OPTION PRICING AND RISK MANAGEMENT

Ferdinand Ernst Zittlau

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Hiermee verklaar ek, Ferdinand Ernst Zittlau, dat die skripsie 'n produk is van my eie arbeid. Geteken te Johannesburg op die .... van ....2001.

F.E. Zittlau

Kommisaris van Ede

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CHAPTER 1

THE PRINCIPLES UNDERLYING THE OPTION PRICING FORMULAE

Over the past two decades, the financial markets have experienced an impressive expansion in terms of securities being traded on the secondary markets. In addition to this, the markets have become more intertwined with one another, allowing almost 24 hour trading around the world in certain commodities and stocks. The advent of the computer and the increase processing power now within the reach of each individual, has led to financial innovation, and the creation of financial derivative instruments. Derivative instruments such as options, futures and swaps have become standard tools available to the risk manager in order to adapt the risk profile of his portfolio.

The benefits of the derivative instruments, and the economic functions performed by them, is often overshadowed by claims that they are responsible for the destabilisation of the economy. This was especially true after the crash of October 1987. It is stated that the derivatives markets trading activities destabilised the cash markets by increasing the volatility of its fundamentals. Derivatives also initially received the blame for the collapse of one of Britain's oldest banks, Barings Bank.

Since then independent studies by the Bank of England and other professional groups (Group of Thirty) have analysed various problems raised by the development of mostly the over the counter (OTC) market for derivatives. They focused more specifically on the size of these markets, the associated sources of risk, the ways to manage these risks and finally possible ways to regulate these markets (CARPENTER, 1991). All these reports clearly showed that at present there is still a lot of areas that remain unexplored within this ever expanding field within the financial markets.

As the size of these markets continue to expand, the responsibilities of the risk managers become even more. It is important for each individual involved in this field to understand the underlying principals of any risk management system, in order for him to better understand the implications of his different actions, and to better interpret the indications obtained from a risk management systems.

1.1 Problem Definition

Up to early 1970's there was relatively little volatility in either exchange rates or interest rates. Commensurately time was thus devoted to either currency risk or interest rate management. Since the Bretton Woods agreement was in place during this time, exchange fluctuations was therefore both small and predictable. A similar scenario existed for interest rates.

During the 1970's the Bretton Woods agreement was replaced by a floating exchange rate, which in turn led to the development of an oil crisis world-wide. Oil prices over the period 1979 - 1981 more than trebled (MERTON, 1986), and this had a dramatic effect on the flow of international. Vast amounts of oil dollars moved around the globe, in search of banks offering the highest return on investment. Oil importing countries tried to compensate for this outflow by trying to attract foreign investment. They also introduced deflationary policies to reduce domestic demand, which led to an even further movement of foreign funds between countries (MERTON, 1986).

A move towards monetaristic policies resulted in policy targets aimed at destabilised interest rates in pursuit of more intermediate monetary aggregates, which in turn increased the volatility of foreign exchange rates as well as world-wide interest rates, and inflation become a more dominant factor in the economic outlook of the world economies.

These changes in essence placed the emphasis on active risk management by identifying exposures and using the new and emerging financial instruments to reduce the possible harmful effects that these increased volatilities might have on the different portfolios. These new instruments brought with them their own type of risk, and risk managers had to develop new techniques to deal with these new risks (GENOTTE, 1990). These risks must however be clearly understood and quantified in order to analyse the effect they would have on the portfolio as a whole.

Various attempts have been made to try and provide a greater understanding of the factors determining the movement of asset prices, the development of accurate derivative pricing models and the implementation of more accurate risk management systems.

A clear study of the risks influencing asset prices, the principles and assumptions made in the development of option pricing models, and the final implementation of a statistical risk management system is essential to any risk manager finding him in the rapidly changing financial markets.

1.2 Objectives

The following are the main objectives of this study:

1.2.1 Scientific research in physics has been instrumental in the development of the Brownian particle movement theory, which form the basis for any stochastic study. No study of option pricing and risk management would be complete without investigating the implications of stochastic theory on price movements within the financial markets.

1.2.2 The critical evaluation of the most commonly used option pricing models. The Black-Scholes and Binomial option pricing models. They are different in their approach in development, but similar in their results obtained, would be derived and their suitability analysed in terms of correctness for application in a real world scenario.

1.2.3 Risk management is ultimately where all theory culminate, and in a final section of this study, the latest statistical management techniques would be investigated, as well as future tendencies within this field would be discussed.

1.3 Methodology

In order to obtain a sound foundation from which to evaluate treasury risk exposure, the study would take the form of a pure literature study, covering the most recent aspect of asset price-movement prediction, option pricing and risk analysis and management. Most of the option risk management techniques are based on one of two option pricing models, namely the Black-Scholes differential and binomial derivative pricing models. These are in turn based on the study of stochastic processes, which have their founding in the statistical mathematics and Brownian particle movement mathematics.

The background against which the study would be done, is the market in which the derivative instruments are traded, and how these factors influence the behaviour of these derivative instruments.

1.4 Economic Value of Derivative Instruments

Evaluating the economic benefits of derivative instruments can not be done by just focusing on derivative instruments and markets. The question is not whether derivative instruments are attractive financial instruments or not, but whether derivatives add additional benefits or risks compared to the next alternative in the underlying spot market. Since derivative instruments have close or almost perfect substitutes in the spot market, the question should be asked whether derivatives add additional benefits or risk diversification to the existing financial market structure. This will explain why a lot of emphasis would be placed on the relation between derivatives and the spot market.

The core function of the financial system is to facilitate the allocation and development of economic resources (GIBSON, 1994), in and ever changing and uncertain environment. This function can be divided into three main dimensions: time, risk and information. Borrowing and saving are the major functions of the financial system, in order to achieve an efficient intertemporal allocation of funds, which in terms implies uncertainty. Through this system, households and industry can accommodate each other to balance their respective earnings and expenses in each period required. It also provides industry with the opportunity to separate their investment and financing decisions, which implies a separation of the ownership and the management of resources.

It is thus clear that risk is, due to the intertemporal nature of the markets, therefore an inherent characteristic of financial decisions. The financial system thus helps in allocating these risk involved in the uncertainty of the markets, to the various market participants. The market provides a wide spectrum of instruments in order to facilitate this diversifying of risk. In order to provide an Pareto-efficient allocation of these risks, markets must provide enough opportunity and liquidity to trade and price these various kinds of risk (GIBSON, 1994). Market prices are indicators of risk, and help the market participants target the amount of risk they are willing to bear, and as such offer valuable information to these participants.

The main dimensions of the financial markets naturally also apply to the derivative instruments available in the markets. As such, no new risks are introduced to the markets, but they are simply structured in a single instrument, and depend on the constitution of the instrument with regards to the underlying instruments.

1.5 Main Risks Associated with Derivatives

1.5.1 Spot Risk

Spot risk refers to the directional risk that is associated with the movement of the spot price of an asset that moves either up or down. These movements take place due to ever changing market conditions. In order to examine these risks, assumptions as to likely price moves and therefore risk scenarios must be made. This is a somewhat arbitrary process, but consistency across components is important (CHEW, 1996).

When evaluating market risk, certain time horizons must be taken into account. These are dependent on the instrument involved, since liquidity plays an important role in the in the determination of these time horizons.

1.5.2 Volatility Risk

Volatility give a clear indication as to the likelihood of an instrument's to move away from its present price. Although all assets go through bursts of volatility, some remain more volatile than others do. In order to compare the volatility of different assets, it must be expressed in a standard manner if it is to impart objective information about the riskiness of different assets.

Volatility is thus best measured by a statistical method imposed on an assets price. In order to ascertain an assets volatility, a study of its prices and the distribution thereof must be made. Since an assets prices are taken to be totally random over a given time horizon, they are assumed to have a normal distribution over a given time horizon. Since asset prices can only fall to zero, and not lower than that, it can be assumed that the distribution can be more log normal than normal (COX, 1981)

When evaluating spot risk, limits as to possible market moves must be set. Most commonly used is a standard deviation of 1,65. This ensures a 95% confidence level. These confidence levels are variable due to the volatility that may exists within the spot instrument (MERTON, 1973:163). Since an asset's price can rise indefinitely, but cannot fall below zero, it can further be assumed that the distribution is more log -normal than normal.

The volatility does not effect the price of the physical asset, but it does influence the price of an option. It does however have its origin in the price movements of the physical asset, and should thus be treated as part of the spot instruments risk profiles.

1.5.3 Information Risk

These risk arise from the imperfect degree of information in financial markets and from the imperfect ability of market participants to correctly estimate and interpret the available information (GENOTTE, 1990:1012). The mere existence of this category of risk emphasises that we have to understand how individuals perceive and respond to risks associated with derivative instruments.

Three factors may generate or increase the amount of perceived risk. They are:

1. Lack of information disclosure and market transparency.
2. Lack of knowledge and educational training among the various market participants
3. Adverse risk incentives provided to the market players by appropriate compensation schemes.

1.5.4 Credit Risk

Credit risk is the loss incurred on a contract if the counterpart is unable to honour his engagements. This type of risk not only effects derivative instruments, but any bilateral agreement involving cash flow transfers across time.

The main difficulty originates from the difficulty to correctly measure default risk. This is due to the fact that ratings are static, subjective and very general, and thus do not provide an accurate measure of a sound credit risk policy (GIBSON, 1994). Credit risk is a dynamic risk which should be measured over time, and should not just be seen as a static risk which would not evolve over time.

Credit risk should thus be approached in the same way one would approach a portfolio, and diversify in order to limit the risk by limiting the weight of any one counter party in the portfolio.

The effectiveness of credit measurement as well as credit risk reduction depends upon the degree of information disclosure and accounting transparency of each party (GIBSON, 1994). This can be obtained by complete disclosure of the asset values used as variables for default risk, and how there values have been obtained. The perceived value can differ substantially from the actual ones, and as such could lead to a incorrect assessment of the counterparty's credit risk.

1.5.5 Liquidity Risk

Liquidity is the ease with which a financial instrument can be traded. It can be measured in terms of cost and time. The more illiquid a instrument is, the more time and/or cost it would take to trade the given instrument.

Illiquidity thus conveys a execution risk (BEINER, 1994), and a uncertainty to transform the instrument into cash or vice versa. This illiquidity would thus demand a higher level of return from the investor in order to invest in the instrument.

Liquidity risk is however not specific to derivative instruments, and depends on the market structure and the transparency of information with regards to the deal. The market structure would determine, through competition the level of liquidity risk. The higher the level of competition for a given instrument, the less would the liquidity risk for the given instrument be. This is directly reflected in the bid ask spread for the instrument. As such one would thus depend on open market interest, bid and ask spreads as well as cross market comparisons to evaluate liquidity risk.

1.5.6 Systemic Risk

The likelihood of a single market participant failing or disrupting the market or a segment thereof, and this ultimately leading to a break down of the financial market, must also be taken into account. Systemic risk is, as with most other risks, not specific to the derivatives market, and an analysis would rather focus on the relative risk introduced into the market by derivative instruments.

Systemic risk originates from panic or confidence loss effect. Thus, a sharp credit, liquidity or market variation that generates or has the potential to generate losses or cause a failure of a financial institution, has the potential to create a systemic crisis.

1.5.7 Option Based Instruments

Options are different from forward based instruments, in that the relationship between their price sensitivities and that of the underlying asset is not linear, but rather convex. The degree of curvature depends on the price of the underlying asset relative to the strike price at that particular point in time, the volatility of the asset and the time to expiry of the option.

1.5.8 Delta of an Option

The delta of an option can be expressed as follows:

$$\Delta = dP/dC$$

Deltas of an option are an indication of how a change in the underlying asset's price will influence the price of the option. Deltas of an option can either be positive (0-100%) or negative (-100 – 0%). A high delta will indicate a high sensitivity to price moves in the underlying asset's price, and vice versa. An option with a delta of 80% would thus change price at a rate of 80% to that of the underlying asset (CHEW, 1996)

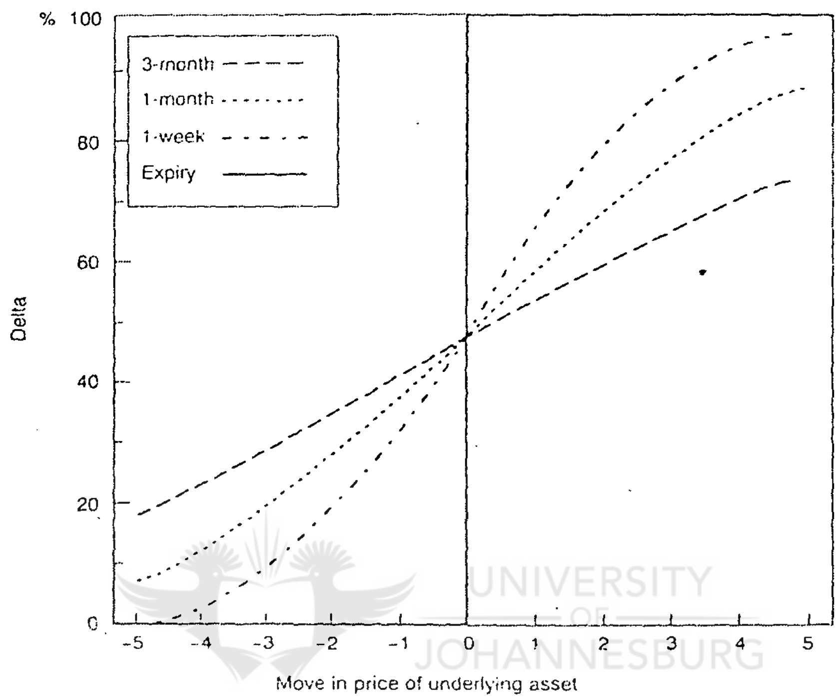
The value of the delta is dependent on the so called state of the option, namely whether it is in-the-money, at-the-money or out-of-the-money. The delta can thus be used t indicate reflect the moneyness of an option, and thus the chance of there being a net financial benefit to be derived from the option (CHEW, 1996).

As can be seen, the delta of an option is not just influenced by the price of the underlying asset, but also by the time horizon of the option, as well as the volatility of the asset's price. Both of these factors contribute to uncertainty, and thus the price of an option.

Graphically the delta of an option can be illustrated for different time horizons, as follows:

Fig 1.1

Change of delta with change of time



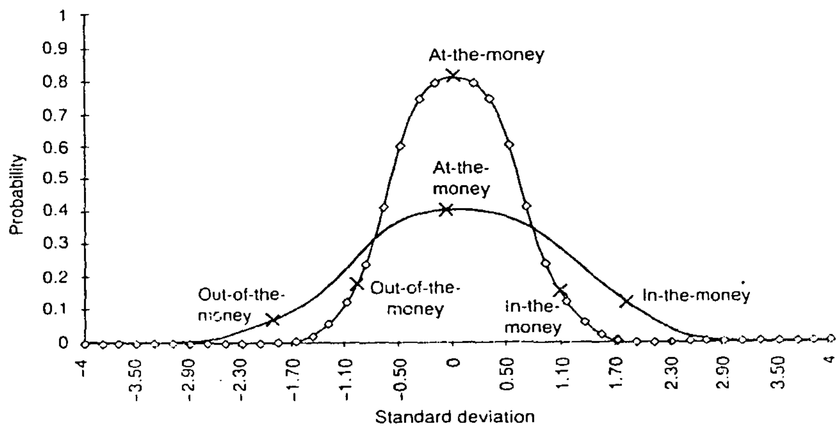
Should we map the S-shaped delta curve into a bell-shaped curve, we find the normal distribution (CHEW, 1996). Derivations can now be made on the bell curve, which can then be applied, to the deltas of the option.

Should we now increase the volatility of the underlying asset's prices, a wider distribution of prices from the mean would result. This in turn would cause a widening in the bell curve, for the same time horizon, due to the increase in uncertainty. Anything that thus causes an increase in uncertainty would cause the bell curve to flatten. Anything that reduces uncertainty would cause the width of the bell-curve to narrow. This is because the expected distribution of prices of the underlying asset will cluster nearer the middle.

The effect of these different scenarios is graphically illustrated in fig 1.2.

Fig 1.2

The relationship between probability and standard deviation



Volatility, prices and the time left till expiry changes daily, thereby resulting in an ever changing bell curve. These influences can all reinforce each other, or they can cancel each other out. It is thus difficult to determine which of these influences would have the strongest effect on the bell curve.

1.5.9 Delta Hedging and Dynamic Replication

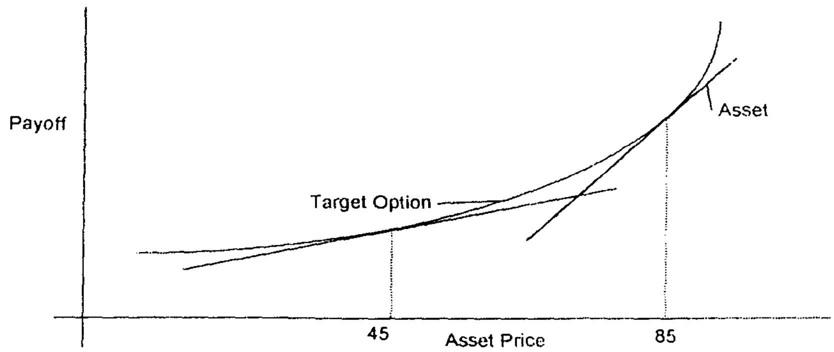
Once a transaction has been entered into, the exposure resulting from the transaction must be neutralised or hedged. The price risk is the single biggest exposure gained from the transaction, and as such must be hedged by taking an equivalent position in the underlying market (RUBINSTEIN, 1980). The size of the position taken in the underlying market is determined by the delta for the option.

The problem arises that the resolved asset flows that results from the option transaction change continuously, due to the dynamic nature of the underlying market. The option delta must thus be calculated on a continuous basis and dynamically adapted to counter the effects of price risk.

The pay-off profile of an option can thus be artificially created through dynamic hedging. This can be illustrated as in fig 1.3. Perfect dynamic hedging is however not possible, due to the fact that assets are traded in certain denominations, and as such hedging can only be done in increments of the denominations.

Fig 1.3

Pay-off profile of an option artificially created



All transactions entered into results in transactional costs. To reduce these cost, the option seller have to make a qualitative assessment on how often these hedge ratios should be adjusted. The less frequent these hedges are adjusted the bigger the possibilities of the hedger missing a major move within the markets. Should an option hedger miss such a move, he will find his dynamic hedge out of line, and since the move has already occurred, the resulting hedge to be inferior to the pay-off profiles of the option itself (CHEW, 1996). The question as to how often to hedge a position resulting from an option transaction now arises. Frequent adjustments would result in higher transactional cost, while infrequent hedging could result in an inferior dynamic hedging strategy.

Theoretically the transactional cost of dynamic replication should equal the option premium. Due to the uncertainty in trying to predict the future volatility and the problems associated in dynamic hedging, most investors prefers the option above dynamic replication, unless the option is seriously mispriced.

1.5.10 Convexity Risk

Convexity is the second order derivative of price, and are contained in most interest rate instruments.

$$\Phi = d^2P/d^2C$$

The magnitude contained in options however overshadow the magnitude of convexity contained in any other interest rate instrument, and as such are treated to be a property unique to options (CHEW, 1996).

The curvature of an option is rooted in the asymmetry of the payoff potentials that characterises the instrument. In one direction of price movement of the underlying asset, the option is in the money, and resembles the underlying asset itself. On the opposite side of the scale, the option is out of the money, and the pay-off approaches zero. The curvature of options and the linearity of the underlying assets means that attempting to dynamically replicate the option curvature necessarily requires continuous adjustments. The more pronounced the curvature, the higher the number of adjustments needed.

Convexity is a measure of how the price sensitivity of an option will change given a change in the underlying. It is also an indication of how quickly an option becomes unhedged. The larger the convexity, the quicker the option becomes unhedged, and vice versa.

A further factor that influences the convexity of an option, is its time to maturity. An at the money option with a short period to redemption, has a higher convexity than an option of similar strike, but with a longer time to maturity. The closer an option is to redemption, the bigger the influence a change in price of the underlying would have on the price of a option, and thus on the convexity of the option. It is due to this that at the money options close to redemption are so difficult to risk manage (FABOZZI, 1989). There is thus an analogy between convexity an the delta of an option. Both bell curves steepen with a decrease in period to maturity. At the day of redemption, this curve alters into a precipice, and an at the money option could swing from in the money to out the money by a slight change in the underlying assets price.

Further to this, there is an inverse relationship between volatility and convexity. The higher the volatility, the more uncertainty regarding future prices, and a flattening of the delta bell curve. This flattening of the delta bell curve implicates reduced convexity. With a decrease in volatility, the is more certainty with regards to future prices, and steepens the delta bell curve. This steepening means an increase in the convexity, since the curve is approaching the shape of a precipice.

By definition, the convexity calculated for an option at a specific price, it is a local measure, and does not provide a global picture with regards to the convexity of the option. This is graphically illustrated in fig. 1.4.

It is for this reason that the current value of the delta and convexity cannot be used to extrapolate for different values of delta.

Hedging of convexity is best done using assets that also have curvature built in. Options are the only instruments with adequate convexity to qualify for adequate hedging. The problem in trying to hedge in options, is that liquidity in the instruments are usually limited, with not to many price makers present in these instruments. It is thus practice to delta hedge option positions, until natural offsetting options or a combination of options can be found.

1.5.11 Time Decay

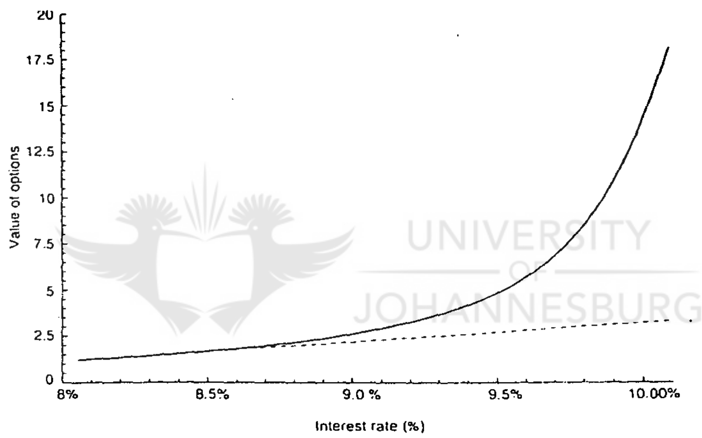
The life of an option can be treated as an exercise in resolving uncertainty (CHEW, 1996). On maturity the uncertainty regarding the asset flow associated with an option is resolved in that is either in the money or out of the money. But the maturity of an option is also a component of its value. The longer the maturity of an option, the higher this value would be.

From the outset, it is a positive value that erodes to zero at maturity, since time runs only in one direction. This value is expressed in point per day, and is illustrated in fig. 1.5.

The intrinsic value of the option is represented by the net value to the buyer, should the option be exercised today. Time value risk refers to the risk that the option would mature, without a movement in the price of the underlying asset, and thus no increase in the intrinsic value of the option. The closer the option is to maturity, the higher the time decay risk on the option.

Fig 1.4

Influence of convexity on an option price

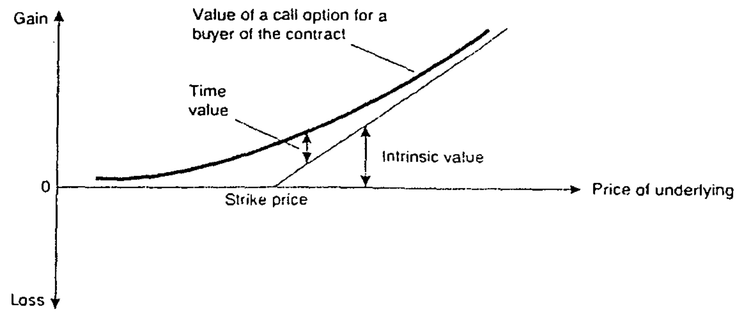


1.5.12 Volatility Smiles and Skews

The assumption has been made that the distribution of prices of the underlying asset around the mean, is normally distributed. In actual fact, there are events during which

Fig 1.5

Time value of an option



the prices move outside the normal distribution, and as such results in a distribution that is different to the normal distribution assumed. These deviations from the normal distribution give rise to volatility smiles, or higher volatilities for far out the money options.

During times when the market is trending in a certain direction, the demand for certain options would be higher than for other options. These differences in demands for different options gives rise to volatilities for option being in demand being higher than for others. This gives rise to volatility skews, another phenomenon that contradicts the assumption of normal distribution of prices around the mean.

1.6 Demarcation of the Research

The study would consist of five chapters. These chapters will each cover the following topics:

Chapter 1

An outline of the scope of the study, as well as a description of the background against which the study would be undertaken.

Chapter 2

Brownian motion theory and binomial trees form the basis for the two most successful option pricing models to date. The two approaches would be studied in detail in order to lay the foundation for chapter 3.

Chapter 3

The Black-Scholes and binomial option pricing models are the most successful option pricing models, and are used widely in the financial markets. Both these model make substantial assumption in order to facilitate the derivation. This chapter would be dedicated to a complete derivation of each model, as well as a discussion regarding the validity of these assumption with reference to their practical applications.

Chapter 4

All theories regarding asset price movements and option pricing theories ultimately culminate in risk management. In this chapter a closer look would be taken of value at risk as a risk management system, as well as practical applications thereof in practise.

Chapter 5

A summary of the major points of discussion regarding all three aspects, as and recommendations made where appropriate.

1.7 Conclusion

An in depth study will be made of the research and literature available regarding asset price movements, option pricing models and how these are affected by real world influences, and the final application of these models in assisting with the management of risk so prevalent in the modern corporate treasury. Since most of the research that are currently being conducted are not yet available in book form, extensive use will be made of the electronic media, as well as recent publications.

CHAPTER 2

THE PRINCIPLES UNDERLYING THE OPTION PRICING FORMULAE

2.1 Introduction

In South Africa, interest-bearing bonds are traded under the auspices of the South African Bond Exchange. They oversee the issuing and trading of the following two basic categories of bonds:

Bonds issued by the Republic of South Africa		74% of volume
Bonds issued by para-statal institutions	Eskom	7% of volume
	Transnet	5% of volume
	Telkom	2% of volume

The most actively traded of the bonds are the government bonds. Of these, the major issues are:

Code	Coupon	Redemption	Issue	
R147	11.50%	30/05/2000	2.672	ZAR (billion)
R162	12.50%	15/01/2002	21.675	ZAR (billion)
R175	9.00%	15/10/2002	10.792	ZAR (billion)
R150	12.00%	28/02/2004-2005-2006	58.388	ZAR (billion)
R184	12.50%	21/12/2006	21.163	ZAR (billion)
R177	9.50%	15/05/2007	8.025	ZAR (billion)
R153	13.00%	31/08/2009-2010-2011	78.032	ZAR (billion)
R157	13.50	15/09/2014-2015-2016	13.555	ZAR (billion)
R186	10.50%	21/12/2026-2027-2028	2.101	ZAR (billion)

2.1.1 Functioning of the Market

The market functions on an over the counter basis. Since April 1998, 11 primary market maker have been appointed, specialising in price making in the government bonds. The appointment of the primary dealers has increased the transparency, liquidity and efficiency of the secondary market.

Volume trade in the market has seen a marked increase during the last two years. During March 1998 ZAR 684 billion changed hands, which is an increase of 117% from the previous year, and 170% up from March 1996.

Foreign involvement in the market has also shown an increase from 20% of volume in 1997 to 38% in 1998.

2.1.2 Conventions in the market

Gilts are traded on yield to maturity (Act/Act). Coupons are paid semi-annually and are calculated Act/365. Certificates are traded cum div up to 1 month before actual coupon payment. During the month before coupon payment, bonds are traded ex dividend, taking into account the coupon not being received by the new buyer during this period of time. The accrued interest during this time is then calculated and is expressed as a negative figure.

2.1.3 Volume and Bid/Ask spreads

Bid/Ask spreads are based on the volume being traded. Typical spreads are 1/2 basis points for 1-3 million up to 3/5 basis points for volumes of 20 million.

2.1.4 Methods of Issuing

Issuing is done via an auction, taking place every week on a Tuesday. The bonds are adjudicated to the best bids, the total volume not exceeding the amount annonced a week earlier. Only primary dealers are allowed to place bids for the bonds, but other parties can participate via one of the primary dealers. The average amount auctioned each week is approx. ZAR 600 million nominal.

The amount of new bonds to be released into the market in a given year, are annonced during the annual budget calculations.

The market process can, over a prolonged period of time, be seen as a stochastic process. In a market as found in South Africa, a bid and offer would always be available in the highly liquid instruments such as the RSA150 bonds. The probability that the offer would be taken is equal to the probability that the bid would be given. There is also the probability that the next transaction might be a compromise between the bid and the offer price, and that there would be traded in the middle.

It is due to this reason that the price movements in a specific instrument can be seen as a stochastic process over a given period of time. After a brief discussion of the random walk theory, it would be indicated that the price movements could be closely approximated by a more specific process, the Wiener process.

2.2 Principles Underlying the Black and Scholes Option Pricing Theory

2.2.1 Simple Random Walk

Suppose the rates are initially at a given rate of X_0 at time n = 0. At a time n = 1 a trade is conducted and the rate undergoes a change Z_1, where Z_1 is a random variable having a given distribution. At time n = 2 another trade takes place, and the rates again undergoes a change Z_2, where Z_2 is independent from the change Z_1, with the same normal distribution of Z_1.

The rates thus moves and after one trade, the rate are at $X_0 + Z_1$, after two trades $X_n = X_0 + Z_1 + Z_2$, and after n trades, the rate can be described as:

$$X_n = X_0 + Z_1 + Z_2 + ...+Z_n$$

By defining $\{Z_i\}$ as a sequence of mutually independent, identically distributed random variables, the rates after n trades can be defined as:

$$X_n = X_0 + Z_1 + Z_2 + \ldots + Z_n$$
$$= X_{n-1} + Z_t \; (n = 1,2,3,.....) \qquad \ldots 2.1$$

If the rate is x at time n, then $X_n = x$, and it is said that the rate is in state x at time n.

Referring to table 1, this would translate into the following:

X_0 = 14,40% and Z_1 = -0,01%,Z_2 = -0,01%..... Z_n with n equalling the trade numbers, resulting in X_n = 14.54%, or the closing rate of the day.

Suppose the changes in the rate can only take the values 1 for a one percentage point increase in the rate, 0 for no change in the rate and -1 for a one percentage point decrease in the rate, the following is now defined:

$$\text{prob}\,(Z_i = 1) = p \qquad \ldots 2.2$$

$$\text{prob}\,(Z_i = 0) = 1 - p - q \qquad \ldots 2.3$$

$$\text{prob}\,(Z_i = -1) = q \qquad \ldots 2.4$$

This process is called a simple random walk. By further assuming that $p + q \leq 1$, with 1 - p - q as the probability of a zero jump in the rate, after a trade has taken place. The movement of the rate is however restricted by certain boundaries, with 0% on the down side, and theoretically unrestricted on the up side. The time value of money restricts the rates of going below 0%.

Unrestricted Movement

Take the rate at time t = 0 to be the origin and $X_0 = 0$, and assume that the rate is free to move indefinitely in either direction. The following situation is then reached:

$$X_n = \sum_{r=1}^{n} Z_r \qquad \ldots 2.5$$

The possible values of the rate at time n are then given by k = 0, $\pm$ 1, ± 2, ± 3,..., $\pm n$.

For the rate to reach a certain value k at time t, the rate has to make r_1 positive move, r_2 negative moves and r_3 zero moves within the time t, r_1, r_2, r_3 may be any non-negative integers satisfying the simultaneous equalities (Cox, p26)

$$r_1 - r_2 = k \qquad \ldots 2.6$$

$$r_3 = n - r_1 - r_2 \qquad \ldots 2.7$$

with n = time t

The probabilities that $X_n = k$ is given by the multinomial probabilities (Cox, p26):

$$\text{prob}\ (X_n = k) = \sum\ [n!/(r_1!r_2!r_3!)] * [p^{r1}(1-p-q)^{r3}q^{r2}] \qquad \ldots 2.8$$

for the values of r_1, r_2 and r_3 satisfying the above mentioned criteria.

In order to illustrate these principles, refer to the rates as given in Table 2.1.

From this data it can be seen that there was a total of 102 trades that day. Of these 102, 52 represented a 1bpt rise from the previous trade, 42 represents a 1 bpt decline from the previous trade, and there were 8 trades that did not cause any change in the rates. The following can thus be calculated from this data.

$n = 102$

$r_1 = 52$

$r_2 = 42$

$r_3 = n - r_1 - r_2$

$= 8$

Table 2.1

Trades Conducted in R150 Government Bond – 17 March 1998

Trade No.	Rate	Trade No.	Rate	Trade No.	Rate
1	14.44%	35	14.46%	69	14.57%
2	14.43%	36	14.45%	70	14.57%
3	14.42%	37	14.46%	71	14.57%
4	14.41%	38	14.47%	72	14.58%
5	14.42%	39	14.47%	73	14.59%
6	14.41%	40	14.48%	74	14.60%
7	14.42%	41	14.49%	75	14.61%
8	14.41%	42	14.48%	76	14.62%
9	14.40%	43	14.49%	77	14.61%
10	14.41%	44	14.48%	78	14.60%
11	14.40%	45	14.49%	79	14.59%
12	14.39%	46	14.50%	80	14.58%
13	14.39%	47	14.51%	81	14.57%
14	14.40%	48	14.52%	82	14.56%
15	14.39%	49	14.53%	83	14.55%
16	14.38%	50	14.52%	84	14.54%
17	14.39%	51	14.51%	85	14.55%
18	14.40%	52	14.52%	86	14.56%
19	14.41%	53	14.51%	87	14.55%
20	14.40%	54	14.52%	88	14.54%
21	14.41%	55	14.53%	89	14.53%
22	14.42%	56	14.54%	90	14.52%
23	14.42%	57	14.55%	91	14.53%
24	14.43%	58	14.56%	92	14.54%
25	14.42%	59	14.55%	93	14.53%
26	14.43%	60	14.55%	94	14.52%
27	14.44%	61	14.56%	95	14.53%
28	14.43%	62	14.57%	96	14.54%
29	14.44%	63	14.56%	97	14.54%
30	14.45%	64	14.55%	98	14.53%
31	14.46%	65	14.56%	99	14.52%
32	14.45%	66	14.57%	100	14.53%
33	14.46%	67	14.58%	101	14.54%
34	14.45%	68	14.59%	102	14.54%

(Source: De Witt Morgan Brokers)

The probability generating function (p.g.f.) of the movement Z_r is (Freund, p79)

$$G(z) = E(z^{Z_r}) = pz + (1 - p - q) + qz^{-1} \qquad \ldots 2.9$$

and hence that of X_n is given by:

$$E(z^{X_n}) = \{G(z)\}^n \qquad \ldots 2.10$$

By letting μ and σ^2 denote the mean and variance of the rate movement, then $\mu = p - q$ and $\sigma^2 = p + q - (p - q)^2$ and thus:

$$E(X_n) = n\mu \qquad \ldots 2.11$$

$$V(X_n) = n\sigma^2 \qquad \ldots 2.12$$

where V(X) denotes the variance of the random variable X. If we however want to calculate the probability that the rate would be at values j, j+1,...,k at time n, with j and k possible values of $X_n(j<k)$, this may include an inconvenient summation of large number of multinomial probabilities. By resorting to an approximation, according to which X_n will be approximately normally distributed (Cox, p27) with mean $n\mu$ and variance $n\sigma^2$ for large n, then:

$$\text{prob}(j \le X_n \le k) \cong (2\pi\sigma^2 n)^{-1/2} \int \exp\{-(x-n\mu)^2/2n\sigma^2\}dx \qquad \ldots 2.13$$

A better approxiamtion is obtained by employing a continuity correction, i.e. by using j-c and k+c as the integration limits, where c =1/2 or c =1. By setting the limits of integration to be j - c and k+ c, an continuity correction is introduced, whereby c = ½ or c = 1 accordingly as p + q < 1 or p + q = 1. After transforming the integral to standard form the following is obtained (Cox, p27):

$$\text{prob}(j \le X_n \le k) \cong \Phi([k + c - n\mu]/[\sigma\sqrt{n}]) - \Phi([j-c-n\mu]/[\sigma\sqrt{n}]) \qquad \ldots 2.14$$

where:

$$\Phi(y) = 1/\sqrt{(2\pi)}\int e^{-\frac{1}{2}x^2}\,dx \qquad \ldots 2.15$$

which is the standard normal distribution function.

In the case where p = q and μ = 0, the central limit theorem states that the rates would be a distance of $\sqrt{n}$ from the origin after n jumps.

2.3 Continuous limit of the random walk: The Wiener process

The above discussion had the limitation that one step transitions were only permitted to the nearest neighbouring state. This type of transition may be regarded as the analogue for discrete states of the phenomenon of continuous changes for continuous states.

Taking small steps of magnitude Δ taking place at small time intervals of length τ, then in the limit as Δ and τ approach zero, it may be expected to obtain a process whose realisations are continuous functions of time.

Consider a rate starting at and arbitrary rate taken to be the origin. In each time interval τ it makes a move Z, where

$$\text{prob}\ (Z = +\Delta) = p \qquad \ldots 2.16$$

$$\text{prob}\ (Z = -\Delta) = q = 1-p \qquad \ldots 2.17$$

If all the steps are mutually independent, then the moment generating function of a single step can be given by the following (Freund, p153):

$$E(e^{-\theta z}) = pe^{-\theta\Delta} + qe^{\theta\Delta} \qquad \ldots 2.18$$

In a time period t there will be n= t/τ movements and the total move in rates X(t) is the sum of all n independent random movements each with the m.g.f. as stated above. The m.g.f. for the total amount of movements can be given by:

$$E\{e^{-\theta X(t)}\} = (pe^{-\theta\Delta} + qe^{\theta\Delta})^n$$
$$= (pe^{-\theta\Delta} + qe^{\theta\Delta})^{t/\tau} \quad ..2.19$$

The mean and variance of X(t) would then be given by (Cox, p205):

$$E\{X(t)\} = t/\tau(p - q)\Delta \quad ...2.20$$

$$V\{X(t)\} = (t/\tau)4pq\Delta^2 \quad ...2.21$$

By letting $\Delta \rightarrow 0$ and $\tau \rightarrow 0$ in such a way that it is a limiting process with a mean μ and variance σ^2 in unit time, they must tend to zero in such a way that:

$$(p - q)\Delta/\tau \rightarrow \mu$$

$$4pq\Delta^2/\tau \rightarrow \sigma^2$$

These conditions will be satisfied if adhere to the following:

$$\Delta = \sigma\sqrt{\tau}$$

$$p = ½(1+ (\mu\sqrt{\tau}/\sigma))$$

$$q = ½(1 - (\mu\sqrt{\tau}/\sigma))$$

The important points about the above relations are that for a small τ, $\Delta=\sigma(\tau^{½})$, and that p and q are each ½ + $\sigma(\tau^{½})$, so that a displacement Δ must be substantially larger than the small time interval τ in which it occurs and p,q must not be to different from ½ if degeneracies are to be avoided.

Substituting into the m.g.f., the following is obtained:

$$E\{e^{-\theta X(t)}\} = [[½(1 + (\mu\sqrt{\tau}/\sigma)e^{-\theta\sigma\sqrt{\tau}})] + [½(1 - (\mu\sqrt{\tau}/\sigma)e^{\theta\sigma\sqrt{\tau}})]]^{t/\tau} \quad ...2.22$$

Since all movements are independent from another, it can thus be taken that movements X(t) - X(0) and X(u) - X(t) are independent from one another, both however normally distributed. The latter with a mean $\mu(u - t)$ and variance $\sigma^2(u - t)$.

For the case where $\mu=0$, the limiting process X(t) were called the Wiener process. For $\mu=0$ the probability generating function of X(t) is symmetrical about the origin for all t. In the case where $\mu\neq0$, a limiting process with drift u and variance o is obtained. For such a process the increment $\Delta X(t) = X(t+\Delta t) - X(t)$ during the small time interval Δt is independent of X(t) and has mean and variance proportional to Δt. For values of t large compared with the intervals between successive rate changes the Wiener process has been found to be a good representation of rate movement.

The representation of the process X(t) in differential form gives a clear view of rate movements. This is analogue to the difference equation defining the simple random walk. The simple random walk was defined as:

$$X_n = X_0 + Z_1 + Z_2 + Z_3 + \ldots\ldots + Z_n$$
$$= X_{n-1} + Z_n$$

The equivalent of a sequence of independent identically distributed random variable is a purely random process Z(t).

The random variable $\{Z(t_j)\}$ at time points $\{t_j\}$ are all mutually independent. If the distribution for Z(t) for any t is normal, a purely Gaussian process is obtained. X(t) can now be defined as follows:

$$X(t + \Delta t) = X(t) + Z(t)\sqrt{\Delta t} \qquad \ldots 2.23$$

Alternatively stated

$$\Delta X(t) = Z(t)\sqrt{\Delta t} \qquad \ldots 2.24$$

The increment in the process in time Δt is a normally distributed random variable with zero mean and variance Δt and is independent to any other increments.

A process with drift u and variance σ^2 per unit time can be written as:

$$\Delta X(t) = \mu\Delta t + \sigma Z(t) \sqrt{\Delta t} \qquad \text{...2.25}$$

This can be rewritten in continuous time as the differential equation for the Wiener process X(t) with drift u as:

$$dX(t) = \mu dt + \sigma Z(t) \sqrt{dt} \qquad \text{...2.26}$$

This equation can be interpreted as follows:

The change in X(t) in a small time interval dt is a normal variable with mean μdt and variance $\sigma^2 dt$ and is independent of X(t) and of the change in any other small time interval. Thus:

$$E\{dX(t)\} = \mu dt \qquad \text{...2.27}$$

$$V\{dX(t)\} = \sigma^2 dt \qquad \text{...2.28}$$

$$C\{dX(t), dX(u)\} = 0 \quad (t \neq u) \qquad \text{...2.29}$$

By analysing the two components of the generalised Wiener process separately, it can be shown that if Z(t) = 0, then $dX(t) = \mu dt$, which implies that:

$$\frac{dX(t)}{dt} = \mu$$

or

$$X(t) = X(t_0) + \mu t \qquad \text{...2.30}$$

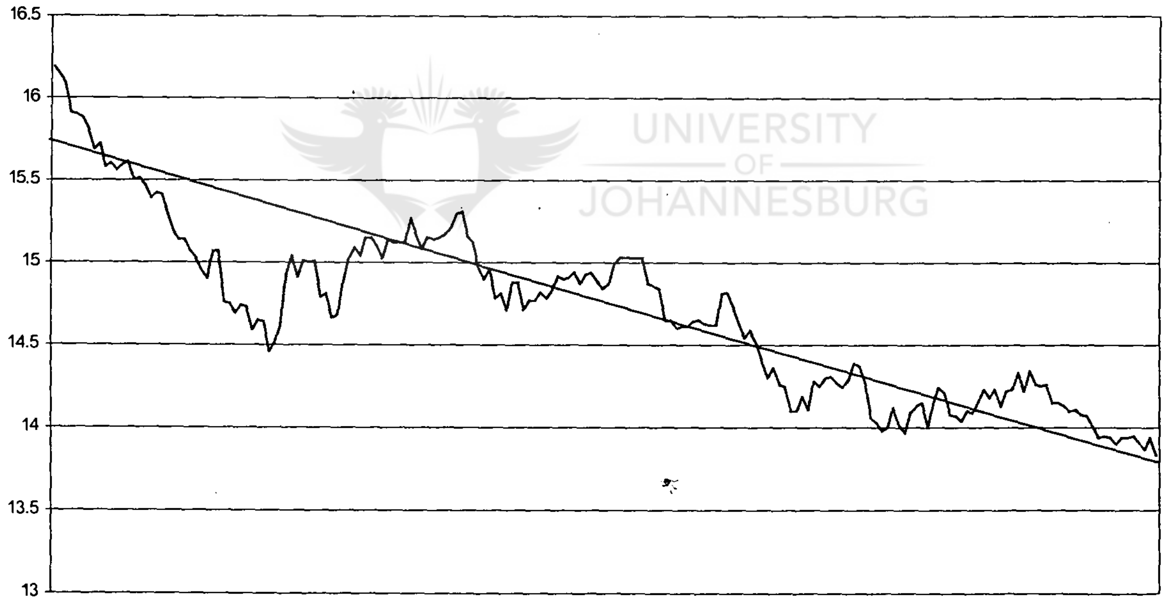
On the other hand, the $\sigma Z(t)\sqrt{dt}$ part of the equation can be regarded as adding noise to the path being followed by X(t). The amount of noise or variability is σ time a Wiener process. The variable σ is usually referred to as the stock rate volatility, while μ represents the expected rate of return demanded by investors. Since the noise is independent from the path followed by X(t), the equation can be rewritten in the following form, and can clearly be graphically illustrated as in Fig. 2.1:

$$dX(t) = \mu dt + \sigma dz \qquad \ldots 2.31$$

with $dz = Z(t)\sqrt{dt}$

Fig 2.1

Graphical Representation of Wiener Process



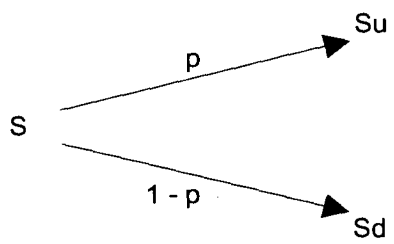
(Source: Boe Securities)

2.4 Principles Underlying the Binomial Option Pricing Theory

Consider the following binomial tree:

Fig 2.2

Basic Buildingblock of the Binomial Theory



(Source: Hull)

Consider the basic process of the markets as illustrated in Fig 2.2. Suppose the price of a bond is S at time t=0. According to the binomial model the price will follow one of two paths in the small time interval Δt. It can move up to price Su with probability P or down to Sd with probability 1 – p. It is clear from the binomial tree that $u > 1$ and $d < 1$, and that p, u and d must give correct values to the mean and variance of the price change during time interval Δt. Over a short period of time, stock prices can assume one of two different values. Over a longer period of time, this assumption is however not accurate, but is sufficient for numerical procedure that was developed to predict bond price movements. Further assumptions that are made, ares: (HULL, 1993)

The expected return from all traded securities is the risk-free interest rate.
Future cash flows can be valued by discounting their expected values at the risk-free interest rate.

From the assumptions it is clear that the expected return on the stock must equal that of the risk-free interest rate, r. From this can be derived that the stock price at the end of the first time interval Δt must equal $Se^{r\Delta t}$. From this follows that:

$$Se^{r\Delta t} = pSu + (1 - p)Sd \qquad ...2.32$$

or

$$e^{r\Delta t} = pu + (1 - p)d \qquad ...2.33$$

It can be proven that the variance of the stock price during a small time interval Δt can be given by $S^2\sigma^2\Delta t$. The variance of a variable Q is further defined by $E(Q^2) - [E(Q)]^2$. From this then follows that: (HULL, 1993)

$$S^2\sigma^2\Delta t = pS^2u^2 + (1 - p)S^2d^2 - S^2[pu + (1 - p)d]^2$$

or

$$\sigma^2\Delta t = pu^2 + (1 - p)d^2 - [pu + (1 - p)d]^2 \qquad ...2.34$$

If it is accepted that:

$u = 1/d$

it can be shown that:(HULL, 1993)

$p = (a - d)/(u - d)$

$u = e^{\sigma\sqrt{\Delta t}}$

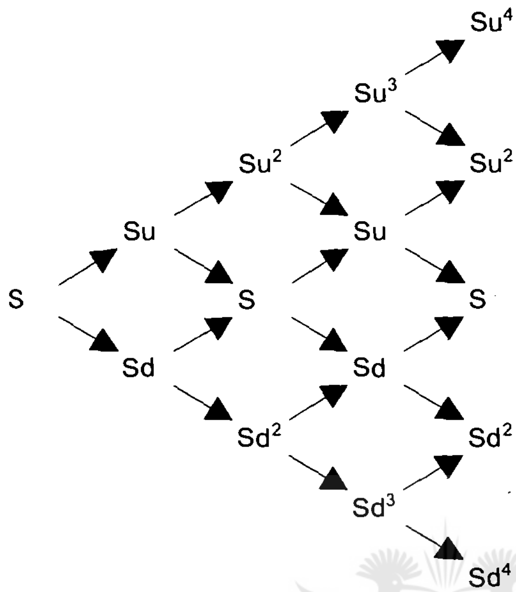
$d = e^{-\sigma\sqrt{\Delta t}}$

where $a = e^{r\sqrt{\Delta t}}$

The tree can now be expanded to the following:

Fig 2.3

Expansion of the Binomial Tree



(Source: Hull)

When the basic binomial tree illustrated in Fig 2.2 is expanded, it can be graphically represented by Fig 2.3. As the stock price moves along the tree, an up and a down movement would cancel each other out. At time iΔt, there have been i + 1 stock price movements, and there are i + 1 stock price values to be considered. These prices can be expressed as:

$$Su^j d^{i-j} \qquad j = 0, 1, 2, 3, \ldots i \qquad \ldots 2.35$$

Because of the relationship u = 1/d being used, Su^2d = Su. The binomial approach would later be expanded further to derive the binomial option pricing model.

2.5 Conclusion

Despite the fact that the Wiener process and the binomial approach vary greatly in their approach to solving the price movement within the financial markets, they form the basis of the two most widely used option pricing models presently employed. As such it is of the utmost importance to clearly understand the reasoning behind these principles, what their assumptions are made, and what are the factors affecting them.

Both these principles do employ statistical methods to try and determine the possible future prices and outcomes, and their relevant probabilities. Since the future can't be predicted with certainty, and outside forces can influence these predictions greatly, these principles will merely be strong guidelines and cannot be treated as a definite in predicting future prices and trends. As such, when deriving the pricing models, all known factors must be taken into account, and the probability of the unforeseen happening, must be made provision for.

CHAPTER 3

OPTION PRICING FORMULAE

3.1 Introduction

In chapter 2 the basic principals underlying the two most widely excepted approaches to option pricing models were discussed. Although these two techniques vary in their approach towards solving the value of an option, they both utilise statistical methods to try and model the price movement of the underlying security. In both instances it is assumed that the statistical distribution of the price movement occurring in the underlying instrument is normally distributed about the present or starting rate, and that there are no arbitrage opportunities available within the markets.

These two assumptions formed the basis for the fundamentals discussed, and these fundamentals will now be used in deriving the full option pricing model for the two different approaches. It will however become clear that despite the differences in the methods used that the end results are very similar.

Volatility, as one of the major factors impacting on the pricing formulas, will be discussed, and the different aspects of volatility discussed with reference to the advantages and disadvantages of the different structures of volatility.

3.2 Ito's Lemma

The price of an option is a function of the rate of the underlying bond and the time to expiry of the option. In more general terms we can state that the price of any derivative instrument on a security is a function of the stochastic process variables underlying the derivative instrument, as well as a time dependant component. Any study of derivative instruments as well as the risk management of these derivative securities, would be incomplete without a thorough study of the Black-Scholes pricing formula, which was derived with the aid of Ito's Lemma.

Consider a security following a Wiener process as derived in the previous chapter:

$$dx = a(x,t)dt + b(x,t)dz$$

with the variables as previously defined. The variable x has a drift of a and variance of b^2. Ito's lemma stipulates that a function G, of x and t follows the following process

$$dG = [\partial G/\partial x \; a + \partial G/\partial t + ½\partial^2 G/\partial x^2 \; b^2]dt + \partial G/\partial x \; bdz$$

where dz is a Wiener process. From this can be deducted that G can also be described as an Ito process. The function G has the following characteristics:

Drift rate: $\partial G/\partial x \; a + \partial G/\partial t + ½\partial^2 G/\partial x^2 \; b^2$

Variance: $(\partial G/\partial x)^2 \; b^2$

The process can also be written as:

$$dS = \mu Sdt + \sigma Sdz$$

where μ and σ are constant. This is a reasonable model of how bond prices react in the market under normal market conditions. From Ito's lemma, it now follows that the stochastic process represented by function G, of S and t, can be given by:

$$dG = [\partial G/\partial S \; \mu S + \partial G/\partial t + ½\partial G^2/\partial S^2 \; \sigma^2 S^2]dt + \sigma S\partial G/\partial S \; Sdz$$

It is clear from the above that both S and G are both affected by the same underlying uncertainty, namely dz.

3.3 Black-Scholes option pricing model

In order to derive the Black-Scholes option pricing model, the assumptions underlying the principles upon which the model was derived, must be considered. They are:

- The bond price follows a Wiener process with μ and σ constant
- No transaction costs are involved. All bonds are perfectly divisible, and no penalty is paid for odd lot trading.
- Short selling of bonds are permitted, with full use of funds available after such a sale.
- There are no coupon payments during the life of the bond.
- Riskless arbitrage opportunities do not exists.
- There is a single borrowing and lending rate.
- Trading is continuous.
- The risk free rate r remains constant for all maturities, thus implying a flat yield curve.

Consider the stochastic process:

$$dS = \mu S dt + \sigma S dz$$

If we take f to be the price of a derivative instrument based on the underlying security S. The price f must follow the same process as the variables S and t, and can thus be represented by the following process:

$$df = [\partial f/\partial S\ \mu S + \partial f/\partial t + \partial^2 f/\partial S^2 - \tfrac{1}{2}\partial^2 f/\partial S^2 \sigma^2 S^2]dt + \partial f/\partial S\ \sigma S dz$$

The discreet versions of the above two expressions are given by:

$$\Delta S = \mu S \Delta t + \sigma S \Delta z \qquad \ldots 3.1$$

and

$$\Delta f = (\partial f/\partial S\ \mu S + \partial f/\partial t + \tfrac{1}{2}\partial^2 f/\partial S^2 \sigma^2 S^2)\ \Delta t + \partial f/\partial S\ \sigma S \Delta z \qquad \ldots 3.2$$

As proved in the previous section the Wiener process underlying both f and S is the same. By choosing a portfolio of both bonds and derivatives, one can eliminate the Wiener process dz in the above equations. This can be obtained by choosing the following portfolio:

derivative = -1

bonds = $+\partial f/\partial S$

This portfolio is thus short one derivative security and long an amount of $\partial f/\partial S$ physical bonds. If we define the value of the portfolio to be Π, we have by definition:

$$\Pi = -f + S\ \partial f/\partial S \qquad \ldots 3.3$$

Should the value of Π change by a small amount of $\Delta\Pi$ during a time Δt, then the new value of the portfolio would be given by:

$$\Delta\Pi = -\Delta f + \partial f/\partial S\ \Delta S \qquad \ldots 3.4$$

If we substitute the equation for the process in the above, we obtain:

$$\Delta\Pi = (-\partial f/\partial t - ½\partial^2 f/\partial S^2\ \sigma^2 S^2)\Delta t \qquad \ldots 3.5$$

Since the equation does not involve Δz, the portfolio does not involve any risk and as such must be riskless during time Δt. From the assumptions previously made, it must be that the portfolio must earn the same rate as the other risk free securities. If it earned more than this, arbitrage opportunities would be present and riskless profit can be made by selling the risk-free securities and using the proceeds to buy the portfolio. If it earned less, a riskless profit could be made be selling the portfolio and buying the riskless securities with these proceeds. It thus follows that:

$$\Delta\Pi = r\Pi\Delta t \qquad \ldots 3.6$$

Where r is the prevailing risk free rate. By substitution we now obtain:

$$(\partial f/\partial t + ½\partial^2 f/\partial S^2\ \sigma^2 S^2)\Delta t = r(f-\partial f/\partial S\ S)\ \Delta t$$

so that:

$$\partial f/\partial t + rS\partial f/\partial S + \tfrac{1}{2}\sigma^2 S^2\, \partial^2 f/\partial S^2 = fr \qquad ...3.7$$

This is the Black-Scholes differential equation. The boundary conditions used determine the particular type of option pricing model obtained. By choosing S and t in such that:

$$f = \max(S-X, 0) \text{ when } 0 < t < T$$

we obtain the expression for a call option. For a put option, chose S and t such that:

$$f = \max(X-S, 0) \text{ when } 0 < t < T$$

By substituting these values into the Black-Scholes differential equation, it can be proven that the option premium is given by:

$$C(S,X,T) = S.N(d_1) - Xe^{-rT}N(d_2) \qquad ...3.8$$

where

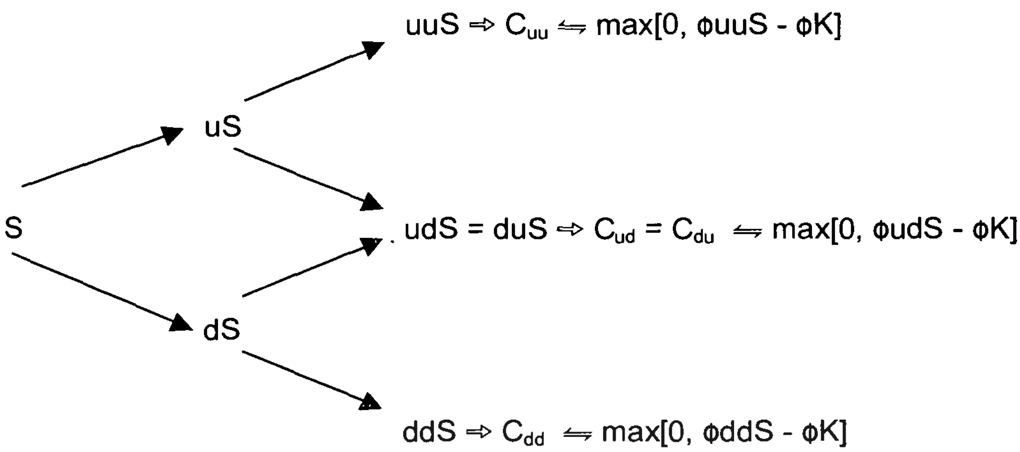
$d_1 = [\ln(S/X) + (r + \sigma^2/2)T]/(\sigma\sqrt{T})$
$d_2 = d1 - \sigma\sqrt{T}$
C = call option premium
S = current asset price
X = exercise price
T = time to expiration
σ^2= volatility
N(.) = cumulative normal distribution function
r = riskless interest rate

3.4 Binomial option valuation

If one assumes that the underlying instrument prices follow a simple binomial random process as outlined in the previous chapter, one can derive another widely accepted model for option valuation. For an option expiring at the end of two periods, the price movement can be illustrated graphically as in Fig 3.1:

Fig 3.1

Graphical Representation of the Binomial Tree



(Source: Rubinstein, 1986)

where u and d are one plus the rates of growth of the underlying asset over each of the different binomial periods. It should be noted that the prices udS and duS will mathematically equate to the same value, but for clarity they are presently left as two separate variables. The value C_{uu}, C_{ud}, C_{du}, C_{dd} are the possible values of the option if held to its expiration date; corresponding to the different possible paths that were followed.

Deriving a model for the value of an option held to expiration is relative simple, but it becomes more difficult if the option is not held for the full length of time. Should the simplest situation; where the asset price has just moved from price S to uS; be considered, the pay-off from the option if it is held for just one more period, can be replicated by holding the underlying asset and cash. (Rubinstein, 1986)

Let Δ_u be the number of units held in the underlying asset, and B_u be the cash value of the replicating investment. Then:

$$C_{uu} = uu\delta\Delta_u + rB_u \qquad ...3.9$$

$$C_{ud} = ud\delta\Delta_u + rB_u \qquad ...3.10$$

δ = one plus the pay-out rate of the underlying asset

r = riskless interest rate earned on cash over each binomial period

By choosing Δ_u and B_u such that the they replicate the pay-off of the option if it is held to expiry (Rubinstein, 1986). By solving the above two equations for the two unknowns, we obtain:

$$\Delta_u = (C_{uu} - C_{ud})/[u(u\text{-}d)\ \delta S] \qquad ...3.11$$

$$B_u = (uC_{ud} - dC_{uu})/(u\text{-}d)r \qquad ...3.12$$

To be consistent with the assumption of no riskless arbitrage, the two investments, if made on the same day and held to expiration, they must have the same value one period later, and also involve the same costs (Jaycobs, 1992). Since the investment is a replica of the option, the value of the unexercised option at the beginning of the second period must equal the cost to establish the replicated asset/cash portfolio. This value amounts to $uS\Delta_u + rB_u$ (Jaycobs, 1992).

Considering the potential of early exercise of the option at the beginning of period two, the value of the option at this time must then be:

$$C_u = \max\{[\phi uS - \phi K],[uS\Delta_u + rB_u]\} \qquad ...3.13$$

since the buyer will exercise the option, and receives $\phi uS - \phi K$. This will be the case if the exercisable value exceeds the holding value $uS\Delta_u + rB$. By substituting the values for Δ_u and B_u, we obtain through algebraic manipulation that:

$$C_u = \max\{[\phi uS - \phi K],[pC_{uu} + (1 - p)C_{ud}]/r\} \qquad ...3.14$$

$$p \equiv ((r/\delta) - d)/(u\text{-}d)$$

When the asset price falls during the first period, the expression becomes: (Rubinstein, 1986)

$$C_u = \max \{[\phi dS - \phi K],[pC_{du} + (1 - p)C_{dd}]/r\} \qquad \ldots 3.15$$

To obtain a value for the option at the beginning of the first period, the value of the option is replicated again one period later with a position in the underlying instrument and the cash market. Choose Δ units in the asset, and B amount of cash; such that: (Jaycobs, 1992)

$$C_u = u\delta S\Delta + rB \qquad \ldots 3.16$$

$$C_d = d\delta S\Delta + rB \qquad \ldots 3.17$$

By solving for Δ and B:

$$\Delta = (Cu - Cd)/[(u - d)\phi S] \qquad \ldots 3.18$$

$$B = (uC_d - dC_u)/(u - d)r \qquad \ldots 3.19$$

By again considering the possibility of an early exercise, but now at the beginning of the first period, the value C of the option must then be:

$$C_u = \max \{[\phi S - \phi K],[pC + (1 - p)C_d]/r\} \qquad \ldots 3.20$$

The value of a standard option has thus been obtained by working backward on the binomial tree. Certain observations will allow the valuation procedure to be generalised or, under certain conditions, simplified. These observations are: (Rubinstein, 1986)

3.5 Observations that will allow procedures to be generalised

3.5.1 Risk Neutral Probability

(p, 1 - p) is a mathematical probability measure. To ascertain that there are no opportunities for riskless arbitrage using only the underlying asset and cash, u and d must be chosen such that $d < (r/\delta) < u$, which in turn again implies that $0 < p < 1$ (Rubenstein, 1986). p can thus be interpreted as a subjective probability of an upward move. If q is the subjective probability, then over one period, the underlying asset price at the beginning of that period must be equal to its expected future value at the end of the period discounted by the interest rate, thus:

$$S = [qu\delta S + (1 - q)d\delta S]/r \qquad \text{...3.21}$$

by solving this equation for q:

$$q = ((r/\delta) - d)/(u - d) = p \qquad \text{...3.22}$$

Due to this result, p is called the risk - neutral probability (Rubenstein, 1986).

3.5.2 Early Exercise

If, as in the case of an European option, the option to exercise early is not possible, then through the previous equations it can be proven that: (Rubinstein, 1986)

$$C = [p2C_{uu} + 2p((1 - p)C_{ud} + (1 - p)^2 C_{dd}]/r^2 \qquad \text{...3.23}$$

This offers the opportunity to value European options.

3.5.3 Similar paths

The size of movements up and down are constant throughout the binomial tree, and thus the pay-out from all possible combinations of the same moves are equal. For many periods wherein there can be many permutations of paths that are the same, it can be stated that: (Jaycobs, 1992)

$$C = \frac{\Sigma c(j,n)p^{i}(1 - p)^{n-j}\max[O,u^{i}d^{n-j}S - K]}{r^{n}} \qquad \ldots 3.24$$

where:

$$c(j,n) \equiv \frac{n!}{j!(n - j)!} \qquad \ldots 3.25$$

3.5.4 Common Greeks

The options sensitivity to the underlying asset, namely the delta value, can also be calculated using the binomial model. This is done using finite difference methods. The binomial delta equals: (Rubinstein, 1986)

$$\Delta \equiv (C_u - C_d)/[(u - d)\ \delta S] \qquad \ldots 3.25$$

which is equal to the number or units of the underlying asset being held in the replicating portfolio. From this the sensitivity of the delta to the price movements in the underlying assets, called the gamma, can be expressed as: (Rubinstein, 1986)

$$\equiv (\Delta_u - \Delta_d)/[(u - d)\delta S] \qquad \ldots 3.26$$

The sensitivity of the option to the reduction in time, called the theta Θ, by one day, can be calculated by reducing the time t to expiration by one day, all other things being equal. This is done using forward finite methods, and can be shown to be: (Rubinstein, 1986)

$$\Theta \equiv (C_{ud} - C)/(2t/n) \qquad \ldots 3.27$$

3.6 Criticism regarding option pricing models

When trading options within the market, it is rare to calculate a price with a model that corresponds to the price being asked within the market. There are a number of different reasons for this discrepancy. The most likely error that may occur, would be an incorrect estimate of the volatility.

Several basic assumptions were made in the derivation of the models, and some of these assumption are unrealistic in their approach to creating a option pricing model. (Black, 1990) The following assumptions that were made, would have the biggest effect on the correctness of the option pricing models:

A stock's volatility is known, and stays constant over the life of the option.
The short-term interest rates never changes.
Anyone can borrow or lend as much as he wants at a single rate, as long as collateral to the value of the amount being borrowed is provided
There are no transaction cost involved in dealing
Stocks pay no interest, and investors are not allowed to exercise their options early.
No events that can end the life of the option early, occur during the life of the option.
The distribution of rates is assumed to be normally distributed around the origin.

Most problems with option pricing models arise from unrealistic assumptions being made, in order to simplify the models. These assumptions mostly involve the volatility of the price of the underlying instrument.

3.6.1 Volatility

Non stationary Volatility

Economic fundamentals change over time, causing the economic outlook to change, and as such it can be expected that volatility would also change over time. Where there is substantial non-stationarity in the variance, any volatility estimation procedure based on the historical approach will be invalid.

Non-uniformity

During times of high activity and substantial rate movements within the markets, it can be expected that the volatility would be higher than inactive trading days. This problem cannot be totally eliminated from volatility calculations, it would however help to switch from a time based calendar to a economic time basis.

Mean Reversion

Most economic time series tend away from extremely high and low value, and revert back to long term trend. In order to correctly estimate the volatility that displays mean reversion, one needs to move away from the Wiener approach for rates movement. If one however matches the sampling horizon to the term of the option being evaluated, one can reduce the estimation error substantially, while still utilising the Wiener process as basis for the rates movement.

In the Black-Scholes option pricing model, volatility is calculated through the following (Jaycobs, 1992), which is the standard deviation for the price range:

$$V = [1/(n-1)]\sum_{i=1}^{n}(X_i - \bar{X})^2 \qquad \ldots 3.28$$

where:

$$X_i = \ln P_i - \ln P_{i-1}$$

$$X = (\sum_{i=1}^{n} X_i)/n$$

P_i= the price of the underlying asset at time i

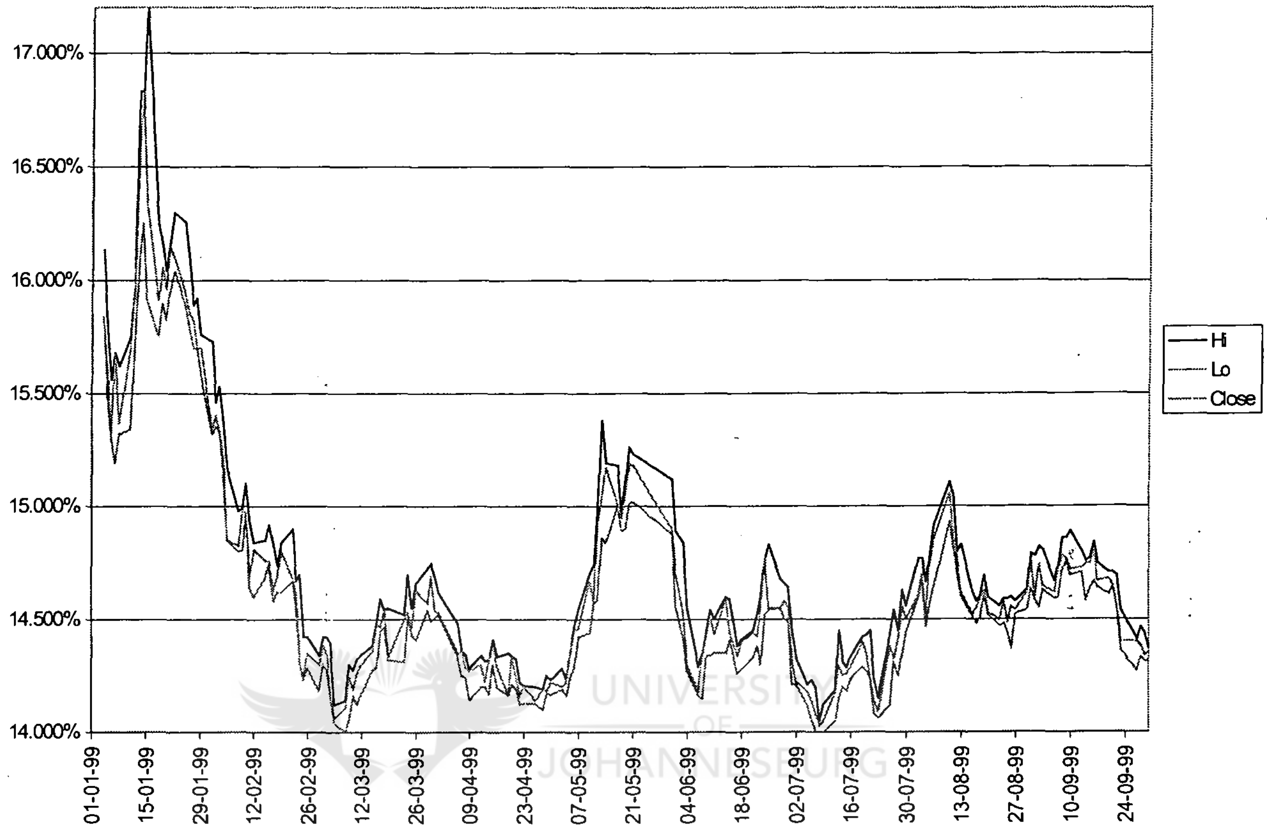
n = the number of observations in the sampling horizon

Further to this one must also consider which rate to observe. Should one make use of the closing rate, or some other alternative. This problem is clearly illustrated in the following graph, where additional to the closing rate, the highest and lowest rates are also included. It is clear from this that in the case where only the closing rate of the day is being used, large peaks as experienced on would not be taken into account. These peaks can represent a substantial risk factor to the institutions involved in options.

The essence for the determining of the volatility, is in order to calculate the risks involved, and to implement an adequate hedging strategy. Although an option can be synthesised using dynamic hedging strategies that maintains delta neutrality, there will be hedge slippage due to lack of convexity in the synthetic option. In an efficient market, the losses thus accumulated will approximately be equal to the option purchase premium (Leong, p83). The more volatile the market, the higher the expected market losses for a synthetic option, and thus a higher premium would be demanded for and option. In this context, the most accurate estimate of volatility is the one that reflects the propensity for hedge slippage for the synthetic option in question, the most accurately, over the time horizon in question.

Fig 3.2

Closing Prices for R150 Government Bond



(Source: De Witt Morgan Brokers)

In order to obtain a efficient adjustment of the hedging strategy, the sampling frequency for the volatility calculation must be adjusted to match the hedge adjustment and risk calculation frequency, thereby avoiding unnecessary adjustments to the portfolio. When working within the Black-Scholes world, the sampling error in the calculation of the variance can be minimised by using as much data as possible. Non stationarity would however cause the use of historical data to be of limited use, if not totally useless. A compromise between these two outlooks is to use only a limited amount of

recent data, thereby partly eliminating the non-stationary element within the volatility, while still offering the advantage of the fact that volatility are basically stable over a longer period of time, due to mean reversion. Volatility models based on these principles are the GARCH and moving average models.

Term structure of volatility.

The present tendency is to move away from assigning a specific volatility to options, irrespective of the time to expiration. Term structure of volatility assigns a different volatility to options of different tenors.

3.6.2 Interest Rate changes

As with volatility, interest rates also changes over time. In contrast to the volatility of a stock that can not be observed, the interest rates and the changes thereof can be observed. This fact makes it much easier to handle interest rates.

3.6.3 Borrowing penalties

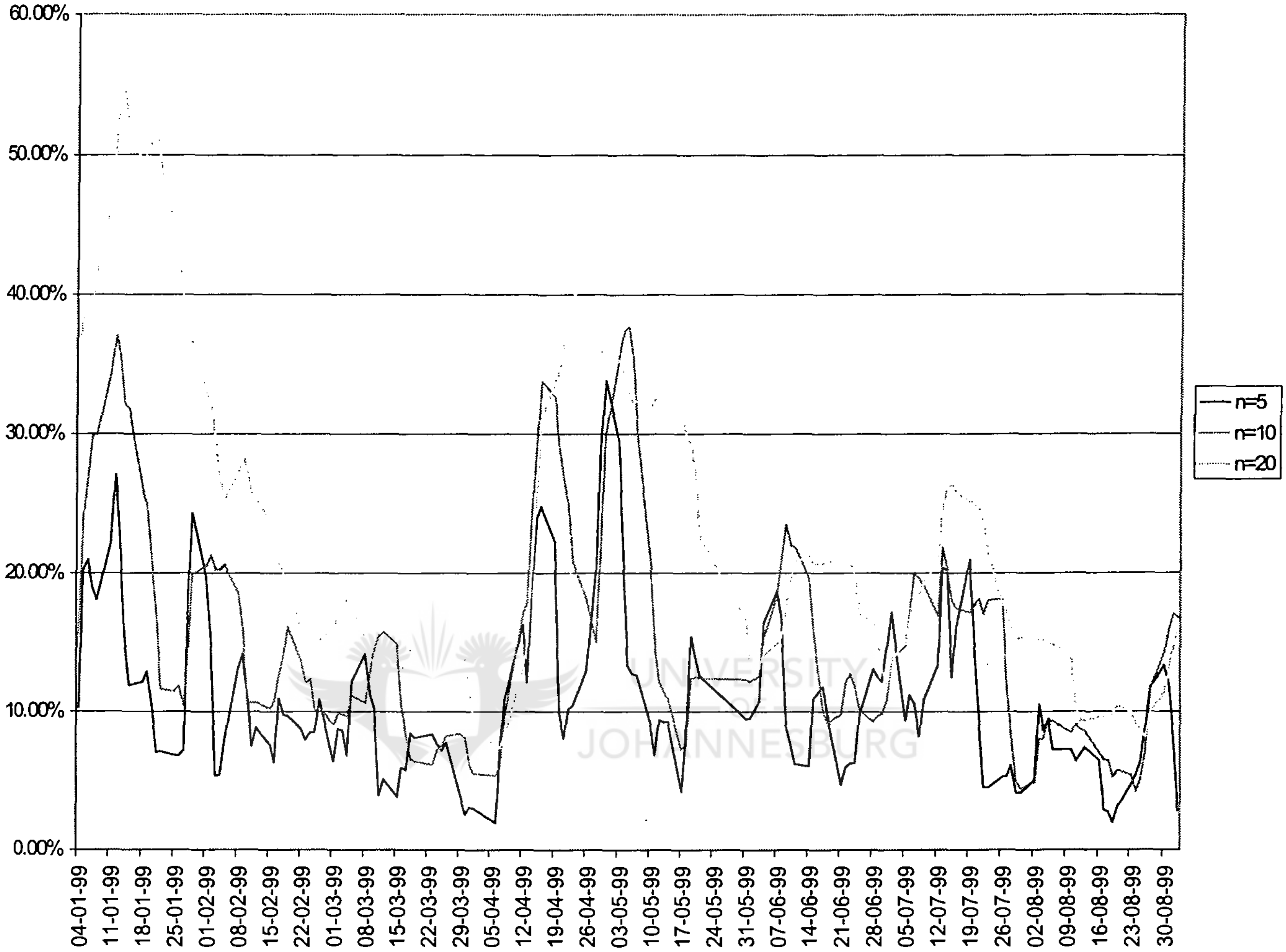
The rate at which investors can borrow, even with adequate collateral as security, is always less than the rate that he can lend. The high borrowing rates and the general limits on the amount that may be borrowed, may cause an increase in the general option values, since options provide leverage that may be used for borrowing.

3.6.4 Transaction costs

Outside investors will have to pay brokerage when trading in options and stock. Traders themselves are subjected to floor and trading levies, and the cost of being a member of the exchange. These charges are sometimes substantial compared to the possible profits that can arise from opportunities that may arise from mispriced options.

Fig 3.3

Volatility calculations for Various Values of n



Due to these costs it is impossible to maintain a neutral hedge continuously, thereby eliminating the rapid changing of the ratio between the stock position and the option position.

The fact that prices may also suddenly gap from a certain level to the other, thereby not offering the opportunity to trade at certain levels, also eliminate the possibility to maintain a perfect neutral hedged portfolio.

3.6.5 Coupon payments

The original Black-Scholes formula does not take the payment of coupons into account. These payments would reduce the value of the call options, and increase the value of the put options, if there is no offsetting adjustment in terms of an option. Coupon payments make early exercise of a call option more likely, and they make early exercise of a put option less likely (Black, p55)

Subsequent to the Black-Scholes development, several other models have been developed, taking these payments into account.

3.6.6 Quantification of time

All option pricing models utilises time to expiry to value the option. The time specified must however be determined correctly, and this is easier said than done. This is due to the fact that the models simplify time, and does not take weekends and holidays into account. During these times volatility is assumed to be homogenous, which could lead to errors in the model.

There are three different ways to quantify time to expiration:

Calendar time: the actual number of days to expiry.
Trading time: The number of trading days to expiry.
Economic time: The number of days on which significant economic data are to be released.

The choice of time horizon used would obviously have a huge impact on the value of the option. Calendar time is mostly used in most commercial software available.

3.7 Conclusion

Black-Scholes have, through the elimination of all non-measurable quantities, succeeded to derive a relatively uncomplicated formulation for the pricing of options. Through the introduction of the synthetic portfolio, the single biggest obstacle namely the level of risk associated with the option, has been identified, quantified and eliminated. The assumptions made in the process does however lead to inaccuracies, especially over longer time horisons, and with strike levels far away from the present strike prices.

Heteroshcedasity in the normal distribution of the rate movements can have a severe distortion effect on longer dated options. These smiles and skews in the distribution should be taken into account when pricing of options, and canto a certain extent be neutralised through manipulation of the volatility use in the pricing model.

Despite these obvious pitfalls in the option pricing models, they are, due to their simplicity, still widely used within the financial markets. Later models that were developed still used these models as their basis, although most attempted to rectify some of these problems.

One must however except that despite the accuracy of these models in the pricing of the options, market will ultimately still determine the price of the option through normal price action of supply and demand. The pricing formulae does however offer the option writer the possibility to gain a clear indication of what the inherent value of an option should be, and also offer him with the necessary instruments in order to hedge the exposure on the position taken by him. This is closely related to the risk management of the position, and will form the basis for the discussion in chapter 4.

CHAPTER 4

RISK MANAGEMENT OF OPTIONS UTILISING VALUE AT RISK

4.1 Introduction

Risk management has in the past few decades develop side by side with the development of the computer technology. As the average computing power available to the risk manager has increased, so has his ability to do up to the minute analysis of the risk the company is exposed to, increased. Where previously a position was risk evaluated only every couple of days, it is now possible to evaluate the risk as and when the position alters.

The risk management technique most frequently utilised by most companies in order to ascertain what their exposure to price risk is, is at present based on the Value at Risk (VAR) technique. VAR measures the maximum potential change in the value of a portfolio of financial instruments with a given probability over a pre-set time horizon. In essence VAR answers the question: How much can the portfolio loose with x% probability over a certain time horizon? The calculations are based on standard deviations and correlation of different financial instruments.

VAR is a number that represents the potential change in a portfolio's future value. How this change is defined depends on:

- The time horizon over which the change in the portfolio's change is measured
- The degree of confidence chosen by the risk manager.

Suppose the risk manager needs to compute the VAR of a portfolio over a 1-day horizon, and the calculation must be within a 5% confidence level. The VAR calculations would consist of the following: (Longerstaey, 1996)

Mark to market the current portfolio = V_o

Define the possible future value of the portfolio as V_1, where $V_1 = V_o e^r$ where r represents the expected return of the portfolio over the time horizon. Should this time horizon be 1 day, it can be assumed that the return be so small, and can be set to zero.

Make a forecast of the 1-day return on the portfolio and set this value to r. Chose a probability such that there is a 5% chance that the actual return will be less than r. This can be expressed as: (Longerstaey, 1996)

$$\text{Prob}(r_A < r) = 5\%$$

If the portfolio's future worst case value is equal to V_1, as $V_1 = V_o e^r$, the VAR can simply be defined as $V_o - V_1$.

Calculating VAR usually involves the decomposing of the instruments into their basic cash flow instruments. The different VAR of these different cash flows are then individually calculated, in order to arrive at the combined VAR of the instrument.

4.2 Measuring the risk of Non linear instruments

When the correlation between the value of the position and the market rates are nonlinear, determining the VAR is not just a simple multiplication of the estimated change in the rates by the sensitivity of the position. In order to calculate this relationship, a measure of the convexity must be taken into account. This analysis can be done using any of two methodologies, namely:

4.2.1 Analytical approximation

The nonlinear relationship is approximated with the aid of a mathematical expression that relates the return on the position to the return on the underlying rates. This is accomplished using a Taylor series expansion.

This approach not only makes use of the delta of the option, but also the gamma, which is the first derivative of the delta, and is commonly referred to as the Delta-Gamma approach.

4.2.2 Structured Monte Carlo simulation

This approach involves the creation of a large number of possible rate scenarios, and applying these rates to revalue the instruments under the different scenarios. VAR is then defined as the 5th percentile of the distribution of the changes in value.

These two approaches differ not only in their approach, but also in their accuracy. The analytical approach uses an approximation in determining the value of the portfolio, while the Monte Carlo simulation accurately evaluates the value of the portfolio.

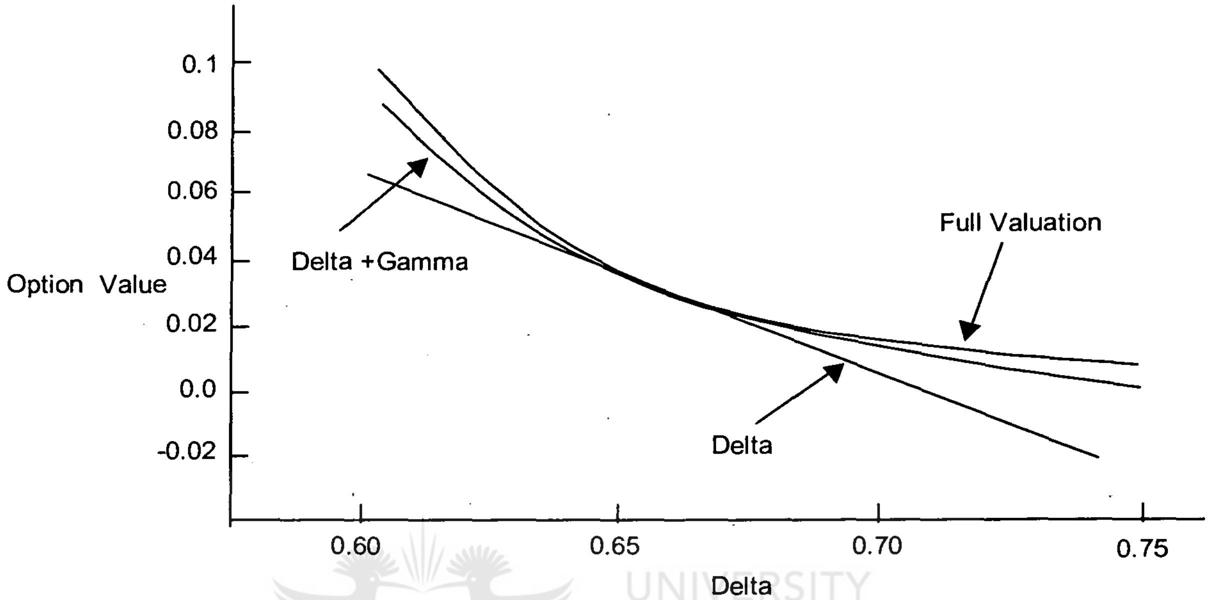
The accuracy of the two approaches can graphically be represented as in Fig. 4.1

4.3 Delta Gamma VAR calculation

Since the pay-off profile for an option is a non-linear function, variables helping to estimate the curvature of the pay-off profile needs to be incorporated in the VAR calculation of an option in order to be more accurate. These variables are the delta, gamma, theta and the interest rate sensitivity rho, of the option, and are commonly referred to as the Greeks. The effect that the incorporation of these variables would have on the pay-off profile, and thus the return on the option portfolio, can be mathematically calculated. This discussion merely serves as a basic background for the detailed Monte Carlo simulation in the next section, and would not be looked at in detail.

Fig 4.1

Graphic Comparison of Delta and Delta Gamma Approach



(Source: Zangari, 1996)

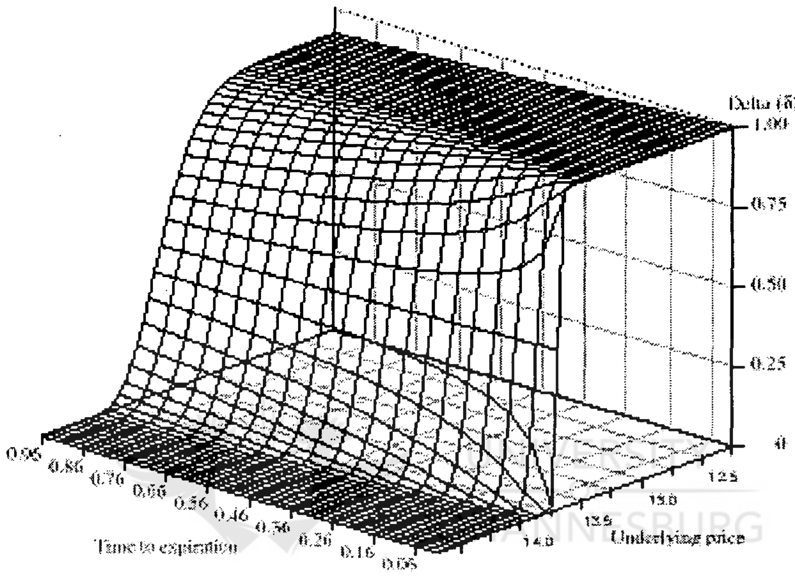
Fig 4.2 and Fig 4.3 is a graphical illustration of the delta and gamma effect as the option nears maturity. It can clearly be seen that the slope of the delta curve steepens as the maturity date is reached, and the rates are close to the strike rate. This effect can also be clearly seen in the price action of the bonds in the days leading up to option close out. The price movements are much more severe, and quickly move between the different strike levels. This can be attributed to the steeper gradient of the curve.

The gamma curve displays what is commonly know as the gamma spike. The gamma's influence becomes more of an influence as the option maturity date comes closer, and the rates are close to the strike rate. Since gamma is the first derivative of delta, this sudden increase in the gamma value is to be expected.

The simplest form of option portfolio, is a single option portfolio. In order to predict the return on the option as accurate as possible, a Taylor series expansion is utilized to calculate this return: (Zangari, 1996)

Fig 4.2

Change in Delta of an Option



(Source: Zangari, 1996)

$$V_{t+n} = V_t + \delta\,(P_{t+n} - P_t) + 0.5\Gamma(P_{t+n} - P_t)^2 + \sigma(\tau_{t+n} - \tau_t) \quad \ldots 4.1$$

This expression can be rewritten as:

$$V_{t+n} - V_t = \delta\,(P_{t+n} - P_t) + 0.5\Gamma(P_{t+n} - P_t)^2 + \sigma(\tau_{t+n} - \tau_t) \quad \ldots 4.2$$

Which, through manipulation, the value of the option and the underlying in relative terms can be rewritten as:

$$R_v = \eta\delta R_p + 0.5(\alpha\Gamma P_t)(R_p)^2 + (\theta/V_t)n \quad \ldots 4.3$$

where

$R_v = (V_{t+n} - V_t)/V_t$

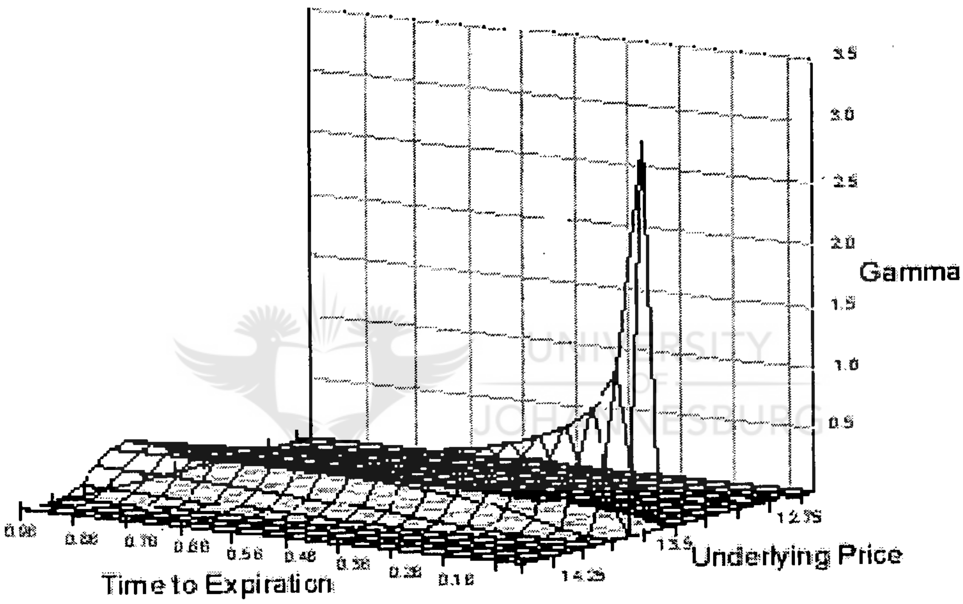
$R_p = (P_{t+n} - P_t)/P_t$

$n = (\tau_{t+n} - \tau_t)$

$= (P_t/V_t)$

Fig 4.3

Change in Gamma of an Option



(Source: Zangari, 1996)

Or alternatively it can be stated as:

$$R_{opt,v} = \delta' R_p + 0.5\Gamma'(R_p)^2 + \theta'(\tau_{t+n} - \tau_t) \qquad \ldots 4.4$$

Where

$\delta' = \eta_1\delta_1$

$T'_1 = n_1 P_{opt,t}\Gamma_1$

$\theta'_1 = \theta_1/V_1$

N = VaR forecast horizon

V_1 = Option's premium

This approximation is accurate enough for VaR calculations, when the Greeks are stable when the underlying price changes. (Zangari, 1996)

In order to demonstrate the effect that gamma and theta has on the return distribution of an option, consider the statistical parameters of a single option in table 4.1:

Table 4.1

Statistical Parameters

Statistical Parameters	Option	Underlying
Return	$r_{opt,t}$	r_t
Mean	$0.5\Gamma'\sigma^2_t + \theta'n$	0
Variance	$\delta'^2\sigma^2_t + 0.5\Gamma'^2\sigma^4_t$	σ^2_t
Skewness	$3\delta'^2\Gamma'\sigma 4_t + \Gamma'^3\sigma^6_t$	0
Kurtosis	$12\delta'^2\Gamma'^2\sigma^6_t + 3T'^4\sigma^8_t + 3\sigma^4_t$	$3\sigma^4_t$

(Source: Zangari, 1996)

In order to determine the numerical values of the moments presented above, an accurate estimate for the values of δ', Γ', θ' and σ^2_t is needed. The first three are easily calculated with the aid of the Black-Scholes valuation model, while σ^2_t can be derived from historical data. Having obtained the four moments of $r_{opt,t}$ distribution, a distribution that matches the moments and of which the distribution is exactly know, must be found. This must be matched to one of the distributions know as Johnson distributions. (Zangari, 1996)

In order to match the moments to a family of distributions, it is required to transform the options return; $r_{opt,t}$; to a return r_t that has a standard normal distribution. Johnson suggested the general transformation:

$$r_t = a + bf\{(r_{opt,t} - c)/d\} \qquad ...4.5$$

where a,b,c and d are parameters that are determined by $r_{opt,t}$, and f() is a monotonic function. The values of a,b,c, and d can be calculated with the aid of the modified Hill, Hill and Holder's algorithm.

Through this transformation, the effect of the incorporation of gamma and theta on the distribution can be mathematically calculated.

The Monte Carlo simulation would include a comparison of the delta and delta-gamma analysis, but these results would not be mathematically verified.

4.4 The Monte Carlo simulation

The Monte Carlo can be used to calculate the risk of a portfolio containing options, which might not be amenable to an mathematical analysis due to the complexity of the non-linear pay off profiles associated with options. This method produces an estimate for the portfolio as a whole, and does not break each instrument into its various cash flow components.
The Monte Carlo simulation consists of three major steps: (Finger, 1996)

- Scenario generation
  Using the volatility estimates for the underlying asset, a large number of future price scenarios are generated through the use of log normal random variables

- Portfolio Valuation.
  The value of the portfolio is computed for each future price scenario generated.

- Summary

The risk measure of the simulation is reported, either as a portfolio distribution, or as a particular risk measure.

4.5 Scenario Generation

Consider a forecast horizon for the simulation of duration of t days. If the price of the underlying instrument is given by P_o, and the estimated volatility of the instrument for the one day to be σ, then the price of the instrument can be mathematically represented by:

$$Pt = P_o e^{\sigma\sqrt{t}Z} \qquad \ldots 4.6$$

where Z is a lognormal random variable. In order to generate scenarios, it is necessary to generate lognormal variables.

To illustrate the Monte Carlo simulation, consider the data set of the option as outlined in Table 4.2:

Table 4.2

Data of Monte Carlo Simulation

Description	Value
Spot Rate	13.50%
Time horizon	7 days
Volatility for time horizon	1.2% over period
No. of Future prices generated	2500 future prices

(Source: Simulation)

The data represented in Table 4.3 are for the obtained from the randomly generated prices for the future, based on the above criteria:

Table 4.3

Statistical Data of Monte Carlo Simulation

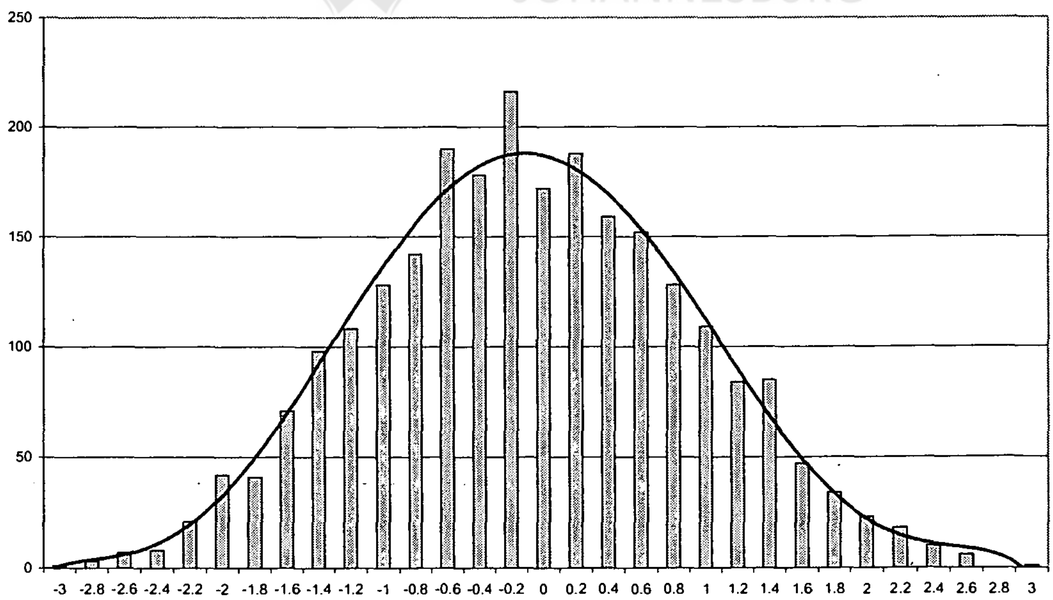
Description	Value
Max future price	15.54 %
Min future price	11.53 %
Average Rate	13.52 %
Mean	13.49 %
Std Dev	0.64 %

(Source: Simulation)

The prices obtained through the simulation; already standardized; is graphically represented in Fig 4.4 and tabled in table 4.4.

Fig 4.4

Frequency Distribution of the Monte Carlo Simulation



(Source: Simulation)

Table 4.4

Standard Deviation, Range and Frequency of Monte Carlo Simulation

Std Dev		Range		Frequency
-3	-2.8	11.57%	11.70%	1.00
-2.8	-2.6	11.70%	11.83%	3.00
-2.6	-2.4	11.83%	11.95%	7.00
-2.4	-2.2	11.95%	12.08%	8.00
-2.2	-2	12.08%	12.21%	21.00
-2	-1.8	12.21%	12.34%	42.00
-1.8	-1.6	12.34%	12.47%	41.00
-1.6	-1.4	12.47%	12.60%	71.00
-1.4	-1.2	12.60%	12.73%	98.00
-1.2	-1	12.73%	12.86%	108.00
-1	-0.8	12.86%	12.98%	128.00
-0.8	-0.6	12.98%	13.11%	142.00
-0.6	-0.4	13.11%	13.24%	190.00
-0.4	-0.2	13.24%	13.37%	178.00
-0.2	0	13.37%	13.50%	216.00
0	0.2	13.50%	13.63%	172.00
0.2	0.4	13.63%	13.76%	188.00
0.4	0.6	13.76%	13.89%	159.00
0.6	0.8	13.89%	14.02%	152.00
0.8	1	14.02%	14.14%	128.00
1	1.2	14.14%	14.27%	109.00
1.2	1.4	14.27%	14.40%	84.00
1.4	1.6	14.40%	14.53%	85.00
1.6	1.8	14.53%	14.66%	47.00
1.8	2	14.66%	14.79%	34.00
2	2.2	14.79%	14.92%	23.00
2.2	2.4	14.92%	15.05%	18.00
2.4	2.6	15.05%	15.17%	10.00
2.6	2.8	15.17%	15.30%	6.00
2.8	3	15.30%	15.43%	0.00
3	3.2	15.43%	15.56%	1.00

(Source: Simulation)

4.5.1 Portfolio Valuation

Taking the prices generated with the aid of the simulation, we can now consider the portfolio specified in Table 4.5:

Table 4.5
Portfolio under Consideration

Description	**Value**
Type of Option	American Call on R150
Transaction Date	3 February 2000
Expiry Date	4 May 2000
Strike Rate	13.50 %
Time to Expiry	3 months
Risk Free Rate	10.50 %
Volatility	7.5 %
Delta	
Gamma	
Theta	

4.5.2 Summary

The value of the portfolio 7 days from the transaction date can now be calculated, using the delta and the delta-gamma valuation, and these can be compared to the full valuation using the Black-Scholes option pricing formula. This is tabulated in Table 4.6.

Using the data obtained in the simulation, the different value at risk parameter can now be read from the processed data. Calculating the 5% worst case loss over the 7 day period, is simply done by choosing the 125^{th} point (5% x 2500 points), and reading the calculated value from there. The different percentiles that are of importance to the risk manager are summarized in table 4.7.

Table 4.6

Statistical Data of Portfolio Under Consideration

Std Dev		Range		Frequency	Average	Full Valuation	Delta	Delta-Gamma
-3	-2.8	11.57%	11.70%	1.00	11.63%	R67,221	R42,587	R67,288
-2.8	-2.6	11.70%	11.83%	3.00	11.76%	R62,391	R40,437	R62,449
-2.6	-2.4	11.83%	11.95%	7.00	11.89%	R57,591	R38,287	R57,641
-2.4	-2.2	11.95%	12.08%	8.00	12.02%	R52,820	R36,136	R52,862
-2.2	-2	12.08%	12.21%	21.00	12.15%	R48,078	R33,986	R48,113
-2	-1.8	12.21%	12.34%	42.00	12.28%	R43,365	R31,836	R43,394
-1.8	-1.6	12.34%	12.47%	41.00	12.41%	R38,681	R29,686	R38,704
-1.6	-1.4	12.47%	12.60%	71.00	12.53%	R34,025	R27,536	R34,043
-1.4	-1.2	12.60%	12.73%	98.00	12.66%	R29,398	R25,385	R29,412
-1.2	-1	12.73%	12.86%	108.00	12.79%	R25,514	R23,235	R25,524
-1	-0.8	12.86%	12.98%	128.00	12.92%	R22,401	R21,085	R22,408
-0.8	-0.6	12.98%	13.11%	142.00	13.05%	R19,516	R18,935	R19,520
-0.6	-0.4	13.11%	13.24%	190.00	13.18%	R16,865	R16,785	R16,867
-0.4	-0.2	13.24%	13.37%	178.00	13.31%	R14,452	R14,634	R14,453
-0.2	0	13.37%	13.50%	216.00	13.44%	R12,276	R12,484	R12,276
0	0.2	13.50%	13.63%	172.00	13.56%	R10,334	R10,334	R10,334
0.2	0.4	13.63%	13.76%	188.00	13.69%	R8,617	R8,184	R8,618
0.4	0.6	13.76%	13.89%	159.00	13.82%	R7,117	R6,034	R7,119
0.6	0.8	13.89%	14.02%	152.00	13.95%	R5,820	R3,883	R5,824
0.8	1	14.02%	14.14%	128.00	14.08%	R4,711	R1,733	R4,717
1	1.2	14.14%	14.27%	109.00	14.21%	R3,774	(R417)	R3,784
1.2	1.4	14.27%	14.40%	84.00	14.34%	R2,991	(R2,567)	R3,005
1.4	1.6	14.40%	14.53%	85.00	14.47%	R2,346	(R4,717)	R2,364
1.6	1.8	14.53%	14.66%	47.00	14.59%	R1,819	(R6,868)	R1,842
1.8	2	14.66%	14.79%	34.00	14.72%	R1,395	(R9,018)	R1,424
2	2.2	14.79%	14.92%	23.00	14.85%	R1,058	(R11,168)	R1,093
2.2	2.4	14.92%	15.05%	18.00	14.98%	R793	(R13,318)	R835
2.4	2.6	15.05%	15.17%	10.00	15.11%	R588	(R15,468)	R638
2.6	2.8	15.17%	15.30%	6.00	15.24%	R430	(R17,619)	R489
2.8	3	15.30%	15.43%	0.00	15.37%	R311	(R19,769)	R379
3	3.2	15.43%	15.56%	1.00	15.50%	R223	(R21,919)	R300

(Source: Simulation)

Table 4.7

Portfolio Value using Greeks and Actual Valuation

	Portfolio future Value (R)					
Percentile %	Full		Delta		Delta-Gamma	
1.0	R	48,078	R	33,986	R	48,113
2.5	R	43,365	R	31,836	R	43,394
5.0	R	36,353	R	28,611	R	36,374
10.0	R	29,398	R	25,385	R	29,412
25.0	R	19,516	R	18,935	R	19,520
50.0	R	12,276	R	12,484	R	12,276
75.0	R	5,820	R	3,883	R	5,824
90.0	R	2,991	R	(2,567)	R	3,005
95.0	R	1,819	R	(6,868)	R	1,842
97.5	R	1,227	R	(10,093)	R	1,259
99.0	R	793	R	(13,318)	R	835

(Source: Simulation)

4.6 Conclusion

Since the markets can be described as a stochastic process where the rates is a variable independent from the previous rates change, risk management of options can be done through a Monte Carlo simulation, provided the random variable generates, are independent and normally distributed. This single feature makes the use of long mathematical calculations unnecessary. The rapid advance of computing technology during the last two decades has brought this on.

Since the Black-Scholes pricing formula can easily be implemented on a computer, the use of the Delta gamma approach can be eliminated, and the contribution of the option to the portfolio as a whole, evaluated. Risk management must however not just be a matter of number crunching, but the underlying principles to the end value must be clearly understood.

Risk management with the aid of value at risk, is a means of stipulating certain risk parameters within which a portfolio must be, and also provides the risk manager with the means to monitor these parameters on a daily, and lately on an hourly basis.

CHAPTER 5

SUMMARY AND CONCLUSION

5.1 Introduction

Chapter 2 discussed the basic principles underlying of the two major option pricing formulae. It clearly showed that two totally different approaches were followed in each case, and yet both arrived at approximately the same value for the price of an option. Both these approaches made certain assumptions in their derivation of the formulae in order to simplify the final expressions, and to produce a more workable solution. They both however made substantial use of statistical probability in order to determine the likelihood of a certain event occurring.

Chapter 3 gave a detailed derivation of both the Black and Scholes and the Binomial tree pricing formulae, as well as the associated criticism and advantages of the respective approaches.

Value at risk, or VaR, was used in determining the statistical probability of a certain portfolio consisting of a specified option losing more than a certain percentage of its value over a given period of time. The resulting number obtained can be used to judge the riskiness of a portfolio in the given market conditions.

All of these formulae are used on a daily basis by financial professionals in the daily operations of a magnitude of different institutions in order to value financial portfolios, the risk associated with these portfolios and the probability of certain events occurring within the portfolios in order to make better decisions and increase the profitability of these institutions, without actually knowing the underlying principles. As such these formulae merely become a number crunching business, and interpretation of these numbers, without realising the pitfalls associated with the approaches in establishing these formulae.

5.2 Simple Random Walk and Wiener Process

The random walk theory for unrestricted movement assumes that at t=0, the rates are at the origin. This can be interpreted as 0%, and instinctively any person would agree that 0% is not possible in any fixed income environment, due to the time value attached to money. Choosing the ruling rate as the origin would be more practical in determining the origin, but care must be taken in assigning probabilities to the up and down movements. At the onset of the problems amongst the emerging markets during 1998, the probability of rates increasing once it reached 17,00% was much higher than that of the rates decreasing. However, barely a month later when the rates had reached its peak at more than 21,00% and were declining again, the probability of the rates increasing once it reached 17,00% again was much lower than that of it decreasing further. This would have a significant effect on the probability generating function, and hence also an effect on the mean and variance thus derived. The probability curve of the rates during these times were also not represented by a standard normal curve, and as such the heteroscedacity of the curve had a major influence on the pricing of options.

During extreme periods both the random walk theory and the Wiener process would be totally skewed, and unreliable answers would be derived from this approach. By adjusting the expression for a non-standard distribution, these problems can be eliminated and an accurate approach once again obtained using this process. Problems that could occur when using this approach to solve inaccuracies would amongst others include the following:

- The incorrect distribution function is being applied for the specific set of conditions prevailing in the market. This is due to the fact that under these abnormal conditions the distribution function can change over a very short period of time.
- Incorrect skews being applied to the distribution function due to fast changing market conditions.
- When to revert back to the normal distribution function.
- It then becomes a question not of an improper analytical approach, but incorrect timing approach.

Since markets mostly perform according to the standardised normal distribution function the Wiener approach hold true for most applications. During periods of increased volatility, adjustments must be incorporated for the uncertainties, and special care must be taken not to utilise the approaches for applications stretching to far into the future.

5.3 Black and Scholes

Fisher Black and Myron Scholes were the first to make use of quantitative differential algebra to quantify the value of an option, utilising a neutrally hedged portfolio consisting of -1 derivative and $\partial f/\partial S$ bonds. By definition, the portfolio is riskless during a small time period Δt. From this principle, through setting of boundary conditions, the Black-Scholes formula for an European call and put option can be obtained.

The assumptions made by Black and Scholes in the derivation of the formula were the following:

- The volatility over the period of the option is known and remains stable.
- Short term interest rates remains constant
- Borrowing and lending of money can be done at a single rate, and is not limited in the quantities that can be borrowed.
- Short selling of bonds would result in the investor having access to the money so obtained.
- There are no transaction costs.
- Trading does not influence the tax the investor pays.
- No coupons or dividends are paid over the period of the option.

These assumptions are unrealistic and far removed from reality. They were however made in order to simplify the expression to arrive at the initial formula. Subsequent pricing formulae based on the Black-Scholes principle have taken into account the varying nature of the markets, and have made provision for varying interest rates based on the forward curves quoted.

Volatility is however the single entity that cannot be projected and fixed for the future, and as such remains the single biggest source of uncertainty in the expressions. Several models such as ARCH and GARCH have been developed in order to try and establish a more accurate measure of volatility. These models are however just statistical models trying to predict an event in the future, and must be treated as such.

The initial Black-Scholes model, despite its obvious pitfalls and unrealistic assumptions, has paved the way for a whole industry of pricing formulae, and has provided the markets with the first scientific measure of the value of an option. This has led to more refined models adjusting for inaccuracies in the initial model, providing market professionals with accurate evaluations of the options they are trading.

5.4 Binomial Tree

The binomial tree assigns a probability to each of the up and down movement expected in the market. It is not dependent on a specific distribution curve, and as such seems to be the better approach to base option pricing on during times of increased volatility. This holds true for the pure binomial approach, but most option pricing software assigns an equal probability to the up and down movements. The reason for this is that these probabilities are just as subjective as pure future volatility. Under normal conditions the difference between these two probabilities are so small that it would not be noticed in the option price once the option price has been rounded to the closest hundred rand. Under extreme volatility this could however make this approach inaccurate if the probability skews cannot be incorporated into the approach.

The accuracy of the binomial tree however also depends on the size of the tree being utilised in the valuation process. With ever more powerful computing platforms being developed, the speed at which these calculations can be done is greatly reduced. This has made it possible to evaluated trees several hundred branches deep to be evaluated. It is thus more and more being used to determine the fair value of option with expiry dates further and further in the future, without modifications being made for some of the basic underlying assumptions. These assumptions are magnified the further the expiry date and strike rate is set from present time and market rates.

Early exercise becomes especially critical in long dated options that are in the money during periods of inverse yield curves. Inverse yield curves would make it cheaper to exercise the option and carry the instrument at a positive carry.

The similar paths assumption becomes more susceptible to horisontal movements in the rate. The rates could move sideways for periods of time, making it difficult to assign an accurate probability to each of the up and down movements in rates. The accuracy with which the option is determined thus becomes doubtful, and it is no longer possible to determine the fair value of the option. This becomes especially relevant for portfolio analysis, arbitrage trading and determining hedging strategies for different portfolios since the value of the various Greeks are directly influenced by this.

A development to try and address these inaccuracies has led to the development of the trinomial tree for option pricing, thereby addressing especially the sideways movement of rates.

5.5 Value at Risk

A number of institutions, including the Group of Thirty and ISDA have recommended that Value at Risk be used as the measure of risk in the markets. The reason for this is that Value at Risk is a quantification of the expected minimum loss (measured in currency units such as Rand) over some time interval for some level of probability selected by the decision maker.

As an example, a portfolio can describe its daily Value at risk as R 20 000 with a 95% probability. This means that over the next 24 hour period, there is a 5% chance that the value of the portfolio can decline by more than R 20 000. The value at risk statement not only describes the probability of losses, but also the portfolio managers aversion to losses. The more conservative the probability levels chosen, the more averse the manager is to losses and risk taking as such.

Value at risk has been developed to measure the levels of risk associated with portfolios. It can be used to determine expected losses, but actual losses can vary greatly from the expected losses calculated. These calculations are all based on standard probabilities of distribution, i.e. standard deviation. As with the Wiener process, the distribution of the movements in the market for the calculation of Value at Risk, is assumed to be normal. The ease with which standard deviations can be calculated, and the wealth of information available from standardised normal curves, has all contributed to the popularity of Value at Risk.

The utility curve of the portfolio has a direct baring on the risk averseness of the portfolio. Managers can thus construct portfolios with specific risk curves to suite the investors need, and institutions can apply Value at Risk to their various portfolios in order to calculate the optimum utilisation of its resources at the lowest level of risk. In order to make a more informed decision, the Value at Risk of the portfolio can be calculated for different probability levels, thereby allowing the decision maker the opportunity to visualise the underlying distributions more accurately.

In order to evaluate portfolios containing various instruments, the portfolio is divided into the different classes of instruments contained in the portfolio. The Value at Risk, and thus the probability of each of these classes of instruments, weighted by portfolio representation, are then calculated separately and summed. This summed result then represents the Value at Risk for the entire portfolio.

Portfolios containing options are relatively difficult to analise mathematically. One way of simplifying the analysis is by constructing a pay-off profile similar to that of an option utilising the different spot instruments in the market. This would generate the same cash flows as an option, albeit with a greater number of instruments rolled into a single package. The Value at Risk for these various instruments are then calculated and summed, giving the total Value at Risk for the option.

With the advent of more powerful computing platforms, the possibility of an accurate analysis using a Monte Carlo simulation offers and alternative to the mathematical approach. A matrix of possible situations would be entered, thereby simulating the most probable events occurring in the markets over the next chosen time period. The answer thus obtained would yield the same answer as the mathematical analysis, without actually breaking the portfolio into its various components, thereby eliminating any possibility of incorrect simulation of for example options. The Monte Carlo simulation also takes into account the non linearity of options, making adjustments for it in its pricing.

5.6 Conclusion

Despite the obvious problems in obtaining a quantitative method for the valuation of options and their associated risk, most of these pitfalls can be eliminated by proper modeling of the factors influencing these models. Volatility is most probably the most inaccurate factor incorporated into the models, and as such needs careful study in the valuation of an option. Historical data coupled to statistical probability functions of future price movements, can bring down the hit and miss approach adopted by so many institutions, and have given rise to models such as GARCH and ARCH for the determination of volatility.

Despite this single factor being able to influence not only the pricing of the option, but also the risk management of the portfolio to a great extend, very accurate valuations of option and their associated risk can be obtained for short periods in the future.

The Black-Scholes and Binomial models has thus succeeded in providing us with ways of effectively pricing these instruments, and Value at Risk with a quantitative method of expressing the risk associated to these instruments.

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