On the chromatic number of commutative rings with identity

by

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THESIS

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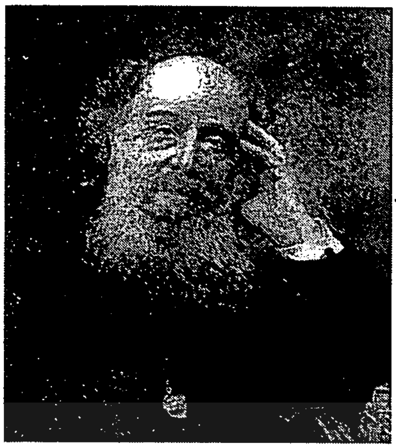
Mathematics is not a book confined within a cover and bound between brazen clasps, whose contents it needs only patience to ransack; it is not a mine, whose treasures may take long to reduce into possesion, but which fill only a limited number of veins and lodes; it is not a soil, whose fertility can be exhausted by the yield of succesive harvests; it is not a continent or an ocean, whose area can be mapped out and its contour defined: it is limitless as that space which it finds too narrow for its aspirations; its possibilities are as infinite as the worlds which are forever crowding in and multiplying upon the astronomer's gaze; it is as incapable of being restricted within assigned boundaries or being reduced to definitions of permanent validity, as the consciousness, the life, which seems to slumber in each monad, in every atom of matter, in each leaf and bud and cell, and is forever ready to burst forth into new forms of vegetable and animal existence.



James Joseph Sylvester (1814–1897)

It is difficult to give an idea of the vast extent of modern mathematics. The word 'extent' is not the right one: I mean extent crowded with beautiful detail — not an extent of mere uniformity such as an objectless plain, but of a tract of beautiful country seen at first in the distance, but which will bear to be rambled through and studied in every detail of hillside and valley, stream, rock, wood, and flower. But, as for every thing else, so for a mathematical theory — beauty can be perceived but not explained.

Arthur Cayley (1821–1895)

Summary

This thesis is concerned with one possible interplay between commutative algebra and graph theory. Specifically, we associate with a commutative ring R a graph and then set out to determine how the ring's properties influence the chromatic and clique numbers of the graph.

The graph referred to is obtained by letting each ring element be represented by a vertex in the graph and joining two vertices when the product of their corresponding ring elements is equal to zero.

The thesis focuses on rings that have a finite chromatic number, where the chromatic number of the ring is equal to the chromatic number of the associated graph. The nilradical of the ring plays a prominent role in these investigations.

Furthermore, the thesis also discusses conditions under which the chromatic and clique numbers of the associated graph are equal. The thesis ends with a discussion of rings with low (≤ 5) chromatic number and an example of a ring with clique number 5 and chromatic number 6.

Opsomming

Hierdie skripsie is gemoeid met een moontlike interaksie tussen kommutatiewe algebra en grafiekteorie. Meer spesifiek neem ons 'n kommutatiewe ring, R, en assosieer hiermee 'n grafiek. Ons bepaal dan hoe die eienskappe van die ring die chromatiese- en kliekgetalle van die grafiek beïnvloed.

Die grafiek waarna verwys word, word verkry deur met elke ringelement 'n punt in die grafiek te assosieer en twee punte in die grafiek te verbind as hulle ooreenstemmende ringelemente se produk nul is.

Die skripsie fokus veral op ringe wat 'n eindige chromatiese getal het, waar die chromatiese getal van die ring gelyk is aan die chromatiese getal van die geassosieerde grafiek. Die nilradikaal speel 'n baie belangrike rol in die verband.

Verder ondersoek die skripsie voorwaardes waaronder die chromatiese- en kliekgetalle van die geassosieerde grafiek gelyk is. Die laaste deel van die skripsie word gewy aan 'n bespreking van ringe met 'n lae (≤ 5) chromatiese getal en 'n voorbeeld van 'n ring met kliekgetal 5 en chromatiese getal 6 word ook gegee.

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Chapter 1

Introduction and background

THE aim of this thesis is to investigate the possible connections that exist between commutative ring theory and graph theory. To a large extent there are not any deep connections with graph theory (yet) and the only graph theoretic tools that are used are a few basic definitions. Thus this thesis is largely algebraic in nature.

After a brief introduction to the thesis, we discuss the background material necessary to be able to read this thesis.

1.1 Introduction

Throughout this thesis all rings will be commutative with identity. A basic reference for ring theory is [11]. The references we found the most useful for commutative rings were [2], [12] and [13]. A good graph theory reference is [4].

We begin by associating with a ring, R, a graph. Every element of the ring becomes a vertex in our graph and two (different) vertices are adjacent if the product of the corresponding (different) ring elements are zero. Specifically then, every nonzero element is adjacent to zero. Note also that our graph will be a simple graph (in contrast to a multigraph), meaning that no loops or multiple edges will be present in the graph. Once we have the graph, we next consider the *chromatic number*, $\chi(R)$, of the graph (or the ring for that matter). This is defined to be the smallest number of colours that can be assigned to R in such a way that adjacent elements have different colours. The colours are usually denoted by integers. Another concept borrowed from graph theory and one that will feature quite often in the sequel is that of a *clique*. A clique is a set of vertices (or ring elements) such that every two vertices from the set are adjacent. From this follows the concept of the *clique number*, $\omega(R)$, of the ring. This is the size (number of vertices) of the largest clique in R.

The whole thesis is concerned with the interplay between ring theoretic properties and the chromatic number of rings. We will see that for certain classes of rings we have that $\chi(R) = \omega(R)$. (We always have $\chi(R) \geq \omega(R)$ — every element in a clique must receive a different colour since it is adjacent to every other element in the clique so that we cannot colour the ring with fewer than $\omega(R)$ colours.)

We now discuss the necessary background material, starting with the terminology. All theorems are given without proof, but we do give complete references to works where the proofs may be located.

1.2 Terminology

As stated R will denote a commutative ring with identity. We will write the nilradical of R as $\mathfrak{B}(R)$. Note that $\mathfrak{B}(R)$ usually denotes the prime radical of the ring R. In the case of a commutative ring, the nilradical and prime radical are equal.

R is *reduced* if $\mathfrak{B}(R) = (0)$. If A and B are subsets of R, then we define $A : B = \{r \in R \mid rB \subseteq A\}$. Further $0 : A = \mathrm{Ann}A$ and if A consists of one element, say x, we write $0 : x = \mathrm{Ann}x$; these ideals are termed *annihilators*.

The set of zero divisors of R will be denoted by $\mathfrak{Z}(R)$. A prime ideal, $\mathfrak{p}$, will also be called an associated prime ideal if $\mathfrak{p} = \mathrm{Ann}x$ for some x in R.

A *finite chain* of prime ideals of a ring R is a finite strictly increasing sequence,

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n,$$

of prime ideals. The *length* of the chain is n. The *dimension* [2] of R is defined to be the supremum of the lengths of all chains of prime ideals, not equal to R, in R and is written as $\dim R$.

The cardinality of a set I will be denoted by $|I|$.

An arbitrary ideal will normally be denoted by a captial letter I, set in the Fraktur typeface, that is as $\mathfrak{I}$ or by a capital letter J, set in the Fraktur typeface, $\mathfrak{J}$.

1.3 Noetherian rings

A *Noetherian Ring* is one in which the ascending chain condition (a.c.c.) on ideals holds. The following result gives equivalent conditions for a ring to be Noetherian.

Proposition 1.1 ([2, 13]). *Let R be a ring. Then the following three conditions are equivalent :*

1. *Every nonempty set of ideals in R has a maximal element.*
2. *Every ascending chain of ideals contains a finite number of ideals.*
3. *Every ideal in R is finitely generated.*

An ideal in a ring R is called a *radical ideal* if it coincides with its radical. Here the radical of an ideal $\mathfrak{I}$ is defined to be the intersection of all prime ideals containing $\mathfrak{I}$ [11, p. 64]. This radical is also denoted as $\mathfrak{B}(\mathfrak{I})$. We also have the following.

Theorem 1.2 ([11]). *An ideal $\mathfrak{q}$ in a ring R is a semi-prime ideal in R if and only if $\mathfrak{B}(\mathfrak{q}) = \mathfrak{q}$.*

Since $\mathfrak{B}(R)$ is semi-prime (intersection of prime ideals), we have that $\mathfrak{B}(\mathfrak{B}(R)) = \mathfrak{B}(R)$. Hence $\mathfrak{B}(R)$ is a radical ideal.

The following result may be found in [10].

Theorem 1.3 ([10]). *In a Noetherian ring every radical ideal has a unique irredundant representation as the intersection of a finite number of prime ideals.*

This in particular shows that in a Noetherian ring $\mathfrak{B}(R)$ is the intersection of a *finite* number of prime ideals.

1.4 Artinian and local rings

An *Artinian Ring* is one in which the descending chain condition (d.c.c.) on ideals holds. A ring R with exactly one maximal ideal, $\mathfrak{m}$, is called a *local ring* and will be written as $(R, \mathfrak{m})$.

The following results will be useful to us ([2] and [13]).

Proposition 1.4 ([2]). *In an Artinian ring R every prime ideal, not equal to R, is maximal.*

Theorem 1.5 ([2]). *A ring R is Artinian $\iff$ R is Noetherian and* $\dim R = 0$.

Theorem 1.6 ([2]). *An Artinian ring is uniquely (up to isomorphism) a finite direct product of Artinian local rings.*

Proposition 1.7 ([13]). *A ring R is local $\Longleftrightarrow$ the set of all the nonunits (i.e the elements that do not have multiplicative inverses) of R forms an ideal.*

Proposition 1.8 ([2]). *Let R be an Artinian local ring. Then the following are equivalent:*

1. *Every ideal in R is principal.*
2. *The maximal ideal $\mathfrak{m}$ is principal.*

1.5 Brauer's theorem

As the section title indicates, this section will be devoted to Brauer's Theorem [7]. We will not use the result itself anywhere in this thesis. The importance of this result lies in its method of proof which will be applied later on in this thesis. We therefore give the full proof of the theorem.

Recall that an ideal $\mathfrak{I}$ is considered to be nilpotent if there exists a positive integer n such that $\mathfrak{I}^n = (0)$. Also, an element r of a ring R is idempotent if $r^2 = r$.

In the proof of the theorem we will also need Hopkins' Theorem [7], which we state without proof.

Theorem 1.9 (**Hopkins' Theorem** [7]). *If R is left (right) Artinian, then every nil left (right) ideal is nilpotent.*

Theorem 1.10 (**Brauer's Theorem** [7]). *Let R be a left (right) Aritinian ring. Any nonnilpotent left (right) ideal in R has a nonzero idempotent element.*

Proof. Let $\mathfrak{I}$ be a nonnilpotent left ideal in R. Since R is left Artinian, the family of all nonnilpotent left ideals of R contained in $\mathfrak{I}$ has a minimal element, say $\mathfrak{I}_1$. Furthermore, $\mathfrak{I}_1$ is not a nil left ideal in R (if it is, Hopkins' Theorem would imply that it is nilpotent which we know it not to be).

Let a be a nonnilpotent element of $\mathfrak{I}_1$ (which we know exits, since $\mathfrak{I}_1$ is not a nil left ideal). Consider Ra. We have $Ra \subseteq \mathfrak{I}_1$, further Ra is nonnilpotent since $a^2 \in Ra$ and a^2 is nonnilpotent. (If a^2 was nilpotent, a would also be nilpotent, which is impossible.) Therefore $Ra = \mathfrak{I}_1$ by the minimality of $\mathfrak{I}_1$. In a similar manner $Ra^2 = \mathfrak{I}_1$. Thus $Ra = Ra^2$.

There exists an $a_1 \in Ra$ such that $a = a_1 a$ ($a \in \mathfrak{I}_1 = Ra$). Now $a_1^2 a = a_1 a = a$, therefore $(a_1 - a_1^2)a = 0$ and $a_1 - a_1^2 \in \{a\}_l \cap Ra$, where $\{a\}_l$ is the set of left annihilators of a.

Let $a_2 = a + a_1 - aa_1$. Then $a_2 a = a^2 + a_1 a - aa_1 a = a^2 + a - a^2 = a$. Also

$$\begin{aligned}(a_1 - a_1^2)a_2 &= a_1 a + a_1^2 - a_1 a a_1 - a_1^2 a - a_1^3 + a_1^2 a a_1,\\ &= a + a_1^2 - aa_1 - a - a_1^3 + aa_1,\\ &= a_1^2 - a_1^3.\end{aligned}$$

Since $a_2 a = a$, a_2 is not nilpotent : assume that a_2 is nilpotent, say $a_2^n = 0$. We now have $a_2^{n-1}(a_2 a) = a_2^{n-1}a$, so that $a_2^n a = a_2^{n-1}a = 0$. In the same way we get $0 = a_2^{n-1}a = a_2^{n-2}a = a_2^{n-3}a = \cdots = a_2^2 a = a_2 a = a$, implying $a = 0$ — this contradicts the fact that a is nonzero. Therefore $Ra_2 = Ra = \mathfrak{I}_1$ and

$$\{a_2\}_l \cap Ra \subseteq \{a\}_l \cap Ra.$$

The last equation follows from the fact that if $b \in \{a_2\}_l$, then $ba_2 = 0$ and since $a_2 a = a$, $ba_2 a = ba = 0$. Thus $b \in \{a\}_l$.

We now either have that $a_1^2 = a_1^3$ or $a_1^2 \neq a_1^3$. If $a_1^2 = a_1^3$, then

$$(a_1^2)^2 = a_1^3 a_1 = a_1^2 a_1 = a_1^3 = a_1^2,$$

so that a_1^2 is idempotent and we are done.

On the other hand, if $a_1^2 \neq a_1^3$, then $(a_1 - a_1^2)a_2 \neq 0$ and $a_1 - a_1^2 \notin \{a_2\}_l \cap Ra$. Thus $\{a_2\}_l \cap Ra \subset \{a\}_l \cap Ra$.

We can now repeat the process with a_2 playing the role of a. We then obtain elements $a_3, a_4 \in \mathfrak{I}_1$ such that either $a_3^2 = a_3^3$ or $a_3^2 \neq a_3^3$ and $\{a_4\}_l \cap Ra \subseteq \{a_2\}_l \cap Ra$. If $a_3^2 = a_3^3$, a_3^2 is the desired idempotent. If $a_3^2 \neq a_3^3$, then the containment above is strict. Therefore if an idempotenet cannot be obtained after a finite number of steps, we have an infinite descending chain of left ideals, contradicting the fact that R is left Artinian. The proof for right Artinian is analogous. □

1.6 Tensor product

The definition of the tensor product is taken from [8]. Let A_R and ${}_RB$ be fixed, right and left R-modules respectively. Consider the formal sums $\sum(a_i, b_i)$ where $a_i \in A$ and $b_i \in B$. If we ignore the order and association of the terms, then the (a_i, b_i)'s determine the sums uniquely. The formal sums, under the operation of concatenation, forms a semigroup S. Recall that a congruence relation on a semigroup S is firstly an equivalence relation $\approx$ and secondly it also satisfies: $r_1 \approx s_1$ and $r_2 \approx s_2 \Rightarrow r_1 + r_2 \approx s_1 + s_2$.

Now let $\approx$ be the smallest congruence relation on S that satisfy

1. $(a_1 + a_2, b) \approx (a_1, b) + (a_2, b)$,

2. $(a, b_1 + b_2) \approx (a, b_1) + (a, b_2)$,

3. $(ar, b) \approx (a, rb)$,

for all $a_1, a_2 \in A$, $b_1, b_2 \in B$ and $r \in R$. The collection of equivalence classes of S with respect to $\approx$ is called the *tensor product of A and B with respect to R* and is denoted by $A \otimes_R B$. The equivalence class that contains the element (a, b) is denoted by $a \otimes b$.

Reference [12] provides a slightly different, although completely equivalent, definition of the tensor product as well.

1.7 Rings of fractions and localisation

Definition 1.11 (Ring of Fractions [13]). Let R be a ring and $S \subseteq R$ a multiplicative set (i.e $1 \in S$ and $st \in S$ for all $s, t \in S$). We introduce the following relation $\sim$ on $R \times S$:

$$(a, s) \sim (b, t) \iff \exists u \in S \text{ such that } u(at - bs) = 0.$$

It can be shown that $\sim$ is an equivalence relation [13]. The *ring of fractions* of R with respect to S, R_S, is

$$S^{-1}R = (R \times S)/\sim,$$

with the ring operations defined as for fractions:

$$\begin{aligned} \frac{a}{s} \pm \frac{b}{t} &= \frac{(at \pm bs)}{st}, \\ \frac{a}{s} \cdot \frac{b}{t} &= \frac{ab}{st}. \end{aligned}$$

Note that above we wrote a/s for the class (a, s). From these definitions it should be clear that the zero of this ring is $0/1$ and the identity is $1/1$.

We also state the following fact as it will be needed later in the thesis. Let $r_1/s_1, r_2/s_2, \ldots, r_n/s_n$ be a finite set of elements from $S^{-1}R$. This finite set may be brought to a common denominator in the following manner. Take the element r_i/s_i from the set and multiply it by $s_1 s_2 \cdots s_{i-1} s_{i+1} \cdots s_n / s_1 s_2 \cdots s_{i-1} s_{i+1} \cdots s_n$. Note that $([s_1 s_2 \cdots s_{i-1} s_{i+1} \cdots s_n], [s_1 s_2 \cdots s_{i-1} s_{i+1} \cdots s_n]) \sim (1, 1)$, so that the multiplication above does not change the element r_i/s_i. The effect of the multiplications is a common denominator of $s_1 s_2 \cdots s_n$. This procedure is the same as the one encountered when dealing with ordinary fractions.

Localisation is a particular case of "ring of fractions". If $\mathfrak{p}$ is a prime ideal then $S = R\backslash\mathfrak{p}$ is a multiplicative set (see for instance [11]) and we set $R_\mathfrak{p} = S^{-1}R$. Further, $\varphi : R \to S^{-1}R$, defined by $r \mapsto r/1$ is a ring homomorphism.

If $\mathfrak{p}$ is a prime ideal in the ring R, then we define the extension of $\mathfrak{p}$, $e(\mathfrak{p})$, to be the ideal generated by the image of $\mathfrak{p}$ (under φ) in $R_\mathfrak{p}$.

Proposition 1.12 **([13]).** *If $\mathfrak{p}$ is a prime ideal in R and $\mathfrak{p} \cap S = \emptyset$ (S a multiplicative set) then $e(\mathfrak{p}) = S^{-1}\mathfrak{p} = (\mathfrak{p} \times S)/\sim$ is a prime ideal of $S^{-1}R$.*

Proposition 1.13 **([13]).** *$a/s \in R_\mathfrak{p}$ is a unit of $R_\mathfrak{p}$ $\iff$ $a \notin \mathfrak{p}$. Therefore $R_\mathfrak{p}$ is a local ring, with maximal ideal $e(\mathfrak{p}) = S^{-1}\mathfrak{p}$.*

If $\mathfrak{I}$ is an ideal of R we will write $\mathfrak{I}R_\mathfrak{p}$ for $(\mathfrak{I} \times S)/\sim$. With this in mind the maximal ideal above, $S^{-1}\mathfrak{p}$, is sometimes also written as $\mathfrak{p}R_\mathfrak{p}$. Later in the thesis we will also employ this notation in the form $\mathfrak{B}(R)R_\mathfrak{p} = (\mathfrak{B}(R) \times S)/\sim$.

The local ring $(R_\mathfrak{p}, \mathfrak{p}R_\mathfrak{p})$ is called the *localisation* of R at P.

Proposition 1.14 **([2]).** *If $\mathfrak{p}$ is a prime ideal of the ring R, the prime ideals of the local ring $R_\mathfrak{p}$ are in one-to-one correspondence with the prime ideals of R contained in $\mathfrak{p}$.*

The one-to-one correspondence referred to above is $\mathfrak{q} \longleftrightarrow S^{-1}\mathfrak{q} = (\mathfrak{q} \times S)\backslash \sim$. Here $\mathfrak{q}$ is a prime ideal contained in $\mathfrak{p}$. Note that every ideal in $R_\mathfrak{p}$ is of the form $S^{-1}\mathfrak{a}$, where $\mathfrak{a}$ is an ideal in R.

Definition 1.15 **(Modules of Fractions [13]).** Let R be a ring, S a multiplicative subset of R and M a left R-module. Then $S^{-1}M$ is the $S^{-1}R$-module defined as follows. We define the equivalence relation $\sim$ on $M \times S$ as before

$$(m, s) \sim (n, t) \iff \exists u \in S \text{ such that } utm = usn,$$

and set $S^{-1}M = (M \times S)/\sim$. The module operations are defined by $m/s \pm n/t = (mt \pm ns)/st$ and $(a/s) \cdot (n/t) = an/st$.

If $S = R\backslash\mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal, then $S^{-1}M$ is a module over the local ring $S^{-1}R = R_\mathfrak{p}$ and is also written as $S^{-1}M = M_\mathfrak{p}$.

If $\mathfrak{I}$ is an ideal of R (and thus a left R-module) we can form the left $S^{-1}R$-module $S^{-1}\mathfrak{I}$. This module will be written using the previous notation, $S^{-1}\mathfrak{I} = \mathfrak{I}R_\mathfrak{p}$.

The following proposition will be used later.

Proposition 1.16 **([2]).** *Let M be a left R-module. Then the $S^{-1}R$ modules, $S^{-1}M$ and $S^{-1}R \otimes_R M$ are isomorphic.*

1.8 Exact sequences

In a number of instances we will make use of the concept of an *exact sequence* [2]. A sequence of R-modules and R-homomorphisms

$$\cdots \longrightarrow M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1} \longrightarrow \cdots$$

is said to be *exact at* M_i if $\mathrm{Im}(f_i) = \mathrm{Ker}(f_{i+1})$. The sequence is *exact* if it is exact at each M_i. We will specifically have use for the following special cases :

$$\begin{aligned} 0 \longrightarrow M' \xrightarrow{f} M \text{ is exact} &\iff f \text{ is injective,} \\ M \xrightarrow{g} M'' \longrightarrow 0 \text{ is exact} &\iff g \text{ is surjective,} \\ 0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0 \text{ is exact} &\iff f \text{ is injective and } g \text{ is surjective.} \end{aligned}$$

1.9 The Peirce decomposition of a ring

Let R be a ring with identity and e an idempotent in R. Then any element in $r \in R$ can be written as $r = er + (r - er)$, so that $R = eR + (1-e)R$, where $(1-e)R = \{r - er \mid r \in R\}$. Also, $eb = b$ for all $b \in eR$ (if $b = er$ then $eb = e^2r = er = b$) and $eb = 0$ for all $b \in (1-e)R$ (if $b = r - er$ then $eb = er - e^2r = er - er = 0$), therefore $eR \cap (1-e)R = (0)$ (if $er_1 = r_2 - er_2 \neq 0$ then $e^2r_1 = er_2 - e^2r_2 = 0$ — a contradiction). Thus $R = eR \oplus (1-e)R$. This is called the right *Peirce decomposition of R relative to e.* We can analogously define a left and two-sided Peirce decomposition as well, see [7, p 83] for more details.

1.10 Some results on finite rings with identity

In some of our further work it is worthwhile to have the following results available [12].

Proposition 1.17. *Let R be a finite ring with identity. If $|R| = char(R)$ then $R \cong \mathbb{Z}_{\mathrm{char}(R)}$.*

Proposition 1.18. *The* only *rings with an identity and four elements are*

1. $\mathbb{Z}_4$,
2. $\mathbb{Z}_2[x]/(x^2)$,
3. $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ *and*
4. $\mathbb{Z}_2[x]/(x^2+1)$.

1.11 Graph theory

This section will be devoted to the graph theoretical terminology and results that will be used in this thesis. See [4] for more details.

A *graph* is a collection of vertices $\{v_1, v_2, \ldots, v_n\}$ and edges $\{e_1, e_2, \ldots, e_m\}$. Each edge may be seen as an unordered pair of vertices, that is $e_i = \{v_k, v_l\}$ if edge e_i joins vertices v_k and v_l. A graph is said to be the *trivial* graph if it has only one vertex.

Let u and v be vertices of a graph (with the possibility that they may be equal). A *u–v walk* of the graph is a finite, alternating sequence,

$$u = u_0, e_1, u_1, e_2, \ldots, u_{k-1}, e_k, u_k = v,$$

of vertices and edges, starting with vertex u and ending with vertex v, such that $e_i = \{u_{i-1}, u_i\}$, $i = 1, 2, \ldots k$. The number k is called the *length* of the walk and is equal to the number of edges in the walk. The walk will often be written by listing only its vertices since the egdes are then obvious. A u–v walk is considered *closed* when $u = v$. A u–v walk with no edge repeated is called a *u–v trail*, while if no vertex is repeated it is called a *u–v path*. A closed trail of a graph is referred to as a *circuit* and a closed path is known as a *cycle*. A cycle is said to be *even* or *odd* depending on whether its length is even or odd, respectively. The vertices that precede and follow the vertex v on a cycle are called the *neighbours* of v. As stated in the introduction all our graphs will be simple. That is, no loops (edges connecting a vertex with itself) or multiple edges (more that one edge between a pair of vertices) are allowed. This, in particular, implies that the length of the smallest odd cycle will be three and the length of the smallest even cycle will be four.

A graph is said to be *bipartite* if it is possible to partition the vertex set, V, of the graph into two subsets, V_1 and V_2, such that the edges of the graph lie only between the two partite sets V_1 and V_2. Thus there are no edges present between the vertices of V_1 and likewise for V_2. We have the following theorem.

Theorem 1.19 ([4]). *A nontrivial graph is bipartite if and only if it contains no odd cycles.*

We will frequently refer to the colouring of a graph in this thesis. By this is meant the assignment of colours (usually denoted by integers) to the vertices of a graph in such a manner that two adjacent vertices receive different colours. Of specific interest is the minimum number of colours that can be assigned to the vertices of a graph. This is known as the chromatic number of a graph and is denoted by $\chi(G)$. Note that a bipartite graph is therefore a graph for which $\chi(G) = 2$. (Assign one colour to the one partite set and another colour to the other partite set.) Also, if C is a cycle of even length, then $\chi(C) = 2$. On the other hand if C is a cycle of odd length, $\chi(C) = 3$. These two facts may be easily verified by drawing cycles of even and odd lengths and trying to colour them with fewer colours. This leads to the following observation. If G is a graph with $\chi(G) \geq 3$, then G contains an odd cycle : assume that G does not contain an odd cycle, then by the theorem above we know that G is then a bipartite graph. This leads to $\chi(G) = 2$ — a contradiction.

1.12 Thesis composition

As stated in the beginning, our main concern will be to determine how the ring theoretic properties of a commutative ring with identity influence its chromatic number. Chapters one through eight are based on the results presented in [3] and chapter nine is based on [1].

Chapter two deals with some examples of rings and their chromatic number. The third chapter deals with a characterisation of rings of finite chromatic number, aptly termed *Colorings*. Chapter four is on the properties of Colorings. The fifth chapter discusses the properties shared by the family of Colorings. Chapter six is devoted to the study of conditions that ensure $\chi(R) = \omega(R)$. The seventh chapter is on rings of low chromatic number (that is $\chi(R) \leq 5$). Chapter eight presents some examples of finite rings with $\chi(R) \leq 3$. Chapter nine discusses an example of a ring with $\omega(R) = 5$ and $\chi(R) = 6$.

Chapter 2

Examples of rings and their chromatic numbers

THE aim of this chapter is to present some examples of rings and to show how their chromatic number is calculated.

The first Proposition follows from the definitions.

Proposition 2.1. $\chi(R) = 1 \iff R = (0)$

Proposition 2.2. $\chi(R) = 2 \iff$ *R is an integral domain,* $R \cong \mathbb{Z}_4$*,* $R \cong \mathbb{Z}_2[x]/(x^2)$ *or* $R \cong \mathbb{Z}_2[x]/(x^2+1)$.

Proof. $\Longrightarrow$ Suppose that $\chi(R) = 2$.
Since products of nonzero elements in an integral domain are always nonzero, the chromatic number of an integral domain is 2. Therefore R may be an integral domain. If R is not an integral domain, we then need to show that either $R \cong \mathbb{Z}_4$, $R \cong \mathbb{Z}_2[x]/(x^2)$ or $R \cong \mathbb{Z}_2[x]/(x^2+1)$.

Therefore suppose that R is not an integral domain.
Then there exist $x, y \in R$, with $x \neq 0$ and $y \neq 0$, but $xy = 0$. In this case $\{0, x, y\}$ forms a clique with three elements, but $\omega(R) \leq \chi(R) = 2$. This implies that $x = y$, so that $x \neq 0$ and $x^2 = 0$. Using this we see that the ideal Rx is a clique ($r_1xr_2x = r_1r_2x^2 = 0$). Now, $0x \in Rx$ and $1x \in Rx$, so that $|Rx| \geq 2$, but since $\omega(R) \leq 2$, $|Rx| = 2$. Also, $Rx \subseteq \mathrm{Ann}x$ ($[rx]x = rx^2 = 0$). Further, $\mathrm{Ann}x \subseteq Rx$: if $z \in \mathrm{Ann}x$, then $\{0, x, z\}$ is a clique, but since $\omega(R) \leq 2$, $z = x$ or $z = 0$. Therefore $z \in Rx = \{0, x\}$, which implies that $\mathrm{Ann}x = Rx$. Consider the exact sequence

$$0 \xrightarrow{f_1} \mathrm{Ann}x \xrightarrow{f_2} R \xrightarrow{f_3} Rx \xrightarrow{f_4} 0, \tag{2.1}$$

where

$$\begin{aligned} f_1 &: 0 \mapsto 0, \\ f_2 &: x \mapsto x \ \forall x \in \mathrm{Ann}x, \\ f_3 &: r \mapsto rx \ \forall r \in R, \\ f_4 &: rx \mapsto 0 \ \forall rx \in Rx. \end{aligned}$$

Clearly $\mathrm{Im}(f_i) = \mathrm{Ker}(f_{i+1})$. Since f_3 is onto Rx, we have by the fundamental theorem on homomorphisms [11], that

$$\begin{aligned} Rx \cong R/\mathrm{Ker}(f_3) &= R/\mathrm{Im}(f_2) = R/\mathrm{Ann}x \\ \therefore |Rx| &= |R|/|\mathrm{Ann}x| \\ \therefore |R| &= |Rx||\mathrm{Ann}x| = 4 \end{aligned} \tag{2.2}$$

A well known corollary from Lagrange's theorem of group theory states that the order of an element divides the order of the group [6]. Also, for a ring with an identity, the characteristic of the ring equals the order of the identity [6]. Therefore the characteristic of R equals the characteristic of 1, which in turn has to divide R. In summary then, the characteristic of R has to divide 4. Therefore the characteristic of R is either 2 or 4. If $\mathrm{char}(R) = 4$, then by Proposition 1.17, $R \cong \mathbb{Z}_4$. If $\mathrm{char}(R) = 2$, then by Proposition 1.18, $R \cong \mathbb{Z}_2[x]/(x^2)$, $R \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ or $R \cong \mathbb{Z}_2[x]/(x^2+1)$. In $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ however, we have a clique of three elements $(\{(0,0),(1,0),(0,1)\})$, but for the present ring $\omega(R) \leq 2$. Therefore if $\mathrm{char}(R) = 2$, then $R \cong \mathbb{Z}_2[x]/(x^2)$ or $R \cong \mathbb{Z}_2[x]/(x^2+1)$.

$\Longleftarrow$ Under the assumption that R is an integral domain, $R \cong \mathbb{Z}_4$, $R \cong \mathbb{Z}_2[x]/(x^2)$ or $R \cong \mathbb{Z}_2[x]/(x^2+1)$ it is easily seen that $\chi(R) = 2$. For ease of reference the corresponding graphs of the rings above are shown in Figure 2.1. □

Proposition 2.3. *Let $p_1, p_2, \ldots, p_k, q_1, q_2, \ldots, q_r$ be different prime numbers and put $N = p_1^{2n_1} p_2^{2n_2} \cdots p_k^{2n_k} q_1^{2m_1+1} q_2^{2m_2+1} \cdots q_r^{2m_r+1}$. Then.*
$\chi(\mathbb{Z}_N) = \omega(\mathbb{Z}_N) = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k} q_1^{m_1} q_2^{m_2} \cdots q_r^{m_r} + r$.

Proof. Put $y_0 = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k} q_1^{m_1+1} q_2^{m_2+1} \cdots q_r^{m_r+1}$. Then $y_0^2 = 0$ in $\mathbb{Z}_N$ and this in turn implies that $y_0\mathbb{Z}_N$ is a clique with $p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k} q_1^{m_1} q_2^{m_2} \cdots q_r^{m_r}$ elements — to see this note that the products between y_0 with all integers from 1 to $p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k} q_1^{m_1} q_2^{m_2} \cdots q_r^{m_r}$, in $\mathbb{Z}_N$, are all distinct.

Let $y_i = y_0/q_i$, $1 \leq i \leq r$. Then the set $C = y_0\mathbb{Z}_N \bigcup \{y_1, y_2, \ldots, y_r\}$ is a clique of size $t = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k} q_1^{m_1} q_2^{m_2} \cdots q_r^{m_r} + r$ elements :

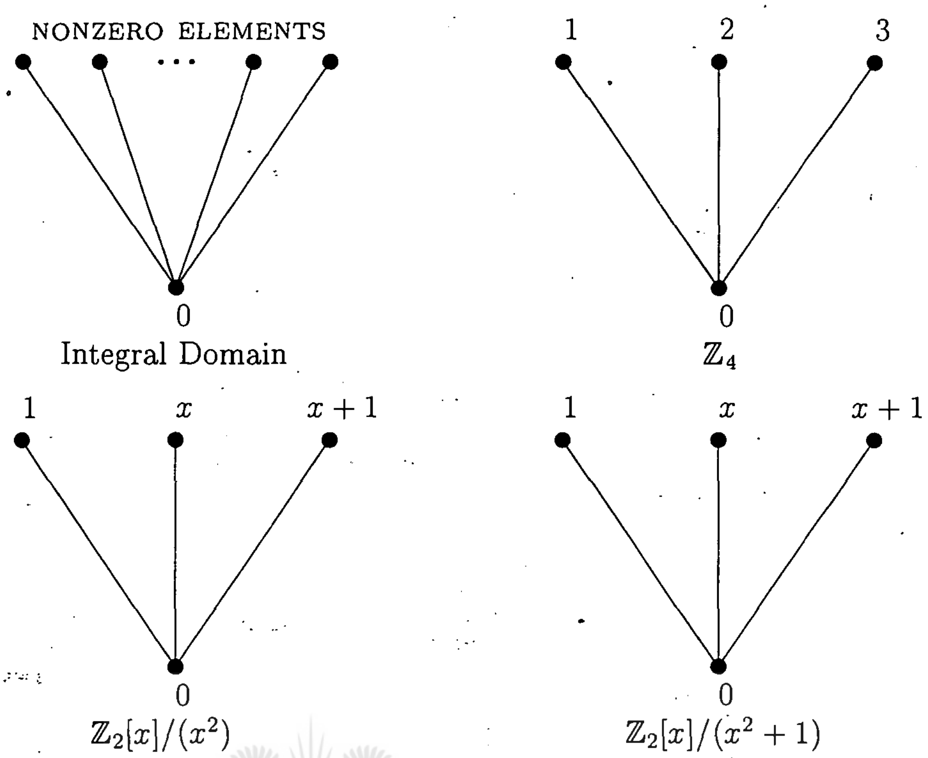


Figure 2.1: Rings With $\chi(R) = 2$.

Firstly, $y_i \notin y_0\mathbb{Z}_N$, $1 \leq i \leq r$, since $y_i < y_0$, $y_i \neq 0$ and $y_0\mathbb{Z}_N$ contains nonzero elements greater than or equal to y_0 together with zero. Note secondly that if $y_0x \in y_0\mathbb{Z}_N$ and $y_i \in \{y_1, y_2, \ldots, y_r\}$, then $y_0xy_i = y_0^2x/q_i = Nq_1q_2\cdots q_{i-1}q_{i+1}\cdots q_rx = 0$ (in $\mathbb{Z}_N$). Thirdly if y_i and $y_j \in \{y_1, y_2, \ldots, y_r\}$ with $i \neq j$, then $y_iy_j = y_0^2/(q_iq_j) =$
$Nq_1q_2\cdots q_{i-1}q_{i+1}\cdots q_{j-1}q_{j+1}\cdots q_r = 0$ (in $\mathbb{Z}_N$).

Therefore $\omega(\mathbb{Z}_N) \geq t$ and in turn $\chi(\mathbb{Z}_N) \geq t$. To show that $\chi(\mathbb{Z}_N) \leq t$ we have to produce a colouring of $\mathbb{Z}_N$ in t colours, the reasoning being that this t-colouring may not be the most optimal one (least number of colours), so that the chromatic number may still be less than or equal to t.

First off we have to colour each element of C with a unique colour of its own (C is a clique). Let $x_i = N/p_i^{n_i}$, $1 \leq i \leq k$. Note that $x_i \in y_0\mathbb{Z}_N$ so that $x_i \in C$ which implies that x_i has been assigned a colour. We will now colour the remaining elements (i.e $\mathbb{Z}_N \backslash C$) of $\mathbb{Z}_N$ as follows ($f(y)$ will denote the colour that we assigned to element y) :

Take $x \notin C$. We assign x a colour as follows

$$f(x) = \begin{cases} f(y_j) & \text{if } p_1^{n_1}p_2^{n_2}\cdots p_k^{n_k} \text{ divides } x \\ & \text{where } j = \min\{i \mid q_i^{m_i+1} \nmid x\}, \\ f(x_j) & \text{if } p_1^{n_1}p_2^{n_2}\cdots p_k^{n_k} \text{ does not divide } x \\ & \text{where } j = \min\{i \mid p_i^{n_i} \nmid x\}. \end{cases} \tag{2.3}$$

We now proceed to show that this results in a valid colouring (i.e. adjacent elements should receive different colours) :

If $p_1^{n_1}p_2^{n_2}\cdots p_k^{n_k}$ divides x, then x receives the same colour as y_j, so we have to ensure that x and y_j are not adjacent.
Recall that $y_j = y_0/q_j = p_1^{n_1}p_2^{n_2}\cdots p_k^{n_k}q_1^{m_1+1}q_2^{m_2+1}\cdots q_j^{m_j}\cdots q_r^{m_r+1}$ and also $q_j^{m_j+1} \nmid x$. Therefore in the product xy_j, the power of q_j can never be greater than or equal to $2m_j+1$, since the power of q_j in x is strictly less than m_j+1. Thus xy_j is never a multiple of N, so that $xy_j \neq 0$, which implies that x and y_j are not adjacent.

If $p_1^{n_1}p_2^{n_2}\cdots p_k^{n_k}$ does not divide x, then there exists at least one p_i such that $p_i^{n_i}$ does not divide x and we chose p_j to be that specific factor such that $j = \min\{i \mid p_i^{n_i} \nmid x\}$. In this case x receives the same colour as x_j. Here $x_j = N/p_j^{n_j} =$
$p_1^{2n_1}p_2^{2n_2}\cdots p_j^{n_j}\cdots p_k^{2n_k}q_1^{2m_1+1}q_2^{2m_2+1}\cdots q_r^{2m_r+1}$. In the product xx_j the power of p_j can never be greater than or equal to $2n_j$, because the power of p_j in x is strictly less than n_j. Therefore xx_j is never a multiple of N implying that $xx_j \neq 0$ with the implication that x and x_j are not adjacent.

In summary, $\chi(\mathbb{Z}_N) \leq t$ so that $\omega(\mathbb{Z}_N) \leq t$. Combining this with our previous results we get $\chi(\mathbb{Z}_N) = \omega(\mathbb{Z}_N) = t$.

□

Chapter 3

Rings with $\chi(R) < \infty$

THIS chapter contains the results that will be needed later on to characterise the rings of finite chromatic number.

Definition 3.1 (Finite element). An element $x \in R$ is said to be *finite* if the ideal Rx is a finite set.

The following lemma plays a key role in the results that follow.

Lemma 3.2. *If R has an infinite number of finite elements then R contains an infinite clique.*

Proof. Let $x_1, x_2, \ldots, x_n, \ldots$ be different finite elements in R. The elements $x_1x_2, x_1x_3, \ldots, x_1x_n, \ldots$ all belong to the finite ideal x_1R. Therefore there exists an infinite subsequence $\{a_n\}$ of $\{2, 3, \ldots, n, \ldots\}$ such that $x_1x_{a_1} = x_1x_{a_2} = \cdots$. As before, the elements $x_{a_1}x_{a_2}, x_{a_1}x_{a_3}, \ldots, x_{a_1}x_{a_n}, \ldots$ belong to the finite ideal $x_{a_1}R$, so that there exists an infinite subsequence $\{b_n\}$ of $\{a_2, a_3, \ldots, a_n, \ldots\}$ with $x_{a_1}x_{b_1} = x_{a_1}x_{b_2} = \cdots$. Continuing in this manner we construct a subsequence $y_1, y_2, \ldots, y_n, \ldots$ of the sequence $x_1, x_2, \ldots, x_n, \ldots$ such that $y_iy_j = y_iy_k$ when $j, k > i$ (all y_j's that follow a specific y_i are still in the same subsequence as the y_i). Here $y_1 = x_1$ and $y_2 = x_{a_1}$.

Define $z_{i,j} = y_i - y_j$. Then if $i < j < k < r$, $z_{i,j}z_{k,r} = (y_i - y_j)(y_k - y_r) = y_iy_k - y_iy_r - y_jy_k + y_jy_r = 0 - 0 = 0$. We are now in a position to construct an infinite clique :

Consider $z_{1,2}z_{3,4} = z_{1,2}z_{3,5} = 0$. We have $z_{3,4} \neq z_{3,5}$ ($z_{3,4} = z_{3,5} \Rightarrow y_4 = y_5$, a contradiction). Thus at least one of $z_{3,4}$ and $z_{3,5}$ is different from $z_{1,2}$. If for example $z_{3,5} \neq z_{1,2}$, then $\{z_{1,2}, z_{3,5}\}$ is a clique with two elements. Further $z_{6,7}$, $z_{6,8}$ and $z_{6,9}$ are pairwise different, so that at least one of them is not equal to $z_{1,2}$ or $z_{3,5}$. Say for example that $z_{6,9} \notin \{z_{1,2}, z_{3,5}\}$, then $\{z_{1,2}, z_{3,5}, z_{6,9}\}$ is a clique with three elements. By repeating the above procedure we obtain an infinite clique. □

Lemma 3.3. *Let $\mathfrak{I}$ be a finite ideal in the ring R. Then R contains an infinite clique $\iff$ $R/\mathfrak{I}$ has an infinite clique.*

Proof. $\Longrightarrow$ Suppose that R contains an infinite clique C.
We will denote the quotient ring $R/\mathfrak{I}$ by $\bar{R}$ and the homomorphic image of C in $\bar{R}$ by $\bar{C}$. Then $\bar{C} = \{c + \mathfrak{I} \mid c \in C\}$. Also, $\bar{C}$ is a clique : $(c_1 + \mathfrak{I})(c_2 + \mathfrak{I}) = c_1c_2 + \mathfrak{I} = 0 + \mathfrak{I} = \mathfrak{I}$, keeping in mind that $\mathfrak{I}$ is the zero element of $\bar{R}$. The fact that $\bar{C}$ is infinite is proved using a contradiction.

If we assume that $\bar{C}$ is finite, then there are only a finite number of different equivalence classes $c + \mathfrak{I}$, $c \in C$. This implies that at least one equivalence class contains an infinite number of elements of C (since C is infinite). Say this class is $c_1 + \mathfrak{I} = c_2 + \mathfrak{I} = \cdots$. Here $c_i \in C$, $c_i \neq c_j$ for $i \neq j$ and $i, j \in K$ where K is an infinite index set. Written differently $c_1 + \mathfrak{I} = c_k + \mathfrak{I}$, $k \in K$. Equivalently, $c_1 - c_k \in \mathfrak{I}$, $\forall\, k \in K$. Furthermore, $c_1 - c_k \neq c_1 - c_l$ for $k \neq l$ (since $c_1 - c_k = c_1 - c_l \Rightarrow c_k = c_l$, a contradiction). Thus we have an infinite number of elements $c_1 - c_k$, $k \in K$ with $c_1 - c_k \in \mathfrak{I}$. This gives the desired contradiction since $\mathfrak{I}$ is finite. Therefore $\bar{C}$ is infinite.

$\Longleftarrow$ Let $\{\bar{x}_i\}_1^\infty$ be an infinite clique in $\bar{R}$ ($\bar{x}_i = x_i + \mathfrak{I}$, $x_i \in R$). Therefore $\bar{x}_i\bar{x}_j = (x_i + \mathfrak{I})(x_j + \mathfrak{I}) = x_ix_j + \mathfrak{I} = \mathfrak{I}$, so that $x_ix_j \in \mathfrak{I}$ for $i \neq j$. Since the products $\{x_ix_j\}_{i \neq j}$ belong to the finite ideal $\mathfrak{I}$, we may apply the same technique as in Lemma 3.2 (where our present ideal $\mathfrak{I}$ plays the role of the ideal Rx_1 in 3.2) to obtain an infinite clique in R. □

Lemma 3.4. *If the ring R contains a nilpotent element which is not finite, then R contains an infinite clique.*

Proof. Assume that $x \in R$ is nilpotent, that is, $x^n = 0$ for some positive integer n and that x is not finite i.e Rx is infinite. The proof is by induction on n. If $x^2 = 0$ and Rx is infinite, then Rx is itself an infinite clique in R. We now assume that the lemma is true for all elements of nilpotency $n - 1$. Let $x^n = 0$, $n \geq 3$ and assume that Rx is infinite. Put $y = x^2$, then $y^{n-1} = (x^2)^{n-1} = x^nx^{n-2} = 0$. If Ry is infinite then we may conclude from the induction assumption that R has an infinite clique. Otherwise if Ry is finite, then $Rx/Ry = \{rx + Ry \mid r \in R\}$ is infinite. (This follows in the same way as for $\bar{C}$ in Lemma 3.3.) Furthermore, Rx/Ry is a clique in R/Ry : $(r_1x + Ry)(r_2x + Ry) = r_1r_2x^2 + Ry = r_1r_2y + Ry = Ry$. Therefore we have the infinite clique Rx/Ry in R/Ry and Ry is finite so that by Lemma 3.3 R has an infinite clique. □

Lemma 3.5. *If the nilradical, $\mathfrak{B}(R)$, of R is infinite, then R has an infinite clique.*

Proof. Assume that $\mathfrak{B}(R)$ is infinite. If every element in $\mathfrak{B}(R)$ is finite, Lemma 3.2 implies that R contains an infinite clique. On the other hand if there is an element in $\mathfrak{B}(R)$ that is not finite, Lemma 3.4 implies that R contains an infinite clique. (The elements in $\mathfrak{B}(R)$ are all nilpotent for a commutative ring R.) □

Remark 3.6. *If R is ring without an infinite clique, then by Lemma 3.5, $\mathfrak{B}(R)$ is finite. Applying Lemma 3.3 we then see that $R/\mathfrak{B}(R)$ also does not have an infinite clique.*

Lemma 3.7. *Let R be a reduced ring (i.e $\mathfrak{B}(R) = (0)$) which does not contain an infinite clique. Then R has the ascending chain condition (a.c.c) on ideals of the form $\mathrm{Ann}x$.*

Proof. Assume that we have an infinite chain of ideals of the form $\mathrm{Ann}x$ (i.e. we are assuming the a.c.c does not hold), that is

$$\mathrm{Ann}a_1 \subset \mathrm{Ann}a_2 \subset \cdots. \tag{3.1}$$

Let $x_i \in \mathrm{Ann}a_i \backslash \mathrm{Ann}a_{i-1}$, $i = 2, 3, \ldots$ and $y_n = x_n a_{n-1} \neq 0$ $n = 2, 3, \ldots$ ($x_n \in \mathrm{Ann}a_n$ and $x_n \notin \mathrm{Ann}a_{n-1}$). Then the y_n's form a clique : $y_n y_m = (x_n a_{n-1})(x_m a_{m-1}) = (x_n a_{m-1})(x_m a_{n-1})$. If we assume, without loss of generality, that $m > n$, then $x_n a_{m-1} = 0$ (since $x_n \in \mathrm{Ann}a_n \subset \mathrm{Ann}a_{n+1} \subset \cdots \subset \mathrm{Ann}a_{m-1} \subset \mathrm{Ann}a_m$, so that $x_n \in \mathrm{Ann}a_{m-1}$), implying that $y_n y_m = 0$. Furthermore, $y_i \neq y_j$ if $i \neq j$: If $y_i = y_j$ then $y_i^2 = y_i y_j$ and $y_j^2 = y_i y_j$, but $y_i y_j = 0$, therefore $y_i^2 = y_j^2 = 0$. This contradicts the fact that $\mathfrak{B}(R) = (0)$. (The nilradical contains all nilpotent elements.)

In summary then, the existence of the infinite chain provided a means to construct an infinite clique (the y_n's), which contradicts our assumption on R that it does not have an infinite clique. Thus the a.c.c holds. □

Lemma 3.8. *Let x and y be elements of the ring R such that* $\mathrm{Ann}x$ *and* $\mathrm{Ann}y$ *are different prime ideals. Then $xy = 0$.*

Proof. The proof is by contradiction. Assuming that $xy \neq 0$, this implies that $x \notin \mathrm{Ann}y$ and $y \notin \mathrm{Ann}x$. Further, $\mathrm{Ann}x : y = \{r \in R \mid ry \in \mathrm{Ann}x\} = \mathrm{Ann}x$ and $\mathrm{Ann}y : x = \{r \in R \mid rx \in \mathrm{Ann}y\} = \mathrm{Ann}y$:

If $r \in \mathrm{Ann}x : y$, then $ry \in \mathrm{Ann}x$ and since $\mathrm{Ann}x$ is a prime ideal, $r \in \mathrm{Ann}x$ or $y \in \mathrm{Ann}x$, but $y \notin \mathrm{Ann}x$ so that $r \in \mathrm{Ann}x$. This proves $\mathrm{Ann}x : y \subseteq \mathrm{Ann}x$. Conversely, if $r \in \mathrm{Ann}x$, then $rx = 0$ and also $(rx)y = (ry)x = 0$ so that $ry \in \mathrm{Ann}x$ which in turn implies that $r \in \mathrm{Ann}x : y$. Therefore $\mathrm{Ann}x : y = \mathrm{Ann}x$. Similarly, $\mathrm{Ann}y : x = \mathrm{Ann}y$.

However $\mathrm{Ann}x : y = \mathrm{Ann}y : x = \mathrm{Ann}(xy)$:

Let $r \in \mathrm{Ann}x : y$, then $ry \in \mathrm{Ann}x$ or $(ry)x = r(xy) = 0$, thus $r \in \mathrm{Ann}(xy)$. Conversely, let $r \in \mathrm{Ann}(xy)$, therefore $r(xy) = (ry)x = 0$, so that $ry \in \mathrm{Ann}x$ from which it follows that $r \in \mathrm{Ann}x : y$. This proves $\mathrm{Ann}x : y = \mathrm{Ann}(xy)$. The proof of $\mathrm{Ann}y : x = \mathrm{Ann}(xy)$ is similar.

All of this implies that $\mathrm{Ann}x = \mathrm{Ann}x : y = \mathrm{Ann}y : x = \mathrm{Ann}y$, but our initial assumption was that $\mathrm{Ann}x$ and $\mathrm{Ann}y$ are different, yielding the contradiction. Therefore $xy = 0$. □

We are now in a position to prove one of the first major results.

Theorem 3.9. *For a reduced ring R the following are equivalent :*

1. *$\chi(R)$ is finite.*

2. *$\omega(R)$ is finite.*

3. *The zero-ideal in R is a finite intersection of prime ideals.*

4. *R does not contain an infinite clique.*

Proof. 1. $\Rightarrow$ 2. This implication follows from $\omega(R) \leq \chi(R)$.

1. $\Rightarrow$ 4. Similar to the implication above.

2. $\Rightarrow$ 4. Obvious.

3. $\Rightarrow$ 1. Let $(0) = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_k$, where $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_k$ are prime ideals. Define a coloring f on R as follows:

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ \min\{i \mid x \notin \mathfrak{p}_i\} & \text{if } x \neq 0. \end{cases} \tag{3.2}$$

We now show that this is a valid colouring by showing that adjacent elements cannot receive the same colour. If x and y are adjacent then $xy = 0$; we will also assume that both x and y are not equal to 0 since 0 receives its own colour. Therefore $xy \in \mathfrak{p}_1$, $xy \in \mathfrak{p}_2, \ldots, xy \in \mathfrak{p}_k$. Since the $\mathfrak{p}_i$'s are prime ideals, this implies

$$x \in \mathfrak{p}_1 \text{ or } y \in \mathfrak{p}_1, x \in \mathfrak{p}_2 \text{ or } y \in \mathfrak{p}_2, \ldots, x \in \mathfrak{p}_k \text{ or } y \in \mathfrak{p}_k. \tag{3.3}$$

If we assume that x and y received the same colour i.e $f(x) = f(y)$, then $f(x) = \min\{i \mid x \notin \mathfrak{p}_i\} = \min\{i \mid y \notin \mathfrak{p}_i\} = f(y)$. This implies that there exists an $i \in \{1, 2, \ldots, k\}$ such that $x \notin \mathfrak{p}_i$ and $y \notin \mathfrak{p}_i$, but this contradicts the equation above. This shows that f is a valid colouring of R. Note that in this case $\chi(R) \leq k + 1$ so that this implies 1.

4. $\Rightarrow$ 3. We assume that R is reduced and that R does not contain an infinite clique. Lemma 3.7 implies that R satisfies the a.c.c on ideals of the form Ann$a$. Let Annx_i, $i \in I$ (I the index set) be the different maximal members of the family $\{\text{Ann}r \mid r \in R,\ r \neq 0\}$. Each Ann$x_i$ is a prime ideal :
Let $xy \in \text{Ann}x_i$ and assume $x \notin \text{Ann}x_i$. Then $xx_i \neq 0$ and $(xy)x_i = 0 = (xx_i)y$. Therefore $y \in \text{Ann}(xx_i)$. But $\text{Ann}(xx_i) \supseteq \text{Ann}x_i$ and Annx_i is a maximal element so that Annx_i cannot be properly contained in $\text{Ann}(xx_i)$, therefore $\text{Ann}(xx_i) = \text{Ann}x_i$. Thus $y \in \text{Ann}x_i$ which proves that Annx_i is prime.
Lemma 3.8 now implies that $|I| < \infty$, because otherwise we would have an infinite clique. We now show that $\bigcap_I \text{Ann}x_i = (0)$:
Assume that $x \in \bigcap_I \text{Ann}x_i$ and that $x \neq 0$. Then $x \in \text{Ann}x_i$ and $xx_i = 0$ for all $i \in I$. Also $\text{Ann}x \subseteq \text{Ann}x_i$ for some $i \in I$: we have two possibilities, $\text{Ann}x \subseteq \text{Ann}x_i$ for some $i \in I$, in which case we are done. Otherwise, $\text{Ann}x \nsubseteq \text{Ann}x_i$ for all $i \in I$, but then Ann$x$ is maximal, i.e. $\text{Ann}x = \text{Ann}x_i$ for an i — a contradiction. From this it follows that $x_i \in \text{Ann}x \subseteq \text{Ann}x_i$. This shows that $x_i^2 = 0$ or in otherwords that x_i is nilpotent and since R is reduced that $x_i = 0$. This contradicts the fact that the x_i's are all nonzero. Thus $\bigcap_I \text{Ann}x_i = (0)$.

□

Theorem 3.10. *Let R be a reduced ring with $\chi(R) < \infty$. Then R has only a finite number of minimal prime ideals. If this number of minimal prime ideals is n, then $\chi(R) = \omega(R) = n + 1$.*

Proof. By Theorem 3.9, (0) is equal to a finite intersection of prime ideals, that is $(0) = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_n$. Every prime ideal, $\mathfrak{p}_i$, contains a minimal prime ideal, $\mathfrak{m}_i$, [13]. Therefore $\mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \cdots \cap \mathfrak{m}_n = (0)$, where each $\mathfrak{m}_i$ is a minimal prime ideal. Note that we are assuming that these minimal prime ideals are different, since there is no point in including the same ideal more than once when forming an intersection. We now show that R has only a finite number of minimal prime ideals :
Assume that R has infinitely many minimal prime ideals. The nilradical is the intersection of all minimal prime ideals [13], so that $\mathfrak{B}(R) = \cap\ \mathfrak{m}_k$, where the intersection is taken over all minimal prime ideals. Since R is reduced, $\mathfrak{B}(R) = (0)$. Using the result above we get

$$\mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \cdots \cap \mathfrak{m}_n = \cap\ \mathfrak{m}_k = (0).$$

With a suitable renumbering of the minimal prime ideals we can rewrite this as

$$\mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \cdots \cap \mathfrak{m}_n = (\mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \cdots \cap \mathfrak{m}_n) \cap (\cap\ \mathfrak{m}'_i),$$

where $\cap\, \mathfrak{m}'_i$ refers to the remainder of the minimal prime ideals. The identity above implies that $(\mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \cdots \cap \mathfrak{m}_n) \subseteq (\cap\, \mathfrak{m}'_i)$, implying $(\mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \cdots \cap \mathfrak{m}_n) \subseteq \mathfrak{m}'_i$ for every i. We also have that $\mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_n \subseteq (\mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \cdots \cap \mathfrak{m}_n)$, so that $\mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_n \subseteq \mathfrak{m}'_i$. Since $\mathfrak{m}'_i$ is prime, $\mathfrak{m}_1 \subseteq \mathfrak{m}'_i$ or $\mathfrak{m}_2 \subseteq \mathfrak{m}'_i$ or $\cdots$ or $\mathfrak{m}_n \subseteq \mathfrak{m}'_i$. Furthermore, $\mathfrak{m}'_i$ is a minimal prime ideal so that $\mathfrak{m}_1 = \mathfrak{m}'_i$ or $\mathfrak{m}_2 = \mathfrak{m}'_i$ or $\cdots$ or $\mathfrak{m}_n = \mathfrak{m}'_i$.

This shows that every minimal prime ideal has to be equal to one of the n original minimal prime ideals that we started with. Thus there is only a finite number of minimal prime ideals.

We now show that the intersection $\mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \cdots \cap \mathfrak{m}_n = (0)$ is also minimal, i.e the removal of any minimal prime ideal from this intersection yields a nonzero intersection : Assume that the intersection is in fact not minimal. Then there exists at least one $\mathfrak{m}_i$ such that $\mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \cdots \cap \mathfrak{m}_n = \mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \cdots \cap \mathfrak{m}_{i-1} \cap \mathfrak{m}_{i+1} \cdots \cap \mathfrak{m}_n = (0)$. From this we may conclude that $\mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \cdots \cap \mathfrak{m}_{i-1} \cap \mathfrak{m}_{i+1} \cdots \cap \mathfrak{m}_n \subseteq \mathfrak{m}_i$. Using the same reasoning as above this leads to $\mathfrak{m}_1 = \mathfrak{m}_i$ or $\mathfrak{m}_2 = \mathfrak{m}_i$ or $\cdots$ or $\mathfrak{m}_n = \mathfrak{m}_i$. Since we assumed these minimal prime ideals to be distinct, this leads to a contradiction implying that the intersection is indeed minimal.

Turning now to the proof of $\chi(R) = \omega(R) = n + 1$, we have as in the implication 3. $\Rightarrow$ 1. of Theorem 3.9 that $\chi(R) \leq n + 1$.

We will now construct a clique with $n + 1$ elements :

$$\bigcap_{\substack{i=1 \\ i \neq j}}^{n} \mathfrak{m}_i \neq (0) \tag{3.4}$$

for every $j = 1, 2, \ldots, n$. Therefore for every $i \in \{1, 2, \ldots, n\}$ we may choose $x_i \neq 0$ such that $x_i \in \mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \cdots \cap \mathfrak{m}_{i-1} \cap \mathfrak{m}_{i+1} \cap \cdots \cap \mathfrak{m}_n$ and $x_i \notin \mathfrak{m}_i$. That is $x_i \in \mathfrak{m}_j$ for all $j \neq i$ and $x_i \notin \mathfrak{m}_i$. Now $x_ix_j = 0$ for $i \neq j$: $x_ix_j \in \mathfrak{m}_j$ for all $j \neq i$ since $x_i \in \mathfrak{m}_j$ for all $j \neq i$ and $x_ix_j \in \mathfrak{m}_i$ for all $i \neq j$ since $x_j \in \mathfrak{m}_i$ for all $i \neq j$. Together, this gives $x_ix_j \in \mathfrak{m}_i$ for all $i = 1, 2, \ldots, n$, i.e $x_ix_j \in \cap_{i=1}^{n} \mathfrak{m}_i = (0)$. Therefore $\{0, x_1, \ldots, x_n\}$ forms a clique of $n + 1$ elements, so that $\omega(R) \geq n + 1$.

Combining our results we see that $\omega(R) \leq \chi(R) \leq n + 1$ and $n + 1 \leq \omega(R) \leq \chi(R)$ imply that $\omega(R) = \chi(R) = n + 1$. □

The following theorem can be considered the main result of this chapter.

Theorem 3.11. *The following conditions are equivalent for a ring R :*

1. *$\chi(R)$ is finite.*

2. $\omega(R)$ is finite.

3. The nilradical in R is finite and equals a finite intersection of prime ideals.

4. R does not contain an infinite clique.

Proof. The following implications follow in the same manner as for Theorem 3.9 : 1. $\Rightarrow$ 2., 1. $\Rightarrow$ 4., 2. $\Rightarrow$ 4.

3. $\Rightarrow$ 1. Let $\mathfrak{B}(R) = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_k$, where $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_k$ are prime ideals and with $\mathfrak{B}(R)$ finite. We can colour the elements outside of $\mathfrak{B}(R)$ as follows : If $x \notin \mathfrak{B}(R)$, assign x the colour $f(x) = \min\{i \mid x \notin \mathfrak{p}_i\}$. This is the same type of colouring as the one defined in Theorem 3.9 so that we know from what we proved there that the elements outside of $\mathfrak{B}(R)$ can be coloured with a finite number of colours. Since $\mathfrak{B}(R)$ is a finite set, we will only need a finite (maybe even zero) amount of additional colours to colour the elements in $\mathfrak{B}(R)$. This shows that $\chi(R) < \infty$.

4. $\Rightarrow$ 3. Assume that R does not have an infinite clique. Then by Lemma 3.5 we see that $\mathfrak{B}(R)$ is finite. Lemma 3.3 then shows that $R/\mathfrak{B}(R)$ does not have an infinite clique. We now apply Theorem 3.9 to $R/\mathfrak{B}(R)$ and conclude that the zero ideal in $R/\mathfrak{B}(R)$ is a finite intersection of prime ideals in $R/\mathfrak{B}(R)$, that is $\{0 + \mathfrak{B}(R)\} = \{\mathfrak{B}(R)\} = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_n$, where the $\mathfrak{q}_i$'s are prime ideals in $R/\mathfrak{B}(R)$. Furthermore we know that there exits a one-to-one, onto mapping between the ideals (prime ideals) in R which contain $\mathfrak{B}(R)$ and the ideals (prime ideals) in $R/\mathfrak{B}(R)$ given by $\mathfrak{p} \mapsto \mathfrak{p}/\mathfrak{B}(R) = \{p + \mathfrak{B}(R) \mid p \in \mathfrak{p}\}$ [11]. Therefore for each of the prime ideals $\mathfrak{q}_i$ above there exists a corresponding prime ideal in R, say $\mathfrak{p}_i$ such that $\mathfrak{p}_i \mapsto \mathfrak{p}_i/\mathfrak{B}(R) = \mathfrak{q}_i$. Thus $\{\mathfrak{B}(R)\} = (\mathfrak{p}_1/\mathfrak{B}(R)) \cap (\mathfrak{p}_2/\mathfrak{B}(R)) \cap \cdots \cap (\mathfrak{p}_n/\mathfrak{B}(R)) = (\mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_n)/\mathfrak{B}(R)$. The second equality follows easily from first principles. The equality as a whole is only possible if $\mathfrak{B}(R) = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_n$. This shows that $\mathfrak{B}(R)$ is a finite intersection of prime ideals, yielding the desired result. □

The following theorem is an application of Theorem 3.11 to a somewhat restricted situation.

Theorem 3.12. *Let R be a ring which contains a finite ideal which is a finite intersection of prime ideals. Then the radical of any finite ideal is finite and equals a finite intersection of prime ideals. Furthermore, the ring has only a finite number of finite ideals.*

Proof. If R contains a finite ideal which is a finite intersection of prime ideals then $\chi(R) < \infty$ by the same procedure used in proving implication 3. $\Rightarrow$ 1. in Theorems 3.9 and 3.11. This also implies that $\omega(R) < \infty$.

Let $\mathfrak{q}$ be any finite ideal in R. Then by Lemma 3.3 $R/\mathfrak{q}$ does not have an infinite clique, since $\omega(R) < \infty$. By Theorem 3.11 we then conclude that $\chi(R/\mathfrak{q}) < \infty$ and also that $\mathfrak{B}(R/\mathfrak{q})$ is finite and equals a finite intersection of prime ideals. Note that $\mathfrak{B}(R/\mathfrak{q}) = \{r + \mathfrak{q} \in R/\mathfrak{q} \mid (r+\mathfrak{q})^n = \mathfrak{q}$ for some positive $n\} = \{r + \mathfrak{q} \in R/\mathfrak{q} \mid r^n + \mathfrak{q} = \mathfrak{q}$ for some positive $n\} = \{r + \mathfrak{q} \in R/\mathfrak{q} \mid r^n \in \mathfrak{q}$ for some positive $n\} = \{r + \mathfrak{q} \in R/\mathfrak{q} \mid r \in \mathfrak{B}(\mathfrak{q})\} = \mathfrak{B}(\mathfrak{q})/\mathfrak{q}$, where $\mathfrak{B}(\mathfrak{q})$ is the radical of $\mathfrak{q}$. That is, $\mathfrak{B}(\mathfrak{q})$ equals the intersection of the prime ideals in R which contain $\mathfrak{q}$. Therefore $\mathfrak{B}(\mathfrak{q})/\mathfrak{q}$ is finite and equal to a finite intersection of prime ideals, say $\mathfrak{B}(\mathfrak{q})/\mathfrak{q} = (\mathfrak{p}_1/\mathfrak{q}) \cap (\mathfrak{p}_2/\mathfrak{q}) \cap \cdots \cap (\mathfrak{p}_n/\mathfrak{q}) = (\mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \cdots \mathfrak{p}_n)/\mathfrak{q}$, so that $\mathfrak{B}(\mathfrak{q}) = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \cdots \mathfrak{p}_n$ (where the $\mathfrak{p}_k$'s are prime ideals in R cf. Theorem 3.11, implication 4. $\Rightarrow$ 3.). Therefore $\mathfrak{B}(\mathfrak{q})$ is also equal to a finite intersection of prime ideals. Since $\mathfrak{B}(\mathfrak{q})/\mathfrak{q}$ and $\mathfrak{q}$ are finite we conclude that $\mathfrak{B}(\mathfrak{q})$ is also finite, since $|\mathfrak{B}(\mathfrak{q})| = |\mathfrak{B}(\mathfrak{q})/\mathfrak{q}|.|\mathfrak{q}|$.

We still need to show that R contains only a finite number of finite ideals. To this end let $\mathcal{A} = \{x \in R \mid x$ is finite$\}$. Since $\omega(R) < \infty$ it follows from Lemma 3.2 that $\mathcal{A}$ is a finite ideal. Also, $\mathcal{A}$ contains every finite ideal :

Suppose $\mathfrak{I}$ be a finite ideal and let $x \in \mathfrak{I}$. Then $xR \subseteq \mathfrak{I}$ and since $|\mathfrak{I}| < \infty$, xR is also finite. Therefore x is a finite element, so that $x \in \mathcal{A}$. Thus $\mathfrak{I} \subseteq \mathcal{A}$, as desired.

Now since $\mathcal{A}$ contains every finite ideal, the number of finite ideals has to be finite.

$\square$

Chapter 4

Properties of rings with $\chi(R) < \infty$ — Colorings

THE previous chapter was devoted to a characterisation of rings of finite chromatic number. The present chapter will be devoted to discussing some of the properties enjoyed by these rings. We first state the following definition :

Definition 4.1 (Coloring). A ring R is called a *Coloring* if $\chi(R)$ is finite.

Lemma 4.2. *If $\mathfrak{I}$ is a finite ideal in a ring R, then $\mathfrak{I} : x/\text{Ann}x$ is a finite R-module.*

Proof. Consider the exact sequence

$$0 \xrightarrow{f_1} \text{Ann}x \xrightarrow{f_2} \mathfrak{I} : x \xrightarrow{f_3} (\mathfrak{I} : x)x \xrightarrow{f_4} 0, \tag{4.1}$$

where

$$\begin{aligned} f_1 &: 0 \mapsto 0, \\ f_2 &: x \mapsto x \;\forall\, x \in \text{Ann}x, \\ f_3 &: r \mapsto rx \;\forall\, r \in \mathfrak{I} : x, \\ f_4 &: r \mapsto 0 \;\forall\, r \in (\mathfrak{I} : x)x. \end{aligned}$$

Clearly $\text{Im}(f_i) = \text{Ker}(f_{i+1})$. Since f_3 is onto, we have by the fundamental theorem on homomorphisms [11] that $(\mathfrak{I} : x)x \cong \mathfrak{I} : x/\text{Ker}(f_3) = \mathfrak{I} : x/\text{Ann}x$. Also $(\mathfrak{I} : x)x \subseteq \mathfrak{I}$, since by definition the product of every element in $(\mathfrak{I} : x)$ with x is in $\mathfrak{I}$. This forces $(\mathfrak{I} : x)x$ to be finite because $\mathfrak{I}$ is finite. This means therefore, that $\mathfrak{I} : x/\text{Ann}x$ is also finite (by the isomorphism above). □

The next lemma will be useful in proving Theorem 4.4.

Lemma 4.3. *If R is a commutative ring with identity, then*

$$(\mathfrak{p} : x) = \begin{cases} R & x \in \mathfrak{p}, \\ \mathfrak{p} & x \notin \mathfrak{p}, \end{cases}$$

where $\mathfrak{p} \neq R$ is a prime ideal and x is any element in R.

Proof. Let $x \in R$ and let $\mathfrak{p} \neq R$ be a prime ideal. If $x \in \mathfrak{p}$, then $(\mathfrak{p} : x) = R$ since the product between x and any $r \in R$ will always be in $\mathfrak{p}$ seeing that $\mathfrak{p}$ is an ideal.

Otherwise if $x \notin \mathfrak{p}$, then for a $y \neq x$ and $y \notin \mathfrak{p}$, $(\mathfrak{p} : x) = (\mathfrak{p} : y)$:
Let $r \in (\mathfrak{p} : x)$, therefore $rx \in \mathfrak{p}$, so that $x \in \mathfrak{p}$ or $r \in \mathfrak{p}$. $\mathfrak{p}$ is prime which implies that $r \in \mathfrak{p}$ ($x \notin \mathfrak{p}$). This leads to $yr \in \mathfrak{p}$ ($\mathfrak{p}$ an ideal), so that $r \in (\mathfrak{p} : y)$, i.e $(\mathfrak{p} : x) \subseteq (\mathfrak{p} : y)$. Similarly we can show that $(\mathfrak{p} : y) \subseteq (\mathfrak{p} : x)$, which proves the assertion above.

Therefore if $x \neq 1$ and $x \notin \mathfrak{p}$, then using the statement above with $y = 1$ ($1 \notin \mathfrak{p}$ since $\mathfrak{p} \neq R$), $(\mathfrak{p} : x) = (\mathfrak{p} : 1) = \mathfrak{p}$. If $x = 1$, then obviously $(\mathfrak{p} : x) = (\mathfrak{p} : 1) = \mathfrak{p}$ also. Note that since $1 \notin \mathfrak{p}$ the possibility $x = 1$ does exist. It does not change the result, though. In summary then

$$(\mathfrak{p} : x) = \begin{cases} R & x \in \mathfrak{p}, \\ \mathfrak{p} & x \notin \mathfrak{p}, \end{cases}$$

so that under the assumptions of the lemma, the possibilities for $(\mathfrak{p} : x)$ are severely limited. $\square$

Note that above we did not use the fact that R was commutative explicitly, thus this result would be valid in a noncommutative ring as well. In the case of a noncommutative ring, though, one would have to formulate the definition of $\mathfrak{p} : x$ more carefully, specifying whether multiplication by elements from $\mathfrak{p} : x$ is to be taken on the left or on the right of x. We choose to circumvent this problem by focusing on a commutative ring.

The following theorem is a generalisation of Lemma 3.7.

Theorem 4.4. *A Coloring has a.c.c on ideals of the form* $\mathrm{Ann}\,a$.

Proof. Let R be a Coloring and assume that we have the infinite chain $\mathrm{Ann}\,y_1 \subset \mathrm{Ann}\,y_2 \subset \cdots$ (i.e the a.c.c does not hold). By Theorem 3.11 we know that $\mathfrak{B}(R)$ is finite. We then remove those $\mathrm{Ann}\,y_i$'s from the chain above which are such that $y_i \in \mathfrak{B}(R)$. This still yields an infinite chain since we are removing at most a finite number of terms from the infinite chain. This produces a chain $\mathrm{Ann}x_1 \subset \mathrm{Ann}x_2 \subset \cdots$ such that $x_i \notin \mathfrak{B}(R)$ for $i = 1, 2, \ldots$. Theorem 3.11 also yields that $\mathfrak{B}(R) = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_n$, where the $\mathfrak{p}_i$'s are prime ideals. For an element $x \in R$ we are then able to write

$$\mathfrak{B}(R) : x = (\mathfrak{p}_1 : x) \cap (\mathfrak{p}_2 : x) \cap \cdots \cap (\mathfrak{p}_n : x), \tag{4.2}$$

using nothing more than the definitions. Applying Lemma 4.3 to each term of the intersection we see that each term can have one of two possible values depending on the location of x. Since we do not know the location of x beforehand, the best that we can do is to say that the intersection will be restricted to one of 2^n possibilities. Each term has 2 possible values and there are n terms. Note further that this set of 2^n possibilities is the same for every $x \in R$. The implication of this is that the family $\{\mathfrak{B}(R) : x \mid x \in R\}$ is finite, specifically $|\{\mathfrak{B}(R) : x \mid x \in R\}| \leq 2^n$. Consequently, there exists a subsequence $\{z_j\}$ of $\{x_i\}$ for which $\mathfrak{B}(R) : z_1 = \mathfrak{B}(R) : z_2 = \cdots$. Consider now the chain $\mathrm{Ann}z_1 \subset \mathrm{Ann}z_2 \subset \cdots$. For each term of the chain we have $\mathrm{Ann}z_i \subseteq \mathfrak{B}(R) : z_i$, but since $\mathfrak{B}(R) : z_i = \mathfrak{B}(R) : z_1$ for all $i = 2, 3, \ldots$, we have $\mathrm{Ann}z_1 \subset \mathrm{Ann}z_2 \subset \cdots \subseteq \mathfrak{B}(R) : z_1$. Now, take $r_1 \in \mathrm{Ann}z_1$, $r_2 \in \mathrm{Ann}z_2 \backslash \mathrm{Ann}z_1$, $r_3 \in \mathrm{Ann}z_3 \backslash \{\mathrm{Ann}z_1 \cup \mathrm{Ann}z_2\} = \mathrm{Ann}z_3 \backslash \mathrm{Ann}z_2$, $\ldots$, then $r_i + \mathrm{Ann}z_1 \neq r_j + \mathrm{Ann}z_1$ for $i < j$:
Assume that $r_i + \mathrm{Ann}z_1 = r_j + \mathrm{Ann}z_1$, then $r_i - r_j \in \mathrm{Ann}z_1$. Say $r_i - r_j = z_1'$ where $z_1' \in \mathrm{Ann}z_1$. Since $\mathrm{Ann}z_1 \subseteq \mathrm{Ann}z_i$, $z_1' \in \mathrm{Ann}z_i$, so that $r_j \in \mathrm{Ann}z_i$ which contradicts the choice of z_j.

This shows that $(\mathfrak{B}(R) : z_1)/\mathrm{Ann}z_1$ is infinite, which contradicts Lemma 4.2. Therefore the a.c.c holds. □

Theorem 4.5. *Let R be a Coloring. Then* Ass$R$ *(the set of associated prime ideals) is finite. Further, we have the following for the set of zerodivisors :*

$$\mathfrak{Z}(R) = \bigcup_{\mathfrak{p} \in \mathrm{Ass}R \backslash \{R\}} \mathfrak{p}.$$

Also, any minimal prime ideal $\mathfrak{q}$, is an associated prime ideal and $R_\mathfrak{q}$ is a field or a finite ring.

Proof. Assume that R is a Coloring, then by Theorem 3.11 we know that $\omega(R) < \infty$. A direct consequence of this and Lemma 3.8 is that Ass$R$ is finite (otherwise we can construct an infinite clique).

Let $x \in \mathfrak{Z}(R)$, then $x \in \mathrm{Ann}r$ for some $r \neq 0$. By Theorem 4.4 we then have that $\mathrm{Ann}r \subseteq \mathrm{Ann}y$ for some maximal $\mathrm{Ann}y$, so that $x \in \mathrm{Ann}y$ for some maximal $\mathrm{Ann}y$.

Furthermore, we saw in Theorem 3.9 that the maximal Ann$y$'s are prime implying that Ann$y$ is an associated prime ideal. This shows that

$$\mathfrak{Z}(R) \subseteq \bigcup_{\mathfrak{p} \in \mathrm{Ass}R\setminus\{R\}} \mathfrak{p}. \tag{4.3}$$

The converse is an easy consequence of the definitions. Note that the union is taken over the set of all associated prime ideals except R — R is an associated prime ideal since $R = \mathrm{Ann}0$. Therefore,

$$\mathfrak{Z}(R) = \bigcup_{\mathfrak{p} \in \mathrm{Ass}R\setminus\{R\}} \mathfrak{p}. \tag{4.4}$$

We now show that every minimal prime ideal is an associated prime ideal. Let $\mathfrak{p}$ be a minimal prime ideal and take $x \notin \mathfrak{p}$. If there does not exist an x such that $x \notin \mathfrak{p}$ then $\mathfrak{p} = R$ and there are no other proper minimal prime ideals. Further, $\mathfrak{p} = \mathrm{Ann}0$ so that $\mathfrak{p} \in \mathrm{Ass}R$ and we are done. Choose Ann$t$ maximal in the family $\{\mathrm{Ann}r \mid \mathrm{Ann}r \subseteq \mathfrak{p}\}$. This family is not empty since $\mathrm{Ann}x \subseteq \mathfrak{p}$:

Let $y \in \mathrm{Ann}x$, then $xy = 0 \in \mathfrak{p}$. If $y \notin \mathfrak{p}$ then $xy \notin \mathfrak{p}$, therefore $y \in \mathfrak{p}$.

This Ann$t$ is prime :

Let $ab \in \mathrm{Ann}t$ and assume that $a \notin \mathrm{Ann}t$ and $b \notin \mathrm{Ann}t$ (i.e we assume that Ann$t$ is not prime). Consider now the ideal Annta.

If $a \notin \mathfrak{p}$ then $\mathrm{Ann}t \subset \mathrm{Ann}ta \subseteq \mathfrak{p}$:

Let $r \in \mathrm{Ann}ta$, then $r(ta) = (ra)t = 0$, so that $ra \in \mathrm{Ann}t \subseteq \mathfrak{p}$. Therefore $ra \in \mathfrak{p}$ and since $\mathfrak{p}$ is prime and $a \notin \mathfrak{p}$, $r \in \mathfrak{p}$. We certainly have $\mathrm{Ann}t \subseteq \mathrm{Ann}ta$. Now $b \in \mathrm{Ann}ta$, since $b(ta) = (ab)t = 0$ ($ab \in \mathrm{Ann}t$), but $b \notin \mathrm{Ann}t$. Therefore $\mathrm{Ann}t \subset \mathrm{Ann}ta$. This contradicts the fact that Ann$t$ is maximal in the family.

If on the other hand $a \in \mathfrak{p}$ and $\mathrm{Ann}ta \subseteq \mathfrak{p}$, then the contradiction is repeated. The contradiction did not depend on a being an element of $\mathfrak{p}$.

Therefore we still need to consider the case $a \in \mathfrak{p}$ and $\mathrm{Ann}ta \not\subseteq \mathfrak{p}$. We now have a $c \in \mathrm{Ann}ta$ and $c \notin \mathfrak{p}$. Here we consider the ideal Anntc and get the contradiction $\mathrm{Ann}t \subset \mathrm{Ann}tc \subseteq \mathfrak{p}$ in the same manner as above ($\mathrm{Ann}t \subset \mathrm{Ann}tc$ since $a \in \mathrm{Ann}tc$ and $a \notin \mathrm{Ann}t$).

Therefore every possibility ends in a contradiction so that Ann$t$ is prime. Since $\mathfrak{p}$ is a minimal prime ideal we need to have $\mathrm{Ann}t = \mathfrak{p}$. This shows that every minimal prime ideal is an associated prime ideal.

Next, we will show that for a minimal prime ideal $\mathfrak{p}$, $R_{\mathfrak{p}}$ is a field or a finite ring. Let $\mathfrak{p}$ be a minimal prime ideal. We know that $\mathfrak{p} = \mathrm{Ann}x$ for some $x \in R$. If $x \notin \mathfrak{p}$, $\mathfrak{p}R_{\mathfrak{p}} = (0)$, i.e the unique maximal ideal in $R_{\mathfrak{p}}$ is the zero ideal :

Recall that $\mathfrak{p}R_\mathfrak{p} = \{p/s \mid p \in \mathfrak{p} \text{ and } s \in R\backslash\mathfrak{p}\}$. Let $p/s \in \mathfrak{p}R_\mathfrak{p}$, then $x(p.1 - s.0) = xp = 0$ ($p \in \mathfrak{p} = \text{Ann}x$), implying that $(p,s) \sim (0,1)$, or in otherwords that the fraction p/s equals the fraction $0/1$ in $R_\mathfrak{p}$; $0/1$ is the zero element in $R_\mathfrak{p}$. This shows that $\mathfrak{p}R_\mathfrak{p} = (0)$. If the unique maximal ideal is (0), it means that the only ideals in $R_\mathfrak{p}$ are (0) and $R_\mathfrak{p}$. Let x be a nonzero element in $R_\mathfrak{p}$, then $xR_\mathfrak{p}$ is a nonzero ideal in $R_\mathfrak{p}$ and so $xR_\mathfrak{p} = R_\mathfrak{p}$. Specifically there exists an element $x' \in R_\mathfrak{p}$ such that $xx' = 1$. This shows that every nonzero element has an inverse so that $R_\mathfrak{p}$ is a field.

Consider now the case $x \in \mathfrak{p}$. We can write $\mathfrak{B}(R) = \mathfrak{p} \cap \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_k$, where $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_k$ are the remaining minimal prime ideals. This is possible since the nilradical is the intersection of all minimal prime ideals. Take $y \in (\mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_k)\backslash\mathfrak{p}$. Then $y\mathfrak{p} \subseteq \mathfrak{B}(R)$, since the product of y with an element in $\mathfrak{p}$ is in $\mathfrak{p}$ as well as in $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_k$ ($\mathfrak{p}$ is an ideal and $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_k$ are also ideals respectively). We now claim that $\mathfrak{p}R_\mathfrak{p} = \mathfrak{B}(R)R_\mathfrak{p}$:

Let $p/s \in \mathfrak{p}R_\mathfrak{p}$, then $(p,s) \sim (yp, ys)$ since $1(pys - syp) = 0$, therefore $p/s = yp/ys$. Now, $yp \in \mathfrak{B}(R)$ ($y\mathfrak{p} \subseteq \mathfrak{B}(R)$) and $ys \in R\backslash\mathfrak{p}$ since $R\backslash\mathfrak{p}$ is a multiplicative set and $y \notin \mathfrak{p}$. This means that every element in $\mathfrak{p}R_\mathfrak{p}$ is equal to some element in $\mathfrak{B}(R)R_\mathfrak{p}$, i.e $\mathfrak{p}R_\mathfrak{p} \subseteq \mathfrak{B}(R)R_\mathfrak{p}$. Further, every element in $\mathfrak{B}(R)R_\mathfrak{p}$ trivially equals some element in $\mathfrak{p}R_\mathfrak{p}$, since $\mathfrak{B}(R) \subseteq \mathfrak{p}$. Thus $\mathfrak{p}R_\mathfrak{p} \supseteq \mathfrak{B}(R)R_\mathfrak{p}$. Combining, $\mathfrak{p}R_\mathfrak{p} = \mathfrak{B}(R)R_\mathfrak{p}$.

Since the ideal $\mathfrak{B}(R)$ is finite, $\mathfrak{p}R_\mathfrak{p}$ is also finite : assume that $\mathfrak{p}R_\mathfrak{p} = \mathfrak{B}(R)R_\mathfrak{p}$ is not finite. Therefore there exist infinitely many pairs r_i/s_i and r_k/s_k in $\mathfrak{B}(R)R_\mathfrak{p}$, with $r_i, r_k \in \mathfrak{B}(R)$ and $s_i, s_k \in R\backslash\mathfrak{p}$, such that $(r_i, s_i) \nsim (r_k, s_k)$. That is for all $u \in R\backslash\mathfrak{p}$, $u(r_is_k - r_ks_i) \neq 0$. Taking $u = 1$ we get that $r_is_k \neq r_ks_i$. Thus we have infinitely many pairs of elements r_is_k and r_ks_i, such that $r_is_k \neq r_ks_i$. Note that r_is_k and r_ks_i are both in $\mathfrak{B}(R)$ since $r_i, r_k \in \mathfrak{B}(R)$. Therefore $\mathfrak{B}(R)$ has infinitely many different elements, but this contradicts the finiteness of $\mathfrak{B}(R)$. Thus $\mathfrak{B}(R)R_\mathfrak{p} = \mathfrak{p}R_\mathfrak{p}$ is finite.

Further, $R/\mathfrak{p} \cong Rx$, the isomorphism being given by $r + \mathfrak{p} = r + \text{Ann}x \mapsto rx$. Since $x \in \mathfrak{p} = \text{Ann}x$, we have that $x^2 = 0$. This implies that Rx is a clique which together with $\omega(R) < \infty$ yields that Rx is a finite; more importantly $R/\mathfrak{p}$ is finite. Now $Rx \otimes_R R_\mathfrak{p} \cong (Rx)R_\mathfrak{p} \subseteq \mathfrak{p}R_\mathfrak{p}$ (the isomorphism is given by Proposition 1.16 and $Rx \subseteq \mathfrak{p}$ since $x \in \mathfrak{p}$), so that $Rx \otimes_R R_\mathfrak{p}$ is finite. Also, $Rx \otimes_R R_\mathfrak{p} \cong R/\mathfrak{p} \otimes_R R_\mathfrak{p} \cong (R/\mathfrak{p})_\mathfrak{p} \cong R_\mathfrak{p}/(\mathfrak{p}R_\mathfrak{p})$ (the second last isomorphism follows from Propostion 1.16 and the last isomorphism will be proved presently).

We now show that $(R/\mathfrak{p})_{\mathfrak{p}} \cong R_{\mathfrak{p}}/(\mathfrak{p}R_{\mathfrak{p}})$ as left $R_{\mathfrak{p}}$-modules. Note that the module operations are given by

$$\begin{aligned}
\frac{r_1+\mathfrak{p}}{s_1}+\frac{r_2+\mathfrak{p}}{s_2} &= \frac{(r_1s_2+\mathfrak{p})+(r_2s_1+\mathfrak{p})}{s_1s_2}, \\
\left(\frac{r_1}{s_1}+\mathfrak{p}R_{\mathfrak{p}}\right)+\left(\frac{r_2}{s_2}+\mathfrak{p}R_{\mathfrak{p}}\right) &= \left(\frac{r_1}{s_1}+\frac{r_2}{s_2}\right)+\mathfrak{p}R_{\mathfrak{p}}, \\
\frac{r'}{s'}\left(\frac{r+\mathfrak{p}}{s}\right) &= \frac{r'r+\mathfrak{p}}{s's}, \\
\frac{r'}{s'}\left(\frac{r}{s}+\mathfrak{p}R_{\mathfrak{p}}\right) &= \frac{r'r}{s's}+\mathfrak{p}R_{\mathfrak{p}}.
\end{aligned}$$

Consider the mapping $(R/\mathfrak{p})_{\mathfrak{p}} \to R_{\mathfrak{p}}/(\mathfrak{p}R_{\mathfrak{p}})$ defined by $(r+\mathfrak{p})/s \mapsto (r/s)+\mathfrak{p}R_{\mathfrak{p}}$ where $r \in R$ and $s \in R\backslash\mathfrak{p}$.

The mapping is **onto**:
If $(r/s)+\mathfrak{p}R_{\mathfrak{p}} \in R_{\mathfrak{p}}/(\mathfrak{p}R_{\mathfrak{p}})$ then obviously, $(r+\mathfrak{p})/s \mapsto (r/s)+\mathfrak{p}R_{\mathfrak{p}}$.

The mapping is also **one-to-one**:
Let $(r_1/s_1)+\mathfrak{p}R_{\mathfrak{p}} = (r_2/s_2)+\mathfrak{p}R_{\mathfrak{p}}$. Therefore $r_1/s_1-r_2/s_2 \in \mathfrak{p}R_{\mathfrak{p}}$. Thus $(r_1s_2-r_2s_1)/s_1s_2 = r'/s'$ where $r' \in \mathfrak{p}$ and $s' \in R\backslash\mathfrak{p}$. That is there exists a $u \in R\backslash\mathfrak{p}$ such that

$$\begin{aligned}
u([r_1s_2-r_2s_1]s'-r's_1s_2) &= 0, \\
\therefore (us')r_1s_2-(us')r_2s_1 = ur's_1s_2 &\in \mathfrak{p}, \\
\therefore u'r_1s_2-u'r_2s_1 \in \mathfrak{p} \text{ where } u' = (us') &\in R\backslash\mathfrak{p}, \\
\therefore u'r_1s_2+\mathfrak{p} &= u'r_2s_1+\mathfrak{p}, \\
\therefore u's_2(r_1+\mathfrak{p}) &= u's_1(r_2+\mathfrak{p}), \\
\therefore \frac{r_1+\mathfrak{p}}{s_1} &= \frac{r_2+\mathfrak{p}}{s_2}.
\end{aligned}$$

The mapping is an R-**homomorphism**:

$$\begin{aligned}
\frac{r_1+\mathfrak{p}}{s_1}+\frac{r_2+\mathfrak{p}}{s_2} &= \frac{(r_1s_2+\mathfrak{p})+(r_2s_1+\mathfrak{p})}{s_1s_2},\\
&= \frac{(r_1s_2+r_2s_1)+\mathfrak{p}}{s_1s_2},\\
&\mapsto \frac{r_1s_2+r_2s_1}{s_1s_2}+\mathfrak{p}R_{\mathfrak{p}},\\
&= \left(\frac{r_1}{s_1}+\frac{r_2}{s_2}\right)+\mathfrak{p}R_{\mathfrak{p}},\\
&= \left(\frac{r_1}{s_1}+\mathfrak{p}R_{\mathfrak{p}}\right)+\left(\frac{r_2}{s_2}+\mathfrak{p}R_{\mathfrak{p}}\right).
\end{aligned}$$

Let $r'/s' \in R_{\mathfrak{p}}$. Then

$$\begin{aligned}
\frac{r'}{s'}\left(\frac{r+\mathfrak{p}}{s}\right) &= \frac{r'r+\mathfrak{p}}{s's},\\
&\mapsto \frac{r'r}{s's}+\mathfrak{p}R_{\mathfrak{p}},\\
&= \frac{r'}{s'}\left(\frac{r}{s}+\mathfrak{p}R_{\mathfrak{p}}\right).
\end{aligned}$$

Now the isomorphism $Rx \otimes_R R_{\mathfrak{p}} \cong R_{\mathfrak{p}}/(\mathfrak{p}R_{\mathfrak{p}})$ implies that $R_{\mathfrak{p}}/(\mathfrak{p}R_{\mathfrak{p}})$ is finite and since $|R_{\mathfrak{p}}| = |\mathfrak{p}R_{\mathfrak{p}}||R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}|$ we have that $R_{\mathfrak{p}}$ is finite. □

Theorem 4.6. *If R is a Coloring and $\mathfrak{p}$ an associated prime ideal in R, then either $R_{\mathfrak{p}}$ is a field or $\mathfrak{p}$ is a maximal ideal.*

Proof. Let $\mathfrak{p}$ be an associated prime ideal. Therefore $\mathfrak{p} = \mathrm{Ann}x$ for some $x \in R$. Suppose firstly that $x \in \mathfrak{p}$, then $x \in \mathrm{Ann}x$ so that $x^2 = 0$. This implies that Rx is a clique and since R is a Coloring i.e $\omega(R) < \infty$, Rx has to be finite. Now, the fact that $\mathfrak{p}$ is prime implies that $R/\mathfrak{p}$ is an integral domain. Also, $Rx \cong R/\mathfrak{p}$ so that $R/\mathfrak{p}$ is a finite integral domain. A finite integral domain is a field, therefore $R/\mathfrak{p}$ is a field. Furthermore, $R/\mathfrak{p}$ is a field if and only if $\mathfrak{p}$ is a maximal ideal. Thus $\mathfrak{p}$ is a maximal ideal.

If $x \notin \mathfrak{p}$, then we conclude in the same manner as in Theorem 4.5 that $\mathfrak{p}R_{\mathfrak{p}} = (0)$ so that $R_{\mathfrak{p}}$ is a field. □

Corollary 4.7. *An associated prime ideal in a Coloring is either a maximal ideal or a minimal prime ideal.*

Proof. From Theorem 4.6 we have that either $\mathfrak{p}$ is a maximal ideal or that $R_{\mathfrak{p}}$ is a field. Therefore one half is already taken care of.

Let $R_{\mathfrak{p}}$ be a field. Recall that the prime ideals of $R_{\mathfrak{p}}$ are in a one-to-one correspondence with the prime ideals of R contained in $\mathfrak{p}$. The correspondence is given by $\mathfrak{q} \leftrightarrow S^{-1}\mathfrak{q} = (\mathfrak{q} \times S)/\sim$, where $\mathfrak{q}$ is a prime ideal contained in $\mathfrak{p}$.

Assume that there exists a prime ideal $\mathfrak{q} \subset \mathfrak{p}$. That is, $\mathfrak{p}$ is not a minimal prime ideal. Then $S^{-1}\mathfrak{q}$ is a prime ideal in $R_{\mathfrak{p}}$. But $S^{-1}\mathfrak{q} \subseteq S^{-1}\mathfrak{p} = \mathfrak{p}R_{\mathfrak{p}} = 0$ because $R_{\mathfrak{p}}$ is a field and $\mathfrak{p}R_{\mathfrak{p}}$ is the unique maximal ideal in $R_{\mathfrak{p}}$. This contradicts the one-to-one correspondence. Thus $\mathfrak{p}$ has to be a minimal prime ideal. $\square$

Chapter 5

Properties of the family of colorings

THE subject of this chapter is the properties shared by the family of Colorings.

The following theorem is rather obvious.

Theorem 5.1. *A subring of a Coloring is itself a Coloring.*

The next theorem is an application of Lemma 3.3 and Theorem 3.11.

Theorem 5.2. *Let $\mathfrak{I}$ be a finite ideal in a Coloring R. Then $R/\mathfrak{I}$ is a Coloring.*

Lemma 5.3. *Let x be an element in a Coloring R. Then $R/\mathrm{Ann}x$ is a Coloring.*

Proof. Let $\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_n$ be a clique in $\bar{R} = R/\mathrm{Ann}x$. That is all elements are distinct and $\bar{r}_i\bar{r}_j = 0$ for $i \neq j$. Stated differently, $(r_i + \mathrm{Ann}x)(r_j + \mathrm{Ann}x) = r_ir_j + \mathrm{Ann}x = \mathrm{Ann}x$, which implies that $r_ir_j \in \mathrm{Ann}x$ or that $r_ir_jx = 0$ for $i \neq j$. Furthermore, the elements $r_1x, r_2x, \ldots, r_nx$ are distinct :
Assume that $r_ix = r_jx$ $(i \neq j)$. Then $(r_i - r_j)x = 0$, therefore $r_i - r_j \in \mathrm{Ann}x$ or $r_i - r_j = r$ where $r \in \mathrm{Ann}x$. Now $r_i + \mathrm{Ann}x = (r_j + r) + \mathrm{Ann}x = r_j + \mathrm{Ann}x$, so that $\bar{r}_i = \bar{r}_j$ which contradicts their initial choice of being all distinct.

This shows that $r_1x, r_2x, \ldots, r_nx$ is a clique in R. Therefore with every clique in $\bar{R}$ we can associate a clique in R of the same size and since the sizes of the cliques in R are bounded by $\omega(R)$ the sizes of the cliques in $\bar{R}$ will also be bounded by $\omega(R)$. Consequently $\omega(\bar{R}) \leq \omega(R) < \infty$. From Theorem 3.11 we then have that $\bar{R} = R/\mathrm{Ann}x$ is also a Coloring. $\square$

Theorem 5.4. *Let $\mathfrak{I}$ be a finite ideal in a Coloring R and $x \in R$. Then $R/(\mathfrak{I} : x)$ is a Coloring.*

Proof. From Lemma 5.3 we have that $R/\mathrm{Ann}x$ is a Coloring. Lemma 4.2 yields that $\mathfrak{I} : x/\mathrm{Ann}x$ is a finite ideal in $R/\mathrm{Ann}x$. Theorem 5.2 then implies that $(R/\mathrm{Ann}x)/(\mathfrak{I} : x/\mathrm{Ann}x)$ is a Coloring. We also have that $(R/\mathrm{Ann}x)/(\mathfrak{I} : x/\mathrm{Ann}x) \cong R/(\mathfrak{I} : x)$ [11], which is the desired result. $\square$

Theorem 5.5. *A finite product of Colorings is a Coloring.*

Proof. We will consider the case of a product of two rings, the general result may be obtained through induction. Let $R = R_1 \times R_2$, where R_1 and R_2 are Colorings. Assume that $\omega(R_1) = n$ and that $\omega(R_2) = m$. Consider any clique C in R. If we project C onto R_1 we see that this projection cannot have more than n different elements as this would yield a clique with more than $n = \omega(R_1)$ elements. The same holds if we project C onto R_2, but in this case there cannot be more than m elements. Since the elements in C are of the form (c_1, c_2) with $c_1 \in R_1$ and $c_2 \in R_2$ we conclude that $|C| \leq nm$. Therefore $\omega(R) \leq nm$ and Theorem 3.11 then implies that R is a Coloring. $\square$

The following theorem is a generalisation of Lemma 5.3.

Theorem 5.6. *If $\mathfrak{I}$ is a finitely generated ideal in a Coloring R, then $R/\mathrm{Ann}\mathfrak{I}$ is a Coloring.*

Proof. Let $\mathfrak{I} = (x_1, x_2, \ldots, x_n)$. Then $\mathrm{Ann}\mathfrak{I} = \mathrm{Ann}x_1 \cap \mathrm{Ann}x_2 \cap \cdots \cap \mathrm{Ann}x_n$:
Let $r \in \mathrm{Ann}\mathfrak{I}$. Since $x_1, x_2, \ldots, x_n \in \mathfrak{I}$, $rx_1 = rx_2 = \cdots = rx_n = 0$ and therefore $r \in \mathrm{Ann}x_1 \cap \mathrm{Ann}x_2 \cap \cdots \cap \mathrm{Ann}x_n$, i.e $\mathrm{Ann}\mathfrak{I} \subseteq \mathrm{Ann}x_1 \cap \mathrm{Ann}x_2 \cap \cdots \cap \mathrm{Ann}x_n$. Conversely, let $r \in \mathrm{Ann}x_1 \cap \mathrm{Ann}x_2 \cap \cdots \cap \mathrm{Ann}x_n$ and let $s \in \mathfrak{I}$. Then $s = \sum_i r_i x_i$ ($r_i \in R$ and $x_i \in \{x_1, x_2, \ldots, x_n\}$), so that $sr = \sum_i r_i x_i r = 0$. Therefore $r \in \mathrm{Ann}\mathfrak{I}$ and $\mathrm{Ann}x_1 \cap \mathrm{Ann}x_2 \cap \cdots \cap \mathrm{Ann}x_n \subseteq \mathrm{Ann}\mathfrak{I}$.

Using the result above we have the injection $R/\mathrm{Ann}\mathfrak{I} \longrightarrow R/\mathrm{Ann}x_1 \times R/\mathrm{Ann}x_2 \times \cdots \times R/\mathrm{Ann}x_n$, given by $r + \mathrm{Ann}\mathfrak{I} \longmapsto (r + \mathrm{Ann}x_1, r + \mathrm{Ann}x_2, \ldots, r + \mathrm{Ann}x_n)$:
Let $(r_1 + \mathrm{Ann}x_1, r_1 + \mathrm{Ann}x_2, \ldots, r_1 + \mathrm{Ann}x_n) = (r_2 + \mathrm{Ann}x_1, r_2 + \mathrm{Ann}x_2, \ldots, r_2 + \mathrm{Ann}x_n)$. Then $r_1 + \mathrm{Ann}x_1 = r_2 + \mathrm{Ann}x_1$, $r_1 + \mathrm{Ann}x_2 = r_2 + \mathrm{Ann}x_2$, $\ldots$ $r_1 + \mathrm{Ann}x_n = r_2 + \mathrm{Ann}x_n$. Thus $r_1 - r_2 \in \mathrm{Ann}x_1$, $r_1 - r_2 \in \mathrm{Ann}x_2$, $\ldots$, $r_1 - r_2 \in \mathrm{Ann}x_n$, so that $r_1 - r_2 \in \mathrm{Ann}x_1 \cap \mathrm{Ann}x_2 \cap \cdots \cap \mathrm{Ann}x_n = \mathrm{Ann}\mathfrak{I}$. Therefore $r_1 + \mathrm{Ann}\mathfrak{I} = r_2 + \mathrm{Ann}\mathfrak{I}$.

By Lemma 5.3 we know that each of the rings $R/\mathrm{Ann}x_i$ is a Coloring so that by Theorem 5.5 $R/\mathrm{Ann}x_1 \times R/\mathrm{Ann}x_2 \times \cdots \times R/\mathrm{Ann}x_n$ is a Coloring. The injection shows that $R/\mathrm{Ann}\mathfrak{I}$ is a subring of $R/\mathrm{Ann}x_1 \times R/\mathrm{Ann}x_2 \times \cdots \times R/\mathrm{Ann}x_n$ and from Theorem 5.1 we can then conclude that $R/\mathrm{Ann}\mathfrak{I}$ is also a Coloring. $\square$

Corollary 5.7. *Let R be a Noetherian ring whose nilradical is finite and let $\mathfrak{I}$ be any ideal in R. Then $\mathfrak{B}(\mathrm{Ann}\mathfrak{I})/\mathrm{Ann}\mathfrak{I}$ is finite.*

Proof. Note that $\mathfrak{B}(\mathrm{Ann}\mathfrak{I})/\mathrm{Ann}\mathfrak{I}$ is the nilradical of $R/\mathrm{Ann}\mathfrak{I}$. That is $\mathfrak{B}(R/\mathrm{Ann}\mathfrak{I}) = \mathfrak{B}(\mathrm{Ann}\mathfrak{I})/\mathrm{Ann}\mathfrak{I}$ (cf. Theorem 3.12 where we had a similar situation). By applying Theorem 3.11 we conclude that R is a Coloring (nilradical is finite and a finite intersection of prime ideals, cf. Theorem 1.3). Since every ideal in a Noetherian ring is finitely generated, Theorem 5.6 implies that $R/\mathrm{Ann}\mathfrak{I}$ is a Coloring. Furthermore, the nilradical of a Coloring is finite (Theorem 3.11), and this implies the result. □

Theorem 5.8. *Let S be a multiplicatively closed set in a Coloring R. Then $R_S = S^{-1}R$ is a Coloring. Moreover, $\chi(R_S) \leq \chi(R)$ and $\omega(R_S) \leq \omega(R)$.*

Proof. Let $\chi(R) = n$. To show that the graph of R_S is n-colourable (i.e that $\chi(R_S) \leq n$), it suffices to show that every finite subset is n-colourable [5].

Let $x_1, x_2, \ldots, x_m$ be a finite subset of R_S. Now any finite set in R_S can be brought to a common denominator (see the discussion in Chapter 1). Therefore we have $x_1 = r_1/s, x_2 = r_2/s, \ldots, x_m = r_m/s$. We will now show that the set $x_1, x_2, \ldots, x_m$ is n-colourable by associating with each element x_i an element $r'_i \in R$. Furthermore, $x_i x_j = 0$ if only if $r'_i r'_j = 0$, so that we may assign the same colours to the x_i's that were assigned to the r'_i's in a colouring of R :

If $x_i x_j = 0$ for $i \neq j$, then $(r_i/s)(r_j/s) = (r_i r_j)/s^2 \sim 0/s$. This means that there exists an $s'_{ij} \in S$ such that $s'_{ij}(r_i r_j s - 0s^2) = 0$, or that $s'_{ij} s r_i r_j \triangleq s_{ij} r_i r_j = 0$ (where $s_{ij} = s'_{ij} s \in S$). Let $t = \prod s_{ij}$, where the product is over all pairs (i, j), $i \neq j$ and $x_i x_j = 0$. Define, $r'_i \triangleq t r_i = (\prod s_{ij}) r_i$. Now $x_i x_j = 0 \iff r'_i r'_j = 0$:
Assume $x_i x_j = 0$, that is $(r_i r_j)/s^2 \sim 0/s$. From our discussion above we know that there exists an $s_{ij} \in S$ such that $s_{ij} r_i r_j = 0$. Therefore $r'_i r'_j = t r_i t r_j = \prod s_{ij} \prod s_{ij} r_i r_j = 0$. The last equality follows by taking the product between r_i, r_j and their corresponding s_{ij} such that $s_{ij} r_i r_j = 0$. Conversely, assume that $r'_i r'_j = (\prod s_{ij} r_i)(\prod s_{ij} r_j) = (\prod s_{ij})^2 r_i r_j = 0$. Therefore there exists an $s' \in S$ such that $s' r_i r_j = 0$ ($s' = (\prod s_{ij})^2$). Thus $s'(r_i r_j s - 0s^2) = 0$, so that $(r_i r_j)/s^2 = (r_i/s)(r_j/s) \sim 0/s$, that is $x_i x_j = 0$.
Furthermore the r'_i's are distinct :
If $r'_i = r'_j$, then $(\prod s_{ij}) r_i = (\prod s_{ij}) r_j$, which implies that $\prod s_{ij}(r_i - r_j) = 0$, so that there exists an $s' \in S$ such that $s'(r_i - r_j) = 0$. Thus $s'(r_i s - r_j s) = 0$. This implies that $r_i/s \sim r_j/s$, which is a contradiction to the fact that the x_i's are distinct.

If we now make the identification $x_i \leftrightarrow r'_i$, we see that we can colour x_i with the same colour as r'_i and still produce a valid colouring of $x_1, x_2, \ldots, x_n$. Since $\chi(R) = n$, we will need at most n colours to colour the set $\{x_1, x_2, \ldots, x_n\}$. This shows that an arbitrary set of R_S is n-colourable implying the theorem. □

Chapter 6

When is $\chi(R) = \omega(R)$?

THIS chapter discusses the conditions under which the chromatic and clique numbers of a ring are equal.

The following interesting fact is a direct consequence of the properties of a prime ideal.

Remark 6.1. *Let $\mathfrak{p}$ be a prime ideal in a Coloring R. If the elements in $\mathfrak{p}$ have been coloured, then we need at most one additional colour to colour the elements in $R\backslash\mathfrak{p}$. To see this we note that no two elements, say $x, y \in R\backslash\mathfrak{p}$, can have a product of zero. If this was the case, $xy = 0$, so that $xy \in \mathfrak{p}$, implying that $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$. Since neither one is in $\mathfrak{p}$ we have a contradiction. Therefore $xy \neq 0$ with the consequence that all elements in $R\backslash\mathfrak{p}$ may be coloured with the same colour.*

Next, we introduce the notion of a separating element to enable us to investigate the question : When is $\chi(R) = \omega(R)$?

Definition 6.2 (Separating Element). An element $x \in R$ is *separating* provided that $x \neq 0$ and $ab = 0$ imply $xa = 0$ or $xb = 0$ for $a, b \in R$.

Definition 6.3 ($\mathfrak{I}$-separating). Let $\mathfrak{I}$ be an ideal. An element $x \in \mathfrak{I}$ is *$\mathfrak{I}$-separating* provided that $x\mathfrak{I} \neq (0)$ and whenever $ab = 0$ $(a, b \in \mathfrak{I})$, then $xa = 0$ or $xb = 0$.

We would like to stress the following points

- The idea of x being separating is equivalent to x being R-separating.
- In the definitions above it is not required that $a \neq b$.
- If x is R-separating and $x \in \mathfrak{I}$, then x fails to be $\mathfrak{I}$-separating if $x\mathfrak{I} = (0)$ (i.e $x \in \mathrm{Ann}\mathfrak{I}$). If, however $x\mathfrak{I} \neq (0)$, then x is also $\mathfrak{I}$-separating.

The first point that we discuss is the existence of separating elements.

Proposition 6.4. *If* $\operatorname{Ann}x$ *is a prime ideal in a ring* R *and* $x \neq 0$, *then* x *is separating.*

Proof. Let $a, b \in R$ and assume $ab = 0$. Then $abx = 0$, so that $ab \in \operatorname{Ann}x$. Since $\operatorname{Ann}x$ is prime, $a \in \operatorname{Ann}x$ or $b \in \operatorname{Ann}x$, that is $xa = 0$ or $xb = 0$, implying that x is separating. □

Proposition 6.5. *A nonzero ideal* $\mathfrak{I}$ *in a Coloring* R *contains a separating element.*

Proof. Consider the family $\{\operatorname{Ann}x_i \mid x_i \in \mathfrak{I} \text{ and } x_i \neq 0\}$. This family is not empty seeing that $\mathfrak{I}$ is nonzero. Since R has a.c.c on annihilators (Theorem 4.4) we conclude that the family has at least one maximal element, say $\operatorname{Ann}x$. It may be verified, as in the proof of Theorem 3.9, that $\operatorname{Ann}x$ is prime. Therefore $\mathfrak{I}$ contains an element x such that $\operatorname{Ann}x$ is prime. By applying Proposition 6.4 we then see that x is separating. □

Theorem 6.6. *Let* $\mathfrak{I}$ *be an ideal in a Coloring* R *such that* $\mathfrak{I}$ *is not contained in the nilradical. Then* $\mathfrak{I}$ *contains an* $\mathfrak{I}$*-separating element.*

Proof. Consider the family $\{\operatorname{Ann}x_i \mid x_i \in \mathfrak{I},\ \mathfrak{I} \not\subseteq \operatorname{Ann}x_i \text{ and } x_i \neq 0\}$. This family is not empty :

Firstly, $\mathfrak{I} \neq (0)$ since $\mathfrak{I} \not\subseteq \mathfrak{B}(R)$. Therefore $\mathfrak{I}$ has nonzero elements. Assume now that $\mathfrak{I} \subseteq \operatorname{Ann}x_i$ for all nonzero x_i in $\mathfrak{I}$. Therefore $x_i\mathfrak{I} = (0)$ for all nonzero x_i in $\mathfrak{I}$. Thus $x_i^2 = 0$ for all nonzero x_i in $\mathfrak{I}$. This implies that every element in $\mathfrak{I}$ is nilpotent, which in turn implies that $\mathfrak{I} \subseteq \mathfrak{B}(R)$ — a contradiction.

Since R has a.c.c on annihilators, we conclude as in Proposition 6.5 that $\mathfrak{I}$ contains an element x such that $\operatorname{Ann}x$ is prime. This x is R-separating. Furthermore, $\mathfrak{I} \not\subseteq \operatorname{Ann}x$, so that $x\mathfrak{I} \neq (0)$. Therefore x is $\mathfrak{I}$-separating. □

Remark 6.7. *If* $\mathfrak{I}$ *is an ideal such that* $\mathfrak{I}^2 = (0)$, *then* $\mathfrak{I}$ *cannot contain any* $\mathfrak{I}$*-separating elements.*

Theorem 6.8. *Let* $\mathfrak{I}$ *be a principal ideal in a Coloring* R. *If* $\mathfrak{I}^2 \neq (0)$, *then* $\mathfrak{I}$ *contains an* $\mathfrak{I}$*-separating element.*

Proof. Let $\mathfrak{I} = Rx$. Note that $x^2 \neq 0$: if $x^2 = 0$ then $\mathfrak{I}^2 = (0)$. Consider the set $\{\operatorname{Ann}x^2r \mid r \in R \text{ and } x^2r \neq 0\}$. This set is not empty since $\operatorname{Ann}x^2$ (i.e $r = 1$) is a member of it. Since R has a.c.c on annihilators, we conclude that the set has a maximal element, say $\operatorname{Ann}x^2t$, which is also prime (following from the maximality, as we have seen before). Then xt is $\mathfrak{I}$-separating :

Let $a, b \in \mathfrak{I}$ and assume that $ab = 0$. Since $\mathfrak{I}$ is principal we can write $a = rx$ and $b = sx$.

Then $ab = rsx^2 = 0$, hence $rs \in \text{Ann}x^2t$. The fact that $\text{Ann}x^2t$ is prime implies that $r \in \text{Ann}x^2t$ or that $s \in \text{Ann}x^2t$. If $r \in \text{Ann}x^2t$, then $(rx)(xt) = a(xt) = 0$. Otherwise $s \in \text{Ann}x^2t$ and $(sx)(xt) = b(xt) = 0$. Furthermore, $(xt)x = x^2t \neq 0$ (following from the choice of t in the set above), so that $(xt)\mathfrak{I} \neq (0)$. □

The following lemma will be used a number of times in the subsequent work and clearly illustrates the importance of separating elements.

Lemma 6.9. *Let $\mathfrak{I}$ be an ideal in a Coloring and assume $x \in \mathfrak{I}$ is $\mathfrak{I}$-separating. Define $\mathfrak{I}' = \text{Ann}x \cap \mathfrak{I}$. Then the following hold :*

1. *If $x^2 = 0$ then $\omega(\mathfrak{I}') = \omega(\mathfrak{I})$ and $\chi(\mathfrak{I}') = \chi(\mathfrak{I})$.*

2. *If $x^2 \neq 0$ then $\omega(\mathfrak{I}') = \omega(\mathfrak{I}) - 1$ and $\chi(\mathfrak{I}') = \chi(\mathfrak{I}) - 1$.*

Proof. Assume first that $x^2 = 0$, therefore $x \in \text{Ann}x$, so that $x \in \mathfrak{I}'$. Let $\omega(\mathfrak{I}) = n$ and choose a maximal clique $C = \{y_1, y_2, \ldots, y_n\}$ in $\mathfrak{I}$.

If $x \in C$, say $x = y_1$, then $xy_2 = xy_3 = \cdots = xy_n = 0$, by the definition of a clique. Hence $y_2, y_3, \ldots, y_n \in \text{Ann}x$, therefore $y_2, y_3, \ldots, y_n \in \mathfrak{I}'$. Thus $C \subseteq \mathfrak{I}'$. $\omega(\mathfrak{I}') \leq \omega(\mathfrak{I})$ since $\mathfrak{I}' \subseteq \mathfrak{I}$ (i.e every clique in $\mathfrak{I}'$ is a clique in $\mathfrak{I}$ with the consequence that the sizes of cliques in $\mathfrak{I}'$ will be bounded by the clique number of $\mathfrak{I}$). Also, $C \subseteq \mathfrak{I}'$ implies that $\omega(\mathfrak{I}') \geq n = \omega(\mathfrak{I})$ (as we have shown that there exists at least one clique of size n in $\mathfrak{I}'$), i.e $\omega(\mathfrak{I}') \geq \omega(\mathfrak{I})$. Put together we get, $\omega(\mathfrak{I}') = \omega(\mathfrak{I})$. The same argument obviously applies when $x = y_i$ for any $i = 1, 2, \ldots, n$.

On the other hand if $x \notin C$, then $xC \neq (0)$ since C is a maximal clique. Assume that $xy_1 \neq 0$. By definition of a clique we have that $y_1y_2 = y_1y_3 = \cdots = y_1y_n = 0$. Since x is $\mathfrak{I}$-separating, $xy_1 = 0$ or $xy_2 = 0$, $xy_1 = 0$ or $xy_3 = 0, \ldots, xy_1 = 0$ or $xy_n = 0$, but since $xy_1 \neq 0$, $xy_2 = 0, xy_3 = 0, \ldots, xy_n = 0$. This shows that $y_2, y_2, \ldots, y_n \in \text{Ann}x$ and subsequently that $y_2, y_2, \ldots, y_n \in \mathfrak{I}'$, but also that $\{x, y_2, \ldots, y_n\}$ is a clique of size n in $\mathfrak{I}'$. By the same reasoning as above one can therefore conclude that $\omega(\mathfrak{I}') = \omega(\mathfrak{I})$. Again the same argument will work if $xy_i \neq 0$ for any $i = 1, 2, \ldots, n$.

Next we consider the chromatic numbers and still work under the assumption that $x^2 = 0$.

The fact that $\mathfrak{I}' \subseteq \mathfrak{I}$ implies that $\chi(\mathfrak{I}') \leq \chi(\mathfrak{I})$ — colour $\mathfrak{I}$ and then use these same colours to colour $\mathfrak{I}'$. To prove that $\chi(\mathfrak{I}) \leq \chi(\mathfrak{I}')$, colour $\mathfrak{I}'$ first. We will now extend the colouring to the whole $\mathfrak{I}$. If $y \in \mathfrak{I} \backslash \mathfrak{I}'$ we can colour y with the same colour as x. This is a valid colouring since :

- x and y are not adjacent : $y \in \mathfrak{I} \backslash \mathfrak{I}'$ implies that $y \notin \text{Ann}x$, therefore $xy \neq 0$.

- $y_1, y_2 \in \mathfrak{I}\backslash\mathfrak{I}'$ are not adjacent : if $y_1y_2 = 0$, then $xy_1 = 0$ or $xy_2 = 0$ (x is $\mathfrak{I}$-separating), but both are impossible ($y_i \notin \text{Ann}x$), so that $y_1y_2 \neq 0$.

- if there is a $z \in \mathfrak{I}'$ with the same colour as x, then any $y \in \mathfrak{I}\backslash\mathfrak{I}'$ is not adjacent to z : firstly $xz \neq 0$ (they were able to receive the same colour), secondly if $yz = 0$, then $xy = 0$ or $xz = 0$ (x is $\mathfrak{I}$-separating). Both are impossible so that $yz \neq 0$.

Therefore we were able to colour $\mathfrak{I}$ using the same set of colours that was used to colour $\mathfrak{I}'$. This shows that $\chi(\mathfrak{I}) \leq \chi(\mathfrak{I})'$. Combining the two inequalities we have that $\chi(\mathfrak{I}') = \chi(\mathfrak{I})$.

Assume now that $x^2 \neq 0$, that is $x \notin \mathfrak{I}'$.

Consider a clique, C', of maximal size ($\omega(\mathfrak{I}')$) in $\mathfrak{I}'$. All of the elements in C' are annihilators of x, so that x may be added to C' to form a clique in $\mathfrak{I}$ of size $\omega(\mathfrak{I}') + 1$. Since this is a clique in $\mathfrak{I}$, $\omega(\mathfrak{I}') + 1 \leq \omega(\mathfrak{I})$, or $\omega(\mathfrak{I}') \leq \omega(\mathfrak{I}) - 1$.

Conversely, let $C = \{y_1, y_2, \dots, y_n\}$ be a maximal clique in $\mathfrak{I}$. If $x \notin C$, then there exists a $y_i \in C$, $i \in \{1, 2, \dots, n\}$, such that $xy_i \neq 0$ (otherwise we can include x in the clique to obtain a clique of size greater than $\omega(\mathfrak{I})$). Without loss of generality we can assume that $y_i = y_1$, i.e $xy_1 \neq 0$. From the clique C we get that $y_1y_2 = y_1y_3 = \dots = y_1y_n = 0$. Since x is $\mathfrak{I}$-separating, $xy_1 = 0$ or $xy_2 = 0$, $xy_1 = 0$ or $xy_3 = 0$, $\dots$; $xy_1 = 0$ or $xy_n = 0$, but $xy_1 \neq 0$, so that $xy_2 = 0 = xy_3 = \dots = xy_n = 0$. This shows that $\{x, y_2, \dots, y_n\}$ is still a clique of maximal size in $\mathfrak{I}$. Therefore x can always be included in a clique of maximal size in $\mathfrak{I}$. Now, if $x \in C$, say $x = y_i$, then $xy_1 = xy_2 = \dots = xy_{i-1} = xy_{i+1} = \dots = xy_n = 0$, since C is a clique. This implies that $y_1, y_2, \dots, y_{i-1}, y_{i+1}, \dots, y_n \in \text{Ann}x$ which implies that $y_1, y_2, \dots, y_{i-1}, y_{i+1}, \dots, y_n \in \mathfrak{I}'$. Therefore $\{y_1, y_2, \dots, y_{i-1}, y_{i+1}, \dots, y_n\}$ is a clique which lies completely in $\mathfrak{I}'$. This clique has size $\omega(\mathfrak{I}) - 1$. Since it is a clique in $\mathfrak{I}'$, $\omega(\mathfrak{I}) - 1 \leq \omega(\mathfrak{I}')$. Combining the inequalities we have that $\omega(\mathfrak{I}') = \omega(\mathfrak{I}) - 1$.

Lastly, we now look at $\chi(\mathfrak{I})$ and $\chi(\mathfrak{I}')$ in the case of $x^2 \neq 0$ or $x \notin \mathfrak{I}'$.

We colour $\mathfrak{I}'$ first and then try to extend the colouring to $\mathfrak{I}$. Assume all elements in $\mathfrak{I}'$ has received a colour. Since $xy = 0$ for all $y \in \mathfrak{I}'$, x has to receive a unique colour (when colouring in $\mathfrak{I}$) i.e one that is different from all colours used in a colouring of $\mathfrak{I}'$. Furthermore if $y_1, y_2 \in \mathfrak{I}\backslash\mathfrak{I}'$ and $y_1, y_2 \neq x$, $xy_1 \neq 0$ and $xy_2 \neq 0$ since $y_1, y_2 \notin \text{Ann}x$. Also if $y_1y_2 = 0$, then $xy_1 = 0$ or $xy_2 = 0$ (x is $\mathfrak{I}$-separating), but neither one is possible so that $y_1y_2 \neq 0$. This shows that the elements in $\mathfrak{I}\backslash\mathfrak{I}'$ are independent (i.e not adjacent). Therefore they can all be assigned the same colour. Thus in colouring $\mathfrak{I}$ we need only one additional colour above those that were used in the colouring of $\mathfrak{I}'$. This shows that $\chi(\mathfrak{I}) \leq \chi(\mathfrak{I}') + 1$ or that $\chi(\mathfrak{I}') \geq \chi(\mathfrak{I}) - 1$.

To prove the converse, colour $\mathfrak{I}$ first. Now use these colours to colour $\mathfrak{I}'$ (i.e colour

$\mathfrak{I}$ and then remove the vertices in $\mathfrak{I}\backslash\mathfrak{I}'$; this leaves us with $\mathfrak{I}'$ that has been coloured). Since $xy = 0$ for all $y \in \mathfrak{I}'$, x had to receive a colour different from all of the elements in $\mathfrak{I}'$ (when colouring in $\mathfrak{I}$). Thus by restricting the colouring of $\mathfrak{I}$ to that of $\mathfrak{I}'$ we see that the colour of x will not appear among the colours found in $\mathfrak{I}'$, therefore we need one less colour. This implies that $\chi(\mathfrak{I}') \leq \chi(\mathfrak{I}) - 1$. The two inequalities together, therefore imply that $\chi(\mathfrak{I}') = \chi(\mathfrak{I}) - 1$. □

Theorem 6.10. *Let $\mathfrak{I}$ be an ideal in a Coloring R and $\{x_1, x_2, \ldots, x_n\}$ be a clique of $\mathfrak{I}$-separating elements. Define $k = |\{x_i \mid x_i^2 \neq 0\}|$ and $\mathfrak{I}' = \mathfrak{I} \cap \operatorname{Ann}(x_1, x_2, \ldots, x_n)$. Then $\omega(\mathfrak{I}') = \omega(\mathfrak{I}) - k$ and $\chi(\mathfrak{I}') = \chi(\mathfrak{I}) - k$.*

Proof. Define $A = \{x_i \mid x_i^2 \neq 0\}$, then $k = |A|$.

Let C' be a maximal clique in $\mathfrak{I}'$. We can adjoin each element of A to C' to form a clique of size $\omega(\mathfrak{I}') + k$ in $\mathfrak{I}$. Therefore $\omega(\mathfrak{I}) \geq \omega(\mathfrak{I}') + k$, that is $\omega(\mathfrak{I}') \leq \omega(\mathfrak{I}) - k$.

Conversely, let $C = \{y_1, y_2, \ldots, y_m\}$ be a maximal clique in $\mathfrak{I}$. If $x_1 \notin C$, then there exists a $y_j \in C$ such that $x_1 y_j \neq 0$. Now $y_j y_i = 0$ for all $i \neq j$ since the y_k's form a clique. Then $x_1 y_j = 0$ or $x_1 y_i = 0$, for each $i \neq j$ since x_1 is $\mathfrak{I}$-separating. Since $x_1 y_j \neq 0$, we have $x_1 y_i = 0$ for all $i \neq j$. Thus $C_1 = \{y_1, y_2, \ldots, y_{j-1}, x_1, y_{j+1}, \ldots, y_n\}$ is still a clique. We now consider x_2 and determine whether $x_2 \in C_1$. If $x_2 \notin C_1$ we can insert it into C_1 in the same manner as we did with x_1. We then get a clique C_2 with both x_1 and x_2 in C_2. Note that x_1 will still be included in C_2, because an element will be removed from C_1 (to get to C_2) only if its product with x_2 is nonzero and since $x_1 x_2 = 0$, x_1 will remain in C. By considering each element in $\{x_1, x_2, \ldots, x_n\}$ one at a time, we are able to form a clique C_n with $\{x_1, x_2, \ldots, x_n\} \subseteq C_n$. We also have $A \subseteq C_n$, since $A \subseteq \{x_1, x_2, \ldots, x_n\}$. This clique will still be of maximal size.

Now, $C_n\backslash A$ is a clique in $\mathfrak{I}'$ (each element in the remaining clique has product of zero with each element in $\{x_1, x_2, \ldots, x_n\}$) of size $\omega(\mathfrak{I}) - k$. Therefore $\omega(\mathfrak{I}') \geq \omega(\mathfrak{I}) - k$. Combining the results we get $\omega(\mathfrak{I}') = \omega(\mathfrak{I}) - k$.

We now discuss the chromatic numbers of $\mathfrak{I}$ and $\mathfrak{I}'$.

Colour $\mathfrak{I}'$ first (using $\chi(\mathfrak{I}')$ colours). We now extend the colouring to $\mathfrak{I}$. Note that every element in A is adjacent to every element in $\mathfrak{I}'$. Therefore when extending the colouring to $\mathfrak{I}$ each element of A will have to receive its own colour (and one that is different from every colour used in $\mathfrak{I}'$). Also $A \cap \mathfrak{I}' = \emptyset$ (since each $x_i \in A$ has $x_i^2 \neq 0$ and so $x_i \notin \operatorname{Ann}(x_1, x_2, \ldots, x_n)$), so that we need k additional colours for the elements in A. Now, consider a $y \in \mathfrak{I}\backslash\mathfrak{I}'$ and $y \notin A$. The fact that y is not in $\mathfrak{I}'$ implies that $y \notin \operatorname{Ann}(x_1, x_2, \ldots, x_n)$. Therefore there exists an x_i, $i \in \{1, 2, \ldots, n\}$, such that $yx_i \neq 0$. Assign y the same colour as this x_i. We still need to check whether $y_1, y_2 \in \mathfrak{I}\backslash\mathfrak{I}'$ and

$y_1, y_2 \notin A$ with y_1 and y_2 assigned the same colour can be adjacent. The fact that y_1 and y_2 were assigned the same colour implies that $y_1x_j \neq 0$ and $y_2x_j \neq 0$ for some x_j, $j \in \{1, 2, \dots, n\}$. Now if $y_1y_2 = 0$ (i.e they are adjacent), then $x_jy_1 = 0$ or $x_jy_2 = 0$ since x_j is $\mathfrak{J}$-separating, but since neither possibility is true we conclude that $y_1y_2 \neq 0$. That is, the y's in $\mathfrak{J}\backslash\mathfrak{J}'$ with $y \notin A$ can all be assigned the same colour as that x_j such that $yx_j \neq 0$.

Therefore to colour the remaining elements of $\mathfrak{J}$ we did not need more than the k additional colours. Thus $\chi(\mathfrak{J}) \leq \chi(\mathfrak{J}') + k$.

Now, colour $\mathfrak{J}$ (using $\chi(\mathfrak{J})$ colours). This automatically assigns colours to the elements of $\mathfrak{J}'$ ($\mathfrak{J}' \subseteq \mathfrak{J}$). Since all possible products between elements in A and elements in $\mathfrak{J}'$ are zero, all these elements are adjacent. Therefore the colours used in A will not appear in the colouring of the elements of $\mathfrak{J}'$. Also, we needed k colours to colour the elements of A since they are all adjacent. Therefore $\mathfrak{J}'$ can be coloured using $\chi(\mathfrak{J}) - k$ colours. That is $\chi(\mathfrak{J}') \leq \chi(\mathfrak{J}) - k$. This in combination with the previous inequality yields $\chi(\mathfrak{J}') = \chi(\mathfrak{J}) - k$. $\square$

Theorem 6.11. *Let $\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_n$ be the minimal prime ideals in a Coloring R. Let $\varepsilon(R) = |\{i \mid R_{\mathfrak{p}_i}$ is a field$\}|$. Then $\omega(R) = \omega(\mathfrak{B}(R)) + \varepsilon(R)$ and $\chi(R) = \chi(\mathfrak{B}(R)) + \varepsilon(R)$.*

Proof. Consider firstly the case in which R is a minimal prime ideal. From the definition of minimality this implies that there are no other prime ideals in R and consequently no other minimal prime ideals in R. Now R_R is not defined since the complement of R (which is used as the multiplicative set in the definition of a ring of fractions) is empty. Therefore $\varepsilon(R) = 0$. Furthermore $\mathfrak{B}(R) = R$, so that we do indeed get the desired equality.

We now consider the case of R not being a minimal prime ideal. By Theorem 4.5 we know that R has only a finite number of minimal prime ideals and further that $\mathfrak{p}_i = \text{Ann}x_i$ for some $x_i \in R$ for every $i = 1, 2, \dots, n$. Further this $x_i \neq 0$, since $\text{Ann}0 = R$, and R is not considered to be a minimal prime ideal. Lemma 3.8 then implies that $\{x_1, x_2, \dots, x_n\}$ is a clique. Furthermore if $a, b \in R$ and $ab = 0$, then $ab \in \text{Ann}x_i$ for $i = 1, 2, \dots, n$. Since $\text{Ann}x_i$ is prime we have that $a \in \text{Ann}x_i$ or $b \in \text{Ann}x_i$ for $i = 1, 2, \dots, n$. Thus $ax_i = 0$ or $bx_i = 0$ for every $i = 1, 2, \dots, n$ and so $\{x_1, x_2, \dots, x_n\}$ are R-separating. Therefore $\{x_1, x_2, \dots, x_n\}$ is a clique of R-separating elements.

We also have that $R_{\mathfrak{p}_i}$ is a field if and only if $x_i^2 \neq 0$:
Assume that $R_{\mathfrak{p}_i}$ is a field. Therefore the element $x_i/1$, with $x_i \neq 0$, is a unit of $R_{\mathfrak{p}_i}$. This implies that $x_i \notin \mathfrak{p}_i = \text{Ann}x_i$ (see Proposition 1.13), so that $x_i^2 \neq 0$. Conversely, if $x_i^2 \neq 0$, then $x_i \notin \text{Ann}x_i = \mathfrak{p}_i$. Recall that the maximal ideal of $R_{\mathfrak{p}_i}$ is $\mathfrak{p}_iR_{\mathfrak{p}_i} = \{p/s \mid p \in \mathfrak{p}_i$ and $s \in R\backslash\mathfrak{p}_i\}$. Let $p/s \in \mathfrak{p}_iR_{\mathfrak{p}_i}$, then $p/s = 0/s$ since $x_i(ps - 0s) = 0$ ($x_ip = 0$ and

$x_i \in R\backslash\mathfrak{p}_i$). Therefore the maximal ideal of $R_{\mathfrak{p}_i}$, $\mathfrak{p}_i R_{\mathfrak{p}_i} = (0)$, so that $R_{\mathfrak{p}_i}$ is a field.

We can now apply Theorem 6.10, noting the equivalence between $x_i^2 \neq 0$ and $R_{\mathfrak{p}_i}$ being a field, to yield the desired result. □

Theorem 6.11 shows that to decide whether $\chi(R) = \omega(R)$ for a given ring, one has to concentrate on the nilradical. The next theorem is an application of this idea to the special case of the nilradical being zero.

Theorem 6.12. *Let R be a reduced ($\mathfrak{B}(R) = (0)$) Coloring. Then $\omega(\mathfrak{I}) = \chi(\mathfrak{I})$ for any ideal $\mathfrak{I} \subseteq R$.*

Proof. If $\mathfrak{I}$ is the zero ideal we trivially have that $\omega(\mathfrak{I}) = \chi(\mathfrak{I})$. Therefore let $\mathfrak{I}$ be a nonzero ideal in R. This implies that $\mathfrak{I} \not\subseteq \mathfrak{B}(R) = (0)$. Theorem 6.6 then yields that $\mathfrak{I}$ has an $\mathfrak{I}$-separating element, say x. Also, $x^2 \neq 0$, since $x^2 = 0$ implies that x is nilpotent and therefore that $x \in \mathfrak{B}(R) = (0)$, i.e $x = 0$; this contradicts the fact that x is $\mathfrak{I}$-separating (specifically $x\mathfrak{I} = (0)$ instead of $x\mathfrak{I} \neq (0)$).

From Lemma 6.9 we now have that $\omega(\mathfrak{I}') = \omega(\mathfrak{I}) - 1$ and $\chi(\mathfrak{I}') = \chi(\mathfrak{I}) - 1$, where $\mathfrak{I}' = \mathfrak{I} \cap \operatorname{Ann} x$. The rest of the proof is by induction on $\omega(\mathfrak{I})$.

If $\omega(\mathfrak{I}) = 1$, the graph of $\mathfrak{I}$ is empty (no lines). This implies that $\chi(\mathfrak{I}) = 1$. Assume now that whenever $\omega(\mathfrak{I}) = n - 1$, $\chi(\mathfrak{I}) = \omega(\mathfrak{I})$. Now let $\omega(\mathfrak{I}) = n$, then $\omega(\mathfrak{I}') = n - 1$, so that $\chi(\mathfrak{I}') = \omega(\mathfrak{I}') = n - 1$ by the induction assumption. Also, $\chi(\mathfrak{I}) - 1 = \chi(\mathfrak{I}') = n - 1$, so that $\chi(\mathfrak{I}) = n$. Therefore $\chi(\mathfrak{I}) = \omega(\mathfrak{I})$. □

Theorem 6.13. *Let R be a Coloring which is a principal ideal ring. Then $\chi(\mathfrak{I}) = \omega(\mathfrak{I})$ for any ideal $\mathfrak{I}$ in R.*

Proof. We will show firstly that we can make a reduction to the case $\mathfrak{I} \subseteq \mathfrak{B}(R)$. Therefore assume that $\mathfrak{I} \not\subseteq \mathfrak{B}(R)$. $\mathfrak{B}(R) = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_n$, where the $\mathfrak{p}_i$'s are the minimal prime ideals — recall that R has only a finite number of minimal prime ideals, since it is a Coloring and also the nilradical is the intersection of all the minimal prime ideals. Then $\mathfrak{I} \not\subseteq \mathfrak{B}(R)$ implies that $\mathfrak{I} \not\subseteq \mathfrak{p}_i$ for at least one i. Therefore there exists an $x \in \mathfrak{I}$ such that $x \notin \mathfrak{p}_i$.

We now have that $\operatorname{Ann} x \subseteq \mathfrak{p}_i$: if $\operatorname{Ann} x \not\subseteq \mathfrak{p}_i$, there exists a $y \in \operatorname{Ann} x$ with $y \notin \mathfrak{p}_i$. Then $xy = 0 \in \mathfrak{p}_i$ and since $\mathfrak{p}_i$ is prime, $x \in \mathfrak{p}_i$ or $y \in \mathfrak{p}_i$, but neither possibility is true and we have a contradiction.

Consider the family $\{\operatorname{Ann} z \mid \operatorname{Ann} z \subseteq \mathfrak{p}_i,\ z \in \mathfrak{I} \text{ and } z \notin \mathfrak{p}_i\}$. We have just shown that it is not empty and since R has a.c.c on ideals $\operatorname{Ann} a$, the family contains a maximal element, say $\operatorname{Ann} z_i$. Furthermore this maximal ideal is prime. (The process of proving this is the same as in Theorem 3.9. Note that the z's appearing above are nonzero since $z \notin \mathfrak{p}_i$.) Since $\mathfrak{p}_i$ is minimal we have that $\mathfrak{p}_i = \mathrm{Ann} z_i$. Now if $a, b \in \mathfrak{I}$ and $ab = 0$, then $ab \in \mathfrak{p}_i$, so that $a \in \mathfrak{p}_i$ or $b \in \mathfrak{p}_i$ since $\mathfrak{p}_i$ is prime. Thus $a \in \mathrm{Ann} z_i$ or $b \in \mathrm{Ann} z_i$. In other words $az_i = 0$ or $bz_i = 0$. Therefore z_i is $\mathfrak{I}$-separating.

Define $\mathfrak{I}_i = \mathfrak{I} \cap \mathrm{Ann} z_i = \mathfrak{I} \cap \mathfrak{p}_i$. Then by Lemma 6.9 $\chi(\mathfrak{I}) = \omega(\mathfrak{I})$ if and only if $\chi(\mathfrak{I}_i) = \omega(\mathfrak{I}_i)$. Therefore the problem of proving $\chi(\mathfrak{I}) = \omega(\mathfrak{I})$ is reduced to the problem of proving $\chi(\mathfrak{I}_i) = \omega(\mathfrak{I}_i)$, where $\mathfrak{I}_i \subseteq \mathfrak{p}_i$.

Recall that we are still working with the situation $\mathfrak{I} \nsubseteq \mathfrak{B}(R)$. This implies that there exists a subset, $\mathfrak{p}_{i_1}, \mathfrak{p}_{i_2}, \dots \mathfrak{p}_{i_l}$, of the minimal prime ideals, $\mathfrak{p}_1, \mathfrak{p}_2, \dots \mathfrak{p}_n$, such that $\mathfrak{I} \nsubseteq \mathfrak{p}_{i_j}$, $j = 1, 2, \dots, l$. We now define a set of ideals as follows

$$\mathfrak{I}_k = \begin{cases} \mathfrak{I} \cap \mathfrak{p}_k \text{ if } \mathfrak{I} \nsubseteq \mathfrak{p}_k, \\ \mathfrak{I} \text{ if } \mathfrak{I} \subseteq \mathfrak{p}_k, \end{cases} \tag{6.1}$$

for $k = 1, 2, \dots, n$. Note also that by the process described above, $\mathfrak{p}_k = \mathrm{Ann} z_k$, where z_k is an $\mathfrak{I}$-separating element, for every $\mathfrak{p}_k$ such that $\mathfrak{I} \nsubseteq \mathfrak{p}_k$. Thus

$$\mathfrak{I}_k = \begin{cases} \mathfrak{I} \cap \mathfrak{p}_k = \mathfrak{I} \cap \mathrm{Ann} z_k \text{ if } \mathfrak{I} \nsubseteq \mathfrak{p}_k, \\ \mathfrak{I} \text{ if } \mathfrak{I} \subseteq \mathfrak{p}_k, \end{cases} \tag{6.2}$$

and $\mathfrak{I}_k \subseteq \mathfrak{p}_k$ for $k = 1, 2, \dots, n$. Therefore $\cap \mathfrak{I}_k \subseteq \mathfrak{B}(R)$. Further, $\cap_{k=1}^{n} \mathfrak{I}_k = \mathfrak{I} \cap (\cap_{t=1}^{l} \mathfrak{p}_{i_t}) = \mathfrak{I} \cap (\mathrm{Ann} z_{i_1} \cap \mathrm{Ann} z_{i_2} \cap \dots \cap \mathrm{Ann} z_{i_l}) = \mathfrak{I} \cap \mathrm{Ann}\{z_{i_1}, z_{i_2}, \dots, z_{i_l}\}$. Also, from Lemma 3.8 $\{z_{i_1}, z_{i_2}, \dots, z_{i_l}\}$ is a clique of $\mathfrak{I}$-separating elements. By Theorem 6.10 it follows (by putting $\cap \mathfrak{I}_k = \mathfrak{I}'$ in the theorem) that $\chi(\cap \mathfrak{I}_k) = \omega(\cap \mathfrak{I}_k)$ if and only if $\chi(\mathfrak{I}) = \omega(\mathfrak{I})$.

Therefore the problem of proving $\chi(\mathfrak{I}) = \omega(\mathfrak{I})$ for an ideal $\mathfrak{I} \nsubseteq \mathfrak{B}(R)$ is equivalent to proving $\chi(\mathfrak{I}) = \omega(\mathfrak{I})$ for an ideal $\mathfrak{I} \subseteq \mathfrak{B}(R)$. This shows that to prove the theorem we can always make a reduction to the case of an ideal contained in the nilradical. We now have to show that equality between the clique and chromatic numbers holds for any ideal in the nilradical.

Assume that $\mathfrak{I}$ is an ideal in R such that $\mathfrak{I} \subseteq \mathfrak{B}(R)$. Since R is a principal ideal ring we may assume that $\mathfrak{I} = Rx$. Now if $\mathfrak{I}^2 = (0)$, then $\mathfrak{I}$ is itself a clique and $\omega(\mathfrak{I}) = |\mathfrak{I}| = \chi(\mathfrak{I})$. If $\mathfrak{I}^2 \neq (0)$, then by Theorem 6.8 $\mathfrak{I}$ contains an $\mathfrak{I}$-separating element, say z_1. Let $\mathfrak{I}_1 = \mathfrak{I} \cap \mathrm{Ann} z_1$, then $\mathfrak{I}_1 \subseteq \mathfrak{I}$. If $\mathfrak{I}_1 = \mathfrak{I}$, then $\mathfrak{I} \cap \mathrm{Ann} z_1 = \mathfrak{I}$. This implies $\mathfrak{I} \subseteq \mathrm{Ann} z_1$, so that $\mathfrak{I} z_1 = (0)$, which is impossible because z_1 is $\mathfrak{I}$-separating. Therefore $\mathfrak{I}_1 \subset \mathfrak{I}$. From Lemma 6.9 it now follows that $\chi(\mathfrak{I}) = \omega(\mathfrak{I})$ if and only if $\chi(\mathfrak{I}_1) = \omega(\mathfrak{I}_1)$. By applying the process above to the ideal $\mathfrak{I}_1$, we will either conclude that $\mathfrak{I}_1$ is a clique (with equal clique and chromatic numbers), so that $\chi(\mathfrak{I}) = \omega(\mathfrak{I})$ or we will find an ideal $\mathfrak{I}_2 \subset \mathfrak{I}_1$ such that $\chi(\mathfrak{I}_1) = \omega(\mathfrak{I}_1)$ if and only if $\chi(\mathfrak{I}_2) = \omega(\mathfrak{I}_2)$. This process can only be repeated a finite number of times since $\mathfrak{B}(R)$ is finite by Theorem 3.11. (Otherwise we could have formed an infinite chain of ideals $\mathfrak{I} \supset \mathfrak{I}_1 \supset \mathfrak{I}_2 \supset \cdots$, contradicting the finiteness of $\mathfrak{B}(R)$.) Therefore we will eventually reach an ideal, $\mathfrak{I}_n$ (which may be the zero ideal), for which $\mathfrak{I}_n^2 = (0)$. This ideal is itself a clique, so that $\chi(\mathfrak{I}_n) = \omega(\mathfrak{I}_n) \Longrightarrow \chi(\mathfrak{I}_{n-1}) = \omega(\mathfrak{I}_{n-1}) \Longrightarrow \cdots \Longrightarrow \chi(\mathfrak{I}_1) = \omega(\mathfrak{I}_1) \Longrightarrow \chi(\mathfrak{I}) = \omega(\mathfrak{I})$. □

Note the following about the proof above. The first part showed that in order to determine whether $\chi(\mathfrak{I}) = \omega(\mathfrak{I})$, it was enough to consider ideals contained in the nilradical. This is valid in any ring (we did not use the fact that R was a principal ideal ring).

By replacing the condition that every ideal in a Coloring be principal (in Theorem 6.13) with another condition, we obtain the following theorem (which uses the same proof as Theorem 6.13).

Theorem 6.14. *Let R be a Coloring with the property that any ideal $\mathfrak{I}$ for which $\mathfrak{I}^2 \neq (0)$ contains an $\mathfrak{I}$-separating element. Then $\chi(\mathfrak{I}) = \omega(\mathfrak{I})$ for any ideal in R.*

Remark 6.15. *Let $\mathfrak{I}$ be the direct sum of two ideals, say $\mathfrak{I} = \mathfrak{I}_1 \oplus \mathfrak{I}_2$. If $\mathfrak{I}_1$ contains an $\mathfrak{I}_1$-separating element x_1, then x_1 is also $\mathfrak{I}$-separating. To see this, let $a = a_1 + a_2$, $b = b_1 + b_2 \in \mathfrak{I}$, where $a_1, b_1 \in \mathfrak{I}_1$ and $a_2, b_2 \in \mathfrak{I}_2$. If $ab = a_1b_1 + a_2b_2 = 0$ (keep in mind that cross products a_ib_j, $i \neq j$ are equal to zero; see [11]), then $a_1b_1 = a_2b_2 = 0$ [11]. Also, since x_1 is $\mathfrak{I}_1$-separating, $a_1x_1 = 0$ or $b_1x_1 = 0$, so that $(a_1 + a_2)(x_1 + 0) = a_1x_1 = 0$ or $(b_1 + b_2)(x_1 + 0) = b_1x_1 = 0$. Therefore $ax_1 = 0$ or $bx_1 = 0$ implying that x_1 is $\mathfrak{I}$-separating.*

The remark above implies that if R is a Coloring that is a finite product of rings, each satisfying the hypothesis of Theorem 6.14, then $\chi(\mathfrak{I}) = \omega(\mathfrak{I})$ for any ideal $\mathfrak{I}$ in R. From this we have the following theorem.

Theorem 6.16. *Let R be a Coloring which is a finite product of reduced rings and principal ideal rings. Then $\chi(\mathfrak{I}) = \omega(\mathfrak{I})$ for any ideal $\mathfrak{I} \subseteq R$.*

Proof. Assume that R is a Coloring and that $R = \bigoplus_{i=1}^n R_i$, where R_i is a reduced ring or a principal ideal ring. Let $\mathfrak{I}$ be an ideal in R, then $\mathfrak{I} = \bigoplus_{i=1}^n \mathfrak{I}_i$, where $\mathfrak{I}_i$ is an ideal in R_i [11]. If $\mathfrak{I}^2 = (0)$, then $\mathfrak{I}$ is itself a clique and $\omega(\mathfrak{I}) = |\mathfrak{I}| = \chi(\mathfrak{I})$. If $\mathfrak{I}^2 \neq (0)$, we can find an $\mathfrak{I}_i$-separating element x_i : note firstly that R_i is a subring of R and therefore itself a Coloring. Now use the proof of Theorem 6.12 or Theorem 6.13, depending on whether R_i is reduced or a principal ideal ring. From the remark preceding the theorem this x_i is also $\mathfrak{I}$-separating. Now from Theorem 6.14 we know that $\chi(\mathfrak{I}) = \omega(\mathfrak{I})$. □

Theorem 6.17. *Let R be a local Coloring whose maximal ideal is a principal ideal. Then R is reduced or a finite principal ideal ring.*

Proof. If R is finite, then R is a local Artinian ring (since any chain of descending ideals can only contain a finite number of ideals because R has only a finite number of elements). For a local Artinian ring we know that if the maximal ideal is principal then every ideal is principal (see Chapter 1). We are given that the maximal ideal is principal, therefore every ideal is principal. Thus R is a finite principal ideal ring.

We now assume that R is not finite and that R is not reduced and derive a contradiction.

Consider the following ideal $\mathfrak{I} = \{x \in R \mid x \text{ is finite i.e } xR \text{ is finite}\}$. Since $\chi(R) < \infty$ we have that $\omega(R) < \infty$, so that by Lemma 3.2 $\mathfrak{I}$ is finite. Also, if $\mathfrak{J}$ is a finite ideal and $x \in \mathfrak{J}$, then $xR \subseteq \mathfrak{J}$, that is xR is finite since $\mathfrak{J}$ is finite. This implies that x is finite and so $x \in \mathfrak{I}$, thus $\mathfrak{J} \subseteq \mathfrak{I}$. Therefore $\mathfrak{I}$ contains all finite ideals. In summary, $\mathfrak{I}$ is finite and is the unique maximal finite ideal.

By Theorem 3.11 we know that $\mathfrak{B}(R)$ is finite. Thus $\mathfrak{B}(R) \subseteq \mathfrak{I}$. By assumption $\mathfrak{B}(R) \neq (0)$ so that $\mathfrak{I} \neq (0)$.

Note also the following : If $\mathfrak{m}$ is the maximal ideal in a local ring R, then $\mathfrak{i} \subseteq \mathfrak{m}$ for all proper ideals $\mathfrak{i}$ of R. If there existed an ideal $\mathfrak{i}$ such that $\mathfrak{i} \nsubseteq \mathfrak{m}$ and $\mathfrak{i} \neq R$, consider the set $\{\mathfrak{J} \mid \mathfrak{J} \text{ is an ideal in } R, \mathfrak{i} \subseteq \mathfrak{J} \text{ and } 1 \notin \mathfrak{J}\}$. This set is not empty since $\mathfrak{i}$ is a member and by Zorn's Lemma we have a maximal element, say $\mathfrak{F}$. This $\mathfrak{F}$ is also a maximal ideal : if there existed an ideal $\mathfrak{G}$ such that $\mathfrak{F} \subset \mathfrak{G}$, then $1 \in \mathfrak{G}$ and $\mathfrak{G} = R$, therefore $\mathfrak{F}$ is maximal. This contradicts our assumption that $\mathfrak{m}$ is the unique maximal ideal, so that all ideals are contained in the maximal ideal. (Note that $\mathfrak{F} \neq \mathfrak{m}$ since $\mathfrak{i} \subseteq \mathfrak{F}$ and $\mathfrak{i} \nsubseteq \mathfrak{m}$.)

Let $B = \mathfrak{I} : t = \{r \in R \mid rt \in \mathfrak{I}\}$. Obviously $\mathfrak{I} \subseteq B$. Also, by the remark above $\mathfrak{I} \subseteq Rt$ (Rt is the unique maximal ideal). Further, $Bt \subseteq \mathfrak{I}$, by the defnition of B. Let $r \in \mathfrak{I}$, then since $\mathfrak{I} \subseteq Rt$, there exists an $r' \in R$ such that $r = r't$. Now if $r' \notin B$ then $r't \notin \mathfrak{I}$, specifically $r't$ cannot be equal to r. Therefore r' has to be in B and $r = r't \in Bt$. Thus $\mathfrak{I} \subseteq Bt$. Put together we have that $\mathfrak{I} = Bt$.

Up to now we have, $\mathrm{Ann}x = \mathfrak{p} = Rt$ $(x \in \mathfrak{I})$. Since $t \in Rt$ (R has unity), $t \in \mathrm{Ann}x$, so that $tx = 0$ or that $x \in \mathrm{Ann}t$ as well. Therefore the map $\mathfrak{I} \longrightarrow t\mathfrak{I}$ $(r \mapsto tr)$ cannot be one-to-one since the kernel has at least one nonzero element namely x. We always have that $t\mathfrak{I} \subseteq \mathfrak{I}$, but since the map is not one-to-one, $t\mathfrak{I} \subset \mathfrak{I}$. Recall also that $\mathfrak{I} \subseteq B$ (by the definition of B). If $\mathfrak{I} = B$, then $t\mathfrak{I} = tB = Bt \subset \mathfrak{I}$. This contradicts $\mathfrak{I} = Bt$, so that $\mathfrak{I} \subset B$.

Let $x_1, x_2 \in \mathrm{Ann}t$. Since $\mathrm{Ann}t \subseteq Rt$ (Rt unique maximal ideal), we have $x_1 = r_1 t$ and $x_2 = r_2 t$. Therefore $x_1 t = r_1 t^2 = 0$ and $x_2 t = r_2 t^2 = 0$, so that $x_1 x_2 = (r_1 t)(r_2 t) =$

$r_1 r_2 t^2 = 0$. Thus Ann$t$ is a clique and since R is a Coloring (with finite clique number), Ann$t$ has to be finite. Lemma 4.2 implies that $\mathfrak{I} : t/\text{Ann}t$ is finite. Also, $|\mathfrak{I} : t| = |\mathfrak{I} : t/\text{Ann}t| \times |\text{Ann}t|$ and since both terms are finite $|B| = |\mathfrak{I} : t|$ is finite, but $\mathfrak{I} \subset B$, contradicting the maximality of $\mathfrak{I}$.

Therefore R is either reduced or finite (and in this case, as we have seen, R is a principal ideal ring). □

Lemma 6.18. *Let R be an indecomposable Coloring. Assume that every maximal ideal which equals* Ann$x$ *for some $x \in \mathfrak{B}(R)$ is principal. Then R is reduced or a finite local principal ideal ring.*

Proof. Assume that R is not finite and that R is not reduced ($\mathfrak{B}(R) \neq (0)$). Using the same technique as in the proof of Brauer's Theorem (Theorem 1.10) we conclude that every finite ideal not contained in the nilradical has an idempotent $e \neq 0$. The use of this technique can be justified as follows. The proof presented in Chapter 1 needs a nonnilpotent element at the start of the proof. This element can be obtained in the present context since the finite ideal above is not completely contained in the nilradical (which contains all the nilpotent elements). Furthermore the proof requires the descending chain condition to obtain a contradiction in terms of an infinite descending chain of ideals that is constructed. In our present situation the finiteness of the ideal above will still yield the same contradiciton. (We cannot form an infinite descending chain of ideals inside a finite ideal.)

Furthermore $e \neq 1$ since the ideal is finite and R is infinite. This idempotent gives us a Peirce decomposition of R relative to e as, $R = eR \oplus (1-e)R$. This is a nontrivial decomposition : if $eR = R$, then $(1-e)R = (0)$. Therefore $(1-e) \times 1 = 0$, so that $e = 1$, which is impossible as stated above. If $(1-e)R = R$, then $eR = (0)$, so that $e \times 1 = e = 0$. Since this is not the case either, we can conclude that the decomposition is nontrivial. This contradicts the fact that R is indecomposable. Therefore every finite ideal is contained in $\mathfrak{B}(R)$, so that $\mathfrak{B}(R)$ is the unique maximal finite ideal.

Using the same idea as in the proof of Theorem 6.17, we can find a maximal ideal $\mathfrak{m} = \text{Ann}x$, $x \in \mathfrak{B}(R)$ (and note that $\mathfrak{m}$ is also prime). From the assumptions of the theorem we can furthermore say that $\mathfrak{m} = Rt$.

We now show that $\text{Ann}t = \text{Ann}\mathfrak{m} \subseteq \mathfrak{B}(R)$:
Assume that $\text{Ann}\mathfrak{m} \not\subseteq \mathfrak{B}(R)$. Then there exists a prime ideal $\mathfrak{p}$, such that $\text{Ann}\mathfrak{m} \not\subseteq \mathfrak{p}$. Thus there exists an $x_1 \in \text{Ann}\mathfrak{m}$ and $x_1 \notin \mathfrak{p}$. Therefore $x_1 rt = 0$ for all $r \in R$ ($\mathfrak{m} = Rt$), so that $x_1(rt) \in \mathfrak{p}$. Thus $x_1 \in \mathfrak{p}$ or $rt \in \mathfrak{p}$ for all $r \in R$, but since $x_1 \notin \mathfrak{p}$, $rt \in \mathfrak{p}$ for all $r \in R$. This implies that $Rt = \mathfrak{m} \subseteq \mathfrak{p}$. Now, $\mathfrak{m}$ is maximal and therefore $\mathfrak{m} \subset \mathfrak{p}$ is impossible. On

the other hand if $\mathfrak{p} = \mathfrak{m}$, then $\text{Ann}\mathfrak{m} \not\subseteq \mathfrak{p} = \mathfrak{m}$. This implies that $\text{Ann}\mathfrak{m} + \mathfrak{m} = R$, since $\mathfrak{m}$ is maximal. Let $x \in \text{Ann}\mathfrak{m} \cap \mathfrak{m}$ and assume that $x \neq 0$. From $\text{Ann}\mathfrak{m} + \mathfrak{m} = R$, we get $1 = y_1 + y_2$, where $y_1 \in \text{Ann}\mathfrak{m}$ and $y_2 \in \mathfrak{m}$. Therefore $x.1 = x(y_1 + y_2) = xy_1 + xy_2 = 0 + 0$ ($x \in \text{Ann}\mathfrak{m} \cap \mathfrak{m}$), so that $x = 0$ — a contradiction. Therefore $\text{Ann}\mathfrak{m} \cap \mathfrak{m} = (0)$. This implies that $R \cong \text{Ann}\mathfrak{m} \oplus \mathfrak{m}$. This decomposition is nontrivial : $\text{Ann}\mathfrak{m} \neq (0)$ since $\text{Ann}\mathfrak{m} \not\subseteq \mathfrak{B}(R)$. If $\text{Ann}\mathfrak{m} = R$, then $\mathfrak{m} = (0)$, that is $\mathfrak{m} = \text{Ann}x = (0)$, where $x \in \mathfrak{B}(R)$. Again $\mathfrak{m} \neq R$ so that $x \neq 0$, also $x \in \mathfrak{B}(R)$ implies that x is nilpotent. Say $x^n = 0$, where $n \in \mathbb{N}$ and n is the smallest such number. Then $x^{n-1} \neq 0$ and $x^{n-1}x = x^n = 0$. Therefore $x^{n-1} \in \text{Ann}x$ — a contradiction. Thus the decomposition is nontrivial. This is impossible since R is indecomposable. Since both cases lead to a contradiction, we have that $\text{Ann}t = \text{Ann}\mathfrak{m} \subseteq \mathfrak{B}(R)$.

This inclusion shows that $\text{Ann}t = \text{Ann}\mathfrak{m}$ is finite since $\mathfrak{B}(R)$ is finite. From Lemma 4.2 it follows that $\mathfrak{B}(R) : t/\text{Ann}t$ is finite and since $\text{Ann}t$ is finite, $\mathfrak{B}(R) : t$ is also finite.

We now show that $\mathfrak{B}(R) \subseteq \mathfrak{m} = \text{Ann}x = Rt$ since we will be using the same technique as in the proof of Theorem 6.17 (which requires this inclusion). There the stated inclusion followed from the fact that R was local, which is not the case in the present situation.

Let $r \in \mathfrak{B}(R)$ and assume that $r \notin \text{Ann}x$. Therefore $rx \neq 0$ so that $\text{Ann}rx \neq R$ (if $\text{Ann}rx = R$, then $1.(rx) = 0$ — a contradiction). We have that $\text{Ann}x \subseteq \text{Ann}rx$, but since $\text{Ann}x$ is maximal, $\text{Ann}x = \text{Ann}rx$. Furthermore $\text{Ann}rx = \text{Ann}r^2x$ for the same reason (as long as $r^2x \neq 0$). Thus $\text{Ann}x = \text{Ann}rx = \text{Ann}r^2x = \cdots$, as long as $r^kx \neq 0$. Since $r \in \mathfrak{B}(R)$, r is nilpotent. Therefore there exists a smallest $n \in \mathbb{N}$ such that $r^nx = 0$ and $r^{n-1}x \neq 0$. Now $r \in \text{Ann}r^{n-1}x$, so that $r \in \text{Ann}x$. Thus $rx = 0$ contradicting our assumption. Therefore $r \in \text{Ann}x = Rt$ and $\mathfrak{B}(R) \subseteq Rt$.

Using the same method as in the proof of Theorem 6.17 we can conclude that $\mathfrak{B}(R) = (\mathfrak{B}(R) : t)t$, $t\mathfrak{B}(R) \subset \mathfrak{B}(R)$ and $\mathfrak{B}(R) \subset \mathfrak{B}(R) : t$ (by putting $\mathfrak{B}(R)$ in this theorem equal to $\mathfrak{J}$ in Theorem 6.17). This is a contradiction since $\mathfrak{B}(R)$ is the unique maximal finite ideal.

This contradiction stems from the assumption made at the start of the proof. Therefore R is finite or reduced.

Now, if R is finite then R is an Artinian ring. Furthermore we know that R is then uniquely (up to isomorphism) a finite direct product of local Artinian rings (see Chapter 1). Our original assumption that R be indecomposable then yields that R itself should be local. Since the maximal ideal, $\mathfrak{m}$, is principal we know that every ideal is principal. Therefore R is a finite local principal ideal ring. □

Chapter 7

Rings of low chromatic number : $\chi(R) \leq 5$

In this chapter we show that $\chi(R) = \omega(R)$ for all $\chi(R) \leq 5$ or $\omega(R) \leq 4$. Using the earlier results we will firstly discuss the finite rings with $\chi(R) \leq 3$.

Proposition 7.1. *Given a Coloring R, then $\chi(R) = \omega(R)$ provided $\omega(R)$, $\chi(R) \leq 2$.*

Proof. Let $\chi(R) = 1$. Then from Proposition 2.1 we know that $R = (0)$ and $\omega(R) = 1$. Now, let $\omega(R) = 1$. This implies that there are no lines in the graph of R. Since 0 is always adjacent to all the nonzero elements, this shows that R does not have nonzero elements. Thus R is again the zero ring and $\chi(R) = 1$.

Let $\chi(R) = 2$. Proposition 2.2 implies that R is an integral domain, $R \cong \mathbb{Z}_4$, $R \cong \mathbb{Z}_2[x]/(x^2)$ or $R \cong \mathbb{Z}_2[x]/(x^2+1)$. For each of these possibilities we have that $\omega(R) = 2$. Consider now the case $\omega(R) = 2$. We always have that $\chi(R) \geq \omega(R)$, therefore $\chi(R) \geq 2$. Assume $\chi(R) > 2$, that is $\chi(R) \geq 3$. Since 0 is adjacent to every nonzero element, it has to receive its own colour. Also, there exist elements x_1 and x_2 in R such that $x_1, x_2 \neq 0$ and $x_1 x_2 = 0$ (for if they did not exist, $\chi(R) \leq 2$). Therefore $0, x_1$ and x_2 form a clique, so that $\omega(R) \geq 3$ — a contradiction. Therefore $\chi(R) = 2$. □

Proposition 7.2. *Let R be a Coloring. Then $\omega(R) = 3 \iff \chi(R) = 3$.*

Proof. It is enough to prove that $\omega(R) \leq 3 \iff \chi(R) \leq 3$. The reason for this is as follows : if $\omega(R) = 3$, then (if we have shown the above) $\chi(R) \leq 3$. Now, $\chi(R) \neq 1, 2$, by Proposition 7.1 ($\chi(R) = 1, 2 \Rightarrow \omega(R) = 1, 2$ — a contradiction). Therefore $\omega(R) = 3$ has to imply that $\chi(R) = 3$. Similarly, $\chi(R) = 3$ will imply $\omega(R) = 3$.

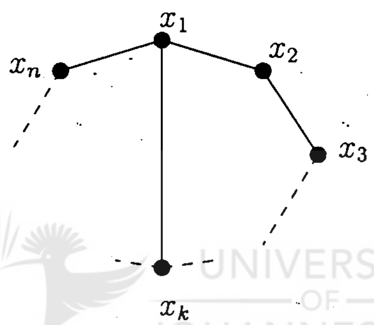
$\Leftarrow$:

Let $\chi(R) \leq 3$. Now it is always true that $\chi(R) \geq \omega(R)$. Therefore $\omega(R) \leq 3$.

$\Rightarrow$:

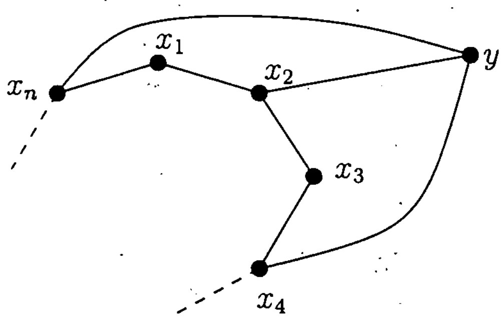
We will prove this assertion using its contrapositive, i.e $\chi(R) > 3 \Rightarrow \omega(R) > 3$. Let $\chi(R) > 3$ and define $R^* \doteq R\backslash\{0\}$. Then $\chi(R^*) \geq 3$, since 0 has its own colour in the graph of R (as it is adjacent to every nonzero element). Since R^* is not 2-colourable it has to contain an odd cycle (R^* is 2-colourable $\iff$ R^* is bipartite $\iff$ R^* does not contain an odd cycle). Let C be an odd cycle of minimum length, say n, in R^* with $C = x_1, x_2, x_3, \ldots, x_n, x_1$.

Assume that $n \geq 5$. We have that $x_1x_2 = x_2x_3 = \cdots = x_nx_1 = 0$. Suppose $x_1x_k = 0$, for some $k \neq 1, 2, n$. Then $x_1, x_2, x_3, \ldots, x_k, x_1$ and $x_1, x_k, x_{k+1}, \ldots, x_n, x_1$ are two cycles of length less than n, one of which has to be odd (for if both were even then C has to be even, which it is not). See the Figure below.



The argument above, using x_1, can obviously be applied to the other points of C as well. Now since C is the smallest odd cycle, no smaller odd cycles can exist, therefore $x_ix_j = 0$ only if x_i and x_j are neighbours (on C).

Now, let $y = x_1x_3$, then $yx_2 = yx_4 = yx_n = 0$.



Therefore y is adjacent to three vertices on C, so that y cannot be on C. (The points on C are only adjacent to two vertices on C i.e its neighbours.) At this point we have that $y, x_4, x_5, \ldots, x_n, y$ is an odd cycle of length $n-2$, but we know that C is the shortest odd cycle. This gives a contradiction to our assumption that $n \geq 5$. Thus $n < 5$, or in otherwords $n \leq 4$.

This shows that R^* has an odd cycle of length 3, say x_1, x_2, x_3, x_1. If we now again consider the graph of R, in which 0 is adjacent to every nonzero element, then we see that we have in fact a clique $\{x_1, x_2, x_3, 0\}$ in R of size four. Therefore $\omega(R) \geq 4$. $\square$

Theorem 7.3. *Let R be a Coloring and k an integer such that $k \leq 4$. Then $\chi(R) = k \iff \omega(R) = k$. Furthermore, $\chi(R) = 5 \implies \omega(R) = 5$.*

Proof. With the same reasoning as in Proposition 7.2, it is enough to show that $\chi(R) \leq k \iff \omega(R) \leq k$.

The first part of the proof of Proposition 7.2 can also be used here. We are therefore only left with the case $\omega(R) \leq k \implies \chi(R) \leq k$. Since the cases $k = 1, 2, 3$ were treated above, we need to show that $\omega(R) \leq 4 \implies \chi(R) \leq 4$. We will do this using the contrapositive, $\chi(R) > 4 \implies \omega(R) > 4$.

If R is reduced, then by Theorem 6.12, $\chi(R) = \omega(R)$. We will therefore assume that $\mathfrak{B}(R) \neq (0)$.

By Theorem 6.11 $\omega(R) = \omega(\mathfrak{B}(R)) + \varepsilon(R)$ and $\chi(R) = \chi(\mathfrak{B}(R)) + \varepsilon(R)$, with $\varepsilon(R)$ as in 6.11. Therefore we need to show that $\omega(\mathfrak{B}(R)) = \chi(\mathfrak{B}(R))$, with the restriction that $\omega(\mathfrak{B}(R)), \chi(\mathfrak{B}(R)) \leq 4$. The reason being that the present theorem only considers values of $\omega(R)$, $\chi(R) \leq 4$ and that $\varepsilon(R) \geq 0$.

Again, all that is left to prove is $\chi(\mathfrak{B}(R)) > 4 \implies \omega(\mathfrak{B}(R)) > 4$. ($\mathfrak{B}(R)$) is itself a Coloring so we may apply Propositions 7.1 and 7.2.)

We show firstly that $\mathfrak{B}(R)$ is nilpotent. Let $\mathfrak{B}(R) = \{r_1, r_2, \ldots, r_n\}$; $\mathfrak{B}(R)$ is finite since R is a Coloring. We know that every element in $\mathfrak{B}(R)$ is nilpotent, so $r_1^{m_1} = r_2^{m_2} = \cdots = r_n^{m_n} = 0$ for some $m_1, m_2, \ldots, m_n \in \mathbb{N}$. Let $m = \max\{m_1, m_2, \ldots, m_n\}$, so that $r_i^m = 0$ for all $i \in \{1, 2, \ldots, n\}$. Consider $\mathfrak{B}(R)^{mn} = \{\sum_i r_{i_1} r_{i_2} \cdots r_{i_{mn}} \mid r_{i_k} \in \mathfrak{B}(R)\}$. Every term in these sums can be written as $r_1^{l_1} r_2^{l_2} \cdots r_n^{l_n}$, by taking $l_i = 0$, if necessary. Now at least one $l_k \geq m$, for if every $l_k \leq m-1$, then every term in the sum will have at most $n(m-1)$ r's instead of nm r's. For this l_k, $r_k^{l_k} = 0$. Thus every term in every sum is zero, so that every possible sum is also zero. Therefore $\mathfrak{B}(R)^{mn} = (0)$.

Let $\mathfrak{J} = \mathfrak{B}(R) \cap \mathrm{Ann}\mathfrak{B}(R)$. Assume that $\mathfrak{J} = (0)$. That is for every nonzero element r of $\mathfrak{B}(R)$, $r\mathfrak{B}(R) \neq (0)$. ($\mathfrak{B}(R)$ does not contain its annihilators.) Since $\mathfrak{B}(R)$ is nilpotent, let $m \in \mathbb{N}$ be the smallest m with $\mathfrak{B}(R)^m = (0)$. That is $\mathfrak{B}(R)^{m-1} \neq (0)$. Let $r' \in \mathfrak{B}(R)^{m-1}$ and $r' \neq 0$. The inclusion $\mathfrak{B}(R)^{m-1} \subseteq \mathfrak{B}(R)$ implies that $r' \in \mathfrak{B}(R)$. From the observation above we should then have that $r'\mathfrak{B}(R) \neq (0)$, but $r'\mathfrak{B}(R) \subseteq \mathfrak{B}(R)^m = (0)$ — a contradiction. Therefore $\mathfrak{J} = \mathfrak{B}(R) \cap \mathrm{Ann}\mathfrak{B}(R) \neq (0)$. Thus $|\mathfrak{J}| \geq 2$.

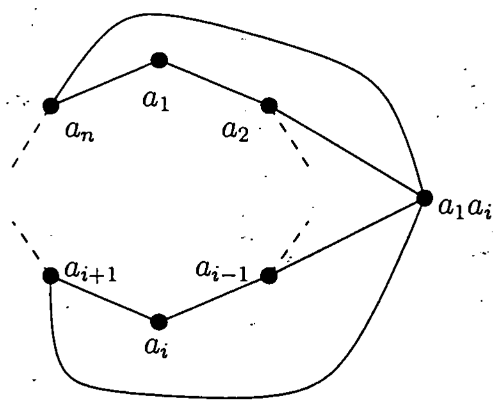
Note that $\mathfrak{J}$ is a clique in $\mathfrak{B}(R)$, since every element in $\mathfrak{J}$ is both in $\mathfrak{B}(R)$ and annihilates $\mathfrak{B}(R)$. If $\mathfrak{J} = \mathfrak{B}(R)$, then $\mathfrak{B}(R)$ is a clique and $\chi(\mathfrak{B}(R)) = \omega(\mathfrak{B}(R))$. If $|\mathfrak{J}| > 4$, then $\omega(\mathfrak{B}(R)) > 4$ and we are done. Therefore we assume that $\mathfrak{I} \subset \mathfrak{B}(R)$ and $|\mathfrak{I}| \leq 4$.

If $|\mathfrak{I}| = 4$, choose $x \in \mathfrak{B}(R) \backslash \mathfrak{I}$. Then $\mathfrak{I} \cup \{x\}$ is a clique with 5 elements so that $\omega(\mathfrak{B}(R)) > 4$.

If $|\mathfrak{I}| = 3$ and $\chi(\mathfrak{B}(R)) > 4$, then there must exist elements x and y in $\mathfrak{B}(R) \backslash \mathfrak{I}$ such that x and y are adjacent to each other as well as to every element in $\mathfrak{I}$. The reason for this is that all three elements in $\mathfrak{I}$ received its own colour ($\mathfrak{I}$ is a clique), but $\chi(\mathfrak{B}(R)) \geq 5$. The elements in $\mathfrak{B}(R) \backslash \mathfrak{I}$ are all adjacent to the elements in $\mathfrak{I}$ and if none of them were adjacent to one another, then four colours would have been enough to colour $\mathfrak{B}(R)$ — a contradiction. Now $\mathfrak{I} \cup \{x, y\}$ forms a clique in $\mathfrak{B}(R)$ with 5 elements. Thus $\omega(\mathfrak{B}(R)) > 4$.

Consider now the case $|\mathfrak{I}| = 2$ and $\chi(\mathfrak{B}(R)) > 4$. Let $\mathfrak{I} = \{0, c\}$. Now $\mathfrak{I}$ is an ideal so that $c + c = 0$. Since $\chi(\mathfrak{B}(R)) \geq 5$, $\mathfrak{B}(R) \backslash \mathfrak{I}$ requires at least 3 distinct colours. Therefore there exists an odd cycle in $\mathfrak{B}(R) \backslash \mathfrak{I}$ (see discussion in Chapter 1) and among all odd cycles let C be one of minimum length. Say $C = a_1, a_2, \ldots, a_n$. If $n = 3$, then $\{a_1, a_2, a_3\} \cup \mathfrak{I}$ is a clique of size 5 and thus $\omega(\mathfrak{B}(R)) > 4$. We may therefore assume that $n \geq 5$.

If $a_i a_k = 0$, where $k \neq i-1, i, i+1$, the cycle decomposes into two smaller cycles. One of these cycles will be odd and since C is the smallest odd cycle this is impossible. Thus the only way that $a_i a_k = 0$ is possible for $i \neq k$ is if a_i and a_k are neighbours on C. As in Proposition 7.2 the element $z = a_i a_j$ (a_i and a_j not neighbours) can not be on C. The reason being that it is adjacent to at least four vertices on C ($a_{i-1}, a_{i+1}, a_{j-1}, a_{j+1}$) instead of the required two (its neighbours). Let $i \neq 1, 2, n$. If i is even $a_1 a_i, a_2, a_3, \ldots, a_{i-1}$ is an odd cycle of length $i - 1 < n$. On the other hand if i is odd then $a_1 a_i, a_{i+1}, a_{i+2}, \ldots, a_n$ is an odd cycle of length $n - i + 1 < n$. See the Figure below.



Since C is the smallest odd cycle in $\mathfrak{B}(R) \backslash \mathfrak{I}$, the point $a_1 a_i$ has to be in $\mathfrak{I}$. Also, $a_1 a_i = 0$

only if a_1 and a_i are neighbours, therefore $a_1 a_i = c$. Thus in general for $i \neq j$

$$a_i a_j = \begin{cases} 0 & \text{only if } a_i \text{ and } a_j \text{ are neighbours,} \\ c & \text{otherwise.} \end{cases}$$

We now prove that $a_i^2 \neq 0$:
Assume that $a_i^2 = 0$ and $a_{i+1} \neq a_i + c$. Then a_i, a_{i+1} and $(a_i + c)$ form a cycle in $\mathfrak{B}(R)\backslash\mathfrak{I}$: clearly, a_i and a_{i+1} are in $\mathfrak{B}(R)\backslash\mathfrak{I}$. Furthermore if $a_i + c \in \mathfrak{I}$, then either $a_i + c = c$ in which case $a_i = 0$ or $a_i + c = 0$ so that $a_i = -c = c \in \mathfrak{I}$. Since both possibilities lead to a contradiction, $a_i + c \in \mathfrak{B}(R)\backslash\mathfrak{I}$. Consider now the possible products between these elements. Firstly $a_i a_{i+1} = 0$ since they are neighbours on C. Secondly $a_i(a_i + c) = a_i^2 + a_i c = 0 + 0$ ($a_i^2 = 0$ and $c \in \operatorname{Ann}\mathfrak{B}(R)$). Lastly $a_{i+1}(a_i + c) = a_{i+1}a_i + a_{i+1}c = 0 + 0$ (a_i and a_{i+1} are neighbours and $c \in \operatorname{Ann}\mathfrak{B}(R)$). This gives a cycle of length 3 in $\mathfrak{B}(R)\backslash\mathfrak{I}$ — a contradiction since C is an odd cycle of minimum length at least 5 in $\mathfrak{B}(R)\backslash\mathfrak{I}$. Thus either $a_i^2 \neq 0$ or $a_{i+1} = a_i + c$. If $a_{i+1} = a_i + c$, then $0 = a_{i+1}a_{i+2} = (a_i + c)a_{i+2} = a_i a_{i+2} + c a_{i+2} = a_i a_{i+2}$ (a_{i+1} and a_{i+2} are neighbours and $c \in \operatorname{Ann}\mathfrak{B}(R)$). Therefore $a_i a_{i+2} = 0$, but this contradicts the fact that a_i and a_{i+2} are not neighbours on C. The only possibility then that is left, is $a_i^2 \neq 0$ for $i \in \{1, 2, \ldots, n\}$.

Let $b = a_1 + a_2 + \cdots + a_{n-2}$. Then

$$\begin{aligned} b a_{n-1} &= a_1 a_{n-1} + a_2 a_{n-1} + \cdots + a_{n-3}a_{n-1} + a_{n-2}a_{n-1}, \\ &= c + c + \cdots + c + 0, \\ &= (n-3)c, \\ &= 0. \end{aligned}$$

The reason for $(n-3)c = 0$ is that $(n-3)$ is even and $c + c = 0$. Also,

$$\begin{aligned} b a_n &= a_1 a_n + a_2 a_n + \cdots + a_{n-3}a_n + a_{n-2}a_n, \\ &= 0 + c + \cdots + c + c, \\ &= (n-3)c, \\ &= 0. \end{aligned}$$

Now $a_n^2 \neq 0$ and $a_{n-1}^2 \neq 0$ so that $b \neq a_n$ and $b \neq a_{n-1}$. Since n is odd we are able to write $n = 2k+1$ for some $k \in \mathbb{Z}$. Consider

$$\begin{aligned} ba_k &= a_1a_k + a_2a_k \cdots + a_{k-1}a_k + a_k^2 + a_{k+1}a_k + \cdots + a_{2k-2}a_k + a_{2k-1}a_k, \\ &= c + c + \cdots + 0 + a_k^2 + 0 + \cdots + c + c, \\ &= 2(k-2)c + a_k^2, \\ &= a_k^2, \\ &\neq 0. \end{aligned}$$

This shows that $b \notin \operatorname{Ann}\mathfrak{B}(R)$ since $a_k \in \mathfrak{B}(R)$. Thus $b \notin \mathfrak{I}$, implying $b \neq 0$ and $b \neq c$.

All of the above leads to the fact that $\{0, a_{n-1}, a_n, b, c\}$ is a clique in $\mathfrak{B}(R)$ of size 5. Therefore $\omega(\mathfrak{B}(R)) > 4$.

We have thus shown that $\chi(\mathfrak{B}(R)) = \omega(\mathfrak{B}(R))$ for $\chi(\mathfrak{B}(R))$, $\omega(\mathfrak{B}(R)) \leq 4$. As discussed above we may now conclude from Theorem 6.11 that $\chi(R) = \omega(R)$ for $\chi(R)$, $\omega(R) \leq 4$.

We now prove the second part of the theorem, i.e $\chi(R) = 5 \Longrightarrow \omega(R) = 5$.

From Theorem 6.11 we have $\chi(R) = \chi(\mathfrak{B}(R)) + \varepsilon(R)$ and $\omega(R) = \omega(\mathfrak{B}(R)) + \varepsilon(R)$. If we can show that $\chi(\mathfrak{B}(R)) = \omega(\mathfrak{B}(R))$ (under the assumption that $\chi(R) = 5$) then $\omega(R) = 5$.

If $\chi(R) = 5$, $\chi(\mathfrak{B}(R)) \leq 5$, since $\mathfrak{B}(R)$ is a subring of R. We already know that if $\chi(\mathfrak{B}(R)) \leq 4$ then $\chi(\mathfrak{B}(R)) = \omega(\mathfrak{B}(R))$. Therefore we only consider the case $\chi(\mathfrak{B}(R)) = 5$. It is always true that $\chi(\mathfrak{B}(R)) \geq \omega(\mathfrak{B}(R))$, thus $\omega(\mathfrak{B}(R)) \leq 5$. From the proof above we know that if $\chi(\mathfrak{B}(R)) > 4$ (which therefore includes the present case), $\omega(\mathfrak{B}(R)) \geq 5$ — we constructed a clique with at least 5 elements. Combining the inequalities we get $\omega(\mathfrak{B}(R)) = 5$. Thus $\chi(\mathfrak{B}(R)) = \omega(\mathfrak{B}(R))$, so that $\chi(R) = 5 \Longrightarrow \omega(R) = 5$. □

Chapter 8

Examples of finite rings with $\chi(R) \leq 3$

IN this chapter we will find some finite rings rings with $\chi(R) \leq 3$.

Propositions 2.1 and 2.2 imply the following:

1. $\chi(R) = 1$ if and only if $R = (0)$.
2. $\chi(R) = 2$ if and only if
   (a) R is an integral domain,
   (b) $R \cong \mathbb{Z}_4$,
   (c) $R \cong \mathbb{Z}_2[x]/(x^2)$ or
   (d) $R \cong \mathbb{Z}_2[x]/(x^2+1)$.

Since we will be restricting our attention to *finite* rings, 2(a) then becomes a finite integral domain. It is well known that a finite integral domain is a finite field.

We now consider the case $\chi(R) = 3$. From Theorem 7.3 we have that $\chi(R) = 3$ if and only if $\omega(R) = 3$. Theorem 6.11 in turn says that $\omega(R) = \omega(\mathfrak{B}(R)) + \varepsilon(R)$, where $\varepsilon(R)$ is the number of minimal prime ideals $\mathfrak{p}$ such that $R_{\mathfrak{p}}$ is a field. Since $\omega(\mathfrak{B}(R)) \geq 1$, the possible values of $\varepsilon(R)$ are $0, 1$ and 2. We examine the cases $\varepsilon(R) = 2$ and $\varepsilon(R) = 1$.

Case $\varepsilon(R) = 2$.

In this case $\omega(\mathfrak{B}(R)) = 1$, so that $\mathfrak{B}(R) = (0)$. (0 is always in $\mathfrak{B}(R)$ and is also adjacent to everything else.) Since R is finite, R is Artinian. Then by Theorem 1.5, $\dim R = 0$.

Theorem 3.10 now implies that $(0) = \mathfrak{p}_1 \cap \mathfrak{p}_2$, where $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are minimal prime ideals. Note that we only have two ideals here since $\omega(R) = 3$ in the present situation, which in turn implies that $n = 2$ in Theorem 3.10. In an Artinian ring every prime ideal is maximal (Proposition 1.4). Therefore $(0) = \mathfrak{m}_1 \cap \mathfrak{m}_2$, where $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are maximal ideals in R (and $\mathfrak{p}_1 = \mathfrak{m}_1$ and $\mathfrak{p}_2 = \mathfrak{m}_2$). Now $\mathfrak{m}_1 + \mathfrak{m}_2 = R$, since $\mathfrak{m}_1 \subset \mathfrak{m}_1 + \mathfrak{m}_2$, $\mathfrak{m}_2 \subset \mathfrak{m}_1 + \mathfrak{m}_2$ and $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are maximal ideals.

We now define an isomorphism, $R \longrightarrow R/\mathfrak{m}_1 \times R/\mathfrak{m}_2$, as follows. Let $r \in R$, then there exist elements $m_1 \in \mathfrak{m}_1$ and $m_2 \in \mathfrak{m}_2$ such that $r = m_1 + m_2$.

$$\begin{aligned}
r = m_1 + m_2 \quad \mapsto \quad & ([m_1 + m_2] + \mathfrak{m}_1, [m_1 + m_2] + \mathfrak{m}_2)\,, \\
= \quad & ([m_1 + \mathfrak{m}_1] + [m_2 + \mathfrak{m}_1], [m_1 + \mathfrak{m}_2] + [m_2 + \mathfrak{m}_2])\,, \\
= \quad & (\mathfrak{m}_1 + [m_2 + \mathfrak{m}_1], [m_1 + \mathfrak{m}_2] + \mathfrak{m}_2)\,, \\
= \quad & ([m_2 + \mathfrak{m}_1], [m_1 + \mathfrak{m}_2])\,.
\end{aligned}$$

This mapping is **onto**:

If $(x + \mathfrak{m}_1, y + \mathfrak{m}_2) \in R/\mathfrak{m}_1 \times R/\mathfrak{m}_2$, $x, y \in R$, then there exist elements $x_1, y_1 \in \mathfrak{m}_1$ and $x_2, y_2 \in \mathfrak{m}_2$ such that $x = x_1 + x_2$ and $y = y_1 + y_2$. Thus

$$\begin{aligned}
(x + \mathfrak{m}_1, y + \mathfrak{m}_2) &= ([x_1 + x_2] + \mathfrak{m}_1, [y_1 + y_2] + \mathfrak{m}_2), \\
&= (x_2 + \mathfrak{m}_1, y_1 + \mathfrak{m}_2).
\end{aligned}$$

Therefore $y_1 + x_2 \mapsto (x_2 + \mathfrak{m}_1, y_1 + \mathfrak{m}_2) = (x + \mathfrak{m}_1, y + \mathfrak{m}_2)$ and $y_1 + x_2 \in R$.

This mapping is also **one-to-one**:

Let $a \mapsto (p + \mathfrak{m}_1, q + \mathfrak{m}_2)$ and $b \mapsto (r + \mathfrak{m}_1, s + \mathfrak{m}_2)$. That is $a \in R$, $a = q + p$, $b \in R$ and $b = s + r$. Note that $q, s \in \mathfrak{m}_1$ and $p, r \in \mathfrak{m}_2$. Assume $(p + \mathfrak{m}_1, q + \mathfrak{m}_2) = (r + \mathfrak{m}_1, s + \mathfrak{m}_2)$. Thus

$$\begin{aligned}
(p + \mathfrak{m}_1, q + \mathfrak{m}_2) \quad &= \quad (r + \mathfrak{m}_1, s + \mathfrak{m}_2), \\
\therefore p + \mathfrak{m}_1 = r + \mathfrak{m}_1 \quad &\text{and} \quad q + \mathfrak{m}_2 = s + \mathfrak{m}_2, \\
\therefore p - r \in \mathfrak{m}_1 \quad &\text{and} \quad q - s \in \mathfrak{m}_2.
\end{aligned}$$

But $p - r \in \mathfrak{m}_2$ and $q - s \in \mathfrak{m}_1$.

$$\begin{aligned}
\therefore p - r \in \mathfrak{m}_1 \cap \mathfrak{m}_2 \quad &\text{and} \quad q - s \in \mathfrak{m}_1 \cap \mathfrak{m}_2, \\
\therefore p - r = 0 \quad &\text{and} \quad q - s = 0, \\
\therefore p = r \quad &\text{and} \quad q = s.
\end{aligned}$$

We therefore have that $a = q + p = s + r = b$.

We will now show that this mapping is also a **homomorphism**:
Let $r_1, r_2 \in R$ with $r_1 = m_1 + m_2$ and $r_2 = m_1' + m_2'$, where $m_1, m_1' \in \mathfrak{m}_1$ and $m_2, m_2' \in \mathfrak{m}_2$. Then

$$\begin{aligned}
r_1 &\mapsto (m_2 + \mathfrak{m}_1, m_1 + \mathfrak{m}_2),\\
r_2 &\mapsto (m_2' + \mathfrak{m}_1, m_1' + \mathfrak{m}_2),\\
r_1 + r_2 &\mapsto ([m_2 + m_2'] + \mathfrak{m}_1, [m_1 + m_1'] + \mathfrak{m}_2),\\
&= ([m_2 + \mathfrak{m}_1] + [m_2' + \mathfrak{m}_1], [m_1 + \mathfrak{m}_2] + [m_1' + \mathfrak{m}_2],\\
&= (m_2 + \mathfrak{m}_1, m_1 + \mathfrak{m}_2) + (m_2' + \mathfrak{m}_1, m_1' + \mathfrak{m}_2).\\
r_1.r_2 &\mapsto ([m_2.m_2'] + \mathfrak{m}_1, [m_1.m_1'] + \mathfrak{m}_2),\\
&= ([m_2 + \mathfrak{m}_1].[m_2' + \mathfrak{m}_1], [m_1 + \mathfrak{m}_2].[m_1' + \mathfrak{m}_2]),\\
&= (m_2 + \mathfrak{m}_1, m_1 + \mathfrak{m}_2).(m_2' + \mathfrak{m}_1, m_1' + \mathfrak{m}_2).
\end{aligned}$$

Therefore $R \cong R/\mathfrak{m}_1 \times R/\mathfrak{m}_2$ and $R/\mathfrak{m}_1$ and $R/\mathfrak{m}_2$ are finite fields since $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are maximal ideals.

In summary, the case $\varepsilon(R) = 2$ corresponds to R being a direct product of two finite fields.

Case $\varepsilon(R) = 1$.

Let $\mathfrak{p}$ be a prime ideal such that $R_\mathfrak{p}$ is a field. (Recall that $\varepsilon(R)$ is the number of prime ideals, $\mathfrak{p}$, such that $R_\mathfrak{p}$ is a field.) As above, we have that $\dim R = 0$. Thus $\mathfrak{p}$ is both a maximal and minimal prime ideal (no chains of length greater than zero exist). Theorem 4.5 now implies that $\mathfrak{p} \in \mathrm{Ass}R$, the set of associated prime ideals. Let $\mathfrak{p} = \mathrm{Ann}x$. Since $R_\mathfrak{p}$ is a field and since $x/1 \in R_\mathfrak{p}$, $x/1$ has an inverse. Proposition 1.13 then implies that $x \notin \mathfrak{p}$.

Since $\mathfrak{p} = \mathrm{Ann}x$, $x\mathfrak{p} = (0)$, so that $x \in \mathrm{Ann}\mathfrak{p}$. Also if $xr \in (x)$ and $xr \in \mathfrak{p}$, then $r \in \mathfrak{p}$ ($x \notin \mathfrak{p}$ and $\mathfrak{p}$ is prime), but then $xr = 0$ ($r \in \mathfrak{p} = \mathrm{Ann}x$). Thus $\mathfrak{p} \cap (x) = (0)$. Furthermore, $\mathfrak{p} \subset \mathfrak{p} + (x)$ ($x \notin \mathfrak{p}$) and every prime ideal in an Artinian ring is maximal (Proposition 1.4). This implies that $\mathfrak{p} + (x) = R$. Also, $(x) \subseteq \mathrm{Ann}\mathfrak{p}$, so that $\mathfrak{p} + \mathrm{Ann}\mathfrak{p} = R$.

Furthermore $\mathfrak{p} \cap \mathrm{Ann}\mathfrak{p} = 0$:
Let $r \in \mathfrak{p} \cap \mathrm{Ann}\mathfrak{p} = 0$. Then $r \in \mathfrak{p}$ and $\mathfrak{p}r = 0$. We have that $\mathfrak{p} + \mathrm{Ann}\mathfrak{p} = R$, so that $1 = x + y$ where $x \in \mathfrak{p}$ and $y \in \mathrm{Ann}\mathfrak{p}$. Multiplying this by r we get $r = rx + ry = 0 + 0$ (the first zero follows from $r \in \mathrm{Ann}\mathfrak{p}$ and the second from $y \in \mathrm{Ann}\mathfrak{p}$).

By the same technique as for the case above, we now find that $R \cong R/\mathfrak{p} \times R/\mathrm{Ann}\mathfrak{p}$. Since $\mathfrak{p}$ is maximal, $R/\mathfrak{p}$ is a finite field, say F. Let $R/\mathrm{Ann}\mathfrak{p} = S$, so that $R \cong F \times S$. From our assumptions about the clique number of R we then have $\omega(F \times S) = 3$. We always have $\omega(F \times S) \leq \omega(F) \cdot \omega(S)$. Since $\omega(F) = 2$, because F is a finite field, $\omega(S) \geq 2$.

If $\omega(S) > 2$, then there exist at least three elements x_1, x_2 and x_3 all in S with $x_1x_2 = x_1x_3 = x_2x_3 = 0$. Now $(1,0), (0,x_1), (0,x_2)$ and $(0,x_3)$ are all in $F \times S$ and form a clique of size four. This contradicts $\omega(F \times S) = 3$. Therefore $\omega(S) = 2$, which implies $\chi(S) = 2$. By assumption $\varepsilon(R) = 1$ and $\omega(R) = 3$, so that $\omega(\mathfrak{B}(R)) = 2$. This shows that R is not reduced. By Proposition 2.2 then, $S \cong \mathbb{Z}_4$, $S \cong \mathbb{Z}_2[x]/(x^2)$ or $S \cong \mathbb{Z}_2[x]/(x^2+1)$.

Therefore $R \cong F \times \mathbb{Z}_4$, $R \cong F \times \mathbb{Z}_2[x]/(x^2)$ or $R \cong F \times \mathbb{Z}_2[x]/(x^2+1)$.

Chapter 9

An example of a ring with $\omega(R) < \chi(R)$

THE results in the previous chapters seem to indicate that $\omega(R) = \chi(R)$ for all Colorings. Indeed this was a conjecture first stated by Beck in [3]. In [1] Anderson and Naseer gave a counterexample to Beck's conjecture. It involves a finite local ring with $\omega(R) = 5$ and $\chi(R) = 6$. This chapter is devoted to a discussion of their counterexample.

The example given in [1] is :

$$R = \mathbb{Z}_4[x, y, z]/(x^2 - 2, y^2 - 2, z^2, 2x, 2y, 2z, xy, xz, yz - 2)$$

To ease in the discussion of R, let $\mathfrak{I} = (x^2 - 2, y^2 - 2, z^2, 2x, 2y, 2z, xy, xz, yz - 2)$. The formation of the factor ring by $\mathfrak{I}$ has the effect of restricting all polynomials in R to that of at most the first degree. The reason for this is the fact that all elements in $\mathfrak{I}$ may be regarded as being equal to zero. Thus $x^2 = 2, y^2 = 2$ and $z^2 = 0$. If a polynomial in R contains a term of the form ax^n, where $a \in \mathbb{Z}_4$ and $n \geq 2$, then $ax^n = ax^2x^{n-2} = 2ax^{n-2}$. This term (if $n - 2 \geq 2$) may be reduced still further. The end result will either be an element in $\mathbb{Z}_4$ (if n is even) or an element of the form bx, where $b \in \mathbb{Z}_4$. The same obviously applies to an element of the form ay^n. Using the same idea a term of the form az^n with $n \geq 2$, will be seen to be equal to zero.

Further, all cross products (i.e xy, xz and yz) will be either zero or reduced to a constant (by the same process as above). Therefore we only consider polynomials of the form

$$a_1x + a_2y + a_3z + a_4,$$

where $a_i \in \mathbb{Z}_4$ for $i = 1, 2, 3, 4$. The presence of $2x, 2y$ and $2z$ in $\mathfrak{I}$ allow for further simplifications, namely that of a_1, a_2 and a_3 not being equal to 2 (as equality implies that such a term equals zero). Also, $3x = 2x + x = 0 + x = x$ (and the same obviously applies to y and z). Therefore, a_1, a_2 and a_3 will not be equal to 3. Thus a_1, a_2 and $a_3 = 0, 1$ and $a_4 = 0, 1, 2, 3$. This implies that there are $2 \times 2 \times 2 \times 4 = 32$ possible elements in R.

Another property of this ring is that it is a local ring with maximal ideal given by

$$\mathfrak{M} = \{0, 2, x, x+2, y, y+2, x+y, x+y+2, z, z+2, x+z, x+z+2, y+z, y+z+2, \\ x+y+z, x+y+z+2\}.$$

The fact that $\mathfrak{M}$ is an ideal may be easily verfied by direct calculation. The other 16 elements of R are the units of R, $U(R)$ (i.e. they have multiplicative inverses). We now examine this fact more carefully. The elements in $R\backslash\mathfrak{M}$ are of the form

$$a_1x + a_2y + a_3z + a_4,$$

where (as before) a_1, a_2 and $a_3 = 0, 1$ and $a_4 = 1, 3$. (Note that the elements in $\mathfrak{M}$ correspond to the case of a_1, a_2 and $a_3 = 0, 1$ and $a_4 = 0, 2$.) Let $a_1x + a_2y + a_3z + 1$ be an element in $R\backslash\mathfrak{M}$ and consider the following product

$$\begin{aligned}(a_1x + a_2y + a_3z + 1)(a_1x + a_2y + a_3z + 3) = \\ a_1^2x^2 + a_1a_2xy + a_1a_3xz + 3a_1x \\ +a_1a_2xy + a_2^2y^2 + a_2a_3yz + 3a_2y \\ +a_1a_3xz + a_2a_3yz + a_3^2z^2 + 3a_3z \\ +a_1x + a_2y + a_3z + 3.\end{aligned}$$

Using the simplifications that are possible because of the ideal $\mathfrak{J}$ in the factor ring, we obtain the following

$$(a_1x + a_2y + a_3z + 1)(a_1x + a_2y + a_3z + 3) = 2a_1^2 + 2a_2^2 + 3.$$

Now, if either a_1 or a_2 equals zero (*not* both), then the product equals 1, indicating that the two elements from $R\backslash\mathfrak{M}$ above are multiplicative inverses of one another. We still need to discuss the case of both a_1 and a_2 being equal to zero or both being equal to 1. Towards this end consider the following product

$$\begin{aligned}(a_1x + a_2y + a_3z + a_4)(a_1x + a_2y + a_3z + a_4) = \\ a_1^2x^2 + a_1a_2xy + a_1a_3xz + a_1a_4x \\ +a_1a_2xy + a_2^2y^2 + a_2a_3yz + a_2a_4y \\ +a_1a_3xz + a_2a_3yz + a_3^2z^2 + a_3a_4z \\ +a_1a_4x + a_2a_4y + a_3a_4z + a_4^2.\end{aligned}$$

Again, the factor ring allows various simplifications, leading to

$$(a_1x + a_2y + a_3z + a_4)(a_1x + a_2y + a_3z + a_4) = 2a_1^2 + 2a_2^2 + a_4^2.$$

In this case if $a_1 = a_2 = 0, 1$ and $a_4 = 1, 3$, the product is 1. Therefore every element of this form is its own multiplicative inverse.

In summary then, an element of the form $a_1x + a_2y + a_3z + a_4$, where $a_1 = 0$, $a_2 = 1$ or $a_1 = 1$, $a_2 = 0$ and $a_4 = 1, 3$, has the element $a_1x + a_2y + a_3z + (a_4 + 2)$ as its multiplicative inverse. On the other hand the element $a_1x + a_2y + a_3z + a_4$, with $a_1 = a_2 = 0, 1$ and $a_4 = 1, 3$ is its own multiplicative inverse. This shows that every element in $R\backslash\mathfrak{M}$ has a multiplicative inverse. The elements in $\mathfrak{M}$ do not have multiplicative inverses. The reason for this is that if at least one element had an inverse, this element times its inverse yields 1, which should then be in $\mathfrak{M}$ since $\mathfrak{M}$ is an ideal. Since $1 \notin \mathfrak{M}$, no element in $\mathfrak{M}$ has a multiplicative inverse. This shows that $R\backslash\mathfrak{M} = U(R)$.

This fact also implies that $\mathfrak{M}$ is maximal since any ideal containing $\mathfrak{M}$ would have to include one of the units of R which would force this ideal to be equal to R. Futhermore, this fact also provides the motivation for $\mathfrak{M}$ being the *unique* maximal ideal. For any other ideal to be different from $\mathfrak{M}$ and to be maximal, it would have to include at least one unit and so will be forced to be equal to R. We also have $U(R) = R\backslash\mathfrak{M} = 1 + \mathfrak{M} = \{1 + m \mid m \in \mathfrak{M}\}$. This may be seen by realizing that $R\backslash\mathfrak{M} \cong \mathbb{Z}_2$ (since $|R| = 32$ and $|\mathfrak{M}| = 16$). Now $1 \notin \mathfrak{M}$ and so the only other equivalence class (apart from $\mathfrak{M}$) in $R\backslash\mathfrak{M}$ is $1 + \mathfrak{M}$. This is precisely all the elements in $R\backslash\mathfrak{M}$.

A multiplication table for $\mathfrak{M}$ is given in Table 9.1 on page 61. Note that 0 and 2 have been omitted from the table since they both annihilate $\mathfrak{M}$. It is easily seen that no other elements in R annihilate $\mathfrak{M}$. For elements in $\mathfrak{M}$ this is clear since there does not exist a column or row, in its multiplication table, entirely made up of zero's (which would indicate an annihilator of $\mathfrak{M}$). For elements in $R\backslash\mathfrak{M} = U(R) = 1 + \mathfrak{M}$ we have that $(1 + m_1)m_2 = m_2 + m_1m_2$, where $(1 + m_1) \in U(R)$ and $m_2 \in \mathfrak{M}$. If $m_1 = 0$, then $(1 + m_1) = 1$, which does not annihilate $\mathfrak{M}$. Further, if $m_2 = 0$, we obviously have $(1 + m_1)m_2 = 0$. This is the trivial case though and is not normally considered when determining annihilators. Therefore consider now the cases of $m_1, m_2 \neq 0$. Here we always have that $m_1m_2 = 0, 2$ (from table), so that $(1 + m_1)m_2 = m_2 + m_1m_2$ is either equal to m_2 or equal to $m_2 + 2$. Thus to have the product equal to zero, we ought to have $m_2 = 0$ or $m_2 = 2$ (so that $m_2 + 2 = 2 + 2 = 0$). The case of $m_2 = 0$ has already been dealt with and the case $m_2 = 2$ implies that $m_1m_2 = 0$ (since 2 annihilates $\mathfrak{M}$) so that the product is in fact $m_2 + m_1m_2 = 2 + 2m_1 = 2$. Therefore no element in $U(R)$ annihilates $\mathfrak{M}$ (or any nonzero element in $\mathfrak{M}$).

Our ultimate goal is to show that $\omega(R) = 5$ and $\chi(R) = 6$. In the first instance this is greatly simplified by the fact that all cliques of R must be contained in $\mathfrak{M}$. This is seen by noting that the product of two elements in $U(R)$ is never zero : let $(1 + m_1)$ and $(1 + m_2)$ be in $U(R)$. Then $(1 + m_1)(1 + m_2) = 1 + m_1 + m_2 + m_1m_2$. Neither m_1 nor m_2 is equal to 1 (they are elements in $\mathfrak{M}$) and $m_1m_2 = 0, 2$ (from table). Therefore the product above is either equal to $1+m_1+m_2$ or equal to $3+m_1+m_2$. To have the product equal to zero would imply that $m_1 + m_2$ would have to be equal to the additive inverse of 1 or 3. Since no two elements in $\mathfrak{M}$ sum to either 3 (the additive inverse of 1) or 1 (the additive inverse of 3), the product cannot be equal to zero. Also, as we have remarked earlier, no product of an element in $U(R)$ with an element in $\mathfrak{M}\backslash\{0\}$ equals zero. Thus all cliques in R are contained in $\mathfrak{M}$. Therefore to show that $\omega(R) = 5$, it suffices to show that $\omega(\mathfrak{M}) = 5$.

A *maximal* clique of R will be a clique that cannot be enlarged. The proof of $\omega(\mathfrak{M}) = 5$ follows from a case by case examination of possible cliques that may exist within $\mathfrak{M}$. This is accomplished by examining the elements in $\mathfrak{M}$ one at a time, with the end result that all cliques containing a given element in $\mathfrak{M}$ will have a maximum size of 5. Most of the following statements follow from the multiplication table.

Observation 9.1. *Every maximal clique of R contains* 0 *and* 2.

This is clear, since 0 and 2 are annihilators of $\mathfrak{M}$.

Observation 9.2. $\{0, 2, x, y, y + z\}$ *is a maximal clique, implying that* $\omega(R) \geq 5$.

Observation 9.3. *Any clique containing x or $x + 2$ has at most* 5 *elements.*

A clique will never contain both x and $x + 2$, since their product is not zero. We may therefore suppose that it contains x (the case of $x + 2$ being similar). The only possible elements that we can include in a clique along with x (besides 0 and 2) are : one of the pair y and $y + 2$ (not both since their product is nonzero), z, $z + 2$ and one of the pair $y + z$ and $y + z + 2$ (again not both since their product is nonzero). If we include z or $z + 2$ (or both) we have to exclude y, $y + 2$, $y + z$ and $y + z + 2$ (since their products with z and $z + 2$ are nonzero). In any case we have a clique size of at most 5.

Observation 9.4. *Any clique containing y or $y + 2$ has at most* 5 *elements.*

As before only one of y and $y + 2$ will be included in a clique. Suppose that a clique contains y. ($y + 2$ may be subjected to the same reasoning.) By Observation 9.3 we may assume that the clique does not contain x or $x + 2$ (if it did, it immediately yields a clique size of at most 5 — by the stated observation). Candidates for the clique, besides 0, 2 and y, include : one of the pair $y+z$ and $y+z+2$, $x+y+z$ and $x+y+z+2$. If we include $y+z$ or $y+z+2$, we must exclude $x+y+z$ and $x+y+z+2$ (the corresponding products are nonzero). This implies that the clique will contain at most 5 elements.

Observation 9.5. *Any clique containing $x+y$ or $x+y+2$ has at most 5 elements.*

We can assume that the clique contains 0, 2, $x+y$ and $x+y+2$. (They all have a product of zero and may be included if they were not originally.) The only possible candidates are : at most one of $x+z$ and $x+z+2$ (their product being nonzero) and at most one of $y+z$ and $y+z+2$. All possible products of the last four elements are nonzero, therefore we can only include at most one of $x+z$, $x+z+2$, $y+z$ and $y+z+2$. Thus the clique has a size of at most 5.

Observation 9.6. *Any clique containing z or $z+2$ has at most 5 elements.*

We may assume that our clique contains 0, 2, z and $z+2$, but that it does not contain x or $x+2$ (by Observation 9.3). The only other candidate element is one of $x+z$ and $x+z+2$. Therefore the clique has at most 5 elements.

Observation 9.7. *Any clique containing $x+z$, $x+z+2$, $y+z$ or $y+z+2$ has at most 5 elements.*

By the previous observations we may assume that the clique does not contain x, $x+2$, y, $y+2$, $x+y$, $x+y+2$, z or $z+2$. By consulting the mutiplication table we find that the clique will have at most 5 elements. (A lot less in some cases, but for the present situation the bound of 5 suffices.)

Observation 9.8. *Any clique containing $x+y+z$ or $x+y+z+2$ has at most 5 elements.*

This is most readily established by keeping in mind that we have already considered all possible elements in the previous observations.

All of the observations combined imply that $\omega(\mathfrak{M}) = \omega(R) \leq 5$. Therefore $\omega(R) = 5$.

Table 9.1: Muliplication Table for $\mathfrak{M}$

	x	$x+2$	y	$y+2$	$x+y$	$x+y+2$	z	$z+2$	$x+z$	$x+z+2$	$y+z$	$y+z+2$	$x+y+z$	$x+y+z+2$
x	2	2	0	0	2	2	0	0	2	2	0	0	2	2
$x+2$	2	2	0	0	2	2	0	0	2	2	0	0	2	2
y	0	0	2	2	2	2	2	2	2	2	0	0	0	0
$y+2$	0	0	2	2	2	2	2	2	2	2	0	0	0	0
$x+y$	2	2	2	2	0	0	2	2	0	0	0	0	2	2
$x+y+2$	2	2	2	2	0	0	2	2	0	0	0	0	2	2
z	0	0	2	2	2	2	0	0	0	0	2	2	2	2
$z+2$	0	0	2	2	2	2	0	0	0	0	2	2	2	2
$x+z$	2	2	2	2	0	0	0	0	2	2	2	2	0	0
$x+z+2$	2	2	2	2	0	0	0	0	2	2	2	2	0	0
$y+z$	0	0	0	0	0	0	2	2	2	2	2	2	2	2
$y+z+2$	0	0	0	0	0	0	2	2	2	2	2	2	2	2
$x+y+z$	2	2	0	0	2	2	2	2	0	0	2	2	0	0
$x+y+z+2$	2	2	0	0	2	2	2	2	0	0	2	2	0	0

We now establish the fact that $\chi(R) = 6$.

Since $\{0, 2, x, y, y+z\}$ forms a clique, we need at least five colours to colour R. We denoted these colours by $1, 2, 3, 4$ and 5 and assign these colours to the clique in this order. Since 0 and 2 are adjacent to every element in $\mathfrak{M}$ only 0 and 2 may receive the colours 1 and 2 (respectively), when considering other elements in $\mathfrak{M}$.

Consider the following subgraph of R : $\{0, 2, x, y, y+z, z, z+2, x+y, x+y+2, x+z\}$. A portion of this subgraph is shown in Figure 9.1. All elements except 0 and 2 are shown. The full subgraph may be obtained by adding 0 and 2 and joining them to every element. Next to each element its colour is indicated in brackets. This colouring is discussed below.

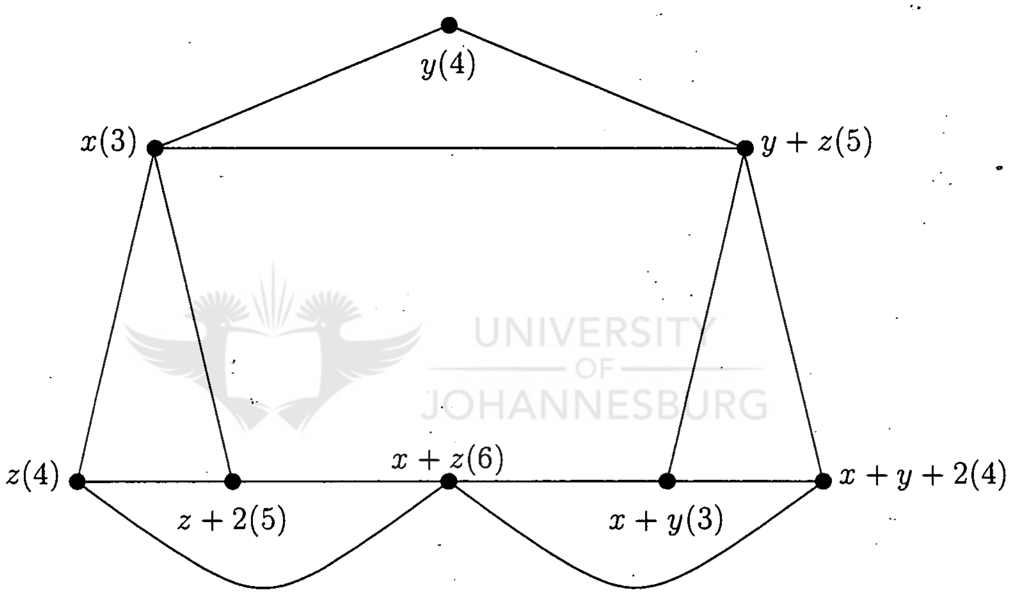


Figure 9.1: Colouring of the subgraph $\{x, y, y+z, z, z+2, x+y, x+y+2, x+z\}$

We show that it is impossible to colour this subgraph with fewer that 6 colours.

Observation 9.9. *Since $xz = x(z+2) = 0$ and $z(z+2) = 0$, we must colour one of the pair z and $z+2$ with 4 and one with 5. (The order is not important.)*

Colour z with 4 and colour $z+2$ with 5.

Observation 9.10. *$(x+y)(x+y+2) = 0$, so $x+y$ and $x+y+2$ must be coloured different colours. Also, $(y+z)(x+y) = (y+z)(x+y+2) = 0$, so again one of $x+y$ and $x+y+2$ must receive the colour 3 and the other the colour 4.*

Colour $x+y$ with 3 and $x+y+2$ with 4.

Observation 9.11. *Since $(x+z)(x+y) = (x+z)z = (x+z)(z+2) = 0$, $x+z$ cannot be coloured with $1, 2, 3, 4$ or 5.*

This necessitates that we assign $x+z$ a new colour (6).
The existence of this subgraph within R implies that $\chi(R) \geq 6$.
The following assignment of colours to the elements of R shows that $\chi(R) \leq 6$.

$$\begin{aligned}
1 &\rightarrow \{0\}\\
2 &\rightarrow \{2\} \cup U(R)\\
3 &\rightarrow \{x, x+2, x+y, x+y+z\}\\
4 &\rightarrow \{y, y+2, z, x+y+2\}\\
5 &\rightarrow \{y+z, y+z+2, z+2, x+y+z+2\}\\
6 &\rightarrow \{x+z, x+z+2\}
\end{aligned}$$

Note that within each colour class (that is the collection of elements that received the same colour) the elements are not adjacent, thus justifiying us assigning them the same colour.

We now have the following theorem.

Theorem 9.12 ([1]). *The ring*

$$R = \mathbb{Z}_4[x, y, z]/(x^2 - 2, y^2 - 2, z^2, 2x, 2y, 2z, xy, xz, yz - 2)$$

has $\omega(R) = 5$ and $\chi(R) = 6$.

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