and disregarded the original questions. The arbitrariness showed learners’ fixation with earlier learnt concepts and procedures.

Item 3.2 Show that the total surface area of the glass can be expressed as

\[ S(r) = \pi r^2 + \frac{400}{r} \]

Responses:
Area = \( \frac{1}{2} \) hb = \( \frac{200}{\pi r} \). r. \( \frac{1}{2} = \pi r^2 + \frac{400}{r} \). This was manipulating towards the correct expression. This is an instance of Brown-Matz-van Lehn Law (Davis, 1984, p.123) that stipulates that higher procedures must run, and that if intervening lower input procedures needed for it to continue are not available, then default procedures will be invented in order for the higher procedure to continue.

\[
S = \pi r^2 + \frac{h}{r}
\]

S = 2. Area of base + \( h \cdot 2r \)

\[
= 2 \pi r^2 + \frac{200 \cdot 2r}{\pi r^2}
\]

\[
S = \pi r^2 + \frac{400}{r^2} \cdot \frac{1}{r^2}.\] This was a product instead of a sum.

S = L. B = 200.106. The L.B implied finding the area of a rectangle.

Area = length. breadth. This was an area of a rectangle are fixation, in which the student thought all areas of figures are found by multiplying length and breadth irrespective of the shape of figure. This fixation came because learners first encountered area of a rectangle as length \( \times \) breadth and could not move from that fixation.

\[
S = \pi r^2 + \frac{2h}{r}
\]

The more capable learners found the area of a closed glass but edited the result so that it suited the given result even without editing their wrong formulation.

One, interestingly wrote \( S = \) area of circle + \( 2 \) \( \text{radius} \).

\[
S = 2.200 + \pi r^2 \] then \( S = \frac{400}{r} + \pi r^2 \text{ this is complete cheating. This was moving towards the answer illogically.}

Others differentiated the correct expression well but could not continue

Area = \( \frac{1}{2} \) absinC = \( \frac{1}{2} \cdot 200.63 \cdot 67.400 \) the idea of area of a triangle occurred more than once. Hybridizing error.

\[
\frac{2}{S} = \frac{2[2a + (n-1)d]}{l}\] seeing the S triggered sum of an AP formulae. Hybridizing error.
These errors also show Feigenbaum’s law of minimum discrimination (Davis, 1984, p.124) where learners are satisfied by not discriminating further than necessary. This often results in errors as learners take things for granted, by over-generalising.

Item 3.3 hence determine the value of r for which the total surface area of the glass is minimum.

Some students cleverly copied $S = \pi r^2 + \frac{400}{r}$, but failed to proceed with the differentiation.

$S = \pi r^2 + \frac{h}{r}$, this implied that $h=400$.

$S(r) = \pi r^2 + \frac{400}{r}$. This was an algebraic error of clearing $r$ from the denominator where this was not appropriate. Here $S(r)$ is regarded as $Sxr$.

$S=\pi r^2 + \frac{400}{r}$. This implied that $r$ was 200, which had been given as the volume.

$R = 1.7$

All was well but the terms were simplified as $\frac{r^{-1}}{r} = r$. This was an arbitrary, structural or executive error.

$S=\pi r^2 + \frac{400}{r}$ cleared to $S=\pi r^3 + 400$

$Hr = 400$

$R= \pm 3.99$

Many students worked the question without any regard to calculus techniques.
Students again showed great struggle in clearing the fraction in $S(r)$, just as they did in item 2 which dealt with differentiation.

Most students just copied the questions as they were, never bothering to write anything more.

In many cases, students chose arbitrary values of $h$ or $r$ which they substituted. For all the three questions on calculus, up to 45% of the students seemed to be completely ignorant of differentiation.

Items 3.1 to 3.3 required the use of mathematical modelling. The modelling had to be based on learners' visualisation of the drinking glass. Then they were required to “cut” the cylinder so that they had a rectangle and a circle and note that the sum of their area was actually the surface area of the drinking glass. It required students to find the value of the radius obtaining when the surface area of a cylinder is maximised (see Figs. 64.1 & 64.2).
The lack of strategy to deal with the problem through correctly setting up the required functions meant that analysis techniques were beyond reach, even though the part of Item 3.3 was accessible to those with procedural knowledge only. This question thus remained inaccessible to students who did not have critical problem solving skills and were not aware that calculus presented a specialised technique to optimise or minimise functions that described the variables of the problem.

The students’ algebraic competency needed to be applied to the real world problem. However, the mathematical symbolism required became a barrier to answering this question. The question required critical analysis and thinking to figure out what the problem was and what was to be done with reference to a function at the derivative at turning points. Some
students seemed to have been confounded because of the function notation $S(r)$ instead of the usual $f(x)$. Analysis also reveals that in as much as 70% of the scripts, students seemed not to show awareness of differentiation beyond writing down symbols and notation used to communicate it.

### 6.7 Conclusion

This chapter began with the statistical analysis of the examination performance of the learners for the whole mathematics paper 1 and then for the calculus items. The Newman hierarchy of error analysis (Clements, 1980) was used as a guideline for analysis. Then the question by question mapping of the mathematical demand of the calculus items was done as a foreground to the errors analysis. Finally the errors and misconceptions in the calculus items were analysed using the mathematics error analytical protocol discussed in Chapter 3. This deductive analysis, where errors were put into those predetermined categories was also aided by vignettes to illustrate the different types of errors learners were having. Besides the vignettes, general analysis on important errors across all scripts was discussed in further detail. The analysis shows that learners grappled with the calculus items as illustrated by the great variety of errors shown in answering the questions. Some errors are deducted from the protocol but some emerged from the data in spite of the analytical protocol. Further discussions on found errors and their relation to literature was undertaken in the following Chapter 7, where a new calculus and mathematics error analysis protocol was introduced.
CHAPTER 7: DISCUSSION, CONCLUSIONS AND RECOMMENDATIONS

7.1 Introduction

Following the data analysis chapter, this concluding chapter is the climax to a lengthy academic study to determine and understand the nature of learner errors in calculus which is also called elementary analysis. Guided by assumed theoretical and conceptual perspectives, student answers in examination scripts were analysed using content analysis methodologies. This chapter concludes the study by wrapping up with discussions, conclusions and recommendations pertinent to it.

The findings, discussions and recommendations of this research are reported in consideration of the theoretical and conceptual perspectives that guided this research. In Chapters 2 and 3 the researcher outlined the theories informing this research. The first theory was on the philosophical question of the nature of mathematics. The researcher presumed that while in the past, absolutism was the dominant philosophy of mathematics; it is now being challenged by fallibilism. The rise of Fallibilism was due to contradictions periodically occurring in supposedly infallible mathematics. These contradictions gave rise to several crises in mathematics. These crises in mathematics showed that mathematical knowledge is oftentimes fallible. These challenges always helped to push ahead the frontiers of mathematical knowledge, as firmly-held mathematical results were revisited and revised (Nesher, 1987). The researcher argued that similarly, learner misconceptions in mathematics when carefully investigated do help learners to advance their mathematical learning. Constructivism was the second theory that guided this research. Constructivism postulates that when learners are constructing knowledge, they do so by incorporating new mathematical ideas onto their earlier knowledge (Jaworski, 1994). As students often come to lessons with semi-constructed realities of what we teach them, their constructed realities are important to the development of teachers’ pedagogical content knowledge. Learners’ constructed realities can become stumbling blocks for learning if teachers do not understand them. The constructed realities can potentially be misconceptions.
So learners' earlier mathematical knowledge can be either an asset or liability depending on what was earlier acquired or constructed, as it is used as lens to interpret new mathematical concepts. The earlier knowledge can support or hinder learners as it is used for making sense of new mathematical phenomena through accommodation, assimilation and equilibration (Piaget, 1968). Then learners build a concept image of a given concept from prior concepts. The third theory from Tall & Vinner (1981) was invoked. Tall and Vinner were disciples of Piaget in that their concept image, concept definition, and procept notions were an elaboration of constructivism.

These notions impinge on this analysis in that they contend that calculus errors occur because of learner concept images that do not match concept definitions. However, the constructed concept image may or may not have errors. If it does not mirror the concept definition; the accepted definition of the concept by the mathematical community, the learner's concept image may be a misconception. The errors and misconceptions that learners have were well-meaning in that learners have rationalised them. The fourth perspective emanated from Vygotsky's (1978) socio-cultural theory of learning. The researcher presumed that learners have calculus amateur concepts just below their Zones of Proximal Development, and that good understanding of these pre-conceptions assists to develop learners' Zones of Proximal Development. This enables teachers to appropriately mediate calculus scientific concepts to academically ready learners. Last, but not least, the mathematical proficiency notion of Kilpatrick et al. (2001) was important to this study in that if mathematics teaching and learning does not encompass all its five strands (conceptual understanding, procedural fluency, adaptive reasoning, strategic competency and productive disposition), learners' mathematics education is famished and leads to the development of many misconceptions and imbalances.

The findings, conclusions and recommendations of this research arise in the background of these theoretical assumptions, while answering the following research questions;

What is the nature of the most common errors that students display in answering Grade 12 mathematics examination questions on introductory differential calculus?

The sub-questions to this question were:-
(a) What are the types of errors made by learners in response to Grade 12 examination Differential Calculus tasks?

(b) How would the common errors and misconceptions the learners made be described?

(c) What typology of errors and misconceptions in calculus is grounded from the study?

7.2 Research findings

To begin with, the researcher reports that the calculus often represents the first time in which students are rigorously confronted with the limiting concept, involving calculations no longer performed through simple arithmetic and algebraic techniques. Learners no longer study only the discrete and finite quantities but also the continuous and increasingly changing. Whereas in arithmetic and algebra direct arguments are used, calculus however is borne of infinite processes negotiated through indirect and intuitive arguments. The highly intuitive nature of calculus particularly the limiting processes present huge barriers for learners in understanding and grasping it. In their attempt to understand and solve calculus problems, learners formulate many alternative conceptions that render them to make many errors.

In general the errors the learners had were a function of their pre-calculus mathematical competency. Learners who had difficulties with pre-calculus topics such as algebra or functions had most of their errors in those areas and therefore made very little engagement with calculus concepts themselves. In many cases, learners had multiple errors in answering any given item. The errors at times snowballed which meant that one small error led to larger ones for example. Some of the learners’ errors were quite ingenious showing their inventive capacities.

The detailed findings are discussed below.
7.2.1 Nature of common errors

✓ Learners’ errors in calculus were exacerbated by lack of competency in pre-calculus concepts and procedures, making worse an already difficult situation.

✓ The research found that in many situations, a learner has different kinds of errors in answering a single examination item. This implies that the process of error making is not static but dynamic.

✓ More than 90% of the students were incapable of doing the simplest routine calculus procedures of differentiating a polynomial. In addition, they met difficulties in answering questions which required higher levels of cognitive thinking, relational thinking and extended abstract.

✓ Many learner errors appeared to be based on over-generalisation of algebraic rules to calculus, such as procedural extrapolation and equation balancing. In such cases, learners differentiated numerators and denominators separately or differentiated products separately.

✓ The students struggled to perform in calculus due to pre-calculus misconceptions. They could not tackle the calculus because they were weighed down by lack of competency in algebra and other pre-calculus topics. For example, they could not simplify algebraic terms in Item 1.2 using laws of indices, to prepare terms for differentiating using the power rule.

✓ Some students who could use the power rule of differentiation had the error of procedural extrapolation of differentiating constants as well by multiplying the constant by the power and reducing the power by 1 (see Fig. 65 below). They did not realise that differentiating a constant gives a zero, that is the derivative/gradient of \( y = c \), a horizontal straight line is zero. This is a good example of using formulae without understanding them. Such an error is also structural.
Figure 65. A learner’s lack of competency in algebra affected his/her differentiation.

This showed that some learners differentiated variables as well as constants, as they treated them as similar mathematical objects. Here, the learner differentiated $6^{-1}x^3$ to $-6^{-2}.3x^4$. The super-procedure in this case was use of the power rule where the term was multiplied by the power and the power reduced by one. This super procedure was followed by a sub-procedure performed on $6^{-1}$ unbeknown that $6^{-1}$ was just a constant. To this learner, the super-procedure dictated that the power rule of differentiation be used whenever a power appeared. There is the possibility that the learner linked this reasoning to the manipulation of powers, such as in laws of indices. This was invoking and applying a procedure where it was not called for; a sort of what I have called a hybridisation error. The learner failed to recognise the essence in $6^{-1}$ and $x^3$ in that one is a constant and the other is a variable and that the derivative of a constant is zero. It was unbeknown to him/her that differentiation using the power rule applied only to variables. If this learner noted that, then s/he would obtain the derivative as $0.-3x^4$ which is 0. That will still be wrong. Indeed, the learner would be in a crux as his
procedure escalated to produce more errors. This is because what is needed in this item if the learner wanted to treat $6^1x^3$ as a product was the use of the product rule for differentiation; namely: $(fg)(x) = g(x)f'(x) + f(x)g'(x)$, which is a procedure that was not in the Grade 12 assessment standards. Again, the learner was not at all entirely wrong, his/her thinking was very insightful, methodical and mathematically acceptable. The learner had that concept image of differentiation. This was a procedural way of thinking devoid of understanding the essential meaning of the derivative. The fact is that the preceding Item 1.1 did not help the learner to think of differentiation conceptually; namely, $\frac{d}{dx}(-3x^2) = -6x$.

This showed that the learner’s knowledge was fragmented as he/she did not think relationally by linking the answer found in Item 1 with the answers in Item 1.2. These did not help the learner to be aware of the big idea tying the different concept images. This was a classical case of instrumentally using procedures (Skemp, 1976).

The same scenario occurred when learners simplified $\frac{1}{6x^3}$ to $6x^{-3}$ in Item 1.2. This error was extremely widespread occurring in at least 80% of cases that attempted this task. It is one of those errors that for some reason is common across different learners. Learners just saw 1 over something and in their concept image was evoked the idea of writing the denominator to the power minus to clear the fraction. In doing this they used the algebraic procedure correctly using the third law of indices. This procedure in itself was good but it was not carried out to its logical conclusion. Learners might not have realised that 6 and $x^3$ were separate terms, which multiplied each other; if they did, they did not regard it important to separate them at this time. Learners failed to do the sub-procedure of detaching 6 from $x^3$. These algebraic errors originated from learners’ frames of reference. They were partially correctly executed and partially wrongly executed. Also a higher procedure dictated that the learners handle that task that way and I predict that without any challenge or mediation, the learners will continue to propagate this error.

That this learner differentiated $6^1x^{-1}$ to $-6^1 \cdot -3x^{-4}$ also showed a pseudo-linearity of differentiating products error.

✓ Learners’ errors in calculus were exacerbated by lack of competency in pre-calculus concepts and procedures. Lack of competency in pre-calculus topics hampered
calculus epistemic access as learners registered many errors in those topics. Learners' entanglement with algebra made access to differentiation difficult.

✓ Competency in pre-calculus topics or lack of it was an important predictor of the errors learners had and therefore of calculus epistemic access.

✓ Thirty five percent of the learners submitted blank responses to the tasks. This suggests that they did not make sense of the questions at all or found the items too difficult to attempt. This situation was particularly prominent in Task 3 in which a certain English Language command level was required to comprehend and correctly interpret the problem. There was evidence that some learners, who attempted that item, ended up finding the surface area of a ‘sealed’ drinking glass. This scenario puts English Language competency in perspective; it seemed to be a key predictor of the difficulties learners encounter in studying mathematics and differential calculus applications. As language is a key mediating artefact in acquiring knowledge (Vygotsky, 1978), students who do not understand the language find that language acts as a translucent instead of a transparent resource that illuminates new mathematical knowledge (Lave & Wenger, 1991). If language structures are similar there is positive transfer if there is no similarity there is negative transfer. As the majority of learners used African languages which are not similar to English, the language of learning and teaching (LoLT) is argued that there was not much positive transfer leading to many errors and misconceptions in mathematics due to lack of comprehending the language. The lack of competency in language led to decoding errors and arbitrary errors. These occurred as learners failed to understand what was required of them in the questions. They interested what was asked incorrectly. The arbitrary errors were closely related to decoding errors as students formulated and solved different questions to those that were given.

✓ Learners hybridised mathematical concepts and procedures. The occurrence of certain mathematical symbols in questions such as \( f(x) \) or \( r \) led learners to retrieve some previously studied concepts and procedures, which did not apply to the present cases and misled learners. For instance the \( r \) made some learners invoke ideas of conditions
for $r$ for the convergence of an infinite geometric series. I referred to these as hybridisation errors.

The $\frac{dy}{dx}$ notation was widely used in the scripts. This notation caused problems for students in Item 3.3 where students were supposed to use $S'(r)$ or $\frac{dS}{dr}$ to maximise the surface area of the glass. Most learners had problems because they were used to working with $\frac{dy}{dx}$ notation. Notation then presented an epistemological obstacle. Also, there is always the problem that learners find $\frac{dy}{dx}$ weird in that they were early on not allowed to treat it as a fraction where $d$ is cancelled and they remain with $y/x$. This prohibition is usually abandoned when students study the integral calculus particularly when solving differential equations. Thus one finding is that the errors learners had were due to inflexibility in the use of different notations in calculus; $\frac{dy}{dx}$, $f'(x)$, $S'(r)$, $y$ and so forth. More than 85% of the students did not show the knowledge that $dS/dr$, $f'(r)$, or $S'(r)$ related to the same notion of the derivative. Ignorance of notation is related to procedural errors. I related these as symbolism errors.

The notation $\lim h \to 0$ and other calculus notations were accepted procedurally rather than through conceptual understanding. This was shown by students who regarded calculus as a form of extended algebra. Such students had all the calculus syntax but deeper analysis showed that they never engaged with calculus concepts and strictly worked in an algebraic plane of reference. This was clearly exhibited in Item 1.1 when learners were supposed to differentiate from first principles. Learners failed to work in an extended abstract to move away from the form of algebraic symbols to the function of the symbols that denoted new calculus knowledge (see Fig. 66 below).
As Tall and Vinner (1981) have indicated, learners form concept images of certain mathematical concepts and procedures in their minds. Analysis showed that these concept images on differentiation and application of differentiation were very incomplete when compared to concept definitions. Often the concept images were inadequate evidenced by the misunderstanding of terminology, and language, where students’ interpretation of terms differed from that used by the mathematical community. For instance, students often confused the gradient of a tangent at a point on a curve to the gradient of a line joining the origin to that point on the curve.

Students were found to make numerous errors in algebra such as failing to apply laws of indices, and failing to manipulate fractional algebraic terms among others. Besides this, students could not factorise quadratic expressions, some could not even solve linear equations. Such students faced calculus in a very ill-empowered and ill-prepared manner as algebra is the language with which calculus is communicated.
It was noted that students seemed to have a process notion of the limit concept (Dubinsky, 1991). It was highly unlikely that they possessed the object-process notion of the limit concept because their understanding was procedural. Students had not yet grasped the highly complex and intricate reasoning embedded in the derivative as the gradient limit of a secant to a curve at point. In this manner, students seemed to have learnt and accepted the notion of derivative as a matter of procedural faith rather than through understanding. This is because the limit is often seen as a process and it is difficult for learners to think of it as an object (Artigue, 1996; Tall, 1996).

Most students learnt mathematics in a second language. This caused them to struggle to access mathematical knowledge through a language which was not their own. The language medium must be an invisible and facilitative resource, so that the mathematics it denotes becomes transparent through it (Adler & Lerman, 2003). In this examination, the language was too hard as to make the mathematics itself invisible to learners. That the language acted as a barrier was amply demonstrated by students’ inability to negotiate the language in Item 3, to model the mathematics problem with mathematical equations in a way that the problem could be solved using differentiation techniques.

In Item 2, students had many errors because of failure to distinguish between the terms; function, derivative, differentiation, limit, gradient, tangent, curve, point, coordinates, turning point, maximum point, minimum point, point of inflexion, equation of tangent and so on. As the study has shown students often interchanged the meaning of these terms resulting in many different kinds of errors.

In many cases, learners tried as much as possible to avoid the idea of a tangent through seeking for alternative interpretations. This showed that learners had not grasped the abstract idea of a tangent to a curve so vital to the idea of a derivative.

One main deficit in learners’ work was inability to appreciate and use the mathematical technique of substitution. Lack of this facility encumbered learners to
take advantage of opportunities to answer the items well. It resulted in many errors being done in evaluating functions or working out algebraic relations needed to answer the items. For example in Item 3 it was necessary to substitute for the volume \( V \) and height \( h \) to come up with the function \( S(r) \) for the surface area on which calculus techniques could then be applied to maximise the surface area and find the corresponding radius.

7.2.2 Types of errors

Many studies on differentiation have shown that globally students often have adequate procedural knowledge of differentiation but do face problems when conceptual understanding of differentiation is needed in problem solving situations (for example Porter & Masingila, 2000). However, in South Africa, well known for its inefficient education system in a modern society (Rusznyak, 2008), performance in mathematics (and science) is among the worst in the world (Reddy, 2006). This poor performance was also confirmed in this study.

The research found that more than 80% of the students lacked adequate expertise in differentiation procedures and had many errors on it. This finding is in contrast with the findings of Orton (1983b) and Porter & Masingila (2000) that students have expertise in differentiation but lack conceptual understanding of it; and are even not aware that there are conceptual underpinnings to the procedures.

Students made a wide range of errors. Some discussed in my conceptual framework and many not. Some of these errors were confirmed from literature but some were grounded from the scripts used in the research. Some of the errors that emerged from the scripts were different from those in literature because of the unique state of mathematics education in South Africa. South Africa is unique in that its low level mathematics education does not emanate from shortage of resources, as South Africa is a country with a modern economy which is the best in Africa. The suggested error protocol (see Fig. 67) can help to improve
expertise in teaching that hinges on analysis of learner mathematical errors identified in this study.
Figure 67. Protocol on generic errors in mathematics/introductory differentiation
7.2.2.1 Explanation of the protocol on error analysis and error types

This section discusses the categories of errors and misconceptions outlined in the protocol (Fig. 67). The types of error learners committed have been extensively described and discussed in Chapter 6. That analysis is the basis of this protocol. It is argued that the errors that learners had were not mutually independent but that they were closely associated and interwoven with each other. As such, error analysis is a dynamic and not a static process. It is quite possible that the errors shown in a learners’ work could be interpreted differently by different researchers. In order to increase validity and reliability of my error categories, they were triangulated with literature. The other errors were also discussed with other researchers at annual research conferences of the Southern African Association of Research in Mathematics, Science and Technology Education (SAARMSTE) in 2010 and 2011. The error categories were also debated at a doctoral seminar at the University of Johannesburg in 2011 and suggestions incorporated in this research.

The errors and their descriptors are discussed below.

1. Systematic error: This type of error occurs due to planning. It is intentional and is caused by an underlying misconception that systematically generates the error. The learner is not aware that the thinking is faulty. Systematic errors are due to incompetency. Sometimes a learner can justify that error for some reasons. Without external interventions, learners may not be able to realise that their thinking is faulty.

2. Unsystematic error: This type of error occurs due to other factors other than incompetency. They could be due to tiredness or anxiety. Learners can readily correct these errors by themselves if they have a re-look at their work.

3. Executive error: This error is due to inability to use a mathematical procedure, such as failure to differentiate a polynomial in \(x\) using the power rule of differentiation. The error is due to lack of understanding of how to use the procedure works.

4. Structural error: This error is conceptual. For example a learner who regards calculus as extended algebra has a structural error. A learner who does not understand that the limit process is the glue that holds calculus together has a structural error.
5. Application errors: Application errors occur when a learner understands concepts and procedures that are needed to solve a mathematics problem but fail to apply it to effect solution to the problem. A learner who can differentiate and understands concepts on differentiation but fails to use this knowledge to solve a problem where these are needed has an application error if he/she makes an error in applying that knowledge. This can also be regarded as a modelling error.

6. Meta-cognition error: This error is due to failure of a learner to reflect about his/her own thinking. The evidence of this error is unreasonable results or contradicting results.

7. Hybridisation errors: These errors occur as a result of an attempt to integrate ideas of mathematics where this does not apply. For example, a learner on seeing \( f(x) \) in a question thinks of transformations of the graph because such notation was previously used in transforming graphs such as vertical or horizontal shifts. At the same time, such a learner fails to notice that \( f(x) \) and \( S(r) \) can be treated the same in differentiation, and so can be hybridised.

8. Logically invalid inference errors: These errors were also related to hybridisation errors and meta-cognition errors. A learner makes a conclusion which is based on faulty reasoning.

9. Interpretation error or decoding errors: In this instance a learner makes a wrong interpretation of a question due to lack of understanding the language or mathematical symbolism. This results in the learner answering a task different to one which is required.

10. Arbitrary error: An arbitrary error occurs when a learner changes a question to a form acceptable to him or her. It could be that the question seems not to make sense to the learner. The learner thus wants to make it in a form familiar to him/her. This change of a question is actually declining the mathematical demand of the task and is often never condoned in mathematics.

11. Careless errors: A careless errors could be due to a psychological state, such as working rapidly; this could be a misreading for example.

12. Random error: A random error is not careless and a learner does not repeat it. It could be due to partial misunderstanding. It is neither systematic nor unsystematic.
13. Procedural extrapolation error: This occurs due to over-generalising previously learnt procedures in ways that produce errors in new mathematical situations.

14. Pseudo-linearity errors: This occurs when learners for example regard the derivative of products as the products of their derivatives.

15. Equation balancing errors: These occur due previously learnt processes of balancing equations. Thus learners would assume that differentiating the numerator polynomial and denominator polynomial with respect to a variable does not cause any errors because the same process has been done to the numerator and denominator.

16. Decoding error: These are errors that occur due to converting problems in everyday language to mathematical symbolism. This is similar to realistic mathematic education’s vertical mathematisation.

17. Encoding error: This is opposite to decoding and occurs when learners fail to link their mathematical results to an everyday problem. Thus learners might be able to optimise a function using differentiation but still fail to interpret the importance of their answer to a real world problem.

18. Syntax error: These are related to mathematical symbolism. Learners do not understand calculus mathematical symbols such as $S'(t)$ or $\lim_{h \to a}$.

19. Fragmentation error: These closely relate to hybridisation errors in that the learners understanding of mathematics concepts is disjoint and not connected.

20. Misread error: These are not intended but occur when a learner is distracted.

21. Theory like error: This error concern mis-understanding on how things work. The treatment of calculus as if it is advanced algebra is an example of a theory like error.

22. Delayed detachment error: these errors occur as learners fail to distinguish between concepts occurring at the same time. Thus learners think $\frac{1}{6x^3}$ as equal to $6x^3$ exhibit a delayed detachment error of 6 and $x^3$.

It is important to inform readers that the error categories in Fig. 67 and discussed above are not mutually independent or distinct. These errors exist in a continuum from serious theory like errors on one end and careless or misread errors on the other end. In between these two extremes there are minor similarities and minor differences. I bring them out to generate
awareness of them as a resource to build teachers’ pedagogical content knowledge (PCK) of teaching calculus and mathematics, even though some already exist in literature.

This error analysis and identification framework is important because it can help to guide teachers when teaching mathematics or calculus. It helps to pinpoint areas of weaknesses and potential errors and misconceptions in mathematics that teachers and others must look out for in a practical way. I argue that the error analysis protocol recommended by this research is an important contribution to calculus and general mathematics education. Fig. 67 collates the errors that I have found occurring in learners’ examination scripts on calculus tasks. Some of the errors listed in this protocol are generic to mathematics but some are specific to calculus and differentiation and its applications. Some were drawn from literature but some were grounded from data and I gave them my own descriptions. It is hoped that this protocol will be a useful tool for improving mathematics and calculus education in South Africa, countries in Africa and other countries around the globe which have problems of low mathematics learning and achievement.

7.2.3 Why do learners have those errors and misconceptions?

The research reports that students had errors because of various reasons spelt out in the following discussions.

✔ The researcher speculates that learners made calculus errors because they did not have “More Knowledgeable Others” (Vygotsky, 1978) to help them learn differentiation. This might be because many teachers in South Africa themselves are reputed to have little expertise in mathematics subject matter knowledge, particularly in calculus and so lack requisite mathematics content knowledge (Howie, 2001; Jansen, 1999b). As Vygotsky has argued, learning begins with interaction in the social plane where learners are passed on society’s valued knowledge from previous generations to create a Zone of Proximal Development. Learners can persist with unscientific or amateurish alternative conceptions if they lack academic support from their teachers that mediates them to scientific concepts.
Of note is that learners often hold strong but wrong beliefs about mathematical knowledge. This forms meta-constant images of the nature of mathematical knowledge. The apartheid legacy did little justice to the teaching and learning of mathematics by the majority of the population and this has the effect on learners today because the majority of black teachers today are a product of that system (Jansen, 1999b; Taylor, Muller, & Vinjevold, 2003). There is a widespread belief that not everyone can do mathematics. This belief is contrary to the NCTM Principles and Standards (1989, 1990) which argued that any learner can do mathematics and that an important feature of an excellent mathematics education programme is that all learners are challenged and encouraged to learn mathematics successfully irrespective of their backgrounds.

The students have misconceptions about the nature of mathematics as they regard it as an extremely difficult subject comprehensible to the very talented. This misconception seems to be serious as learners are already defeated before they start learning mathematics because of the myth that mathematics is difficult and incomprehensible. This misconception related to attitude towards mathematics is related to the fifth mathematical proficiency strand of 'productive disposition' (Kilpatrick et al., 2001).

One of the problems affecting learners is the rearguard mathematical philosophy of Fallibilism very popular nowadays with educational researchers. Fallibilism purports that it is fashionable to make mathematical errors and have misconceptions. Fallibilists point out that even arguably the most famous mathematician of today; Andrew Wiles; who undisputedly pushed forward the frontiers of mathematical knowledge by finally proving Fermat’s last theorem (Ball, 2003), made errors in his original proof. Thus a straw-man argument is made that it is fashionable to make mathematical errors when learning it. Therefore fallibilism may imply that it is acceptable for mathematics teachers not to be fully knowledgeable of mathematics, as mathematical knowledge itself is fallible after all. This philosophy seems to tolerate teachers who are not mathematically competent. It is also bulwarked by the student.

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Footnote: For over 500 years mathematicians had failed to prove the Diophantine, called Fermat’s last theorem that stated that there are no integer values n, greater than 2 such that $x^n + y^n = z^n$, where, x, y and z are also integers.
centred learning where there is the implication that teachers must not lecture but be facilitators of students discovering mathematical knowledge by themselves. This philosophy of teaching seems to be against the grain that teachers must be more knowledgeable others in order to teach properly.

✓ The researcher argues that at times doing mathematics, even by expert mathematics might produce errors. But mathematics is a thorough academic discipline that keenly admits errors and readily revises its ideas. That does not mean that all mathematical ideas are faulty, or beyond critical reflection.

✓ In particular, socially and academically challenged students learning mathematics (and its specialised branches such as calculus) need more direction and support from their teachers (Lubiensky, 1988). Therefore, it is far much better to present the absolutist view of mathematics where exercise and drill feature prominently in students’ work, so that students at first imbibe mathematical knowledge before they make tenuous statements of the fallibility of mathematical knowledge.

✓ Constructivists argue that learners are not explicitly taught the errors and misconceptions they have (see for instance Confrey, 1987; Davis, 1984; Smith et al., 1993). Rather, learners build misconceptions by themselves as they strive to interpret new experiences and give meaning to them. These misconceptions are often concept images used to interpret mathematical knowledge. Making errors and misconceptions, then is an acceptable stage in learning, although it interferes and slows down learning. This means that errors need to be seen in a positive light so that through probing, learners can be guided to refine their thinking in order to formulate good mathematics concepts.
7.3 Discussion

According to Davis (1984), the errors learners make in mathematics are generally of two types; errors common to different learners, and those specific to individual learners. Error patterns that are common across different learners suggest that learners possess similar concept images with respect to certain mathematical concepts and procedures. The other types of errors were those idiosyncratic, unpredictable or consistent and predictable to a particular individual learner. This report mainly dealt with those errors that were common across different learners, but errors held by individual learners are also discussed.

Errors made by students were analyzed for their patterns and nature. It appeared that the origins of errors were: intuitive assumptions (Smith et al., 1993), failure to understand the syntax of algebra, analogies with other familiar symbol systems such as the alphabet and interference from arithmetic. Students also created their own procedures which made sense from their point of view. As Davis (1984) has reported, students can have many procedures in working out a mathematical task. The students can have a super-procedure and one or more sub-procedures. It is possible that when errors are made, the super-procedure or one or more sub-procedures may be wrong. Sometimes the procedures being applied, while correct in their own right, were applied where they were not called for. This is a case of misapplication of a procedure. Sometimes learners call upon previously learnt procedures and modify it inappropriately. These scenarios have been found in this study where students made several errors on one task. The procedures that students created were sometimes very insightful and well-meaning even though they were wrong.

Orton (1983a) argued that lack of conceptual understanding of the relationship between limit, function and derivative seen as isolated facts may be mainly due to a learning and teaching approach that emphasises to a large extent the procedural aspects of the calculus, and neglects a solid grounding in the understanding of the conceptual underpinnings of the calculus. This could explain the above error common to all learners.

It is important to report that learners need to understand the pre-calculus topic of co-ordinate geometry. The notions of Cartesian plane, x axis, y axis, ordered pairs and coordinates, equations of straight lines, gradients of straight lines, the gradients of a line joining two given points, the geometry of a circle, tangents of circles and curves, secants of circles and curves,
the transformation of rotation of straight lines and the centre of rotation. The static (object) and dynamic (process) concept of an angle is also quite important. The concepts of curves, rotations (including centre of rotation), secants and tangents, and so forth, are important to help explain that the limiting gradient of a tangent occurs when the rotating secant collides with the tangent to the curve at a point, which happens to be the centre of rotation. Such explanations help a lot to avoid the misconceptions that, at a point the secant would have become shorter and shorter before disappearing (Orton, 1983a), for example.

From the analysis, it can be interpreted that examinees learnt differentiation with little real understanding. They had formed concept images of calculus concepts and techniques that were too fragmented as to give them little power to do the calculus. One of the causes of these errors is the pressure exerted on teachers to get through the curriculum (Littler & Jirotkova, 2008), and the insistence that a good teacher is one whose learners pass the examination. This pressure leads teachers to seek for quick fix approaches, usually use of procedures geared to help learners obtain correct answers to mathematics tasks as rapidly as possible.

The quick fix techniques are usually successful on specific tasks but cannot be adapted for general tasks for which learners had not formed adequate conceptual structures. Errors and misconceptions are seen in the form of inconsistent procedures of solving problems across mathematics tasks and within tasks as learners consider tasks individually failing to see the common thread among them. Another observation is that learners do not check if their answers are sensible due to lack of meta-cognition. One of the reasons why learners develop misconceptions is that they are not given sufficient experiences of a concept (Littler & Jirotkova, 2008; Suh, 2007). It is important that learners are given opportunities and time to experiment and conjecture results so that they prove for themselves what makes sense. This is because teaching them by giving them quick fix methods almost always leads to providing learners with formulae, and formulae are second hand knowledge (Littler & Jirotkova, 2008).

As second hand knowledge, learners do not know how to flex it to match contexts different for which it was learnt, (Boaler, 1998) mainly because formulae knowledge is specific to specific contexts. Thus formula knowledge such as the power rule of differentiation is superficial knowledge that cannot be applied at strategic level, for those learners who have not experienced its constraints and limits.
As the constructivist epistemology suggests, students organise and integrate new concepts in a way that the new ideas equilibrate with prior knowledge. Students regard new knowledge with the lens of earlier acquired knowledge. Students do not see the need for revision of their misconceptions if their ideas are not counter-poised and challenged for students to realise the critical need to revisit them. Otherwise for lack of belief in the necessity of new concepts, students are not convinced of the need to change their alternative conceptions, not least because they have rationalised the concepts themselves. This rationalisation does not occur in a vacuum, but in a context in which the utility of the concept is verified. Therefore students did not realise the need for revision of their misconceptions if they were not convinced that their conceptions were no longer adequate for new tasks.

Misconceptions occurred due to intuitive models that learners built from prior knowledge. As some incorrect strategies yield correct answers in limited cases, misconceptions can be deeply ingrained and extremely difficult to eradicate (Smith et al., 1993; Nesher, 1987). This is because they are ingrained in learners’ minds in the form of concept images. The concept images are periodically reinforced by some correct answers. Thus errors in differentiation seem to be partially caused by learner constructions in which valid procedures are intuitively extrapolated to new contexts. The researcher thus noted that misconceptions resulted mainly from intuitive extrapolations of valid prior knowledge, but some misconceptions occurred because of lack of prior knowledge altogether.

Two strategies are suggested to help learners reconcile persistent misconceptions. These are cognitive conflict and analogy. Cognitive conflict was suggested in the work of Piaget (1968). Flavell (1977) claimed that cognitive conflict results in the growth of understanding by the learner and occurs as learners:

- Notice two conflicting elements;
- Appreciate the inherent conflict of the two;
- Search for a resolution; and
- Achieve a conceptualisation that resolves the conflict (p.34).
The method of busting errors through cognitive conflict relies on the ingenuity of the teacher in maximising cognitive conflict by the selection of shrewd examples that helps learners to appreciate that there is real conflict and that their thinking is inadequate and unreasonable. Learners need to be convinced that there really is cognitive conflict otherwise they would accept like Galileo to 'an earth-centred astronomy' (Davis, 1984, p. 45).

Under cognitive conflict, learners are confronted with conflicting situations that clearly show that their strategies do not work in all cases. Confronting tasks need to be selected carefully. It is very important that the learners are not told that their strategies do not work. This is because their strategies are ingrained in the learners’ conceptual frames, and telling them that their strategies do not work undermines their inner confidence. Because of that learners will in future no longer trust the teacher because the teacher does not also respect learners’ ideas. The crucial issue is for learners to discover for themselves that their strategy has limited usefulness after all. Giving plenty of examples of cognitive conflict is the best way for learners to gradually realise for themselves the limits of their misconception (Hejný, 2005). Yet as Flowers et al. (2005) have found, while some learners are fairly quick to get a cognitive shift; that is have conceptual change as a result of exposure to cognitive conflict, many others need extended repetitions of the conflict before eventually noticing the need to change. Unfortunately, Green et al. (2008) have found, similar to Piaget (1968) on children’s achievement formal operations, that a few learners never seem capable of achieving cognitive conflict no matter how long they are induced to the conflict. Piaget (1968) in his stages of cognitive development theory stated that some people no matter their biological age never reach the highest stage of formal operations where they could reason independent of concrete models.

Misconceptions can also be overcome through teaching by analogy (Tirosh, Tsamir, & Hershkovitz, 2008) through a technique which is similar to scaffolding. Teaching by analogy begins by presenting learners with a foolproof anchoring task designed to elicit a correct response. Then learners are presented with a series of bridging tasks each with a factor or factors regarded as misleading becoming progressively more evident. Finally, a target task likely to elicit an incorrect response is introduced. Such a technique was seen to overcome misconceptions that occur through intuitive models (Clement, 1993; Stavy & Tirosh, 2000). The learners gradually learn that their intuition is actually wrong. In this model, it is important to start with really simple examples that learners can follow and understand. It is
not wise to rush over the method, as its power lies in convincing the learner that their intuitive conception is actually wrong.

From the above, it seems important to give learners tasks that are likely to lead and elicit erratic responses in differentiation concepts to induce cognitive perturbation. Hence it is important for educators to learn and be aware of the most important mathematical situations that are likely to elicit differentiation errors in learners. The errors that learners make are then used as springboards for exploration of the nature of the errors and how learners can learn from them (Riccomini, 2005). The teachers facilitate conceptual change by providing cognitive conflict or analogies. These depend on the teachers' intimate knowledge of the learners' errors in calculus.

For all the three calculus examination tasks (i.e. in NSC Paper 1, 2008; Que. 8, 9 and 10); a surprisingly high number of learners (70%) showed little or no awareness of differentiation notions. All they did was to scribble meaningless algebraic symbols in their scripts. These findings are important in that they showed that learners held many misconceptions on the nature of calculus. Learners could also not connect fragmented mathematical ideas, hence the blind application of procedures. Examples of such answers were giving the derivative in Task 1.2 and Tasks 3 in terms of logarithms. Although answers to the tasks could be found by also using logarithmic differentiation, learners clearly did not have the mathematical maturity to handle this procedure at this time. Besides, this was not in the syllabus. Thus, while the logarithmic differentiation super-procedure called upon was fine, learners lacked the sub-procedures to complete the differentiation tasks to their logical conclusions. Also it is a measure of lack of mastery of the subject in that learners might want to call upon more complicated procedures when easier ones are available. The researcher thinks that such notions of logarithmic differentiation came from exposure to the concept without any understanding.

Analysis of performance on calculus items from different centres showed that while in some centres the introduction of differentiation by way of the difference quotient; 
\[ \lim_{k \to 0} \frac{f(x + k) - f(x)}{k} \]; was fairly grasped, students found it hard to tackle calculus questions
beyond this one. Subsequent procedures of differentiation such as the power rule, and the application of differentiation to solve problems was poorly done and often left unanswered.

An important finding following the above scenario is that the calculus topic was taught better in some schools than others. In other schools it seemed to have been poorly taught. This position is argued in that in any group of learners, performance in an examination or test usually follows a normal distribution (Crawshaw and Chambers, 2008) in that while many learners may weakly achieve educational outcomes, and very few do not achieve them at all, there are also some, even though they will also be very few, who achieve the objectives. If that does not occur then the problem must lie elsewhere and not with the group of learners.

7.4 Recommendations

From this study the following recommendations are suggested:

✓ Teachers and students must establish trust between them that it is okay to make mistakes in mathematics. They must all come to realise that errors offer an invaluable opportunity for exploration and growth when they are exposed and explained (van de Walle, 2004). All students must believe that their ideas whether right or wrong will be met with high level of trust. If this trust is not there many useful ideas will not be shared and everyone loses the opportunity to learn.

✓ Students and teachers need to appreciate that mathematics makes sense and so the correctness or validity resides in the mathematics itself and not with some authoritative source such as the teacher or the answer book. Therefore it is important to help learners to realise that it is acceptable to make errors in mathematics. What is important is that learners must see errors as aids to formulation of better mathematics ideas.

✓ Teachers must train themselves to listen attentively and actively to learners' contributions whether spoken or written in order to find out what students think, how students think and what they know. It may be necessary to probe or interview learners to elicit their thinking.
✓ Teachers must teach the big ideas, the fundamental concepts of calculus; finding the gradient of a tangent on a curve through drawing and helping learners appropriate the correct mathematical terminology and symbolism; for example the gradient of the line at point x, can also be regarded as \( \frac{dy}{dx} \). Students who have learned conceptual ideas in a relational manner are likely to avoid many errors and misconceptions.

✓ I recommend use of multiple representations in teaching calculus in order to avoid learner errors and misconceptions in calculus. Multiple representations of the big ideas of mathematics such as functions are needed for learners to fully understand them. These representations could be verbal, numerical, visual and algebraic. The verbal is the description of a function through use of words. The numerical occurs through the use of number patterns and tables. The visual occurs through the use of graphs or sets (domain and range) whereas the symbolic occurs through the use of explicit algebraic formula. These mathematical models help learners to understand the function phenomena from a variation of standpoints. As the Tulane Conference in 1986 has argued, the teaching of calculus must focus on conceptual understanding, and the ‘rule of three’ that suggest that the topics should be presented geometrically, numerically and algebraically. Visualisation, numerical and graphical investigation, and other approaches can influence conceptual understanding in fundamental ways. Thorough understanding of the function concept has the potential to dispel the many errors and misconceptions that affect handling calculus as has been discussed in this research. Understanding of functions is crucial because calculus notions are conveyed through them.

✓ Limits also need to be treated from descriptive, graphical, numerical, and algebraic points of view. This multi-model approach helps learners to view the limit phenomena from various viewpoints which lead to more understanding and dispelling of errors and misconceptions on a critical calculus notion.
I recommend that visual illustrations must be introduced first to facilitate students' understanding of calculus concepts and should be used more often by teachers in their lessons.

The power of information and communication technologies is with us and can no longer be ignored in education anymore. It must be harnessed as a mediating artefact to promote learning of calculus. A first approach to differentiation must be quite informal and may be based largely on numerical and graphical explorations assisted by an electronic calculator and a micro-computer. The micro-computer or calculator can be used to investigate the limit of gradient or rate of change or derivative at a point on the curve either numerically and graphically. The computer can simulate functions and the limiting processes. This helps learners to build good visually based concept images of calculus which helps to avoid defective notions. By such means students may discover the gradient formulae for a variety of polynomial functions and thus discover the general principle involved. At this stage an algebraic proof might still not be necessary though the way is then clear for more able and mature students to consider a more general proof.

Students need to draw curves and manually estimate the gradient of a tangent at a point as this construction bootstraps and prepares them for more formal calculus techniques. These approaches may be included in the curriculum at even earlier levels such as grade 10.

Communication skills in mathematics include reading, writing, speaking, modelling, and so on. These are important skills as they force students to express their ideas clearly and to organize them coherently. Hence teachers must be in the habit of asking students to write down what they think about certain concepts or ideas and then discuss them with their peers. Students could also be asked to set questions as this requires higher order thinking skills. These questions could then be used as a platform for a class discussion where other students can comment on these questions' viability and difficulty level.
Teachers must have the content knowledge and ability to provide appropriate and focused instruction to correct students' misconceptions and errors. Improving the ability of teachers to recognize error patterns and plan more appropriate instruction can be addressed through pre-service programs and professional development opportunities in mathematics where error analysis is emphasised.

In no small way in-service education and training (INSET) is needed to cater for not only the subject matter knowledge of current teachers but also cater for their pedagogical content knowledge so that the teachers really become the more knowledgeable others who can judiciously assess the ZPD of their learners and use semiotic mediation to scaffold them so that they can progress rather than be stunted in their mathematics education. The INSET programs must take into consideration the common errors and misconceptions learners often have in mathematics. All mathematics teacher education programs must have a course for mathematics errors analysis and diagnosis. Teachers need to learn how to do Error Analysis and Diagnosis (EAD) for errors and misconceptions peculiar to learners on specific topics. Teachers must be empowered to develop strategies that address the mathematical errors and misconceptions that their learners experience.

As analysis shows that errors are also due to lack of understanding of calculus terminology and symbolism; with respect to terminology, it might be important to research whether teaching of calculus in learners' home languages (see Nkambule & Setati, 2007) may not circumvent this problem and reduce errors.

From the study, further research themes emerged. These are:

1. How can errors and misconceptions in mathematics identified in this study be used as resources to improve the learning of mathematics or calculus?

2. What impact would teaching anchoring on the error analysis protocol created from this study have in the teaching and learning of calculus in high schools?

3. In what ways do learners’ errors in algebra affect the learning of differentiation?
4. What impact would the use of learners' home language have on the errors they make when learning differentiation?

7.5 Conclusion

The researcher concludes that learners encounter substantive challenges in learning introductory differentiation. These challenges manifested themselves through a variety of errors that learners showed in their scripts. Generally students had many not-on-task errors, that is; they actually had many errors on non-calculus concepts than they had on actual calculus concepts. When they could handle the pre-requisite concepts, they met on-task challenges related to the indirect reasoning required in calculus as when taking limits.

The researcher learnt a number of things from this study. Firstly he learnt that many students do not understand differentiation procedures (see analysis of Item 1.2) and concepts (see analysis of Item 1.1), let alone its usage as a tool for optimisation functions (see analysis of Items 2), and simulated problems amenable to its application, such as the water glass problem (see analysis of Items 3). Secondly, the study showed that learners have many basic errors and misconceptions on differentiation. This is exemplified by differentiating constants (see fig. 65), where a learner blindly used a differentiation procedure without understanding its conceptual basis. Thirdly, there were clear indications from the study that lack of competence in algebra constituted an enormous epistemological barrier to accessing differentiation and its applications. Fourthly, most learners treated calculus just as just like algebra, they held a very superficial understanding of what differentiation was, thereby failing to grasp its essential substance. To them the calculus was adding, subtracting, multiplying, and dividing variables or unknowns such as \( x, r, h, \delta, S, \infty \) and so on. As a result, the learners failed to dissociate and delineate the essence of the calculus from algebra, thereby loosing big time by failing to realise the big idea that calculus is. Such errors are theory-like (Chi, 2005), because they influence learners to think in a non-standard way about a subject. It seems the learners' incompetence in algebra blurred them from differentiating calculus from it. Therefore, the researcher noted that students' competence or lack of it in algebra had a bearing on the success or failure in learning calculus. The errors and misconceptions learners had in calculus were closely aligned to algebra. Thus a learner could not settle the derivative of a function
because he/she could not change expressions in surd form to index form, which is absolutely necessary before differentiating using the power rule in such cases (see errors on Item 1.2 for example). Also those learners who could not subtract or multiply directed numbers; whole or rational could not make any headway in answering that item. Algebraic competences assumed in differentiation are multifaceted. The same applies to ability to manipulate formula functions. Learners who did not have the ability to manipulate functions could not answer Item 3.

The competences of learners in algebra and functions, hinge upon the mathematical "constructs" of substitution and variable. The ability to substitute is a key competence in algebraic thinking as it enables learners to manipulate and handle variables by substituting them with others whenever an equation links them. It can be argued that those learners who have not grasped the principle of substitution; the interchangeability of either variables with variables or variables with numbers, cannot do any mathematics beyond primary school, let alone calculus. Substitution and use of variables that comes along with it, are central to the 'mathematics game'. They are the verbs and nouns of the mathematics language. Many learners had errors because of failure to use substitution and interchange variables in functions. That inca"pability locked them out of understanding and doing calculus. So learners had great difficulties before they had even begun. Ultimately, many learners could not have calculus epistemic access due to lack of competence in algebra and other topics, such as functions, in which the study of calculus depends and is embedded. As a result, these learners left high school without fully appreciating calculus as an important tool to further help them understand mathematics itself as well as an aid to understand other school subjects and in general problem solving. For example, the use of calculus in analysing pendulum motion in physics helps learners to realise its value in scientific investigations. It can be argued that learners who have errors and misconceptions on calculus are denied the opportunity to experience the beauty of mathematics and what it offers them in understanding and controlling their environment. Errors and misconceptions in mathematics can be followed by a negative disposition and negative philosophy of mathematics, borne by experiencing mathematics as an insensible, disjoint and incomprehensible subject, itself a misconception about the true nature of mathematics.
Fifthly, learners found it tough comprehending the terminology of the calculus, such as limit, approaching zero, tangent, point of inflexion etc. In particular, students found it very difficult to conceptualise the notion of a tangent to a curve. Whenever they were required to refer to the notion of a tangent they attempted very much to avoid it by forming alternative concepts of it. This avoidance of comprehending a fundamental notion of the derivative meant that learners' understanding of calculus was very superficial. This resulted in procedural learning; learning bereft of meaning. Such learning results in so many errors and misconceptions which have been discussed in detail in this study. Sixthly, related to terminology, are calculus symbols. Some learners were mystified by calculus symbols. The lack of understanding the meaning of symbols resulted in many decoding and encoding errors. It is important to be able to interpret mathematical symbols, as they cannot be separated from the concepts they portray.

Seventhly and lastly but not least, the researcher learnt that it is crucial to study and understand the errors and misconceptions learners face at all levels in mathematics education as learner errors in written work usually follow a particular line of thinking that can be deciphered, although it is time consuming to do that. Though time consuming, once the understanding takes place, both the teacher and learner will be rewarded many times in the future as understood concepts are used to assimilate and accommodate new concepts into a learners' schema. So the time invested in error analysis is a lot of time saved rather, though at first it might not seem so. Also the same errors are often shared by many learners because the quality of reasoning the learners have is often similar. At the same time there are idiosyncratic errors particular to individuals based on their personal knowledge and experience. The research urges that it is important to observe learners as they do mathematics and encourage them to talk about what they do so that the errors they have are shared with other learners and the teacher. The research pleads with all teachers to be curious, inquisitive and observant when dealing with learners' errors; never to rush to blame learners when they give errorful responses. It urges rather, for teachers to diagnose those errors by script analysis or otherwise to discover learners thinking patterns and reasoning, in order to engage with those patterns and reasons to help learners see for themselves the errors they have. This can promote learning mathematics with understanding.
When a database of learner errors in mathematics is compiled and research helps to explain how learners come to have them, then it becomes easier for teachers to teach in ways that increase learners' academic opportunities. This can only result in better achievement in mathematics. The researcher suggests that the error analysis protocol (see Fig. 67) offers an important contribution to the wider mathematics education community in South Africa, Africa, and beyond, as it identifies some new errors learners showed in mathematics and calculus, and the explanations for them. Worldwide, mathematics education practitioners can reflect on them as they teach mathematics and calculus.

The study puts that learner competencies on calculus prerequisite concepts of geometry, coordinate geometry, algebra, and functions have the greatest bearing on the errors and misconceptions learners have in calculus. These errors in calculus seem directly proportional to the errors they have in these topics. Also as recommended, the use of computer based approaches may help learners to dynamically visualise important differential processes and so circumvent misconceptions about basic calculus notions, on which calculus procedures are based. The researcher noted that learners' errors and misconceptions on calculus concepts and procedures are very sophisticated and complex. As a result, learners had many different errors in answering even a single item. These errors reinforce and entangle each other rendering learning mathematics a hugely unsuccessful and stressful task. This brings to the fore the need to discover learner errors to deal with them one at a time.

The researcher hopes that this study helps to shed light and afford teachers and other stakeholders greater understanding on learner errors and misconceptions in mathematics and calculus. Such an understanding would hopefully enable teachers to be on the look out for learner errors in their own classes, and to be reflective about learners' wrong answers to mathematics questions; for each wrong answer a learner gives has a message. If this message is explored, this will empower both the learner and the teacher by rewarding them with greater understanding; for the teacher, how to teach the learner better, and for the learner overcoming his/her error to begin to learn maths more successfully. The researcher hopes that this research will kick start improvement in the teaching not only of calculus but of other topics related to it such as algebra, geometry, coordinate geometry and functions which
learner errors on these topics featured uninvited in this study; for the errors in introductory calculus and differentiation proved inseparable from those made on these pre-requisite topics.
REFERENCES


APPENDICES

APPENDIX A: THE NCS FET MATHEMATICS CURRICULUM

National Curriculum Statement
Grades 10-12
(General)
MATHEMATICS

Learning Outcome 2 (Continued)

Functions and Algebra
The learner is able to investigate, analyse, describe and represent a wide range of functions and solve related problems.

The approach to the content of this Learning Outcome should ensure that learning occurs through investigating the properties of functions and applying their characteristics to a variety of problems. Functions and algebra are integral to modelling and so to solving contextual problems.

Problems which integrate content across Learning Outcomes and which are of a non-routine nature should also be used. Human rights, health and other issues which involve debates on attitudes and values should be involved in dealing with models of relevant contexts.

Differential calculus:
- an intuitive understanding of the limit concept in the context of approximating the rate of change or gradient of a function at a point;
- the derivatives of the following functions from first principles:
  \[ f(x) = b \]
  \[ f(x) = x \]
  \[ f(x) = x^2 \]
  \[ f(x) = x^3 \]
  \[ f(x) = \frac{1}{x} \]
  - the derivative of \( f(x) = x^n \) (proof not required);
  - the following rules of differentiation.
\[
\frac{d}{dx} [f(x)\pm g(x)] = \frac{d}{dx} [f(x)] \pm \frac{d}{dx} [g(x)]
\]
\[
\frac{d}{dx} [k \cdot f(x)] = k \frac{d}{dx} [f(x)]
\]

- the equations of tangents to graphs;
- sketch graphs of cubic and other suitable polynomial functions using differentiation to determine the stationary points (maxima, minima and points of inflection) and the factor theorem and other techniques to determine the intercepts with the x-axis;
- practical problems involving optimisation and rates of change.

- Average rate of change of a function between two values of the independent variable.
- Average gradient between two points on a curve and the gradient of a curve at a point.
APPENDIX B: MATHEMATICS ASSESSMENT STANDARDS

Learning Outcome 2: Functions and Algebra

When solving problems, the learner is able to recognise, describe, represent and work confidently with numbers and their relationships to estimate, calculate and check solutions.

12.2.7

(a) Investigate and use instantaneous rate of change of a variable when interpreting models of situations:
   • demonstrating an intuitive understanding of the limit concept in the context of approximating the rate of change or gradient of a function at point;
   • establishing the derivatives of the following functions from first principles:

\[
\begin{align*}
  f(x) &= b \\
  f(x) &= x \\
  f(x) &= x^2 \\
  f(x) &= x^3 \\
  f(x) &= \frac{1}{x}
\end{align*}
\]

and then generalise to the derivative \( f(x) = x^n \)

(b) Use the following rules of differentiation:

\[
\frac{d}{dx}[f(x) \pm g(x)] = \frac{df}{dx} \pm \frac{dg}{dx}
\]

\[
\frac{d}{dx}[k \cdot f(x)] = k \cdot \frac{df}{dx}
\]

(c) Determine the equations of the tangents to graphs.

(d) Generate sketch graphs of cubic functions using differentiation to determine the stationary points (maxima, minima and points of inflection) and the factor theorem and other techniques to determine the intercepts with the x-axis.

(e) Solve practical problems involving optimisation and rates of change.
This question paper consists of 10 pages, an information sheet and 2 diagram sheets.
**QUESTION 8**

8.1 Determine \( f'(x) \) from first principles if \( f(x) = -3x^2 \).

8.2 Determine, using the rules of differentiation:

\[
\frac{dy}{dx} \text{ if } y = \frac{\sqrt{x}}{2} - \frac{1}{6x^3}
\]

Show ALL calculations.
QUESTION 9

Sketched below is the graph of \( g(x) = -2x^3 - 3x^2 + 12x + 20 = -(2x - 5)(x + 2)^2 \)
A and T are turning points of \( g \). A and B are the \( x \)-intercepts of \( g \).
P\((-3, 11)\) is a point on the graph.

9.1 Determine the length of \( AB \).  
9.2 Determine the \( x \)-coordinate of \( T \).  
9.3 Determine the equation of the tangent to \( g \) at \( P(-3, 11) \), in the form \( y = \ldots \)  
9.4 Determine the value(s) of \( k \) for which \(-2x^3 - 3x^2 + 12x + 20 = k \) has three distinct roots.  
9.5 Determine the \( x \)-coordinate of the point of inflection.
QUESTION 10

A drinking glass, in the shape of a cylinder, must hold 200 ml of liquid when full.

10.1 Show that the height of the glass, \( h \), can be expressed as \( h = \frac{200}{\pi r^2} \). (2)

10.2 Show that the total surface area of the glass can be expressed as \( S(r) = \pi r^2 + \frac{400}{r} \). (2)

10.3 Hence determine the value of \( r \) for which the total surface area of the glass is a minimum. (5)

[9]
QUESTION 8

8.1

\[ f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \]

\[ = \lim_{h \to 0} \frac{-3(x + h)^2 - (-3x^2)}{h} \]

\[ = \lim_{h \to 0} \frac{-3x^2 - 6xh - 3h^2 + 3x^2}{h} \]

\[ = \lim_{h \to 0} \frac{-6xh - 3h^2}{h} \]

\[ = \lim_{h \to 0} h(-6x - 3h) \]

\[ = -6x \]

\[ \checkmark \checkmark \text{ definition} \]

\[ \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \]

\[ \checkmark -3(x + h)^2 \]

\[ \checkmark \text{ substitution of } -3x^2 \]

\[ \checkmark \text{ correct answer} \]

(5)

Note:
Penalty 1 for incorrect notation
If a candidate has used the rules only: 0/5

8.2

\[ y = \sqrt{x} - \frac{1}{2} \frac{1}{6x^3} \]

\[ y = \frac{1}{2} x^{1/2} - \frac{1}{6} x^{-3} \]

\[ \frac{dy}{dx} = \frac{1}{4} x^{-1/2} + \frac{3}{6} x^{-3} \]

\[ \frac{dy}{dx} = \frac{1}{4} x^{-1/2} + \frac{1}{2} x^{-3} \]

\[ \frac{dy}{dx} = \frac{1}{4 \sqrt{x} + 2x^4} \]

Note:
If removed coefficients, or moved the numbers from the denominator to the numerator:

\[ \checkmark \text{ Simplification} \]

\[ \frac{1}{4} x^{-1/2} \]

\[ \checkmark \frac{1}{2} x^{-4} \text{ or } \frac{3}{6} x^{-4} \]

\[ \checkmark \text{ correct derivative} \]

Max 2/3

If leave out \( \frac{dy}{dx} \) penalise 1 mark.

(3)
**QUESTION 9**

<table>
<thead>
<tr>
<th>Question</th>
<th>Equation</th>
<th>Solution</th>
<th>Correctness</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>9.1</strong></td>
<td>$-(2x-5)(x+2) = 0$</td>
<td>$x = \frac{5}{2}$ or $-2$</td>
<td>✓ answer (2)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$AB = 4.5$ units</td>
<td>✓ answer (2)</td>
</tr>
<tr>
<td></td>
<td>OR</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$-(2x-5)(x+2) = 0$</td>
<td>$x = \frac{5}{2}$ or $-2$</td>
<td>✓ answer (2)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$AB = \sqrt{(2,5 - (-2)^2 + (0 - 0)^2}$</td>
<td>✓ answer (2)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$AB = 4,5$ units</td>
<td>✓ answer (2)</td>
</tr>
<tr>
<td><strong>9.2</strong></td>
<td>$g'(x) = 0$</td>
<td>$-6x^2 - 6x + 12 = 0$</td>
<td>✓ factorisation</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$x^2 + x - 2 = 0$</td>
<td>✓ answer (4)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(x + 2)(x - 1) = 0$</td>
<td>✓ factorisation</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$x = -2$ or $x = 1$</td>
<td>✓ answer (4)</td>
</tr>
<tr>
<td></td>
<td>at $T$, $x = 1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>9.3</strong></td>
<td>$g'(x) = -6x^2 - 6x + 12$</td>
<td>$g'(-3) = -6(-3)^2 - 6(-3) + 12$</td>
<td>✓ ✓ method of setting up straight line equation</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$g'(-3) = -54 + 18 + 12$</td>
<td>✓ substitution of point ($-3; 11$)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$g'(-3) = -24$</td>
<td>✓ ✓ answer in equation form (5)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$y = ax + q$</td>
<td>✓ ✓ method of setting up straight line equation</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$11 = -24(-3) + q$</td>
<td>✓ substitution of point ($-3; 11$)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$q = -61$</td>
<td>✓ ✓ answer in equation form (5)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$y = -24x - 61$</td>
<td>✓ ✓ method of setting up straight line equation</td>
</tr>
<tr>
<td></td>
<td>OR</td>
<td></td>
<td></td>
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<tr>
<td></td>
<td>$g'(x) = -6x^2 - 6x + 12$</td>
<td>$g'(-3) = -6(-3)^2 - 6(-3) + 12$</td>
<td>✓ ✓ method of setting up straight line equation</td>
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<tr>
<td></td>
<td></td>
<td>$g'(-3) = -54 + 18 + 12$</td>
<td>✓ substitution of point ($-3; 11$)</td>
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<tr>
<td></td>
<td></td>
<td>$g'(-3) = -24$</td>
<td>✓ ✓ answer in equation form (5)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$y - 11 = -24(x + 3)$</td>
<td>✓ substitution of point ($-3; 11$)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$y - 11 = -24x - 72$</td>
<td>✓ ✓ method of setting up straight line equation</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$y = -24x - 61$</td>
<td>✓ ✓ method of setting up straight line equation</td>
</tr>
</tbody>
</table>
### QUESTION 10

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
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</tr>
</thead>
</table>
| 10.1 | $V = \pi r^2 h$ | ✓ formula  
200 = $\pi r^2 h$  
$h = \frac{200}{\pi r^2}$ | ✓ substitution  
(2) |
| 10.2 | Surface Area = $2\pi rh + \pi r^2$ | ✓ formula  
$S(r) = \pi r^2 + \frac{200}{\pi r^2} 2\pi r$  
$S(r) = \pi r^2 + \frac{400}{r}$ | ✓ substitution  
(2) |
| 10.3 | $S(r) = \pi r^2 + 400r^{-1}$ | ✓ exponents correct  
$\frac{dS}{dr} = 2\pi r - 400r^{-2}$  
At minimum: $\frac{dS}{dr} = 0$  
$2\pi r - \frac{400}{r^2} = 0$  
$\pi r^3 - 200 = 0$  
$r^3 = \frac{200}{\pi}$  
$r = 3.99 \text{ cm}$ | ✓  
(5) |

Note:  
If did not put $\frac{dy}{dx} = 0$, penalise 1 mark  
If notation is $\frac{dy}{dx}$, ignore notation.  

[9]