VECTOR FIELD DECOMPOSITION AND FIRST INTEGRALS WITH APPLICATIONS TO NON-LINEAR SYSTEMS

by

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Summary

Roels [1] showed that on a two dimensional symplectic manifold, an arbitrary vector field can be locally decomposed into the sum of a gradient vector field and a Hamilton vector field. The Roels decomposition was extended to be applicable to compact even dimensional manifolds by Mendes and Duarte [2]. Some of the limitations of local decomposition are overcome by incorporating modern work on Hodge decomposition. This leads to a theorem which, in some cases, allows an arbitrary vector field on an even m-dimensional non-compact manifold to be decomposed into one gradient vector field and up to m-1 Hamiltonian vector fields. The method of decomposition is condensed into an algorithm which can be implemented using computer algebra. This decomposition is then applied to chaotic vector fields on non-compact manifolds [3]. This extended Roels decomposition is also compared to Helmholz decomposition in $\mathbf{R}^3$. The thesis shows how Legendre polynomials can be used to simplify the Helmholz decomposition in non-trivial cases. Finally, integral preserving iterators for both autonomous and non-autonomous first integrals are discussed [4]. The Hamilton vector fields which result from Roels' decomposition have their Hamiltonians as first integrals.

Contents

Chapter 1

Introduction

This thesis investigates the vector field decomposition given by Roels, and also the way this theory has been extended. Roels decomposition is compared to Helmholz decomposition in $\mathbf{R}^3$. The thesis also covers geometric integration with a focus on the preservation of first integrals. The relationship between vector decomposition and integral preserving iterators is discussed.

Roels [1] showed in 1974 that an arbitrary vector field on a two dimensional symplectic manifold can be locally decomposed into a Hamilton vector field and a gradient vector field. His work was used in 1981 by Mendes and Duarte [2] who extended the Roels' local decomposition to vector fields on even dimensional compact manifolds. The background to the work of the above authors is presented here. The theorems are elucidated and a number of related theorem are proved. The decomposition theory is extended to show that some vector fields can be globally decomposed on Euclidean spaces.

In modern differential geometry, one finds a number of different notations. This can be confusing to readers unfamiliar with a particular notation. Chapter two covers some of the most important mathematical concepts used throughout the rest of the paper. The notation used throughout the paper is clearly defined so as to avoid ambiguities. Chapter two also provides a few uncommon examples. The concepts of differential geometry presented here are used throughout the rest of the thesis.

Chapter three examines the theory of vector fields on smooth manifolds. Of particular importance are the Hamilton and gradient vector fields. This chapter presents some of the special properties of Hamilton and gradient vector fields. This is of importance as later chapters will show how an arbitrary vector field is decomposed into the sum of fields with special properties. It is these properties which provide the basis for applications of the theory in modern physics and applied mathematics.

Roels' decomposition theorem implicitly uses the Hodge de Rham decomposition theorem. The Hodge de Rham decomposition theorem provides the basis for Roels' decomposition in higher dimensions given by Mendes and Duarte [2]. Hodge de Rham theory is highly technical. Chapter four thus covers only certain aspects of this theory, including homology, de Rham cohomology and Hodges' decomposition theorem. Hodge's original decomposition theorem applies only to compact manifolds. Recently, his decomposition has been extended to some non-compact manifolds. Again, this work is highly technical and so only some aspects of the theory are covered which pertain to the Hodge decomposition in Euclidean space.

Using modern notation, Chapter five presents the extension of Roels' decomposition theorem to manifolds of any even dimension due to Mendes and Duarte. The full proof of Mendes and Duarte's local decomposition theorem is given for compact manifolds of any even dimension. A version for decomposition on non-compact Euclidean manifolds is presented. I also give a general step-by-step decomposition algorithm which can be implemented using computer algebra. This algorithm is then applied to the van der Pol system. Even the decomposition of a simple vector field into the Hamiltonain and gradient components requires long calculations. Therefore, the lengthy calculations for decompositions in the rest of the chapters are omitted.

Helmholz decomposition in three dimensions is well known to physicists. The actual decomposition of vector fields according to Helmholz's theorem proves very difficult in all but the simplest of cases. In chapter six Helmholz's theorem is given without proof. I show how Legendre polynomials can be used in the Helmholz decomposition of general vector fields [3]. I investigate analytic systems of autonomous first order differential equations and decompositions of the corresponding analytic vector fields according the decomposition algorithm given in chapter five. The Helmholz and Roels decompositions of a vector field in $\mathbf{R}^3$ are compared, and from this we see that they are distinct. In particular, I apply the decomposition theorem to dynamical systems with chaotic behaviour.

Chapter seven relates to the preservation of first integrals [4]. Many nonlinear dynamical systems expressed as autonomous systems of first order ordinary differential equations admit first integral and explicitly time-dependent first integrals. An example is a Hamilton system, where the Hamiltonian is a first integral. Under numerical integration these first integrals should be preserved. I discuss how this preservation can be extended to explicitly time-dependent first integrals. Also, I look at a method for finding iterators of increasing order of precision. This method can be applied to the Hamiltonian vector field component of any vector field on an even manifold, once the decomposition theorem has been applied.

Chapter 2

Mathematical concepts

This chapter provides an overview of the most important mathematical concepts used in later chapters. A number of different notations are used in modern differential geometry. One of the purposes of the mathematical overview presented in this chapter is to establish the notation used throughout the rest of the thesis. Properties and definitions given in this chapter are necessary for the understanding of the following chapters. In a some sections, applications of these concepts are given by way of example.

2.1 Important concepts and definitions

Definition. Let M_1 and M_2 be two topological spaces. A **homeomorphism** is a map

$$f : M_1 \to M_2$$

that is continuous and has an inverse

$$f^{-1} : M_2 \to M_1$$

that is also continuous. The spaces M_1 and M_2 are then said to be homoemorphic.

Definition. An m-dimensional **topological manifold** M is a second countable Hausdorff space satisfying the condition that for every point $\mathbf{x} \in M$, $\exists$ a neighbourhood U which is homeomorphic with an open subset of $\mathbf{R}^m$.

2.2 Vectors and vector fields

Let C be a smooth curve on a differentiable manifold of dimension m. If U is an open interval in $\mathbf{R}$ containing $\{0\}$, then the parameterization is given by

$$\gamma: U \rightarrow M.$$

Using local coordinates $\mathbf{x} = (x_1, x_2, ..., x_m)$, C is given by the smooth function $\gamma(t) = (\gamma_1(t), \gamma_2(t), \ldots, \gamma_m(t))$ for $t \in U$. At each point $\gamma(t)$ along curve C there is a tangent vector to the curve given by $\frac{d\gamma}{dt}(t)$. In local coordinates we have the coordinate representation of the tangent vector given as

$$\frac{d\gamma(t)}{dt} \equiv \left(\frac{d\gamma_1(t)}{dt}, \frac{d\gamma_2(t)}{dt}, \ldots, \frac{d\gamma_m(t)}{dt} \right).$$

Definition. The **tangent space** $TM|_{\mathbf{x}}$ to M at $\mathbf{x}$ is the vector space of tangent vectors to all possible smooth curves passing through point $\mathbf{x}$.
If the local coordinate chart $\mathbf{x} = \{x_i\}$ is used for the basis, then the basis vectors for $TM|_{\mathbf{x}}$ are given by

$$\left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_m} \right].$$

The tangent space $TM|_{\mathbf{x}}$ is not part of the manifold and can be thought of as a vector space attached to the manifold at $\mathbf{x}$.
Definition. The **tangent bundle** TM consists of all tangent spaces corresponding to all points $\mathbf{x} \in M$ i.e. $TM = \bigcup_{\mathbf{x} \in M} TM|_{\mathbf{x}}$. The tangent bundle TM is a smooth manifold with dimension $2m$.
Definition. A vector field Z on a manifold is a rule which assigns to each point $\mathbf{x} \in M$ a single element of $TM|_{\mathbf{x}}$ i.e. it assigns a vector to each point on M.

$$\begin{aligned} Z : M &\rightarrow TM \\ : \mathbf{x} &\mapsto TM|_{\mathbf{x}} \end{aligned}$$

Using local coordinates we can express the vectors of a vector field in terms of the coordinate basis. If $\mathbf{x} \in M$ and $\mathbf{Z} \in TM|_{\mathbf{x}}$, then we have

$$\mathbf{Z}(\mathbf{x}) = \xi_1(\mathbf{x}) \frac{\partial}{\partial x_1} + \cdots + \xi_m(\mathbf{x}) \frac{\partial}{\partial x_m}$$

where each ξ_i is a smooth real function on $\mathbf{x}$ and is called a coordinate function i.e.

$$\xi_i : M \to \mathbf{R}.$$

Definition. An **integral curve** of a vector field Z is a smooth parameterized curve $\phi(t)$ on M whose tangent vector at each point is equal to the vector assigned by the vector field at that point. So we have

$$\frac{d\phi}{dt}(t) = Z(\phi(t)).$$

The integral curve is thus a solution to the system of ordinary differential equations

$$\frac{d\phi_i}{dt}(\mathbf{x}) = \xi_i(\mathbf{x}); \quad i = 1, \ldots, m.$$

2.3 Differential forms

Definition. Let $TM_{\mathbf{x}}$ be the tangent space at $\mathbf{x} \in M$. Then the **dual space** or the **cotangent space** $T^*M|_{\mathbf{x}}$ is the space of differential 1-forms, each element of which is a function on $TM|_{\mathbf{x}}$. Let $\omega \in T^*M|_{\mathbf{x}}$, then

$$\begin{aligned} \omega &: TM \to \mathbf{R} \\ &: \mathbf{X} \mapsto \omega(\mathbf{X}) \equiv < \omega, \mathbf{X} > . \end{aligned}$$

Definition. Let $TM_{\mathbf{x}}$ be the tangent space at $\mathbf{x} \in M$. The space of **differential k-forms** at $\mathbf{x}$ is the vector space consisting on all k-linear alternating functions

$$\omega : TM|_{\mathbf{x}} \times TM|_{\mathbf{x}} \times \cdots \times TM|_{\mathbf{x}} \to \mathbf{R}$$

$$: (\mathbf{Z}_1 \times \mathbf{Z}_2 \times \cdots \times \mathbf{Z}_k) \mapsto < \omega; \mathbf{Z}_1, \mathbf{Z}_2, \cdots, \mathbf{Z}_k > .$$

The map ω is linear in each argument meaning

$$< \omega; \mathbf{Z}_1, \ldots, c\mathbf{Z}_i, \ldots, \mathbf{Z}_k > = c < \omega; \mathbf{Z}_1, \ldots, \mathbf{Z}_i, \ldots, \mathbf{Z}_k >$$

and

$$< \omega; \mathbf{Z}_1, \ldots, \mathbf{Z}_i + \mathbf{Z}_j, \ldots, \mathbf{Z}_k > = < \omega; \mathbf{Z}_1, \ldots, \mathbf{Z}_i, \ldots, \mathbf{Z}_k > + < \omega; \mathbf{Z}_1, \ldots, \mathbf{Z}_j, \ldots, \mathbf{Z}_k > .$$

Since ω is alternating it is an anti-symmetric tensor of type T_k^0. This gives

$$< \omega; \mathbf{Z}_{i_1}, \ldots, \mathbf{Z}_{i_k} >= \epsilon_{i_1 \ldots i_k} < \omega; \mathbf{Z}_1, \ldots, \mathbf{Z}_k >$$

where $\epsilon_{1\ldots k} = 1$ is a completely anti-symmetric tensor.
A 0-form is a smooth real valued function $f \in C(M)$.
Definition. A smooth **differential** k**-form** ω on M is a collection of smoothly varying k-linear maps $\{\omega\}_{\mathbf{x}}$ with $\omega_{\mathbf{x}} \in \bigwedge_k T^*M$ at each $\mathbf{x} \in M$, where we have the requirement that given k smooth vector fields $Z_1, Z_2, \ldots, Z_k$ on M, $< \omega; Z_1, \ldots, Z_k >$ is a smooth real-valued function on M i.e.

$$< \omega; Z_1, \ldots, Z_k >: M \to \mathbf{R}.$$

We say the form has rank k.
Definition. The **cotangent bundle** is the union of all cotangent vector spaces on M

$$T^*M = \bigcup_{\mathbf{x} \in M} T^*M|_{\mathbf{x}}$$

Using local coordinates $\{x_i\}$ on m-dimensional differentialble manifold M, the basis vectors for the cotangent space are denoted

$$\{dx_1, dx_2, \ldots, dx_m\}.$$

This set of basis vectors is dual to the basis vectors of the vector space $TM|_{\mathbf{x}}$, which as discussed above are

$$\left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_m} \right\}.$$

This means that we have chosen the vectors $\{dx_i\}$ so that

$$< dx_i; \frac{\partial}{\partial x_j} >= \delta_{ij}$$

where δ_{ij} is the Kronecker delta defined as $\delta_{ij} = 1$ when $i = j$ and $\delta_{ij} = 0$ when $i \neq j$.
Therefore, using local coordinates, any differential 1-form ω can be expressed as a linear combination of the basis vectors

$$\omega(\mathbf{x}) = \sum_{k=1}^{m} h_k(\mathbf{x}) dx_k$$

where each coordinate functions $h_i \in C^\infty$. For a vector field given by

$$\mathbf{Z}(\mathbf{x}) = \sum_{k=1}^{m} \xi(\mathbf{x}) \frac{\partial}{\partial x_k}$$

and a differentiable 1-form $\omega(\mathbf{x}) = \sum_{k=1}^{m} h_k(\mathbf{x}) dx_k$, we have the scalar product

$$< w; \mathbf{Z} > (\mathbf{x}) = \sum_{k=1}^{m} h_k(\mathbf{x}) \xi(\mathbf{x})$$

which is a smooth real-valued function on M.
Let f be a C^1 function on M. Then we can form the differential 1-form df and this is expressed as

$$df(\mathbf{x}) = \sum_{k=1}^{m} \frac{\partial f}{\partial x_k}(\mathbf{x})\ dx_k.$$

So we have

$$< df; \mathbf{Z} > (\mathbf{x}) = \mathbf{Z}(f)(\mathbf{x}).$$

Differential p-forms are commonly described using 2 notations. Let ω be a p-form. Then it is expressed in the 2 notations as:

1.

$$\omega = \sum_{i_1 < i_2 < \ldots < i_p}^{m} a_{i_1 \ldots i_p}\ dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_p}$$

or as

2.

$$\omega = \sum_{i_1, i_2, \ldots, i_p = 1}^{m} \alpha_{i_1 \ldots i_p} dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_p}$$

where $\alpha_{i_1 \ldots i_r \ldots i_s \ldots i_p} = -\alpha_{i_1 \ldots i_s \ldots i_r \ldots i_p}$ for any $1 \leq r \leq m, 1 \leq s \leq m$.

The relationship between the coefficients $a_{i_1 \ldots i_p}$ and $\alpha_{i_1 \ldots i_p}$ is determined by the fact that

$$\omega = \sum_{i_1 < i_2 < \ldots < i_p}^{m} a_{i_1 \ldots i_p} dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_p} = \frac{1}{p!} \sum_{i_1, i_2, \ldots, i_p = 1}^{m} \alpha_{i_1 \ldots i_p} dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_p}.$$

$\frac{1}{p!}$ is called the scaling coefficient. We see that $\alpha_{ij} = \frac{a_{ij}}{p!}$.

2.4 Exterior product

There is an algebraic operation $\wedge$, called the **exterior product**, that acts on forms of different ranks to produce forms of higher rank. Thus we have at $\mathbf{x} \in M$

$$\wedge : \bigwedge_j T^*M|_{\mathbf{x}} \times \bigwedge_k T^*M|_{\mathbf{x}} \to \bigwedge_{j+k} T^*M|_{\mathbf{x}}.$$

Since the result is a differential form, $\wedge$ is linear in each argument and alternating. Given a collection of k 1-forms $\{\omega_1, \ldots, \omega_k\}$, we can form a differential k-form

$$\omega = \omega_1 \wedge \ldots \wedge \omega_k.$$

This form acts on k vector fields $Z_1, \ldots, Z_k$ as follows

$$< \omega_1 \wedge \ldots \wedge \omega_k; Z_1, \ldots, Z_k >= \det(< \omega_j; Z_k >).$$

In local coordinates, the vector space $\bigwedge_k T^*M$ is spanned by the basis k-form

$$dx_I \equiv dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_k}$$

where $1 \leq i_1 < i_2 < \ldots < i_k \leq m$. The dimension of space $\bigwedge_k T^*M$ is $\binom{m}{k}$.

Any smooth differential form on a manifold M has a local coordinate expression

$$\omega(\mathbf{x}) = \sum_I \alpha_I(\mathbf{x}) dx_I$$

where I is a strictly increasing multi-index and each α_I is a smooth real valued function on M. Suppose

$$\omega = \omega_1 \wedge \ldots \wedge \omega_k, \qquad \sigma = \sigma_1 \wedge \ldots \wedge \sigma_l$$

where ω_i and σ_j are 1-forms for $i = 1, \ldots k$ and $j = 1, \ldots l$. Then their exterior product is the $(k+l)$-form

$$\omega \wedge \sigma = \omega_1 \wedge \ldots \wedge \omega_k \wedge \sigma_1 \wedge \ldots \wedge \sigma_l.$$

We also have

$$(c\omega + c'\omega') \wedge \sigma = c(\omega \wedge \sigma) + c'(\omega' \wedge \sigma)$$

$$\omega \wedge (c\sigma + c'\sigma') = c(\omega \wedge \sigma) + c'(\omega \wedge \sigma')$$

where $c, c' \in \mathbf{R}$.

The exterior product is associative and anti-commutative. Given the k-form ω and the l-form σ, we have

1. $\omega \wedge (\sigma \wedge \theta) = (\omega \wedge \sigma) \wedge \theta$

2. $\omega \wedge \sigma = (-1)^{kl} \sigma \wedge \omega$.

2.5 Contraction

Let $\omega \in \bigwedge_k T^*M$ and let Z be a smooth vector field on M. The inner product $\rfloor$ is a mapping from k-forms to $(k-1)$-forms. Let $Z(M)$ represent the set of all smooth vector fields on M. Let Z be a vector field on M i.e. $Z \in Z(M)$. Let $\mathbf{Z} \in TM|_{\mathbf{x}}$ be the vector given by the vector field Z at some $\mathbf{x} \in M$. Then

$$\rfloor : \bigwedge_k T^*M|_{\mathbf{x}} \times TM|_{\mathbf{x}} \to \bigwedge_{k-1} T^*M|_{\mathbf{x}}$$

$$: (\omega, \mathbf{Z}) \mapsto \mathbf{Z} \rfloor \omega.$$

We have

$$< Z \rfloor \omega; Z_1, \ldots, Z_{k-1} > = < \omega; Z, Z_1, \ldots, Z_{k-1} >$$

for every set of vector fields $Z_1, \ldots, Z_{k-1} \in Z(M)$. If $Z \in Z(M)$ and $f \in C^\infty(M)$ then define

$$Z \rfloor f = 0.$$

In term of basis vectors, we have

$$\frac{\partial}{\partial x_j} \rfloor dx_i = \delta_{ij}.$$

Let $f_i \in C^\infty(M)$ for $i = 1, 2, 3, 4$ and let $Y, Z \in Z(M)$. Then it follows from the linearity of k-forms that

$$(f_3 Z \rfloor + f_4 Y \rfloor)(f_1 \omega + f_2 \sigma) = f_1 f_3 (Z \rfloor \omega) + f_1 f_4 (Y \rfloor \omega) + f_2 f_3 (Z \rfloor \sigma) + f_2 f_4 (Y \rfloor \sigma).$$

Let $Z, Y \in Z(M), \omega \in \bigwedge_k T^*M, \sigma \in \bigwedge_l T^*M$. Then the properties below follow:

1. $Z \rfloor (Z \rfloor \omega) = (Z \rfloor Z) \rfloor \omega = 0$;

2. $Z \rfloor (\omega \wedge \sigma) = (Z \rfloor \omega) \wedge \sigma + (-1)^k \omega \wedge (Z \rfloor \sigma)$;

3. $Z \rfloor Y \rfloor \omega = -Y \rfloor Z \rfloor \omega$.

2.6 Exterior derivative

The exterior derivative d is a linear differential operator that takes k-forms to $k+1$-forms i.e.

$$d : \bigwedge_k T^*M \to \bigwedge_{k+1} T^*M$$

$$: \omega \mapsto d\omega.$$

Consider any smooth differential k-form ω on M, expressed in local coordinates as

$$\omega = \sum_I \alpha dx_I.$$

Then

$$d\omega = \sum_I d\alpha \wedge dx_I = \sum_I \left(\sum_j \frac{\partial \alpha_I}{\partial x_j} \right) \wedge dx_I = \sum_{I,j} \frac{\partial \alpha_I}{\partial x_j} \wedge dx_I.$$

The exterior derivative has the following properties:

1. Let $f \in \bigwedge_0 T^*M|_{\mathbf{x}}$. Then $df \in \bigwedge_1 T^*M|_{\mathbf{x}}$. If $\mathbf{Z} \in TM|_{\mathbf{x}}$ then

$$df(\mathbf{Z}) = \mathbf{Z}(f).$$

2. d is a linear map. Let $\omega \in \bigwedge_k T^*M|_{\mathbf{x}}$ and $\sigma \in \bigwedge_k T^*M|_{\mathbf{x}}$ with $c \in \mathbf{R}$ for some $\mathbf{x} \in M$. Then

$$d(\omega + \sigma) = d\omega + d\sigma$$

$$d(c\omega) = cd(\omega).$$

3. d is an anti-derivative. This means that d is not a Leibniz differential operator. Let $\omega \in \bigwedge_k T^*M|_{\mathbf{x}}$ and $\sigma \in \bigwedge_l T^*M|_{\mathbf{x}}$, then

$$d(\omega \wedge \sigma) = (d\omega) \wedge \sigma + (-1)^k \omega \wedge (d\sigma).$$

4. Poincaré Lemma. For any $\omega \in \bigwedge_k T^*M_{\mathbf{x}}$, we have

$$d(d\omega) = 0.$$

   If for any form σ, we have $d\sigma = 0$, then σ is said to be **closed**.

Let $\omega \in \bigwedge_k T^*M|_{\mathbf{x}}, k \geq 1$. If there exists some $\sigma \in \bigwedge_{k-1} T^*M|_{\mathbf{x}}$ such that $d\sigma = \omega$, then ω is said to be **exact**.

2.7 Lie derivative

The **Lie Derivative** gives the infinitesimal change of a geometrical object under the flow $e^{\epsilon Z}$ induced by a vector field Z on a differential manifold M. The geometrical object can be a smooth function, a smooth vector field, a smooth differential form or a smooth tensor field. So the Lie derivative is always associated with a vector field on the manifold and the Lie derivative with respect to vector field Z is denoted by the operator L_Z.

Definition. Let Z be a vector field defined on M and let $\{\phi(t)\}$ be the local 1-parameter transformation group generated by Z. The Lie derivative with respect to Z of a C^∞ function f on M is defined as

$$(L_Z f)(\mathbf{x}) := \lim_{t\to 0} \frac{(\phi^*(t)f)(\mathbf{x}) - f(\mathbf{x})}{t} = \frac{d}{dt} f(\phi(\mathbf{x},t))\Big|_{t=0}$$

where $\phi^*(t)$ denotes the pull-back map.

Definition. Let Z be a vector field defined on M and let $\{\phi(t)\}$ be the local 1-parameter transformation group generated by Z. The Lie derivative with respect to Z of a k-form ω on M is the k-form

$$(L_Z \omega)(\mathbf{x}) := \lim_{t\to 0} \frac{(\phi^*(t)\omega)(\mathbf{x}) - \omega(\mathbf{x})}{t}$$

where $\phi^*(t)$ denotes the pull-back map.

The Lie derivative of a k-form can be written in terms of the exterior derivative d and the inner product $\rfloor$ as

$$L_Z\omega = Z\rfloor d\omega + d(Z\rfloor\omega).$$

If $f \in \bigwedge_0 T^*M$ then

$$L_Z f = Z\rfloor df + d(Z\rfloor f)) = Z\rfloor df = Zf.$$

Let $X, Y \in Z(M); \omega, \sigma \in \bigwedge_k T^*M; \theta \in \bigwedge_k T^*M f, g \in \bigwedge_0 T^*M; c_1, c_2 \in \mathbf{R}$, then we have

1. $L_Z(fg) = (L_Z f)g + f(L_Z g)$.

2. The operator L_Z commutes with the exterior derivative d i.e.

$$L_Z(d\omega) = d(L_Z\omega).$$

3. The L operator is linear in both of its arguments

$$L_Z(c_1\omega + c_2\sigma) = c_1 L_Z\omega + c_2 L_Z\sigma$$

and

$$L_{(c_1Z+c_2Y)}\omega = c_1 L_Z\omega + c_2 L_Y\omega.$$

4. The L operator obeys the Leibniz rule, i.e.

$$L_Z(\omega \wedge \theta) = (L_Z\omega) \wedge \theta + \omega \wedge (L_Z\theta).$$

From (4) it follows that

$$L_Z(f\omega) = (Zf)\omega + f(L_Z\omega).$$

5. $L_{fZ}\omega = f(L_Z\omega) + df \wedge (Z\rfloor\omega)$.

6. $L_{[Z,Y]}\omega = (L_Z L_Y - L_Y L_Z)\omega = [L_Z, L_Y]\omega$.

Let $Z, Y \in Z(M)$. The Lie derivative of Y with respect to Z is defined so that we have

$$L_Z(Y\rfloor\omega) = L_Z Y\rfloor\omega + Y\rfloor L_Z\omega.$$

It can be shown that

$$L_Z Y = [Z, Y].$$

2.8 Metric tensor fields

Let f be any r-linear function at $\mathbf{x} \in M$

$$f : TM|_\mathbf{x} \times \cdots \times TM|_\mathbf{x} \to \mathbf{R}$$

$$: (\mathbf{Z}_1, \ldots, \mathbf{Z}_r) \mapsto f(\mathbf{Z}_1, \ldots, \mathbf{Z}_r)$$

and let g be any s-linear function at $\mathbf{x} \in M$

$$g : T^*M|_\mathbf{x} \times \cdots \times T^*M|_\mathbf{x} \to \mathbf{R}$$

$$: (\mathbf{Z}_1, \ldots, \mathbf{Z}_r) \mapsto g(\mathbf{Z}_1, \ldots, \mathbf{Z}_r).$$

So for f we have the linear properties:

1. $f(\mathbf{Z}_1, \ldots, c\mathbf{Z}_i, \ldots, \mathbf{Z}_r) = cf(\mathbf{Z}_1, \ldots, \mathbf{Z}_i, \ldots, \mathbf{Z}_r)$.

2. $f(\mathbf{Z}_1, \ldots, \mathbf{Z}_i + \mathbf{Z}_j, \ldots, \mathbf{Z}_r) = f(\mathbf{Z}_1, \ldots, c\mathbf{Z}_i, \ldots, \mathbf{Z}_r) + f(\mathbf{Z}_1, \ldots, c\mathbf{Z}_j, \ldots, \mathbf{Z}_r)$.

Similarily, these apply to function g.

Definition. The **tensor product** of f and g defined above is the map

$$f \otimes g : TM|_{\mathbf{x}} \times \cdots \times TM|_{\mathbf{x}} \times T^*M|_{\mathbf{x}} \times \cdots \times T^*M|_{\mathbf{x}} \to \mathbf{R}$$

$$: (\mathbf{Z}_1, \ldots, \mathbf{Z}_r, \omega_1, \ldots, \omega_s) \mapsto f \otimes g(\mathbf{Z}_1, \ldots, \mathbf{Z}_r, \omega_1, \ldots, \omega_s)$$

where

$$(f \otimes g)(\mathbf{Z}_1, \ldots, \mathbf{Z}_r, \omega_1, \ldots, \omega_s) = f(\mathbf{Z}_1, \ldots, \mathbf{Z}_r) g(\omega_1, \ldots, \omega_s).$$

Definition. A type (s, r) **tensor** T_r^s defined at $\mathbf{x} \in M$ is a multilinear function

$$T_r^s : (TM|_{\mathbf{x}})^r \times ((T^*M|_{\mathbf{x}})^s \to \mathbf{R}$$

Definition. A **tensor bundle** of type (s, r) over M is given by

$$T_r^s = \bigcup_{\mathbf{x} \in M} T_r^s|_{\mathbf{x}}.$$

For the classical representation of a tensor, suppose we have local coordinate chart $\{x_i\}$. Then as stated above we have the basis vecors

$$\left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_m} \right\}$$

for $TM|_{\mathbf{x}}$ and dual basis

$$\{dx_1, dx_2, \ldots, dx_m\}$$

for $T^*M|_{\mathbf{x}}$. Then any tensor T of type (s, r) has local coordinate representation

$$T(\mathbf{x}) = \sum_{i_1, \ldots, i_s = 1}^{m} \sum_{j_1, \ldots, j_r = 1}^{m} t_{j_1, \ldots, j_r}^{i_1, \ldots, i_s}(\mathbf{x}) \frac{\partial}{\partial x_{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x_{i_s}} \otimes dx_{j_1} \otimes \cdots \otimes dx_{j_r}$$

where at some fixed $\mathbf{x} \in M$

$$t_{j_1, \ldots, j_r}^{i_1, \ldots, i_s} = T\left(\frac{\partial}{\partial x_{i_1}}, \ldots, \frac{\partial}{\partial x_{i_s}}, dx_{j_1}, \ldots, dx_{j_r}\right).$$

Definition. The **metric tensor field** is a covariant tensor field denoted in local coordinates by

$$g(\mathbf{x}) = \sum_{i=1}^{m}\sum_{j=1}^{m} g_{ij}(\mathbf{x})dx_i \otimes dx_j.$$

The metric tensor field is symmetric i.e. $g(\mathbf{X},\mathbf{Y}) = g(\mathbf{Y},\mathbf{X})$ where $\mathbf{X},\mathbf{Y} \in TM|_{\mathbf{x}}$. The metric g is an isomorphism between $TM|_{\mathbf{x}}$ and $TM^*|_{\mathbf{x}}$. Its inverse g^{-1} exists and is expressed as g^{ij} in local coordinates. We have

$$g^{-1}(\mathbf{x}) = \sum_{i=1}^{m}\sum_{j=1}^{m} g^{ij}(\mathbf{x})\frac{\partial}{\partial x_i} \otimes \frac{\partial}{\partial x_j}.$$

The action of g on a vector to give a 1-form is often referred to as lowering the indices. When used in this way, g is often denoted as $g^\flat$. Using coordinates, this action on a vector $\mathbf{X} = \sum_{i=1}^{m} \mathbf{X}_i \frac{\partial}{\partial x_i}$ can be expressed as

$$\omega_i = \sum_{j=1}^{m} g_{ij}\mathbf{X}_j, \quad i = 1, ..., m.$$

The action of g^{-1} on a 1-form to give a vector is often referred to as raising the indices. When used in this way, g^{-1} is often denoted as $g^\sharp$. Using coordinates, this action on a 1-form $\omega_i = \sum_{i=1}^{m} \omega_i dx_i$ can be expressed as

$$\mathbf{X}_i = \sum_{j=1}^{m} g^{ij}\omega_j, \quad i = 1, ..., m.$$

Definition. A **Riemannian manifold** is a differentiable manifold M together with a metric g on M with $\det(g_{ij}) \geq 0$. If $\det(g_{ij}) \leq 0$ then the manifold is a called a **pseudo-Riemannian manifold**. This means that g is a symmetric positive definite bilinear form.

Definition. A **volume form** on an m-dimensional manifold M is an m-form defined on M.

Definition. An m dimensional manifold M is said to be **orientable** $\iff$ there is a volume form Ω defined on M such that $\Omega|_{\mathbf{x}} \neq 0 \ \forall \ \mathbf{x} \in M$.

Definition. The Riemannian metric can be used to define a volume form on a Riemannian manifold called the **Riemannian volume form** given in local coordinates as

$$\Omega(\mathbf{x}) = \sqrt{|\det(g_{ij})|}\ dx_1 \wedge \ldots \wedge dx_m.$$

2.9 Curvature

Definition. The **Christoffel symbols of the first kind** on a charted Riemannian manifold M of dimension m are defined as the m^3 functions

$$\Gamma_{ijk} := \frac{1}{2}(-g_{ij,k} + g_{jk,i} + g_{ki,j})$$

where $g_{rs,t} := \frac{\partial g_{st}}{\partial x_r}$.

It is easily seen that $\Gamma_{ijk} = 0 \ \forall i,j,k$ if all g_{rs} are constant. There is also symmetry in the first two indices i.e.

$$\Gamma_{ijk} = \Gamma_{jik} \ \forall \ i,j,k \leq m.$$

Christoffel Symbols of the first kind are not tensors.

Definition. The **Christoffel symbols of the second kind** on a charted Riemannian manifold M of dimension m are defined as the m^3 functions

$$\Gamma^i_{jk} := g^{ir}\Gamma_{jkr}.$$

Again we have $\Gamma^i_{jk} = 0 \ \forall i,j,k$ if all g_{rs} are constant. There is symmetry in the lower two indices i.e.

$$\Gamma^i_{jk} = \Gamma^i_{kj} \ \forall \ i,j,k \leq m.$$

Definition. The components of the **Riemannian tensor of the second kind** on a charted Riemannian manifold M of dimension m are defined as the m^4 functions

$$R^i_{jkl} := \frac{\partial \Gamma^i_{jl}}{\partial x_k} - \frac{\partial \Gamma^i_{jk}}{\partial x_l} + \Gamma^r_{jl}\Gamma^i_{rk} - \Gamma^r_{jk}\Gamma^i_{rl}.$$

Definition. The components of the **Riemannian tensor of the first kind** on a charted Riemannian manifold M of dimension m are defined as the m^4 functions

$$R_{ijkl} := g_{ir}R^r_{jkl}.$$

The following properties hold for R_{ijkl}:

$$R_{ijkl} = -R_{jikl}, \qquad R_{ijkl} = -R_{ijlk}, \qquad R_{ijkl} = R_{klij},$$

$$R_{ijkl} + R_{iklj} + R_{iljk} = 0.$$

Definition. The components of the **Ricci tensor of the first kind** on a charted Riemannian manifold M of dimension m are defined as the m^2 functions

$$R_{ij} := R^k_{ijk}.$$

Definition. The components of the **Ricci tensor of the second kind** on a charted Riemannian manifold M of dimension m are defined as the m^2 functions

$$R^i_j := g^{is} R_{sj}.$$

Definition. The **Ricci or scalar curvature** is the invariant defined as

$$R := R^i_i.$$

Example. The parameterized torus τ^2 in $\mathbf{R}^3$ will be examined. Let

$$f(\theta, \phi) = ((a + b\cos\phi)\cos\theta, (a + b\cos\phi)\sin\theta, b\sin\phi)$$

where $a > b > 0$, and $0 \leq \phi \leq 2\pi$ and $0 \leq \theta \leq 2\pi$. Then f is a 2 dimensional parameterization of τ^2.

We use the Euclidean metric on $\mathbf{R}^3$ to find g on the torus. The Eucidean metric tensor field is given by

$$g = dx_1 \otimes dx_1 + dx_2 \otimes dx_2 + dx_3 \otimes dx_3.$$

We have

$$x_1(\phi, \theta) = (a + b\cos\phi)\cos\theta,$$
$$x_2(\phi, \theta) = (a + b\cos\phi)\sin\theta,$$
$$x_2(\phi, \theta) = b\sin\phi.$$

Thus

$$dx_1(\phi, \theta) = (-b\sin\phi\cos\theta)d\phi - (a + b\cos\phi)\sin\theta\ d\theta,$$
$$dx_2(\phi, \theta) = (-b\sin\phi\sin\theta)d\phi + (a + b\cos\phi)\cos\theta\ d\theta,$$
$$dx_3(\phi, \theta) = b\cos\phi\ d\phi.$$

So it follows that

$$\begin{aligned}
dx_1 \otimes dx_1 = &(-b\sin\phi\cos\theta)^2 d\phi \otimes d\phi \\
&-((a+b\cos\phi)\sin\theta)(-b\sin\phi\cos\theta)d\theta \otimes d\phi \\
&-((a+b\cos\phi\sin\theta))(-b\sin\phi\cos\theta)d\phi \otimes d\theta \\
&+((a+b\cos\phi)\sin\theta)^2 d\theta \otimes d\theta, \\
dx_2 \otimes dx_2 = &(-b\sin\phi\sin\theta)^2 d\phi \otimes d\phi \\
&-(b\sin\phi\sin\theta)((a+b\cos\phi)\cos\theta)d\theta \otimes d\phi \\
&-(b\sin\phi\sin\theta)((a+b\cos\phi)\cos\theta)d\phi \otimes d\theta \\
&+((a+b\cos\phi)\cos\theta)^2 d\theta \otimes d\theta, \\
dx_3 \otimes dx_3 = &b^2(\cos\phi)^2 d\phi \otimes d\phi.
\end{aligned}$$

By gathering coefficients, we find the expression for g on the torus:

$$g|_{\tau^2} = (a+b\cos\phi)^2 d\theta \otimes d\theta + b^2 d\phi \otimes d\phi.$$

Thus $g|_{\tau^2}$ has matrix representation

$$g|_{\tau^2} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

$$= \begin{pmatrix} (a+b\cos\phi)^2 & 0 \\ 0 & b^2 \end{pmatrix}$$

Therefore

$$(g|_{\tau^2})^{-1} = \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{(a+b\cos\phi)^2} & 0 \\ 0 & \frac{1}{b^2} \end{pmatrix}.$$

The Christoffel symbols of the second kind found using the explicit formula

$$\Gamma^i_{jk} = \frac{1}{2} g^{ir}\left(\frac{\partial g_{jr}}{\partial x_k} + \frac{\partial g_{kr}}{\partial x_j} + \frac{\partial g_{jk}}{\partial x_r}\right).$$

We find

$$\Gamma^1_{11} = 0, \qquad \Gamma^2_{22} = 0, \qquad \Gamma^1_{22} = 0, \qquad \Gamma^2_{12} = 0,$$

and

$$\Gamma^1_{12} = -\frac{b\sin\phi}{(a+b\cos\phi)},$$

$$\Gamma^2_{11} = -\sin\phi(a - bcos\phi).$$

All the others are found directly using the symmetry conditions. We have

$$R_{11} = \frac{\partial\Gamma^2_{11}}{\partial\phi} - \frac{\partial\Gamma^2_{21}}{\partial\theta} + (\Gamma^2_{21}\Gamma^1_{11} + \Gamma^2_{22}\Gamma^2_{11} - \Gamma^2_{11}\Gamma^1_{12} - \Gamma^2_{12}\Gamma^2_{12}),$$

$$R_{22} = \frac{\partial\Gamma^1_{22}}{\partial\theta} - \frac{\partial\Gamma^1_{12}}{\partial\phi} + (\Gamma^1_{11}\Gamma^1_{22} + \Gamma^1_{12}\Gamma^2_{22} - \Gamma^1_{21}\Gamma^1_{21} - \Gamma^2_{21}\Gamma^1_{22}).$$

This gives

$$R_{11} = \frac{1}{b}\cos\phi(a+b\cos\phi)$$

$$R_{22} = \frac{b\cos\phi}{(a+b\cos\phi)}.$$

So

$$\begin{aligned}
R &= g^{11}R_{11} + g^{22}R_{22} \\
&= \frac{1}{(a+b\cos\phi)^2}\frac{1}{b}\cos\phi(a+b\cos\phi) + \frac{1}{b^2}\frac{b\cos\phi}{(a+b\cos\phi)} \\
&= \frac{\cos\phi}{b(a+b\cos\phi)^2} + \frac{\cos\phi}{b(a+b\cos\phi)^2} \\
&= \frac{2\cos\phi}{b(a+b\cos\phi)^2}.
\end{aligned}$$

$\spadesuit$

From this we see that the Ricci curvature is not always positive.

2.10 Volume forms

If a m dimensional manifold M is endowed with a metric $g = \sum_{i,j=1}^{m} g_{ij}dx_i \otimes dx_j$, then there exists a natural volume form.

$$\Omega(\mathbf{x}) := \sqrt{|g|}dx_1 \wedge dx_2 \wedge \ldots \wedge dx_m$$

Where $g = \det g_{ij}$ and $|.|$ is the modulus.
It can be shown that this volume form is invariant under coordinate changes. If our manifold is covered by overlapping charts, then we know that where any two charts $(\mathbf{O}, \mathbf{x})$ and $(\mathbf{U}, \mathbf{y})$ intersect, we have

$$\Omega|_y = \Omega|_{\mathbf{x}}.$$

If $\{\theta_i\}$ forms a non-coordinate basis for $T^*M|_{\mathbf{x}}$, then the invariant volume form can be expressed as

$$\Omega(\mathbf{x}) = \theta_1 \wedge \theta_2 \wedge \ldots \wedge \theta_m.$$

2.11 Hodge star operator

If there is a metric g associated with the manifold M, then the Hodge star operator defines a natural isomorphism .The Hodge star operator is an f-linear isomorphism between the spaces $\bigwedge_p T^*M$ and $\bigwedge_{m-p} T^*M$ i.e

$$* : \bigwedge_p T^*M \rightarrow \bigwedge_{m-p} T^*M$$

Since $*$ is an isomorphism, we have that $\bigwedge_p T^*M$ is isomorphic to $\bigwedge_{m-p} T^*M$. This relationship is expressed as

$$\bigwedge_p T^*M \approx \bigwedge_{m-p} T^*M.$$

Let $\omega, \sigma \in \bigwedge_p T^*M$. Then as a result of linearity we have

$$*(f\omega + g\sigma) = f(*\omega) + g(*\sigma)$$

for all $f, g \in \bigwedge_0 T^*M$. The $*$ operator applied to a basis vector of a p-form with $p \leq m$ is given by

$$*(dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_p}) := \sum_{j_1 \ldots j_m = 1}^{m} g^{i_1 j_1} \ldots g^{i_p j_p} \frac{1}{(m-p)!} \frac{g}{\sqrt{|g|}} \epsilon_{j_1 \ldots j_m} dx_{j_{p+1}} \wedge \ldots \wedge dx_{j_m}$$

where $g \equiv \det(g_{ij})$, $\epsilon_{j_1 \dots j_m}$ is the totally antisymmetric tensor. Also, g^{ij} is the inverse matrix of g_{ij} i.e. $\sum_{j=1}^{m} g^{ij} g_{jk} = \delta_{ik}$.

We can use non-coordinate basis $\{\theta_i\}$, where each non-coordinate basis vector is formed as linear combination of local basis vectors i.e. $\theta_i = \sum_{k=1}^{m} e_k^i dx^k$, $e_k^i \in \mathbf{R}$. Using non-coordinate basis, the operation becomes

$$*(\theta_{i_1} \wedge \theta_{i_2} \wedge \dots \wedge \theta_{i_p}) := \sum_{j_1 \cdots j_m = 1}^{m} g^{i_1 j_1} \dots g^{i_p j_p} \frac{1}{(m-p)!} \epsilon_{j_1 \dots j_m} \theta_{j_{p+1}} \wedge \dots \wedge \theta_{j_m}.$$

For $\omega \in \bigwedge_p T^* M$ we have

$$** \omega = (-1)^{p(m-p)} \omega.$$

For $\omega, \sigma \in \bigwedge_p$ we have

$$\omega \wedge (*\sigma) = \sigma \wedge (*\omega).$$

The inverse of * is given by

$$(*)^{-1} := (-1)^{p(m-p)}.$$

if (M, g) is Riemannian, and

$$(*)^{-1} := (-1)^{1+p(m-p)}.$$

if (M, g) is Lorentzian. We also have that $*1$ is the invariant volume form when g is positive definite.

$$\begin{aligned} *1 &= \sum_{j_1 \cdots j_m = 1}^{m} \frac{1}{m!} \frac{g}{\sqrt{|g|}} \epsilon_{j_1 \dots j_m} dx_{j_1} \wedge \dots \wedge dx_{j_m} \\ &= \sqrt{|g|}\, dx_1 \wedge \ldots \wedge dx_m. \end{aligned}$$

When g is positive definite, we obviously have

$$\frac{g}{\sqrt{|g|}} = \frac{g}{\sqrt{|g|}} \times \frac{\sqrt{|g|}}{\sqrt{|g|}} = \sqrt{|g|}.$$

2.12 Inner products of p-forms

Consider two p-forms expressed using basis vectors, with scaling coefficients

$$\omega = \frac{1}{p!}\sum_{i_1 \dots i_p}^{m} \omega_{i_1 \dots i_p} dx_{i_1} \wedge \dots \wedge dx_{i_p}$$

$$\eta = \frac{1}{p!}\sum_{i_1 \dots i_p}^{m} \eta_{i_1 \dots i_p} dx_{i_1} \wedge \dots \wedge dx_{i_p}.$$

Now $*\eta$ is an $(n-p)$-form, so that $\omega \wedge *\eta$ is an m-form. We find that

$$\omega \wedge *\eta = \frac{1}{p!}\sum_{i_1 \dots i_p}^{m} \omega_{i_1 \dots i_p} \eta_{i_1 \dots i_p} \sqrt{|g|}\; dx_1 \wedge \dots \wedge dx_m$$

and so $\omega \wedge *\eta = \eta \wedge *\omega$. For non-coordinate basis $\{\theta_i\}$, have

$$\omega = \frac{1}{p!}\sum_{i_1 \dots i_p}^{m} \omega_{i_1 \dots i_p} \theta_{i_1} \wedge \dots \wedge \theta_{i_p}$$

$$\eta = \frac{1}{p!}\sum_{i_1 \dots i_p}^{m} \eta_{i_1 \dots i_p} \theta_{i_1} \wedge \dots \wedge \theta_{i_p}$$

and

$$\omega \wedge *\eta = \frac{1}{p!}\sum_{i_1 \dots i_p}^{m} \omega_{i_1 \dots i_p} \eta_{i_1 \dots i_p} \theta_1 \wedge \dots \wedge \theta_m.$$

The inner product of two p-forms on compact manifold M is the function

$$(,) : \bigwedge_p T^*M \times \bigwedge_p T^*M \to \mathbf{R}$$

where

$$\begin{aligned}(\omega, \eta) &:= \int_M \omega \wedge *\eta \\ &= \frac{1}{p!}\int_M \sum_{i_1 \dots i_p}^{m} \omega_{i_1 \dots i_p} \eta_{i_1 \dots i_p} \sqrt{|g|}\; dx_1 \wedge \dots \wedge dx_m.\end{aligned}$$

The inner product is symmetric i.e. $(\omega, \eta) = (\eta, \omega)$. The inner product is positive definite if the manifold M is Riemannian i.e.

$$(\omega, \omega) \geq 0$$

and $(\omega, \omega) = 0 \Leftrightarrow \omega = 0$.

2.13 Codifferential

The codifferential is an operator on an m dimensional manifold M that takes p-forms to $(p-1)$-forms i.e.

$$\delta : \bigwedge_p T^*M|_{\mathbf{x}} \to \bigwedge_{(p-1)} T^*M|_{\mathbf{x}}$$

$$: \omega \mapsto \delta\omega$$

where the action is defined as

$$\delta\omega = (-1)^{mp+m+1} * d * \omega.$$

If f is a 0-form, then $\delta f = 0$.

Theorem. Suppose (M, g) is a unbounded compact orientable manifold. Let $\omega \in \bigwedge_p T^*M$ and $\eta \in \bigwedge_{(p-1)} T^*M$. Then

$$(d\eta, \omega) = (\delta\eta, \omega).$$

The proof of this theorem can be found in [1]. Due to this theorem, d and δ may be said to be adjoint to each other.

2.14 Laplacian

Definition. The **Laplacian operator** Δ is defined as

$$\Delta := d\delta + \delta d.$$

So we have

$$\Delta : \bigwedge_p T^*M \to \bigwedge_p T^*M$$

$$: \omega \mapsto \Delta\omega = d\delta\omega + \delta d\omega.$$

Δ is self adjoint in the sense that

$$(\Delta\omega, \eta) = (\omega, \Delta\eta).$$

Δ is also a positive definite operator in the sense that

$$(\Delta\omega, \omega) \geq 0.$$

A p-form ω is called harmonic if $\Delta\omega = 0$. A p-form ω is called closed if $d\omega = 0$. A p-form ω is called coclosed if $\delta\omega = 0$.

Theorem. A p-form is harmonic if and only if it is closed and coclosed. Proof in [1].

The space of harmonic r-forms on manifold M is denoted $\mathrm{Harm}^r(M)$.

Chapter 3

Theory of vector fields

This chapter covers sections of vector field theory which are relevant to the decomposition of vector fields. The definitions of a symplectic metric and manifold are given. The concept of a symplectic manifold is used in chapter five. We will be examining a method for decomposing vector fields into Hamilton and gradient vector fields and so we will be looking at some of the special properties of these systems. It is these properties that make the decomposition interesting in the first place. The flow-box theorem is also stated as it is referred to in chapter five.

3.1 Symplectic group and metric

The symplectic metric $s(,)$ is a real anti-symmetric bilinear form. Therefore

$$s : TM|_{\mathbf{x}} \times TM|_{\mathbf{x}} \to \mathbf{R}$$

$$: (\mathbf{X}, \mathbf{Y}) \mapsto s(\mathbf{X}, \mathbf{Y})$$

Anti-symmetry gives

$$s(\mathbf{X}, \mathbf{Y}) = -s(\mathbf{Y}, \mathbf{X}).$$

Linearity gives

$$s(\mathbf{X}, \mathbf{Y} + \mathbf{Z}) = s(\mathbf{X}, \mathbf{Y}) + s(\mathbf{X}, \mathbf{Z})$$

and

$$s(\mathbf{X}, c\mathbf{Y}) = cs(\mathbf{X}, \mathbf{Y})$$

where $c \in \mathbf{R}$.

The elements of the symmetric group always represent transformations on even dimensional vector spaces $\mathbf{R}^{2m}$. We can represent the action of the symplectic metric using the $2m \times 2m$ matrix S as follows:

$$s(\mathbf{X}, \mathbf{Y}) = \mathbf{X}^T S \mathbf{Y} = \sum_{i=1}^{2m} \sum_{j=1}^{2m} X_i S_{ij} Y_j = \sum_{i=1}^{m} X_i Y_{m+i} - \sum_{i=1}^{m} X_{m+i} Y_i.$$

We can always choose the basis so that S has a particularly simple matrix representation

$$S = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}.$$

S is non-singular so s_{ij} has inverse s^{ij} giving

$$\sum_{i=1}^{2m} \sum_{j=1}^{2m} s_{ij} s^{jk} = \delta_i^k.$$

3.2 Symplectic manifolds

This is a even dimensional manifold M endowed with an anti-symmetric tensor field s. The symplectic structure s is closed symplectic 2-form i.e. $ds = 0$.
The inverse of s_{ij} is s^{ij} and is used in the definition of the Poisson bracket. The bracket acts on two scalar fields on M to produce a third.

$$\{,\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

defined by

$$\{\theta, \phi\} = -\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} s^{ij} \nabla_i \theta \nabla_j \phi$$

If we have 2 symplectic manifolds of the same dimension, then they are locally completely identical. This means if we have any point p in one of the manifolds and any point q in the other, we can find open sets containing p and q that are identical [1].

3.3 Hamilton vector fields and Hamiltons equations

Hamiltons equations of motion are a particular formulation of the classical equations of motion. We have the canonical coordinates $(\mathbf{q}, \mathbf{p}) = (q_1, ..., q_m, p_1, ..., p_m)$ in the $\mathbf{R}^{2m}$ phase space.

Let $H(\mathbf{p},\mathbf{q})$ be the Hamilton for a system. Then Hamiltons equations are given by

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \tag{3.1}$$

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \tag{3.2}$$

where $i = 1, 2, \ldots, m$. Suppose ω is a closed 2-form of maximal rank on some even dimensional manifold. If H is a real valued function, then $X(\mathbf{x}) = \sum_{i=1}^{m} X_i(\mathbf{x})\frac{\partial}{\partial x_i}$ is the Hamiltonian vector field for H if it satisfies

$$dH = -X \rfloor \omega$$

where $\omega = \sum_{i=1}^{m} dp_i \wedge dq_i$. Here q_i and p_i represent the position and momentum of the ith particle respectively.
In $\mathbf{R}^m$ have

$$\begin{aligned} -\sum_{i=1}^{m}\left(\frac{\partial H}{\partial q_i}dq_i + \frac{\partial H}{\partial p_i}dp_i\right) &= \sum_{i=1}^{m} X\rfloor(dp \wedge dq) \\ &= \sum_{i=1}^{m}\left((X\rfloor dp_i) \wedge dq_i + (-1)dp_i \wedge (X\rfloor dp_i)\right) \\ &= \sum_{i=1}^{m}\left(X_{p_i}dq_i - X_{q_i}dp_i\right). \end{aligned}$$

By associating coefficients and using Hamiltons equations of motion (3.1) and (3.2), we find

$$-\frac{\partial H}{\partial q_i} = X_{p_i} = \frac{dp_i}{dt}.$$

$$\frac{\partial H}{\partial p_i} = X_{q_i} = \frac{dq_i}{dt}.$$

Another formulation of a Hamiltonian system is as follows. Consider the vector field on the phase space given by

$$\frac{dq_i}{dt} = G_i(\mathbf{q},\mathbf{p}), \qquad \frac{dp_i}{dt} = F_i(\mathbf{q},\mathbf{p}). \tag{3.3}$$

Definition. [2] A vector field on the phase space $\mathbf{R}^{2m}$ is a **Hamiltonian system** if the differential 1-form α given by

$$\alpha(\mathbf{q},\mathbf{p}) = \sum_{i=1}^{m} (G_i(\mathbf{q},\mathbf{p})dp_i - F_i(\mathbf{q},\mathbf{p})dq_i)$$

is a closed form i.e. $d\alpha = 0$.

Proposition. [2] If the right hand sides of the equations (3.1) and (3.2) satisfy the Helmholtz conditions [3-5] in the phase space, then the system is a Hamilton system. The Helmholtz conditions for (3.2) are given as

$$\frac{\partial G_i}{\partial p_j} - \frac{\partial G_j}{\partial p_i} = 0,$$

$$\frac{\partial G_j}{\partial q_i} - \frac{\partial F_i}{\partial p_j} = 0,$$

$$\frac{\partial F_i}{\partial q_j} - \frac{\partial F_j}{\partial q_i} = 0.$$

The proof of the above proposition can be found in [2].
Some Hamiltonian systems can be defined by a unique function, the Hamilton function.

Proposition. [2] A vector field on the phase space $\mathbf{R}^{2m}$ is a Hamilton system uniquely defined by the continuously differentiable Hamilton function $H(\mathbf{q},\mathbf{p})$ if the differential 1-form

$$\alpha(\mathbf{q},\mathbf{p}) = \sum_{i=1}^{m} (G_i(\mathbf{q},\mathbf{p})dp_i - F_i(\mathbf{q},\mathbf{p})dq_i)$$

is an exact form such that $\alpha = dH$. See [2] for the proof.

3.4 Gradient vector fields

Suppose $\phi(\mathbf{x})$ is a scalar field on manifold M

$$\phi : M \to R$$

$$: \mathbf{x} \mapsto \phi(\mathbf{x}).$$

We now have a scalar associated with every point of the manifold. It is possible to construct a vector field from this scalar field. This vector field is given by

$$\frac{d\mathbf{x}}{dt} = -\nabla\phi(\mathbf{x})$$

and can be expressed as $X(\mathbf{x}) = \sum_{i=1}^{m} \frac{\partial\phi}{\partial x_i}\frac{\partial}{\partial x_i}$.

Definition. A vector field on a manifold M of dimension m given by

$$\frac{dx_i}{dt}(\mathbf{x}) = G_i(\mathbf{x})$$

is a **gradient vector field** if the 1-form

$$\omega(\mathbf{x}) = \sum_{i=1}^{m} G_i(\mathbf{x})dx_i \tag{3.4}$$

is an exact form $\omega = -d\phi$ where ϕ is a continuously differentiable function on M [2]. Taking the exterior derivative of (3.4) gives

$$d\omega = \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\left(\frac{\partial G_i}{\partial x_j} - \frac{\partial G_j}{\partial x_i}\right)dx_j \wedge dx_i. \tag{3.5}$$

Then using (3.5), the condition $d\omega = 0$ gives $\frac{\partial G_i}{\partial x_j} - \frac{\partial G_j}{\partial x_i} = 0$ for $i, j = 1, 2, \ldots, m$. A form 2-form which is exact is closed due to

$$\frac{\partial^2\phi}{\partial x_i \partial x_j} = \frac{\partial^2\phi}{\partial x_j \partial x_i}.$$

Poincaré's lemma that on a contractable open subset of U of $\mathbf{R}^m$, any smooth p-form defined on U which is exact, is also closed [2].

Proposition. [2] If a smooth vector field $X(\mathbf{x}) = \sum_{i=1}^{m} X_i(\mathbf{x})\frac{\partial}{\partial x_i}$ satisfies the relations

$$\frac{\partial G_i}{\partial x_j} - \frac{\partial G_j}{\partial x_i} = 0, \qquad i, j = 1, 2, \ldots m$$

on a contractable open subset U of $\mathbf{R}^m$, then the vector field is a gradient system such that

$$\frac{dx_i}{dt} = -\frac{\partial\phi}{\partial x_i}(\mathbf{x}).$$

The line integral along any closed path in such a field is zero. This is as a result of the Fundamental theorem for gradients, which states that

$$\int_{\mathbf{a}}^{\mathbf{b}} \nabla\phi \cdot d\mathbf{l} = \phi(\mathbf{b}) - \phi(\mathbf{a}).$$

Thus gradient system can be fully defined by a scalar function ϕ. Gradient fields are also called conservative fields.

3.5 Flow-box theorem

The flow-box theorem applies to smooth vector fields, and states that the dynamics of the vector field near a non-equilibrium point are toplogically conjugate with translation. Close to a non-degenerate equilibrium point, linearizing the field by differentiation allows for relatively simple characterisation of most of the local dynamics. Specifically, this allows for the characterisation of the local behavior of solutions and flows for smooth vector fields.

Flow-box theorem. If X is a smooth C^1 vector field on manifold M, with $\mathbf{x}_0 \in M$ not an equilibrium point (i.e. $X(\mathbf{x}_0) \neq 0$), then there exists a diffeomorphism which transfers the vector field near $\mathbf{x}_0$ to a constant vector field. That is, the local flow of X is conjugate to translation, via a diffeomorphism [6].

Chapter 4

de Rham cohomology and Hodge decomposition

Hodge de Rham decomposition is used in the decomposition theorems due to Roels, Mendes and Duarte [4,5]. To understand these decomposition theorems, an understanding of Hodge's decomposition theorem is essential. To state Hodge's decomposition theorem without ambiguity, the basic concepts underpinning this theory are detailed in this chapter. Some modern developments pertaining to Hodge decomposition in Euclidean space are discussed at the end of the chapter.

4.1 Homology groups

4.1.1 Simplexes and related groups

Simplexes form the building blocks of polyhedrons. Suppose $\mathbf{p}_0, ..., \mathbf{p}_r$ are geometrically independent points in $\mathbf{R}^m$ with $m \geq r$. Then the r-simplex $\sigma_r = < \mathbf{p}_0, ..., \mathbf{p}_r >$ is expressed as

$$\sigma_r := \left\{ \mathbf{x} \in \mathbf{R}^m \mid \mathbf{x} = \sum_{i=0}^{r} c_i \mathbf{p}_i, \ \sum_{i=0}^{r} c_i = 1, \ c_i \geq 0 \right\}.$$

Let K be a finite set of simplexes in $\mathbf{R}^m$. The dimension of K is the greatest dimension of simplexes in K. K is called a simplicial complex if the individual simplexes are 'nicely' fitted together. This means:

1. An arbitrary face of a simplex contained in K belongs to K;

2. If σ and σ' are two simplexes in K, then the intersection of the simplexes $\sigma \bigcap \sigma'$ is either the empty set or a face common to both σ and σ'.

If each simplex σ_i of simplicial complex $K = \{\sigma_1, ..., \sigma_r\}$ is regarded as a subset of $\mathbf{R}^m$, then $\bigcup_{i=1}^{r} \sigma_i = |K| \in \mathbf{R}^m$. $|K|$ is the polyhedron of simplicial complex K. We have $\dim |K| = \dim K$.

Definition. Suppose X is a topological space. X is said to be **triangulable** if there exists a simplicial complex K and a homeomorphism $f : |K| \to X$. The pair (K, f) is the triangulation of topological space X.

Definition. Consider $r+1$ independent points $\mathbf{p_i}(i = 0, ..., r)$ in $\mathbf{R}^m$. If $P' = \{\mathbf{p_{i_0}}, ..., \mathbf{p_{i_r}}\}$ is a permutation of set $P = \{\mathbf{p}_0, ..., \mathbf{p}_r\}$, then we define P' equivalent to P if P' is an even permutation of P. This represents an equivalence relation and the equivalence class is called and **oriented r-simplex**.

The equivalence relation above gives rise to two equivalence classes, consisting of the even and odd permutations of the set $\{\mathbf{p}_0, ..., \mathbf{p}_r\}$. The equivalence class of even permutations is denoted $\sigma_r = (\mathbf{p}_0\mathbf{p}_1...\mathbf{p}_r)$. The equivalence class of odd permutations is denoted $-\sigma = -(\mathbf{p}_0\mathbf{p}_1...\mathbf{p}_r)$. For $r = 0$, we define an oriented 0-simplex to be a point $\sigma_0 = \mathbf{p}_0$.

Definition. The **r-chain group** $C_r(K)$ of a given simplicial complex K is a free Abelian group generated by the oriented r-simplexes contained in K. If $\dim K < r$, then $C_r(K)$ is defined to be 0. The elements of $C_r(K)$ are known as r-chains.

Let I_r be the number of r-simplexes in K. Then each of the r simplexes in denoted by $\sigma_{r,i}$ with $i = 1, ..., I_r$. Then $c \in C_r(K)$ can be expressed as

$$c = \sum_{i=1}^{I_r} c_i \sigma_{r,i} \qquad c_i \in \mathbf{Z}.$$

So $C_r(K)$ is a free abelian group with rank I_r, and we can write

$$C_r(K) \cong \mathbf{Z} \oplus \mathbf{Z} \oplus \cdots \oplus \mathbf{Z}$$

where $\oplus$ denotes the direct sum. The boundary of an r-simplex σ_r is denoted $\partial_r \sigma_r$.

Definition. Consider the oriented r-simplex $\sigma_r(\mathbf{p}_0...\mathbf{p}_r)$. Then the **boundary** $\partial_r \sigma_r$ of σ_r is an $(r-1)$-chain given by

$$\partial_r \sigma_r \equiv \sum_{i=0}^{r} (-1)^i (\mathbf{p}_0\mathbf{p}_1...\overline{\mathbf{p}_i}...\mathbf{p}_r)$$

where the line over the element $\mathbf{p}_i$ means that it is not present. $\partial_o \sigma_0$ is defined to be zero. The boundary operator defines a homomorphism

$$\partial_r : C_r(K) \to C_{r-1}(K)$$

Examples.

1. A 0-simplex (point) has no boundary, so $\partial_0 \mathbf{p}_0 = 0$.

2. For a line we have
$$\partial_1(\mathbf{p}_0\mathbf{p}_1) = \mathbf{p}_1 - \mathbf{p}_0.$$

3. For a 2-simplex we have
$$\partial_2(\mathbf{p}_0\mathbf{p}_1\mathbf{p}_2) = (\mathbf{p}_1\mathbf{p}_2) - (\mathbf{p}_0\mathbf{p}_2) + (\mathbf{p}_0\mathbf{p}_1).$$ ♠

Suppose K is an n dimensional simplicial complex. Then there exists a chain of free Abelian groups and homomorphisms

$$C_n(K) \longrightarrow C_{n-1}(K) \longrightarrow \ldots \longrightarrow C_1(K) \longrightarrow C_0(K) \longrightarrow 0.$$

The sequence above is known as the chain complex associated with K. It is denoted $C(K)$.

Definition. If $c \in C_r(K)$ satisfies

$$\partial_r c = 0,$$

then c is referred to as an r-**cycle**. The set of r-cycles, denoted $Z_r(K)$ is a subgroup of $C_r(K)$ and is known as the r-**cycle group**. So $Z_r(K) = \ker \partial_r$ i.e. the kernel of ∂_r.

Definition. Let $c \in C_r(K)$. If there exists an element $d \in C_{r+1}(K)$, such that

$$c = \partial_{r+1} d$$

then c is called an r-**boundary**. The set of r-boundries, denoted $B_r(K)$, is a subgroup of $C_r(K)$ and is called the r-**boundary group**. We have $B_r = \text{im}\, \partial_{r+1}$ i.e. the image of ∂_{r+1}.

From the fundamental theorem of homomorphisms, we know that $Z_r(K)$ and $B_r(K)$ are subgroups of $C_r(K)$.

Lemma. The composite map given by $\partial_r \circ \partial_{r+1} : C_{r+1}(K) \to C_{r-1}(K)$ is a zero map. See [1] for proof.

Theorem. [1] If $Z_r(K)$ is the r-cycle group and $B_r(K)$ is the r-boundary group of $C_r(k)$, then

$$B_r \subset Z_r(K).$$

This follows directly from the above lemma.

4.1.2 Homology groups

Definition. Consider n-dimensional simplicial complex K. Then the rth **homology group** $H_r(K), 0 \leq r \leq n$ asscoiated with K is defined

$$H_r(K) \equiv Z_r(K)/B_r(K).$$

So $H_r(K)$ is a quotient group, and we have $H_r(K) \equiv \{[z]|z \in Z_r(K)\}$, where each equivalence class is called a **homology class**. The group can be defined with $\mathbf{Z}$ and $\mathbf{R}$ coefficients.

Theorem. [1] Homolgy groups are topological invariants. So, if X is homeomorphic to Y, and if (K, f) and (L, g) are triangulations of X and Y respectively, then

$$H_r(K) \cong H_r(L) \qquad r = 1, 2, ...$$

This theorem tells us that the homology groups of a topological space are independent of the particular triangulation (K, f).

4.1.3 Properties of homology groups

Theorem. [1] Suppose K is the disjoint union of N connected components, i.e. $K = \sum_{i=1}^{N} K_i$ with $K_i \bigcap K_j = \emptyset;\ i \neq j$. Then

$$H_r(K) = H_r(K_1) \oplus H_r(K_2) \oplus \cdots \oplus H_r(K_N).$$

See [1] for proof.

$Z_r(K)$ and $B_r(K)$ are subgroups of $C_r(K)$, and are therefore free Abelian groups. It does not follow that $H_r(K)$ is a free Abelian group. The most general form of $H_r(K)$ is

$$H_r(K) \cong \underbrace{\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}}_{d} \oplus \mathbf{Z}_{k_1} \oplus \cdots \oplus \mathbf{Z}_{k_p}.$$

The first d factors form a free Abelian group of rank d.

Definition. Suppose K is a simplicial complex. Then the rth **Betti number** $b_r(K)$ is defined as

$$b_r(K) \equiv \dim H_r(K; \mathbf{R}).$$

That means that $b_r(K)$ is the rank of the free Abelian part of $H_r(K; \mathbf{Z})$.

4.1.4 Stokes' Theorem

Below the integration of an r-form over an r-simplex in Eucldean space is defined. The **standard n-simplex** $\overline{\sigma}_r = (\mathbf{p}_0 ... \mathbf{p}_r)$ in $\mathbf{R}^r$, where

$$\begin{aligned} \mathbf{p}_0 &= (0, 0, ..., 0), \\ \mathbf{p}_1 &= (1, 0, ..., 0), \\ &\cdots \\ \mathbf{p}_r &= (0, 0, ..., 1). \end{aligned}$$

Suppose $\{x_i\}$ are coordinates for $\mathbf{R}^r$, then $\overline{\sigma}_r$ is expressed as

$$\overline{\sigma}_r = \left\{ (x_1, ..., x_r) \in \mathbf{R} \mid x_i \geq 0, \sum_{i=1}^{r} x_i \leq 1 \right\}.$$

A general r-form ω in $\mathbf{R}^r$ can be written as

$$\omega = w(x) dx_1 \wedge dx_2 \wedge ... \wedge dx_r.$$

Then the integration of ω over $\overline{\sigma}_r$ is defined as

$$\int_{\overline{\sigma}_r} \omega \equiv \int_{\overline{\sigma}_r} w(x) dx_1 \wedge dx_2 \wedge ... \wedge dx_r.$$

Definition. Suppose we have an m-dimensional manifold M and suppose σ_r is an r-simplex in $\mathbf{R}^r$. Let $f : \sigma_r \rightarrow M$ be a smooth function which is not necessarily invertible. Then the image $f(\sigma_r) \in M$ is denoted s_r and is called an **singular** r**-simplex** in M. $\{s_{r,i}\}_{1 \leq i \leq r}$ is the set of r-simplexes in M.

Definition. [1] Consider $\{s_{r,i}\}_{1\leq i\leq r}$ as defined above. Then an r**-chain** in M is defined as

$$c = \sum_{i=1}^{r} a_i s_{r,i} \qquad a_i \in \mathbf{R}.$$

The r-chains in manifold M give the **chain group** $C_r(M)$. Under the function $f : \sigma_r \to M$, the boundary $\partial\sigma_r$ of the r-simplex σ is mapped to a subset of M. We have $\partial s_r \equiv f(\partial\sigma_r)$ is a set of $(r-1)$-simplexes in M and is known as the boundary of s_r. This gives the function

$$\partial : C_r(M) \to C_{r-1}(M).$$

We have $\partial^2 = 0$.

Definition. Let M be a manifold, and let $\partial : C_r(M) \to C_{r-1}(M)$ be defined as above. Let $c_r \in C_r(M)$. Then if $\partial c_r = 0$, then c_r is an r**-cycle**. The set of r-cycles is called the r**-cycle group** $Z_r(M)$.

Definition. Let M be a manifold, and let $\partial : C_r(M) \to C_{r-1}(M)$ be defined as above. Let $c_r \in C_r(M)$. If there exists $d_{r+1} \in C_{r+1}(M)$ such that $\partial d_{r+1} = c_r$, then c_r is called an r**-boundary**. The set of r-boundies is called the r**-boundary group** $B_r(M)$.

Definition. Let $B_r(M)$ and $Z_r(M)$ be defined as above. Then the rth **singular homology group** on manifold M is defined as

$$H_r(M) \equiv Z_r(M)/B_r(M).$$

Let M be a manifold and $f : \overline{\sigma}_r \to M$ be a smooth map such that we have $s_r = f(\overline{\sigma}_r)$. The integration of an r-form ω on an r-simplex s_r on manifold M is defined as

$$\int_{s_r} \omega = \int_{\overline{\sigma}_r} f^*\omega.$$

For a general r-chain $c = \sum_{i=1}^{r} a_i s_{r,i} \in C_r(M)$, the integration is defined as

$$\int_c \omega = \sum_{i=1}^{r} \int_{s_{r,i}} \omega.$$

Stokes' Theorem. Let ω be a $(r-1)$-form on manifold M, and let $c \in C_r(M)$. Then

$$\int_c d\omega = \int_{\partial c} \omega.$$

4.1.5 de Rham cohomology groups and de Rham's theorem

The cohomology group is the space dual to the homology group. The duality is given by Stokes' Theorem.

Definition. Suppose M is an m-dimensional differentiable manifold. The set of closed r-forms is known as the rth **cocycle group**, denoted $Z^r(M)$. The set of exact r-forms is known as the rth **coboundry group**, denoted $B^r(M)$. Both $Z^r(M)$ and $B^r(M)$ are vector spaces with real coefficients.

It is also the case that $B^r(M) \subset Z^r(M)$. This is because $d^2 = 0$. The fact that $d^2 = 0$, is a result of the symmetry of second derivatives. This is easily seen using a simple example.

Definition. The rth **de Rham cohomology group** is given by the expression

$$H^r(M; \mathbf{R}) \equiv Z^r(M)/B^r(M).$$

Suppose $\omega \in Z^r(M)$. Then the equivalence class $[\omega] \in H^r(M)$ is given by the set $\{\sigma \in Z^r(M) | \sigma = \omega + d\eta, \eta \in \bigwedge_{r-1} T^*M\}$. Two r-forms are said to be cohomologous if they differ by an exact form.

Let M be an m-dimensional manifold and let $C_r(M)$ be the associated chain group. Let $c \in C_r(M)$ and $\omega \in \bigwedge_r T^*M$, with $1 \leq r \leq m$. We can define the inner product

$$(\, , \,) : C_r(M) \times \bigwedge_r T^*M \to \mathbf{R}$$

$$: c, \omega \mapsto (c, \omega) \equiv \int_c \omega.$$

The map defined above is bilinear, and for a given $c \in C_r(M)$ and $\omega \in \bigwedge_r T^*M$, we can consider the seperate linear maps

$$(c, \,) : \bigwedge_r T^*M \to \mathbf{R}$$

$$(\, , \omega) : C_r(M) \to \mathbf{R}.$$

Stokes' theorem can now be written in the form $(c, d\omega) = (\partial c, \omega)$.

The inner product defined above induces an inner product

$$\Lambda : H_r(M) \times H^r(M) \to \mathbf{R}$$
$$: ([c], [\omega]) \mapsto \Lambda([c], [\omega]) \equiv (c, \omega) = \int_c \omega.$$

The function above is independent of the choice of representative element. Now $\Lambda(\ , [\omega])$ is a linear map from $H_r(M) \to \mathbf{R}$ and $\Lambda([c],\ )$ is a linear map from $H^r(M) \to \mathbf{R}$.

Showing that $\Lambda(\ , [\omega])$ has maximal rank proves the duality of the spaces $H_r(M)$ and $H^r(M)$.

de Rham's theorem. [1] Consider a compact manifold M. Then $H_r(M)$ and $H^r(M)$ are finite dimensional. Also, the map

$$\Lambda : H_r(M) \times H^r(M) \to \mathbf{R}$$

is a bilinear and non-degenerate map giving that $H_r(M)$ is dual to $H^r(M)$.

So we have the chain complex $C(M)$ and the de Rham complex $\bigwedge_* T^*M$. This can be depicted graphically as

$$\longleftarrow C_{r-1}(M) \longleftarrow C_r(M) \longleftarrow C_{r+1}(M) \longleftarrow$$
$$\longrightarrow \bigwedge_{r-1} T^*M \longrightarrow \bigwedge_r T^*M \longrightarrow \bigwedge_{r+1} T^*M \longrightarrow$$

where the rth homology group given by $H_r(M) = Z_r/B_r(M) = \ker \partial_r/\text{im } \partial_{r+1}$ and the rth de Rham cohomology group given by $H^r(M) = Z^r/B^r(M) = \ker d_{r+1}/\text{im } d_r$.

4.2 Hodge decomposition theorem

We denote the set of harmonic p-forms on manifold M as H^pM. We denote the set of exact p-forms on manifold M as $d(\bigwedge_{p-1} T^*M)$. Also, we denote the set of coexact p-forms on manifold M as $\delta(\bigwedge_{p+1} T^*M)$.

Hodge decomposition theorem. [1] Let (M, g) be a compact orientable Riemannian manifold. Then the space $\bigwedge_p T^*M$ is decomposed uniquely as

$$\bigwedge_p T^*M = d(\bigwedge_{p-1} T^*M) \oplus \delta(\bigwedge_{p+1} T^*M) \oplus H^pM.$$

This means that if ω is any p-form, then there exists a $(p-1)$-form α and a $(p+1)$-form β and harmonic form γ such that we have

$$\omega = d\alpha + \delta\beta + \gamma$$

where the forms $d\alpha, \delta\beta, \gamma$ are unique.

Example. Consider the manifold $M = \mathbf{R}^2$ and the metric tensor field $g = dx_1 \otimes dx_1 + dx_2 \otimes dx_2$.

Let $\omega = \omega_1(\mathbf{x})dx_1 + \omega_2(\mathbf{x})dx_2$ with $\omega_1, \omega_2 \in C^\infty(\mathbf{R}^2)$ be a differential 1-form in M. Here it is shown that ω can be written as

$$\omega = d\alpha + \delta\beta + \gamma$$

where α is a $C^\infty(\mathbf{R}^2)$ function, β is a 2-form given by $\beta = b(\mathbf{x})dx_1 \wedge dx_2$, and $\gamma = \gamma_1 dx_1 + \gamma_2 dx_2$ is an harminic 1-form i.e. $(d\delta + \delta d)\gamma = 0$.

We have $d\alpha = \frac{\partial\alpha}{\partial x_1}dx_1 + \frac{\partial\alpha}{\partial x_2}dx_2$.

Using $\delta\beta = (-1) * d * \beta, *dx_1 \wedge dx_2 = 1, *dx_{1=dx_2}$ and $*dx_2 = -dx_1$, we obtain

$$\delta\beta = \frac{\partial b}{\partial x_1}dx_1 - \frac{\partial b}{\partial x_2}dx_2.$$

From the requirement that γ be an harmonic 1-form, we obtain

$$\begin{aligned}(d\delta + \delta d)\gamma &= d(\delta\gamma) + \delta(d\gamma) \\ &= -\left(\frac{\partial^2\gamma_1}{\partial x_1^2} + \frac{\partial^2\gamma_1}{\partial x_2^2}\right)dx_1 - \left(\frac{\partial^2\gamma_2}{\partial x_1^2} + \frac{\partial^2\gamma_2}{\partial x_2^2}\right)dx_2.\end{aligned}$$

Then by comparing coefficients for dx_1 and dx_2 in $\omega = d\alpha + \delta\beta + \gamma$, we find

$$\begin{aligned}\omega_1(\mathbf{x}) &= \frac{\partial\alpha}{\partial x_1} + \frac{\partial b}{\partial x_2} + \gamma_1(\mathbf{x}), \\ \omega_2(\mathbf{x}) &= \frac{\partial\alpha}{\partial x_2} - \frac{\partial b}{\partial x_1} + \gamma_1(\mathbf{x})\end{aligned}$$

and that γ is an harmonic 1-form

$$\frac{\partial^2\gamma_1}{\partial x_1^2}+\frac{\partial^2\gamma_1}{\partial x_2^2}=0, \qquad \frac{\partial^2\gamma_2}{\partial x_1^2}+\frac{\partial^2\gamma_2}{\partial x_2^2}=0. \qquad \spadesuit$$

Example. Consider the 1-form $\omega = x_1x_2\, dx_1 + x_1^2 dx_2$ on manifold $M = \mathbf{R}^2$ with Euclidean metric $g = dx_1 \otimes dx_1 + dx_2 \otimes dx_2$. In this case we can decompose the form on $\mathbf{R}^2$, even though $\mathbf{R}^2$ is not compact.
If we choose

$$\alpha = \frac{x_1^2 x_2}{2} - 2x_2, \qquad \beta = -\frac{x_1^3}{6}\, dx_1 \wedge dx_2, \qquad \gamma = 2dx_2,$$

then we have $\omega = d\alpha + \delta\beta + \gamma$. First we show that γ is closed and coclosed, i.e. harmonic. We have

$$d\gamma = 0$$

and

$$\delta\gamma = - * d * \gamma = - * d * (2dx_2) = * \; d\,(2dx_1) = 0.$$

So γ is an harmonic 1-form.
We have $d\alpha = x_1x_2\, dx_1 + \left(\frac{x_1^2}{2} - 2\right) dx_2$ and

$$\delta\beta = - * d * \left(-\frac{x_1^3}{6}\, dx_1 \wedge dx_2\right) = *d\left(\frac{x_1^3}{6}\right) = *\left(\frac{x_1^2}{2} dx_1\right) = \frac{x_1^2}{2}\, dx_2.$$

Then

$$d\alpha + \delta\beta + \gamma \;=\; x_1x_2\, dx_1 + \left(\frac{x_1^2}{2} - 2\right) dx_2 + \frac{x_1^2}{2}\, dx_2 + 2dx_2 = \omega. \qquad \spadesuit$$

4.3 Hodge's theorem

Each element of $H^r(M)$ has a unique harmonic representation. This is contained in the following theorem.

Hodge's theorem. On a compact orientable Riemannian manifold (M, g), the de Rham cohomology group $H^r(M)$ is isomorphic to set of harmonic forms $\mathrm{Harm}^r(M)$ i.e. $H^r(M) \cong \mathrm{Harm}^r(M)$.

The isomorphism is given by taking $[\omega] \in H^r(M)$ to $P\omega \in \text{Harm}^r(M)$, where

$$P : \bigwedge_r T^*M \to \text{Harm}^r(M)$$

is a projection operator to the space of harmonic r-forms on M. The proof of Hodge's theorem is given in [1].

4.4 Hodge theory on non-compact manifolds

The proof of Hodge's decomposition theorem for non-compact requires the use of Sobolev spaces. Sobolev spaces provide the setting for the idea of square integrability, as well as for the concept of the convergence of a Cauchy sequence of forms to some limit form. Using Sobolev scales allows some of the requirements for compact support to be overcome. Below is a cursory overview of some of the mathematical tools and ideas used to extend Hodge theory to non-compact manifolds.

4.4.1 Sobolev space

Definition. A **Sobolev space** is a normed function space, where we have a finite $L^p, p \geq 1$ norm for a function f and all of its derivatives up to some order of k. More formally, for $n \geq 1$, and U an open subset of $\mathbf{R}^n, p \in [1; +\infty]$ and $k \in \mathbf{N}$, the Sobolev space $W^{k,p}(U)$ is defined as

$$W^{k,p}(U) := \{f \in L^p(U) \mid \forall\ |\alpha| \leq k, \partial_x^\alpha f \in L^p(U)\}$$

where $\alpha = (\alpha_1, ..., \alpha_n)$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and the derivatives

$$\partial_x^\alpha f := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} f$$

are taken in a weak sense [2].

When the Sobolev space is endowed with a norm, it is a Banach space. Sobolev spaces admit a natural norm, given by:

$$||f||_{k,p} = \sum_{\alpha=0}^{k} ||\partial_x^\alpha f||_p = \sum_{\alpha=0}^{k} \left(\int |\partial_x^\alpha f(t)|^p \, dt \right)^{\frac{1}{p}}.$$

The norm is actually uniquely defined by using the first and last terms in the above sum, i.e.

$$||f||_{k,p} = ||f||_p + ||\partial_x^k f||_p.$$

4.4.2 Hodge theory on smoothly bounded domain in Euclidean space

Classical Hodge theory discussed above on a compact domain $\Omega \in \mathbf{R}^{m+1}$ is based on the chain given by repeated application of the operator d:

$$\bigwedge_0 T^*M \to \bigwedge_1 T^*M \to \bigwedge_2 T^*M \to ...$$

In the Hilbert space $L^2(\Omega)$, the domain of existence and the form of the Hilbert space adjoint d^* of d can be found. The Hilbert space adjoint d^* equals δ, the formal adjoint of d on the domain of d^*.
We thus have the second order self-adjoint operator $\Box = dd^* + d^*d$. Calculated on a region of space Ω, it is found that $\Box$ is the negative of the Laplacian Δ, discussed above. $\Box$ has a closed range in $L^2(\Omega)$. The Neumann operator N is the inverse of $\Box$ for the d chain complex.
Below are some theorems which develop the Hodge theory for the operator d in a Sobolev space $W^1(\Omega)$ topology. $W^1(\Omega)$ is given as the Banach space of functions

$$W^1(\Omega) = \{f \in L^2(\Omega) \mid f^{(1)} \in L^2(\Omega)\}.$$

If Ω is now fixed as a smoothly bounded domain $\Omega \in \mathbf{R}^{m+1}$. Let $\bigwedge_r T^*\overline{\Omega}$, denote f-forms with $C^\infty(\mathbf{R}^{m-1})$ coefficients, such that the intersection of the support of the form with $\overline{\Omega}$ is compact in $\overline{\Omega}$.
Proposition. [3] Let Ω be defined as above and let $q = 0, 1, ..., m$. Then

$$\text{dom } d^* \bigcap \bigwedge_{q-1} (\overline{\Omega}) = \left\{ \phi \in \bigwedge_{q-1} (\overline{\Omega}) : \ \nabla_{\overline{\mathbf{n}}} \phi \rfloor \overline{\mathbf{n}}|_{\partial\Omega} = 0 \right\}.$$

Above, $\nabla_{\overline{\mathbf{n}}}\phi$ is the covariant derivative in the normal direction of the form ϕ. dom d^* is the domain of d^*.
Proposition. Let Ω be a smoothly bounded domain. Then on dom d^*,

$$d^* = \delta + K$$

where δ is the formal adjoint of d, and K is an operator taking $(q-1)$-forms to q-forms. Components of $K\phi$ are solutions of the boundary value problem:

$$\begin{cases} (-\Delta + I)(k\phi)_I & = 0 \text{ on } \Omega \\ \frac{\partial}{\partial n}(K\phi)_I & = T_2\phi \rfloor \mathbf{n} \text{ on } \partial\Omega \end{cases}$$

where T_2 is a second order differential operator on forms whose top order terms equal those of the Laplacian on the boundary [3].

Definition. For a smoothly bounded domain Ω, put

$$G_\Omega = Kd + dK.$$

Now $\Box = dd^* + d^*d = -\Delta + G_\Omega$.

The boundary value problem becomes

$$\begin{cases} (-\Delta + G_\Omega)\phi & = \alpha \text{ on } \Omega \\ \phi \in \text{dom } d^* \\ d\phi \in \text{dom } d^*. \end{cases} \tag{4.1}$$

Theorem. [3] Let $W_q^s(\Omega)$ denote the Sobolev space of q-forms. Let Ω be a smoothly bounded domain. Consider the boundary value problem (4.1). Let $s \geq \frac{1}{2}$. Then there exists a finite dimensional subspace (which is the harmonic space), denoted H_q of $\bigwedge_q T^*\overline{\Omega}$ and a constant $c = c_s > 0$, such that if $\alpha \in W_q^s(\Omega)$ is orthogonal to H_q, then (4.1) has a unique solution ϕ which is orthogonal to H_q in the W^1 topology such that

$$|\phi|_{s-2} \leq c|\alpha|_s.$$

Theorem. [3] Let Ω be a smoothly bounded domain in $\mathbf{R}^{m+1}$. Let $W_q^1(\Omega)$ denote the 1-Sobolev space of q-forms. Then there exists a strong orthogonal decomposition

$$W_q^1 = dd^*(W_q^1) \oplus d^*(dW_q^1) \oplus H_q$$

where H_q is a finite dimensional subspace of W_q^1.

Chapter 5

Symplectic manifolds and lemma

5.1 Introduction

A dynamical system is represented by a vector field Z on a differentiable manifold M. In this section, it will be shown that a vector field on a manifold M of dimension m is locally the sum of a gradient vector field and up to $m-1$ Hamilton vector fields. Most of the theoretical aspects of the work contained in this chapter were formulated by Roels [1] and Duarte and Mandes [2]. The theorems given by Roels, Duarte and Mendes are based on local decomposition. I have proved propositions contained in the works of the above authors and also filled in details ommited by the authors. I have used the fact that some vector fields can be globally decomposed by Hodge decomposition to extend the theory to non-compact Euclidean manifolds. The proof is cleaner than that given in [2] as pull-back maps are not used. I also formulate an algorithm for the decomposition of vector fields, and explicitly decompose the van der Pol vector field on non-compact space $\mathbf{R}^2$ according to this algorithm.

5.2 Symplectic manifolds

5.2.1 Codifferentials of 2-forms

The following proposition is used in a later proof.

Proposition. Consider Euclidean space $\mathbf{R}^m$ with the Euclidean metric $g = \sum_{i,j=1}^{m} \delta_{ij}\, dx_i \otimes dx_j$. Then the codifferential of a 2-form

$$\omega = \sum_{i,j=1}^{m} \alpha_{ij} dx_i \wedge dx_j$$

is given by

$$2(-1)^m \sum_{i,j=1}^{m} \frac{\partial \alpha_{ij}}{\partial x_j} dx_i.$$

Note. When we write ω in the form

$$\omega = \sum_{i_r<i_s}^{m} a_{i_r i_s} dx_{i_r} \wedge dx_{i_s}$$

then

$$\delta\omega = (-1)^m \sum_{i,j=1}^{m} \frac{\partial \alpha_{ij}}{\partial x_j} dx_i.$$

In what follows, a line over any particular basis element means that it is omitted from the expression.

Proof. Let $\alpha = \sum_{i,j=1}^{m} \alpha_{ij} dx_i \wedge dx_j$. Then

$$\begin{aligned}
\delta\alpha &= - * d * \alpha \\
&= - * d \sum_{i,j=1}^{m} \alpha_{ij} * (dx_i \wedge dx_j) \\
&= - * d \sum_{i,j=1}^{m} \alpha_{ij} \left(\frac{1}{(m-2)!} g^{i\nu_1} g^{j\nu_2} \varepsilon_{\nu_1 \ldots \nu_m} dx_{\nu_3} \wedge \ldots \wedge dx_{\nu_m} \right) \\
&= - * d \sum_{i,j=1}^{m} \alpha_{ij} \left(\frac{1}{(m-2)!} g^{i\nu_1} g^{j\nu_2} \varepsilon_{ij\nu_3 \ldots \nu_m} dx_{\nu_3} \wedge \ldots \wedge dx_{\nu_m} \right) \\
&= - * d \sum_{i,j=1}^{m} \alpha_{ij} \left(\varepsilon_{ij12\ldots N} dx_1 \wedge \ldots \wedge \overline{dx_i} \wedge \ldots. \wedge \overline{dx_j} \wedge \ldots dx_m \right) \\
&= - * \sum_{i,j=1}^{m} \left(\varepsilon_{ij12\ldots m} (d\alpha_{ij}) \, dx_1 \wedge \ldots \wedge \overline{dx_i} \wedge \ldots. \wedge \overline{dx_j} \wedge \ldots dx_m \right) \\
&= - * \sum_{i,j=1}^{m} \left(\varepsilon_{ij12\ldots m} \left(\sum_{k=1}^{m} \frac{\partial \alpha_{ij}}{\partial x_k} dx_k \right) dx_1 \wedge \ldots \wedge \overline{dx_i} \wedge \ldots. \wedge \overline{dx_j} \wedge \ldots dx_m \right) \\
&= - * \sum_{i,j=1}^{m} \left(\varepsilon_{ij12\ldots m} \frac{\partial \alpha_{ij}}{\partial x_i} dx_i \wedge dx_1 \wedge \ldots \wedge \overline{dx_i} \wedge \ldots. \wedge \overline{dx_j} \wedge \ldots dx_m \right) \\
&= - \sum_{i,j=1}^{m} \left(\varepsilon_{ij12\ldots.m} \left(\frac{\partial \alpha_{ij}}{\partial x_i} \varepsilon_{i12\ldots mj} dx_j + \frac{\partial \alpha_{ij}}{\partial x_j} \varepsilon_{j12\ldots mi} dx_i \right) \right)
\end{aligned}$$

$$= -\sum_{i,j=1}^{m}\left(\varepsilon_{ij12....m}\varepsilon_{i12...mj}\frac{\partial\alpha_{ij}}{\partial x_i}dx_j + \varepsilon_{ij12....m}\varepsilon_{j12...mi}\frac{\partial\alpha_{ij}}{\partial x_j}dx_i\right)$$

$$= -\sum_{i,j=1}^{m}\left(\varepsilon_{ij12....m}\varepsilon_{i12...mj}\frac{\partial\alpha_{ij}}{\partial x_i}dx_j + \varepsilon_{ji12....m}\varepsilon_{i12...mj}\frac{\partial\alpha_{ji}}{\partial x_i}dx_j\right)$$

$$= -\sum_{i,j=1}^{m}\left(\varepsilon_{ij12....m}\varepsilon_{i12...mj}\frac{\partial\alpha_{ij}}{\partial x_i}dx_j + \varepsilon_{ij12....m}\varepsilon_{i12...mj}\frac{\partial\alpha_{ij}}{\partial x_i}dx_j\right)$$

$$= -\sum_{i,j=1}^{m}\left(\varepsilon_{ij12....m}(-1)^{m-2}\varepsilon_{ij12...m}\frac{\partial\alpha_{ij}}{\partial x_i}dx_j + \varepsilon_{ij12....m}(-1)^{m-2}\varepsilon_{ij12...m}\frac{\partial\alpha_{ij}}{\partial x_i}dx_j\right)$$

$$= -2\sum_{i,j=1}^{m}\varepsilon^2_{ij12....m}(-1)^{m-2}\frac{\partial\alpha_{ij}}{\partial x_i}dx_j$$

$$= -2(-1)^{m-2}\sum_{i,j=1}^{m}\frac{\partial\alpha_{ij}}{\partial x_i}dx_j$$

$$= 2(-1)^{m-2}\sum_{i,j=1}^{m}\frac{\partial\alpha_{ji}}{\partial x_i}dx_j$$

$$= 2(-1)^{m-2}\sum_{i,j=1}^{m}\frac{\partial\alpha_{ij}}{\partial x_j}dx_i.$$

Example. Consider $\alpha = a_{13}(\mathbf{x})dx_1 \wedge dx_3 + a_{24}(\mathbf{x})dx_2 \wedge dx_4$ on $\mathbf{R}^4$. Then, using $a_{ij} = -a_{ji}$, we can directly write

$$\delta\alpha = \frac{\partial a_{13}}{\partial x_3}dx_1 + \frac{\partial a_{31}}{\partial x_1}dx_3 + \frac{\partial a_{24}}{\partial x_4}dx_2 + \frac{\partial a_{42}}{\partial x_2}dx_4. \qquad \spadesuit$$

5.2.2 2-form existence lemma

Lemma. [2] Consider $\mathbf{R}^m$ endowed with the Euclidean metric $g = \sum_{j=1}^{m} dx_j \otimes dx_j$. Let $\mathbf{p} \in M$ and let U be a compact neighbourhood of $\mathbf{p}$. Let Ω represent the volume form on $\mathbf{R}^m$. Then there exist $m-1$ non-degenerate 2-forms α_i on U such that

1. All the α_i's are closed i.e.

$$d\alpha_i = 0\ ;\quad i = \{1, \ldots, m-1\}$$

2.

$$\alpha_i \wedge \alpha_i \wedge \ldots \wedge \alpha_i = \frac{m}{2}\,\Omega$$

3.

$$*\alpha_i = \frac{1}{(m/2-1)!}\ \alpha_i \wedge \ldots \wedge \alpha_i$$

where the exterior product using the α_i's is formed $\left(\frac{m}{2}-2\right)$ times.

4.

$$*(\alpha_i \wedge \alpha_j) = 0; \quad i \neq j.$$

Now let ω any $C^\infty(\mathbf{R}^m)$ 2-form. Then there exist $m-1$ $C^\infty(\mathbf{R}^m)$ functions f_i and a 2-form α with

$$\alpha = \sum_{i=1}^{m-1} f_i \alpha_i$$

such that locally $\delta\alpha = \delta\omega$.

Proof. Consider the m dimensional real manifold with Euclidean coordinates. The $m-1$ non-degenerate 2-forms α_i discussed above are formed as follows

$$\begin{aligned}
\alpha_1 &= dx_1 \wedge dx_2 + dx_{i_{13}} \wedge dx_{i_{14}} + \cdots + dx_{i_{1(m-1)}} \wedge dx_{i_{1m}} \\
\alpha_2 &= dx_1 \wedge dx_3 + dx_{i_{23}} \wedge dx_{i_{24}} + \cdots + dx_{i_{2(m-1)}} \wedge dx_{i_{2m}} \\
\ldots &= \ldots\ldots\ldots \\
\alpha_{m-1} &= dx_1 \wedge dx_m + dx_{i_{(m-1)3}} \wedge dx_{i_{(m-1)4}} + \cdots + dx_{i_{(m-1)(m-1)}} \wedge dx_{i_{(m-1)m}}
\end{aligned}$$

where for α_q the numbers $1, p+1, i_{p3}, i_{p4}, \ldots, i_{pm}$ are an even permutation of the set $\{1, 2, \ldots, m\}$ and no basic 2 form appears more than once in the system above.

Now suppose we are given any smooth 2-form ω. The $m-1$ functions f_i need to be found such that

$$\delta\alpha = \delta\omega \tag{5.1}$$

where

$$\alpha = \sum_{i=1}^{m-1} f_i \alpha_i.$$

It was shown above that if α is a two form written as $\alpha = \sum_{1<j_1<j_2<m} \alpha_{j_1 j_2} dx_{j_1} \wedge dx_{j_2}$, then

$$\delta\alpha = \sum_{i,j=1}^{m} \frac{\partial \alpha_{ij}}{\partial x_j} dx_i$$

where $\alpha_{ij} = -\alpha_{ji}$.

Writing (1) out explicitly and equating coefficients leads to a system of partial differential equations which needs to be solved to find the functions f_i. The system can be written as

$$\sum_{j=1}^{m-1} M_{ij} f_i = \sum_{j=1}^{m} \frac{\partial \omega_{ij}}{\partial x_j} \qquad (5.2)$$

where $i = 1, \ldots m$, with matrix elements $M_{ij} = \frac{\partial}{\partial x_\nu}$ with $\nu \in \{1, 2, ..., m\}$. It can be shown that if the 2-form $dx_i \wedge dx_j$ occurs in α_k, the matrix elements are

$$M_{ik} = \frac{\partial}{\partial x_j}$$

and

$$M_{jk} = \frac{\partial}{\partial x_i}.$$

By a smooth truncation of the functions on the right hand side of system (5.2) ($\sum_{j=1}^{m} \frac{\partial \omega_{ij}}{\partial x_j}$) outside of the compact neighbourhood U, the above system of partial differential system can be written as

$$\sum_{j=1}^{m-1} M_{ij} f_i = u_i \qquad (5.3)$$

where u_i are C^∞ functions on $\mathbf{R}^m$, for example polynomials, that coincide with $\sum_{j=1}^{m} \frac{\partial \omega_{ij}}{\partial x_j}$ in U. Then solutions of the system (5.2) and (5.3) will coincide on U. System (5.3) consistes of the $m-1$ unknown functions f_i and m equations. By the construction

$$\sum_{i=1}^{m} \eta_i \, \overline{M}(\eta)_{ij} = 0 \quad i = 1, \ldots, m$$

it is seen that the system is not independent.

Eliminating one of the rows in matrix of partial derivatives gives a system of $m-1$ equations in $m-1$ unknowns

$$\sum_{j=1}^{m-1} \overline{M}_{ij} \; f_i = u_i \qquad (5.4)$$

with $i = 1, \ldots (m-1)$ such that $\det(\overline{M}_{ij})$ not identically zero.

The existence of a solution to (5.4) follows from the existence of a fundamental solution for operators with constant coefficients $\det(\overline{M}(\partial))$.
From the fundamental solution, by convolution with the functions u_i, it is proved that the $C^\infty(U)$ solution functions f_i to system (5.1) exist. ♠
Example. The lemma is now explicitly verified for $m = 4$. Choose

$$\alpha_1 = dx_1 \wedge dx_2 + dx_3 \wedge dx_4,$$

$$\alpha_2 = dx_1 \wedge dx_3 + dx_4 \wedge dx_2,$$

$$\alpha_3 = dx_1 \wedge dx_4 + dx_2 \wedge dx_3.$$

Then for (1)

$$d\alpha_1 = d(dx_1 \wedge dx_2) + d(dx_3 \wedge dx_4) = 0.$$

For (2)

$$\begin{aligned}\alpha_1 \wedge \alpha_1 &= (dx_1 \wedge dx_2 + dx_3 \wedge dx_4) \wedge (dx_1 \wedge dx_2 + dx_3 \wedge dx_4)\\ &= dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 + dx_3 \wedge dx_4 \wedge dx_1 \wedge dx_2\\ &= dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 + (-1)^4 dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4\\ &= 2dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4\\ &= \frac{1}{2}\,\Omega.\end{aligned}$$

For (3) we have

$$*\alpha_1 = *(dx_1 \wedge dx_2 + dx_3 \wedge dx_4) = *(dx_1 \wedge dx_2) + *(dx_3 \wedge dx_4).$$

Now

$$\begin{aligned}*(dx_1 \wedge dx_2) &= \sum_{j_1,\ldots,j_4=1}^{4} g^{1j_1} g^{2j_2} \frac{1}{(4-2)!} \frac{g}{\sqrt{|g|}} \varepsilon_{j_1} \ldots j_4 dx_{j_3} \wedge dx_{j_4}\\ &= \frac{1}{2}\varepsilon_{1234} dx_3 \wedge dx_4 + \frac{1}{2}\varepsilon_{1243} dx_4 \wedge dx_3\\ &= \frac{1}{2}(dx_3 \wedge dx_4 - (-1)^1 dx_3 \wedge dx_4 = dx_3 \wedge dx_4.\end{aligned}$$

similarly, we have $*(dx_3 \wedge dx_4) = dx_1 \wedge dx_2$.
So $*\alpha_1 = \alpha_1$.
For (4)

$$\begin{aligned}
*(\alpha_1 \wedge \alpha_2) &= *\left((dx_1 \wedge dx_2 + dx_3 \wedge dx_4) \wedge (dx_1 \wedge dx_3 + dx_4 \wedge dx_2)\right) \\
&= *(dx_1 \wedge dx_2 \wedge dx_1 \wedge dx_3 + dx_1 \wedge dx_2 \wedge dx_4 \wedge dx_2 \\
&\quad + dx_3 \wedge dx_4 \wedge dx_1 \wedge dx_3 + dx_3 \wedge dx_4 \wedge dx_4 \wedge dx_2) \\
&= 0.
\end{aligned}$$

This is a direct consequence of the alternating nature of $\wedge$.
Now consider general 2-form η in where $m = 4$.. We can write

$$\begin{aligned}
\eta(\mathbf{x}) &= \eta_{12}(\mathbf{x})\ dx_1 \wedge dx_2 + \eta_{13}(\mathbf{x})\ dx_1 \wedge dx_3 + \eta_{14}(\mathbf{x})\ dx_1 \wedge dx_4 \\
&\quad + \eta_{23}(\mathbf{x})\ dx_2 \wedge dx_3 + \eta_{24}(\mathbf{x})\ dx_2 \wedge dx_4 + \eta_{34}(\mathbf{x})\ dx_3 \wedge dx_4.
\end{aligned}$$

Then we have

$$\begin{aligned}
\delta\eta &= \frac{\partial\eta_{12}}{\partial x_2}dx_1 + \frac{\partial\eta_{21}}{\partial x_1}dx_2 + \frac{\partial\eta_{13}}{\partial x_3}dx_1 + \frac{\partial\eta_{31}}{\partial x_1}dx_3 + \frac{\partial\eta_{14}}{\partial x_4}dx_1 + \frac{\partial\eta_{41}}{\partial x_1}dx_4 \\
&\quad + \frac{\partial\eta_{23}}{\partial x_3}dx_2 + \frac{\partial\eta_{32}}{\partial x_2}dx_3 + \frac{\partial\eta_{24}}{\partial x_4}dx_2 + \frac{\partial\eta_{42}}{\partial x_2}dx_4 + \frac{\partial\eta_{34}}{\partial x_4}dx_3 + \frac{\partial\eta_{43}}{\partial x_3}dx_4 \\
&= \left(\frac{\partial\eta_{12}}{\partial x_2} + \frac{\partial\eta_{13}}{\partial x_3} + \frac{\partial\eta_{14}}{\partial x_4}\right)dx_1 + \left(\frac{\partial\eta_{21}}{\partial x_1} + \frac{\partial\eta_{23}}{\partial x_3} + \frac{\partial\eta_{24}}{\partial x_4}\right)dx_2 \\
&\quad + \left(\frac{\partial\eta_{31}}{\partial x_1} + \frac{\partial\eta_{32}}{\partial x_2} + \frac{\partial\eta_{34}}{\partial x_3}\right)dx_3 + \left(\frac{\partial\eta_{41}}{\partial x_1} + \frac{\partial\eta_{42}}{\partial x_2} + \frac{\partial\eta_{43}}{\partial x_3}\right)dx_4.
\end{aligned}$$

Now let

$$\begin{aligned}
\alpha(\mathbf{x}) &= a_1(\mathbf{x})\alpha_1 + a_2(\mathbf{x})\alpha_2 + a_3(\mathbf{x})\alpha_3 \\
&= a_1(\mathbf{x})(dx_1 \wedge dx_2 + dx_3 \wedge dx_4) + a_2(\mathbf{x})(dx_1 \wedge dx_3 + dx_4 \wedge dx_2) \\
&\quad + a_3(\mathbf{x})(dx_1 \wedge dx_4 + dx_2 \wedge dx_3) \\
&= a_1(\mathbf{x})dx_1 \wedge dx_2 + a_1(\mathbf{x})dx_3 \wedge dx_4 + a_2(\mathbf{x})dx_1 \wedge dx_3 + a_2(\mathbf{x})dx_4 \wedge dx_2 \\
&\quad + a_3(\mathbf{x})dx_1 \wedge dx_4 + a_3(\mathbf{x})dx_2 \wedge dx_3.
\end{aligned}$$

Then we have

$$\delta\alpha = -\frac{\partial a_1}{\partial x_1}dx_2 + \frac{\partial a_1}{\partial x_2}dx_1 - \frac{\partial a_1}{\partial x_3}dx_4 + \frac{\partial a_1}{\partial x_4}dx_3 - \frac{\partial a_2}{\partial x_1}dx_3 + \frac{\partial a_2}{\partial x_3}dx_1$$

$$-\frac{\partial a_2}{\partial x_4}dx_2 + \frac{\partial a_2}{\partial x_2}dx_4 - \frac{\partial a_3}{\partial x_1}dx_4 + \frac{\partial a_3}{\partial x_4}dx_1 - \frac{\partial a_3}{\partial x_2}dx_3 + \frac{\partial a_3}{\partial x_3}dx_2$$

$$= \left(\frac{\partial a_1}{\partial x_2} + \frac{\partial a_2}{\partial x_3} + \frac{\partial a_3}{\partial x_4}\right) dx_1 + \left(-\frac{\partial a_1}{\partial x_1} - \frac{\partial a_2}{\partial x_4} + \frac{\partial a_3}{\partial x_3}\right) dx_2$$

$$+ \left(\frac{\partial a_1}{\partial x_4} - \frac{\partial a_2}{\partial x_1} - \frac{\partial a_3}{\partial x_2}\right) dx_3 + \left(-\frac{\partial a_1}{\partial x_3} + \frac{\partial a_2}{\partial x_2} - \frac{\partial a_3}{\partial x_1}\right) dx_4$$

Putting $\delta\eta = \delta\alpha$ and equating coefficients we find

$$\left(\frac{\partial a_1}{\partial x_2} + \frac{\partial a_2}{\partial x_3} + \frac{\partial a_3}{\partial x_4}\right) = \left(\frac{\partial \eta_{12}}{\partial x_2} + \frac{\partial \eta_{13}}{\partial x_3} + \frac{\partial \eta_{14}}{\partial x_4}\right),$$

$$\left(-\frac{\partial a_1}{\partial x_1} - \frac{\partial a_2}{\partial x_4} + \frac{\partial a_3}{\partial x_3}\right) = \left(\frac{\partial \eta_{21}}{\partial x_1} + \frac{\partial \eta_{23}}{\partial x_3} + \frac{\partial \eta_{24}}{\partial x_4}\right),$$

$$\left(\frac{\partial a_1}{\partial x_4} - \frac{\partial a_2}{\partial x_1} - \frac{\partial a_3}{\partial x_2}\right) = \left(\frac{\partial \eta_{31}}{\partial x_1} + \frac{\partial \eta_{32}}{\partial x_2} + \frac{\partial \eta_{34}}{\partial x_3}\right),$$

$$\left(-\frac{\partial a_1}{\partial x_3} + \frac{\partial a_2}{\partial x_2} - \frac{\partial a_3}{\partial x_1}\right) = \left(\frac{\partial \eta_{41}}{\partial x_1} + \frac{\partial \eta_{42}}{\partial x_2} + \frac{\partial \eta_{43}}{\partial x_3}\right).$$

This system can be expressed as follows:

$$\frac{\partial a_1}{\partial x_2} + \frac{\partial a_2}{\partial x_3} + \frac{\partial a_3}{\partial x_4} = \sum_{j=1}^{3} \frac{\partial \eta_{1j}}{\partial x_j},$$

$$-\frac{\partial a_1}{\partial x_1} - \frac{\partial a_2}{\partial x_4} + \frac{\partial a_3}{\partial x_3} = \sum_{j=1}^{3} \frac{\partial \eta_{2j}}{\partial x_j},$$

$$\frac{\partial a_1}{\partial x_4} - \frac{\partial a_2}{\partial x_1} - \frac{\partial a_3}{\partial x_2} = \sum_{j=1}^{3} \frac{\partial \eta_{3j}}{\partial x_j},$$

$$-\frac{\partial a_1}{\partial x_3} + \frac{\partial a_2}{\partial x_2} - \frac{\partial a_3}{\partial x_1} = \sum_{j=1}^{3} \frac{\partial \eta_{4j}}{\partial x_j}.$$

This coincides with equation (5.1). We can write the above using matrix notation as

$$\begin{pmatrix} \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_4} \\ -\frac{\partial}{\partial x_1} & -\frac{\partial}{\partial x_4} & \frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_4} & -\frac{\partial}{\partial x_1} & -\frac{\partial}{\partial x_2} \\ -\frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} & -\frac{\partial}{\partial x_1} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{3} \frac{\partial \eta_{1j}}{\partial x_j} \\ \sum_{j=1}^{3} \frac{\partial \eta_{2j}}{\partial x_j} \\ \sum_{j=1}^{3} \frac{\partial \eta_{3j}}{\partial x_j} \\ \sum_{j=1}^{3} \frac{\partial \eta_{4j}}{\partial x_j} \end{pmatrix}.$$

Consider $\alpha_1 = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$. Set $k = 1$. Then for $dx_1 \wedge dx_2$ we set $i = 1$ and $j = 2$. This gives

$$M_{ik} = M_{11} = \frac{\partial}{\partial x_2},$$

$$M_{jk} = M_{21} = -\frac{\partial}{\partial x_1}.$$

For $dx_3 \wedge dx_4$, we set $i = 3$ and $j = 4$. This gives

$$M_{ik} = M_{31} = \frac{\partial}{\partial x_4},$$

$$M_{jk} = M_{41} = -\frac{\partial}{\partial x_3}.$$

We find agreement with equation (5.2) above. ♠

5.2.3 Coordinate free proof for 4 dimensions

The above lemma can be proved in a coordinate free manner for $m = 4$.

Proof. Given $\eta \in \bigwedge_2 T^*M$, we use the Hodge de Rham theorem to decompose η as

$$\eta = d\beta + \delta\gamma$$

where $\beta \in \bigwedge_1 T^*M$ and $\gamma \in \bigwedge_3 T^*M$.

Choose $\alpha = d\beta + *d\beta$. Then we need to show firstly that α is self dual, and secondly that $\delta\eta = \delta\alpha$.

$$*\alpha = *(d\beta + *d\beta) = *d\beta + **d\beta = *d\beta + d\beta = \alpha.$$

We have $\delta\eta = \delta d\beta + \delta\delta\gamma = \delta d\beta$.
and for α we have

$$\begin{aligned}\delta\alpha &= \delta d\beta + \delta * \beta \\ &= \delta d\beta + (-1)^{8+4+1} * d * (*d\beta) \\ &= \delta d\beta - *d(* * d\beta) \\ &= \delta d\beta - (-1)^4 * dd\beta \\ &= \delta d\beta \\ &= \delta\eta.\end{aligned}$$

♠

5.3 Decomposition theorems

Below are two identities which are used in the proofs that follow. The proofs of the identities are given.
Let α_i be the 2-forms discussed above and Z be a vector field on $\mathbf{R}^m$ with Euclidean metric $g = \sum_{i,j=1}^{m} \delta_{ij}\ dx_i \otimes dx_j$ and volume form $v = dx_1 \wedge \ldots \wedge dx_m$. Then

1. $*Z\rfloor v = (-1)^{m+1} g^\flat(Z)$.

   Proof. On the right hand side we know that in general, using metric g to lower indices, we have

$$\sum_{j=1}^{m} g_{ij} Z_j = \omega_i$$

but since $g_{ij} = \delta_{ij}$ we have $\omega_i = Z_i$. Thus $g^\flat Z = \sum_{i=1}^{m} Z_i dx_i$. On the left hand side, we have

$$Z\rfloor v = \sum_{i=1}^{m} (-1)^{i-1} Z_i dx_1 \wedge dx_2 \ldots \wedge \overline{dx_i} \wedge \ldots \wedge dx_m.$$

So

$$\begin{aligned}*(Z\rfloor v) &= \sum_{i=1}^{m} (-1)^{i-1} Z_i * (dx_1 \wedge dx_2 \ldots \wedge \overline{dx_i} \wedge \ldots \wedge dx_m) \\ &= \sum_{i=1}^{m} (-1)^{i-1} Z_i \frac{1}{(m-(m-1))!} \varepsilon_{1\ldots\bar{i}\ldots mi} dx_i\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{m} (-1)^{i-1} Z_i \varepsilon_{1\ldots\bar{i}\ldots mi} dx_i \\
&= \sum_{i=1}^{m} (-1)^{i-1+m-i} Z_i dx_i \\
&= \sum_{i=1}^{m} (-1)^{m+1} Z_i dx_i \\
&= (-1)^{m+1} g^{\flat}(Z). \qquad \spadesuit
\end{aligned}
$$

2. $Z \rfloor (\alpha_i \wedge \ldots \wedge \alpha_i) = \frac{m}{2} \alpha_i \wedge \ldots \wedge \alpha_i \wedge (Z \rfloor \alpha_i)$

   where on the left side there are $\frac{m}{2}$ occurances of α_i in the product.

   Proof.

$$
\begin{aligned}
Z \rfloor (\alpha_i \wedge \alpha_i \wedge \ldots \wedge \alpha_i) = {} & (Z \rfloor \alpha_i) \wedge \alpha_i \wedge \ldots \wedge \alpha_i + (-1)^2 \alpha_i \wedge (Z \rfloor \alpha_i) \wedge \ldots \wedge \alpha_i + \\
& (-1)^4 \alpha_i \wedge \alpha_i \wedge (Z \rfloor \alpha_i) \wedge \ldots \wedge \alpha_i + \ldots + (-1)^{2(\frac{m}{2}-1)} \alpha_i \wedge \alpha_i \wedge \ldots \wedge (Z \rfloor \alpha_i) \\
= {} & (Z \rfloor \alpha_i) \wedge \alpha_i \wedge \ldots \wedge \alpha_i + (-1)^{2} (Z \rfloor \alpha_i) \wedge \alpha_i \wedge \ldots \wedge \alpha_i + \\
& (-1)^4 (Z \rfloor \alpha_i) \wedge \alpha_i \wedge \ldots \wedge \alpha_i + \ldots + (-1)^{2(\frac{m}{2}-1)} (Z \rfloor \alpha_i) \wedge \alpha_i \wedge \ldots \wedge \alpha_i \\
= {} & \frac{m}{2} (Z \rfloor \alpha_i) \wedge \alpha_i \wedge \ldots \wedge \alpha_i \\
= {} & \frac{m}{2} \alpha_i \wedge \alpha_i \wedge \ldots \wedge (Z \rfloor \alpha_i). \qquad \spadesuit
\end{aligned}
$$

Below are the decomposition theorems that can be applied to different manifolds. These are coordinate based proofs and offer constructive methods for the actual decomposition.

Theorem. Consider manifold $\mathbf{R}^m$, where m is even. Let X vector field on $\mathbf{R}^m$, whose 1-form equivalent can be globally decomposed on $\mathbf{R}^m$ using Hodge's decomposition theorem. Then X can be globally decomposed into one gradient vector field and $(m-1)$ Hamiltonian vector fields on $\mathbf{R}^m$.

Proof. Let $g^{\flat} : TM \rightarrow \bigwedge_1 T^*M$ be the isomorphism from vector fields to 1-forms on $\mathbf{R}^m$, induced by g. Let $g^{\sharp} : \bigwedge_1 T^*M \rightarrow TM$ be the inverse. Using Poincare's Lemma and given the Hodge de Rham decomposition for the non-compact manifold $\mathbf{R}^m$, we can write

$$g^\flat(X) = dS + \delta\eta$$

where S is a real valued function on $\mathbf{R}^m$ and η is a 2-form on $\mathbf{R}^m$. Therefore

$$X = g^\sharp(dS) + g^\sharp(\delta\eta).$$

So $g^\sharp(dS)$ is a gradient vector field on $\mathbf{R}^m$. From the above lemma, we can choose the α_i so that we have

$$\delta\eta = \delta \sum_{i=1}^{m-1} a_i\alpha_i.$$

Therefore, we can write

$$g^\sharp(\delta\eta) = g^\sharp(\delta(\sum_{i=1}^{m-1} a_i\alpha_i)) = \sum_{i=1}^{m-1} g^\sharp(\delta(a_i\alpha_i)).$$

It needs to be shown that each $g^\sharp(\delta(a_i\alpha_i))$ is a Hamiltonian vector field with respect to the symplectic form α_i. So we need to show

$$g^\sharp(\delta(b_i\alpha_))\rfloor\alpha_i = db_i.$$

We have

$$\begin{aligned} g^\sharp\delta(b_i\alpha_i)\rfloor\alpha_i &= -\left[g^\sharp * (db_i \wedge *\alpha_i)\right]\rfloor\alpha_i \\ &= - * (*\alpha_i \wedge *(db_i \wedge *\alpha_i)) \\ &= - * \left[\left(\frac{1}{(m/2-1)!}\underbrace{\alpha_i \wedge \alpha_i \wedge \ldots \wedge \alpha_i}_{(\frac{m}{2}-1)}\right) \wedge *(db_i \wedge *\alpha_i)\right] \\ &= -\frac{1}{(m/2-1)!} * (\alpha_i \wedge \alpha_i \wedge \ldots \wedge \alpha_i \wedge *(db_i \wedge *\alpha_i)) \\ &= -\frac{1}{(m/2-1)!} * (\alpha_i \wedge \alpha_i \wedge \ldots \wedge \alpha_i \wedge g^\sharp(db_i)\rfloor\alpha_i) \\ &= -\frac{1}{(m/2)!} * (g^\sharp bd_i)\rfloor(\alpha_i \wedge \ldots \wedge \alpha_i) \\ &= - * (g^\sharp db_i)\rfloor v \\ &= db_i. \end{aligned}$$

The proof is complete. ♠

Note. The following was used in the proof. Using the above equalities, we have

$$\begin{aligned} g^{\sharp}(db_i)\rfloor\alpha_i &= *(*\alpha_i \wedge g^{\flat}(g^{\sharp}(db_i)) \\ &= *(*\alpha_i \wedge db_i) \\ &= (-1)^{(m-2)} * (db_i \wedge *\alpha_i) \\ &= *(db_i \wedge *\alpha_i) \end{aligned}$$

as m is even.

Below is a theorem which allows any vector field to be locally decomposed into a gradient and Hamilton parts.

Theorem. [2] Given an m-dimensional C^{∞} manifold M, where m is even. Then for every $\mathbf{x} \in M$, a open neighbourbood Ω of $\mathbf{x}$ can be found, such that every vector field defined on Ω can be decomposed into one gradient vector field and $(m-1)$ Hamiltonian vector fields.

Proof. Let $\phi : \Omega \to \mathbf{R}^m$ be a chart around $\mathbf{x} \in M$ so that $\phi(\mathbf{x}) = 0$. Let

$$\widehat{g} = \phi^* g$$

where ϕ^* is the pull back map and $g = \sum_{i,j=1}^{m} \delta_{ij}\, dx_i \otimes dx_j$ is the Euclidean metric defined on $\mathbf{R}^m$. Also, let

$$\widehat{\alpha_i} = \phi^* \alpha_i$$

where the α_i are the 2-forms discussed in the previous lemma. Let $\widehat{g^{\flat}} : T\Omega \to \bigwedge_1 T^*\Omega$ be the isomorphism from vector fields to 1-forms on Ω, induced by $\widehat{g}$. Let $\widehat{g^{\sharp}} : \bigwedge_1 T^*\Omega \to T\Omega$ be the inverse.

Also, let $\widehat{*}$ and $\widehat{v}$ be the Hodge Star operator and the volume form associated with $\widehat{g}$ on Ω, respectively.

Using Poincare's Lemma and the Hodge-De Rham Theorem, we can write

$$\widehat{g^{\flat}}(X) = dS + \delta\widehat{\eta}$$

where S is a real valued function on Ω and $\widehat{\eta}$ is a 2-form on Ω with η a two form on $\mathbf{R}^n$ such that $\phi^*\eta = \widehat{\eta}$. Therefore

$$X = \widehat{g^{\sharp}}(dS) + \widehat{g^{\sharp}}(\delta\widehat{\eta}).$$

So $\widehat{g^\sharp}(dS)$ is a gradient vector field on Ω. We have

$$\delta\widehat{\eta} = \delta\phi^*\eta = \phi^*(\delta\eta) = \phi^*(\delta\sum_{i=1}^{m-1} a_i\alpha_i) = \delta\sum_{i=1}^{m-1} a_i\phi^*\alpha_i = \delta\sum_{i=1}^{m-1} a_i\widehat{\alpha_i}.$$

Therefore, we can write

$$\widehat{g^\sharp}(\delta\widehat{\eta}) = \sum_{i=1}^{m-1} \widehat{g^\sharp}(\delta(a_i\widehat{\alpha_i})).$$

It needs to be shown that each $\widehat{g^\sharp}(\delta(a_i\widehat{\alpha_i}))$ is a Hamiltonian vector field with respect to the symplectic form $\widehat{\alpha_i}$. So we need to show

$$\widehat{g^\sharp}(\delta(a_i\widehat{\alpha_i}))\rfloor\widehat{\alpha_i} = da_i.$$

$$\begin{aligned}
\widehat{g^\sharp}\delta(b_i\widehat{\alpha_i})\rfloor\widehat{\alpha_i} &= -\left[\widehat{g^\sharp} * (db_i \wedge *\widehat{\alpha_i})\right]\rfloor\widehat{\alpha_i} \\
&= - * (*\widehat{\alpha_i} \wedge *(db_i \wedge *\widehat{\alpha_i})) \\
&= - * \left[\left(\frac{1}{(m/2-1)!}\underbrace{\widehat{\alpha_i}\wedge\widehat{\alpha_i}\wedge\ldots\wedge\widehat{\alpha_i}}_{(\frac{m}{2}-1)}\right)\wedge *(db_i\wedge *\widehat{\alpha_i})\right] \\
&= -\frac{1}{(m/2-1)!} * (\widehat{\alpha_i}\wedge\widehat{\alpha_i}\wedge\ldots\wedge\widehat{\alpha_i}\wedge *(db_i\wedge *\widehat{\alpha_i})) \\
&= -\frac{1}{(m/2-1)!} * (\widehat{\alpha_i}\wedge\widehat{\alpha_i}\wedge\ldots\wedge\widehat{\alpha_i}\wedge \widehat{g^\sharp}(db_i)\rfloor\widehat{\alpha_i}) \\
&= -\frac{1}{(m/2)!} * (\widehat{g^\sharp}bd_i)\rfloor(\widehat{\alpha_i}\wedge\ldots\wedge\widehat{\alpha_i}) \\
&= - * (\widehat{g^\sharp}db_i)\rfloor v \\
&= db_i.
\end{aligned}$$

The proof is complete. ♠

Decomposition Theorem for Riemannian manifolds. [2] Let X be a vector field on a Riemannian manifold (M, g). Then for each $\mathbf{x} \in M$, there exists a neighbourhood Ω of $\mathbf{x}$ and a symplectic form ω_X on Ω such that vector field X can be decomposed as the sum of one gradient and one Hamiltonian vector field.

Proof. Let $\mathbf{x} \in M$. Then either we have $X(\mathbf{x}) = 0$ or $X(\mathbf{x}) \neq 0$. If $X(\mathbf{x}) \neq 0$, then by the flow box theorem there exists a neighbourhood Ω and a local diffeomerphism $\phi : \Omega \to \mathbf{R}^m$ where $\phi(y) = (y_1, y_2, ..., y_m)$ such that $\phi_*(X) = \frac{\partial}{\partial y_1}$. Then, $\phi_*(X)$ is a Hamiltonian for the canonical symplectic form

$$\omega = \sum_{i=1}^{m/2} dy_{2i-1} \wedge dy_{2i}$$

in $\mathbf{R}^m$. Then X is a Hamiltonian in Ω for the form $\phi^* = \omega_X$.
If $X(\mathbf{x}) = 0$, then take X_∇ any gradient field for the metric g such that $X_\nabla(\mathbf{x}) \neq 0$. Then

$$Y = X + X_\nabla$$

does not vanish at point $\mathbf{x}$. Then we can apply the above argument so that Y is a Hamiltonian where

$$X = X_\nabla - Y.$$

Decomposition Theorem for Symplectic manifolds. [2] Let X be a vector field on a symplectic manifold M_ω. Then for each $\mathbf{x} \in M$, there is an open neighbourhood and a Riemannian metric g_X on Ω, such that X is decomposed into one gradient vector fiels and one Hamiltonian vector field.
Proof. Consider $\mathbf{x} \in M$. Then either $X(\mathbf{x}) = 0$ or $X(\mathbf{x}) \neq 0$. If $X(\mathbf{x}) \neq 0$, by the flow box theorem, we can find a neighbourhood of $\mathbf{x}$ and a metric g_X on Ω such that X is a gradient field. If $X(\mathbf{x}) = 0$, Then we can choose a Hamiltonian vector field X_ω such that $X_\omega(\mathbf{x}) \neq 0$. Then take $Y = X + X_\omega$ and apply the same argument again.

5.4 General decomposition algorithm

For simplicity we work with $\mathbf{R}^m$ and Euclidean metric $g = \sum_{i,j=1}^{m} \delta_{ij}\ dx_i \otimes dx_j$. Suppose we are given a vector field X. Then we use g to create the 1-form field $g^\flat(X)$. By Hodge de Rham theorem we write

$$g^\flat = dS + \delta\eta \tag{5.5}$$

where we know S is a function on $\mathbf{R}^m$ and $\eta \in \bigwedge_2 T^*M$. By taking the codifferential of both sides of (5.5), we get

$$\delta g^{\flat}(X) = \delta dS.$$

Now taking the Laplacian of S gives

$$\Delta S = (d\delta + \delta d)S = d\delta S + \delta dS = \delta dS,$$

as $d\delta S = 0$. So S is found by solving

$$\Delta S = \delta g^{\flat}(X).$$

Once we have found S, we have the gradient field given by $g^{\sharp}(dS)$. We substitute the expression for S into (5.5) and this gives

$$\delta\eta = g^{\flat}(X) - dS$$

to find an expression for $\delta\eta$. We then implicitly choose the α_i as described in the lemma above, which gives rise to the system of partial differential equations (5.1). The system is then solved to give the coefficients a_i. The $m-1$ Hamiltonian vector fields are then given by

$$H_i = g^{\sharp}(\delta(a_i\alpha_i)) \tag{5.5}$$

for $i = 1...(m-1)$.

Example. Consider the van der Pol system in $\mathbf{R}^2$

$$\begin{aligned}\frac{du_1}{dt} &= u_2\\ \frac{du_2}{dt} &= \alpha(1-u_1^2)u_2 - u_1.\end{aligned}$$

Then the vector field can be expressed as

$$X(u_1,u_2) = u_2\frac{\partial}{\partial u_1} + (\alpha(1-u_1^2)u_2 - u_1)\frac{\partial}{\partial u_2}.$$

We need to find a function ϕ such that $\Delta\phi = \delta g^{\flat}(X)$. With the Euclidean metric $g = du_1 \otimes du_1 + du_2 \otimes du_2$, we find

$$g^\flat(X) = u_2 du_1 + (\alpha(1-u_1^2)u_2 - u_1)du_2.$$

Using the fact that $*du_1 = du_2$ and $*du_2 = -du_1$, we have

$$\begin{aligned}
\delta g^\flat(X) &= - * d * (g^\flat(X)) \\
&= - * d \left(u_2 du_2 - (\alpha(1-u_1^2)u_2 - u_1)du_1 \right) \\
&= - * \left(-\frac{\partial}{\partial u_2}(\alpha(1-u_1^2)u_2 - u_1) \right) du_2 \wedge du_1 \\
&= - * \left(\alpha(1-u_1^2) \right) du_1 \wedge du_2 \\
&= \alpha(1-u_1^2).
\end{aligned}$$

A solution to the Poisson equation $\Delta\phi = \alpha(1-u_1^2)$ is

$$\phi(u_1,u_2) = \alpha\left(\frac{u_1^2}{2} - \frac{u_1^4}{12}\right).$$

This gives

$$d\phi = \alpha\left(u_1 - \frac{u_1^3}{3}\right) du_1.$$

According to the above algorithm, we put $\delta\eta = g^\flat(X) - d\phi$, and from this

$$\begin{aligned}
\delta\eta &= u_2 du_1 + \left(\alpha(1-u_1^2)u_2 - u_1\right) du_2 - \alpha\left(u_1 - \frac{u_1^3}{3}\right) du_1 \\
&= (u_2 - \alpha\left(u_1 - \frac{u_1^3}{3}\right))du_1 + \left(\alpha(1-u_1^2)u_2 - u_1\right) du_2.
\end{aligned}$$

Now consider the 2-form $\omega(u_1,u_2) = \omega_{12}(u_1,u_2)du_1 \wedge du_2$. Then

$$\delta\omega = \frac{\partial\omega_{12}}{\partial u_2}du_1 + \frac{\partial\omega_{21}}{\partial u_1}du_2$$

where $\omega_{21} = -\omega_{12}$. We now put $\delta\eta = \delta\omega$ and equate coefficients. This gives the system of partial differential equations

$$\frac{\partial \omega_{12}}{\partial u_2} = u_2 - \alpha u_1 + \alpha \frac{u_1^3}{3}$$
$$\frac{\partial \omega_{21}}{\partial u_1} = \alpha u_2 - \alpha u_1^2 u_2 - u_1$$

To solve the system, we use an ansatz polynomial $a = \sum_{i,j=0}^4 a_{ij} u_1^i u_2^j$. Taking partial derivatives of this gives

$$\frac{\partial a}{\partial u_1} = \sum_{i,j=0}^4 i a_{ij} u_1^{(i-1)} u_2^j$$
$$\frac{\partial a}{\partial u_2} = \sum_{i,j=0}^4 j a_{ij} u_1^i u_2^{(j-1)}$$

where terms with negative exponents are zero. Equating these equations and using the fact that $a_{21} = -a_{12}$ gives

$$u_2 - \alpha u_1 + \frac{\alpha}{3} u_1^3 = 2a_{02} u_2 + a_{11} u_1 + a_{31} u_1^3$$

and

$$-\alpha u_2 + \alpha u_1^2 + u_1 = a_{11} u_2 + 3a_{31} u_1^2 u_2 + 2a_{20} u_1.$$

From this we find

$$a_{02} = \frac{1}{2}, \qquad a_{11} = -\alpha, \qquad a_{31} = \frac{\alpha}{3}, \qquad a_{20} = \frac{1}{2}.$$

So we have $\omega_{12} = \frac{u_1^2}{2} + \frac{u_2^2}{2} - \alpha\left(u_1 - \frac{u_1^3}{3}\right) u_2$. This is the single Hamiltonian H for the decomposition and we put $H = \omega_{12}$. We use equation (5.5) to find the Hamiltonian vector field. First we find $\delta\left(\omega_{12}\, du_1 \wedge du_2\right)$. So

$$\delta\left(\left(\frac{u_1^2}{2} + \frac{u_2^2}{2} - \alpha\left(u_1 - \frac{u_1^3}{3}\right) u_2\right) du_1 \wedge du_2\right)$$

$$= (-1)^7 * d * \left(\left(\frac{u_1^2}{2} + \frac{u_2^2}{2} - \alpha \left(u_1 - \frac{u_1^3}{3} \right) u_2 \right) du_1 \wedge du_2 \right)$$

$$= - * \left((u_1 - \alpha u_2 + \alpha u_1^2 u_2) du_1 + \left(u_2 - \alpha \left(u_1 - \frac{u_1^3}{3} dx_2 \right) \right) du_2 \right)$$

$$= -(u_1 - \alpha u_2 + \alpha u_1^2 u_2) du_2 + \left(u_2 - \alpha \left(u_1 - \frac{u_1^3}{3} dx_2 \right) \right) du_1 .$$

Then

$$g^\sharp \left(\delta \left(\omega_{12}\, du_1 \wedge du_2 \right) \right) = \left(u_2 - \alpha \left(u_1 - \frac{u_1^3}{3} dx_2 \right) \right) \frac{\partial}{\partial u_1} - (u_1 - \alpha u_2 + \alpha u_1^2 u_2) \frac{\partial}{\partial u_2}$$

$$= \left(\frac{\partial H}{\partial u_2} \right) \frac{\partial}{\partial u_1} - \left(\frac{\partial H}{\partial u_1} \right) \frac{\partial}{\partial u_2} .$$

and so the Hamiltonian system can be written as

$$\frac{du_1}{dt} = \frac{\partial H}{\partial u_2}, \qquad \frac{du_2}{dt} = -\frac{\partial H}{\partial u_1}.$$

Then the van der Pol system can be written as the sum of the gradient field and the Hamiltonian field as follows:

$$\frac{du_1}{dt} = \frac{\partial \phi}{\partial u_1} - \frac{\partial H}{\partial u_2}, \qquad \frac{du_2}{dt} = \frac{\partial \phi}{\partial u_2} + \frac{\partial H}{\partial u_1}.$$

♠

Chapter 6

Decomposition of vector fields

This chapter demonstrates practical aspects of the theory discussed thus far. The Helmholz vector field decomposition theorem is given and I also show how Legendre polynomials can be used when applying this theorem. A simple formula is given which is equivalent to the Roels decomposition in $\mathbf{R}^2$. The decomposition of the van der Pol system into a gradient and one Hamilton vector field is given graphically near the origin. The Lorenz system is decomposed using the Helmholz method and the extended Roels decomposition. We see that these two decompositions are not equivalent. It is the extended Roels method that gives rise to vector fields with special properties. Decompositions of the Rikitake and the Rössler systems are presented [12].

6.1 Helmholz Theorem

Suppose we have a vector field X whose curl and divergence go to zero at infinity. Then the vector field may written as the sum of a solenoidal component and an irrotational (gradient) component i.e.

$$X = \nabla\phi + \nabla \times \mathbf{A}$$

where

$$\phi = -\frac{1}{4\pi}\int_X \frac{\nabla . X}{|\mathbf{r}' - \mathbf{r}|} d^3\mathbf{r}'$$

$$\mathbf{A} = \frac{1}{4\pi}\int_X \frac{\nabla \times X}{|\mathbf{r}' - \mathbf{r}|} d^3\mathbf{r}'.$$

This implies that we can reconstruct any vector field if it meets the specified boundary conditions and we know its divergence and curl at each point. This decomposition is not unique.

6.2 Application of decomposition

In physics and engineering many dynamical system are gradient systems. A gradient system on the open set $\mathbf{R}^n$ is a dynamical system which takes the form

$$\frac{d\mathbf{u}}{dt} = \text{grad}\Phi(\mathbf{u}) \tag{1}$$

where $\Phi : \mathbf{R}^n \to \mathbf{R}$ is an analytic function (potential energy function).
The corresponding vector field to the first order autonomous system (1) is given by

$$V_\Phi = \sum_{j=1}^{n} \frac{\partial \Phi}{\partial u_j} \frac{\partial}{\partial u_j} . \tag{2}$$

Gradient systems [1] have special properties that make their flow rather simple. For example we have

$$\frac{d\Phi}{dt} = \|\text{grad}\Phi(\mathbf{u})\|^2 .$$

Furthermore a gradient system cannot have a limit cycle or show chaotic behaviour. In physics and engineering we also find dynamical systems which are Hamilton systems, i.e., they can be derived from a Hamilton function $H(\mathbf{p}, \mathbf{q})$ as discussed above. The differential equations are

$$\frac{dq_j}{dt} = \frac{\partial H}{\partial p_j}, \qquad \frac{dp_j}{dt} = -\frac{\partial H}{\partial q_j} \tag{3}$$

where $j = 1, 2, \ldots, n$. Thus the vector field takes the form

$$V_H = \sum_{j=1}^{n} \left(\frac{\partial H}{\partial p_j} \frac{\partial}{\partial q_j} - \frac{\partial H}{\partial q_j} \frac{\partial}{\partial p_j} \right) . \tag{4}$$

A typical example is the pendulum. A Hamilton system also does not posses limit cycle behaviour. However we find Hamilton system with chaotic behaviour ($n \geq 2$), for example the Hénon-Heiles model.

However in most cases the dynamical system $d\mathbf{u}/dt = V(\mathbf{u})$, $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ is neither a pure gradient nor a pure Hamilton system. We assume that $V_j : \mathbf{R}^n \to \mathbf{R}$ are analytic functions. It was shown above that we can decompose the vector field V into a gradient part and a Hamilton part using up to $n-1$ Hamilton functions. For $n = 3$ there is also the Hodge-Helmholtz decomposition. These decompositions provide new insight into the behaviour of nonlinear dynamical systems. It also will be useful in the integration of the dynamical system using the Lie series technique [2,3,4].

Consider first the case $n = 2$. In a two-dimensional symplectic manifold every vector field is locally the sum of a Hamilton vector field and a gradient field. This can be extended to $\mathbf{R}^n$. Thus each autonomous system of first order in $\mathbf{R}^2$ can be written as

$$\frac{du_1}{dt} = \frac{\partial \Phi}{\partial u_1} + \frac{\partial H}{\partial u_2} \tag{5a}$$

$$\frac{du_2}{dt} = \frac{\partial \Phi}{\partial u_2} - \frac{\partial H}{\partial u_1} \tag{5b}$$

where H is the Hamilton function (u_1 is identified with p and u_2 is identified with q) and Φ is the potential energy function. A typical example is the van der Pol equation

$$\frac{du_1}{dt} = u_2$$

$$\frac{du_2}{dt} = b(1-u_1^2)u_2 - u_1$$

where $b > 0$ for a stable limit cycle system. For the van der Pol equation we find

$$\Phi(u_1, u_2) = b\left(\frac{u_1^2}{2} - \frac{u_1^4}{12}\right), \quad H(u_1, u_2) = \frac{u_1^2}{2} + \frac{u_2^2}{2} - b\left(u_1 - \frac{u_1^3}{3}\right)u_2 .$$

This decomposition is clearly seen when the separate vector fields are drawn seperately. For the graphs, $b = -2$. Figure 6.1 is the van der Pol system. Figure 6.2 is the gradient vector field. Figure 6.3 is the Hamiltonian vector field. Figure 6.3 is the sum of the gradient and Hamiltonian vector fields which corresponds with van der Pol system as expected.

Consider the manifold $\mathbf{R}^2$ with the metric tensor field $g = du_1 \otimes du_1 + du_2 \otimes du_2$. This implies the differential volume form [6] $\Omega = du_1 \wedge du_2$. The f-linear Hodge

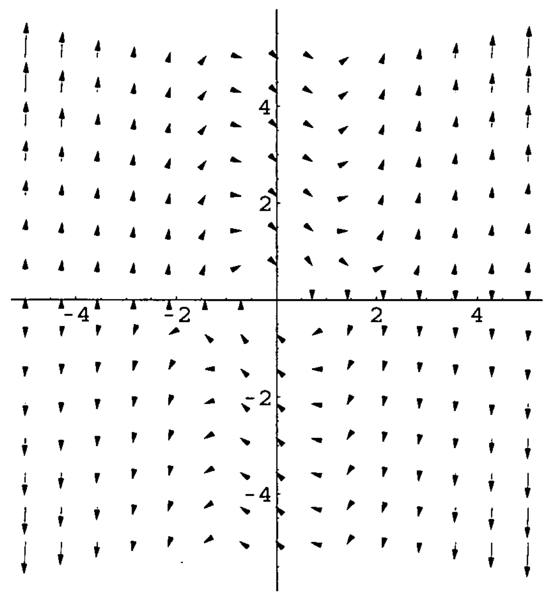

Figure 6.1: van der Pol system with b = -2

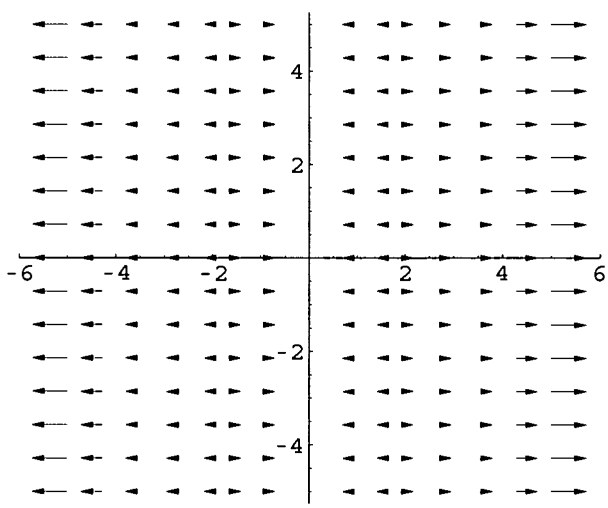

Figure 6.2: van der Pol gradient vector field

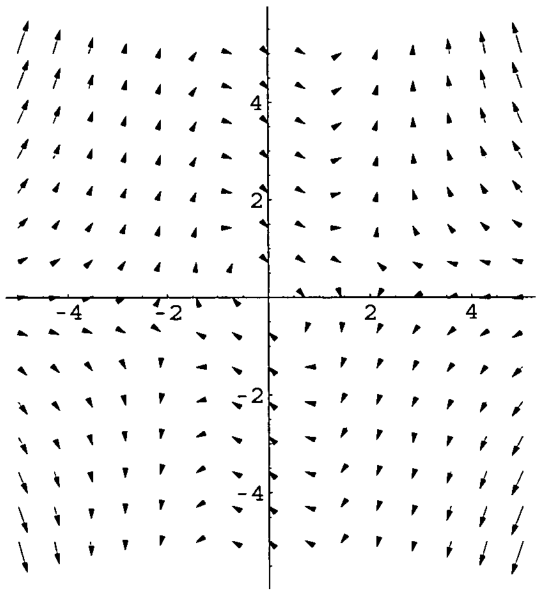

Figure 6.3: van der Pol Hamiltonian vector field

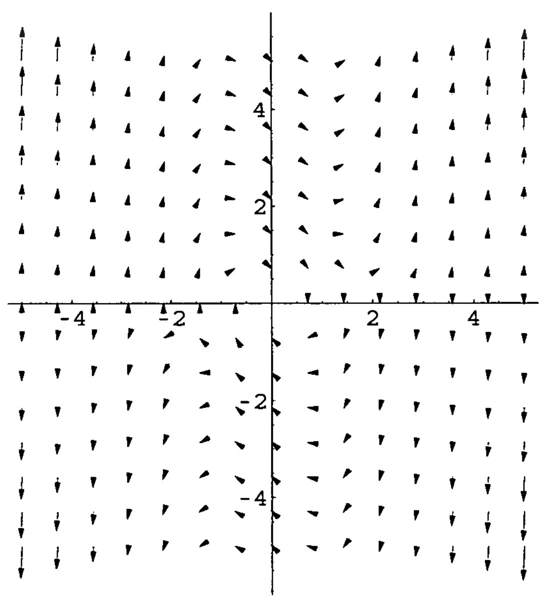

Figure 6.4: sum gradient and Hamiltonian vector fields

star operator [6] provides $*1 = du_1 \wedge du_2$, $*du_1 \wedge du_2 = 1$, and $*du_1 = du_2$ and $*du_2 = -du_1$. The coderivative δ is defined as

$$\delta\alpha := (-1)^{np+n-1} * d * \alpha$$

where α is a p-form $(0 \leq p \leq n)$ and $n = 2$ in the present case. Let

$$V = V_1(u_1, u_2)\frac{\partial}{\partial u_1} + V_2(u_1, u_2)\frac{\partial}{\partial u_2}$$

be a vector field on the manifold. Then system (5) can be expressed as

$$*(V \rfloor \Omega) = -d\Phi - \delta\eta$$

where $\eta := H(u_1, u_2) du_1 \wedge du_2$ and $\rfloor$ denotes the inner product $(\partial/\partial u_j \rfloor du_i = \delta_{ij})$. Since

$$*(V \rfloor \Omega) = *(V_1(u_1, u_2)du_2 - V_2(u_1, u_2)du_1) = -V_1(u_1, u_2)du_1 - V_2(u_1, u_2)du_2$$

and

$$\delta\eta = - * d * \eta = - * dH(u_1, u_2) = -\frac{\partial H}{\partial u_1}du_2 + \frac{\partial H}{\partial u_2}du_1$$

we obtain system (5). Taking the divergence of (5) we find $\Delta_2\Phi = \mathrm{div}V$, where Δ_2 the Laplace operator in two dimensions. This decomposition into a gradient part and a Hamilton part can be extended to higher dimensions.

For $n = 3$ we also have the Helmholtz-Hodge decomposition theorem of a given vector field V related to the autonomous system $d\mathbf{u}/dt = V(\mathbf{u})$ as [4,5,7,8]

$$V = \mathrm{grad}\Phi + \mathrm{curl}W \qquad (6)$$

where $\mathrm{grad}\Phi$ is the potential part. Note that (6) can also be formulated using differential forms. We have $\mathrm{div}(\mathrm{curl}W) = 0$. Let $D = \mathrm{div}V$ and $C = \mathrm{curl}V$. Then the condition is that $D(\mathbf{u})$ and $C(\mathbf{u})$ tend to zero sufficiently rapidly for $|\mathbf{u}| \to \infty$. This condition would not apply to most of our cases studied, for example the Lorenz model and the Rikitake two-disc dynamo. However any analytic vector field in $\mathbf{R}^3$ regardless of its behaviour at infinity can be written in the form (6), but this more general result does not follow directly from the Helmholtz-Hodge theorem. We have that $\mathrm{curl}W$ is a solenoid vector field S. Thus we can write

$V = \text{grad}\Phi + S$, where $\text{div}S = 0$. The decomposition is not unique. Since only the gradient of Φ is relevant to V, V only determines Φ up to an additive constant. Similarly, since only the curl of W is relevant to V, one can add the gradient of any function g to W without changing V. Modifying W in such a way is called making a gauge transformation. Specifying the divergence of W is called choosing a gauge. Obviously V is not affected by gauge transformations. We could additionally impose the condition that $\text{div}W = 0$.
Taking the divergence of the two sides of (6) it follows that $\text{div}V = \Delta_3\Phi$, where Δ_3 is the Laplace operator in three dimension and we used $\text{div}(\text{curl}W) = 0$. Taking the curl of both sides of (6) we obtain

$$\text{curl}V = \text{curl}(\text{curl}W) \equiv \text{grad}(\text{div}W) - \Delta_3 W$$

where we used $\text{curl}(\text{grad}\Phi) = \mathbf{0}$. If $\text{div}W = 0$, we have $\Delta_3 W = -\text{curl}V$.

Instead of a gradient part and a curl part we can also express V by a grad part and a part expressed with two of the three Hamilton functions $H_{12}(u_1, u_2)$, $H_{13}(u_1, u_3)$, $H_{23}(u_2, u_3)$, where the first independent variable will be the role of the coordinate and the second independent variable will play the role of the momentum. Examples will be given later.

If the condition described above for $D(\mathbf{u})$ and $C(\mathbf{u})$ is satisfied we can find the scalar potential $\Phi(\mathbf{u})$ and vector potential $W(\mathbf{u})$ via integration with the additional condition $\text{div}W = 0$. Let

$$d^2 := (u_1 - u_1')^2 + (u_2 - u_2')^2 + (u_3 - u_3')^2 = r^2 + r'^2 - 2(u_1 u_1' + u_2 u_2' + u_3 u_3')$$

where $r^2 = u_1^2 + u_2^2 + u_3^2$ and $r'^2 = u_1'^2 + u_2'^2 + u_3'^2$. Then we have

$$\Phi(\mathbf{u}) = -\frac{1}{4\pi} \iiint \left(V_1(\mathbf{u}') \frac{\partial d^{-1}}{\partial u_1} + V_2(\mathbf{u}') \frac{\partial d^{-1}}{\partial u_2} + V_3(\mathbf{u}') \frac{\partial d^{-1}}{\partial u_3} \right) du_1' du_2' du_3'$$

and

$$W_1(\mathbf{u}) = \frac{1}{4\pi} \iiint \left(V_3(\mathbf{u}') \frac{\partial d^{-1}}{\partial u_2} - V_2(\mathbf{u}') \frac{\partial d^{-1}}{\partial u_3} \right) du_1' du_2' du_3'$$

$$W_2(\mathbf{u}) = \frac{1}{4\pi} \iiint \left(V_1(\mathbf{u}') \frac{\partial d^{-1}}{\partial u_3} - V_3(\mathbf{u}') \frac{\partial d^{-1}}{\partial u_1} \right) du_1' du_2' du_3'$$

$$W_3(\mathbf{u}) = \frac{1}{4\pi} \iiint \left(V_2(\mathbf{u}') \frac{\partial d^{-1}}{\partial u_1} - V_1(\mathbf{u}') \frac{\partial d^{-1}}{\partial u_2} \right) du_1' du_2' du_3' .$$

Introducing spherical coordinates $u_1 = r\sin\theta\cos\phi$, $u_2 = r\sin\theta\sin\phi$, $u_3 = r\cos\theta$, where $0 \le \phi < 2\pi$, $0 \le \theta < \pi$, we obtain using the spherical cosine theorem

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')$$

the potential

$$\Phi(\mathbf{u}) = -\frac{1}{4\pi}\iiint (V_1(\mathbf{u}')(u_1'-u_1)+V_2(\mathbf{u}')(u_2'-u_2)+V_3(\mathbf{u}')(u_3'-u_3))K(r,r',\gamma)r'^2\sin\theta' dr' d\theta' d\phi'$$

and the vector potential

$$W_1(\mathbf{u}) = \frac{1}{4\pi}\iiint (V_3(\mathbf{u}')(u_2'-u_2) - V_2(\mathbf{u}')(u_3'-u_3))K(r,r',\gamma)r'^2\sin\theta' dr' d\theta' d\phi'$$

$$W_2(\mathbf{u}) = \frac{1}{4\pi}\iiint (V_1(\mathbf{u}')(u_3'-u_3) - V_3(\mathbf{u}')(u_1'-u_1))K(r',r',\gamma)r'^2\sin\theta' dr' d\theta' d\phi'$$

$$W_3(\mathbf{u}) = \frac{1}{4\pi}\iiint (V_2(\mathbf{u}')(u_1'-u_1) - V_1(\mathbf{u}')(u_2'-u_2))K(r,r',\gamma)r'^2\sin\theta' dr' d\theta' d\phi'$$

where

$$K(r,r',\gamma) := \frac{1}{(r^2 + r'^2 - 2rr'\cos\gamma)^{3/2}}.$$

Using $(r^2 + r'^2 - 2rr'\cos\gamma)^{3/2} = r^3(1 + t^2 - 2t\cos\gamma)^{3/2}$ where $t = r'/r$ with $r' < r$ we have the expansion with Legendre polynomials

$$\frac{1-t^2}{(1-2t\cos\gamma + t^2)^{3/2}} = \sum_{n=0}^{\infty}(2n+1)t^n P_n(\cos\gamma)$$

and

$$P_n(\cos\gamma) = P_n(\cos\theta)P_n(\cos\theta') + 2\sum_{m=1}^{\infty}\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta)P_n^m(\cos\theta')\cos(m(\phi-\phi')).$$

The case $r' > r$ is considered analogously. Using these expansions we can do the integration to find Φ, W_1, W_2 and W_3. Applications of this decompositions are in three-dimensional vector tomography [9], in current-density functional theory [10] and obviously in hydrodynamics [8].

The integration described above cannot be applied to the following examples since the integrals diverge. We could treat the problem in the sense of generalized functions. In $\mathbf{R}^n$ (hence in the absence of boundary conditions) the Hodge decompositions on Sobolev and Besov scales can be deduced from potential theoretic representation formulas and classical Calderon-Zygmund theory. However since the V_j are polynomials in our example we can make a polynomial ansatz for Φ and W and then determine the coefficients of the ansatz using computer algebra [11].

Let us now consider some examples. The Lorenz model is given by

$$\frac{du_1}{dt} = -\sigma u_1 + \sigma u_2$$

$$\frac{du_2}{dt} = -u_1 u_3 + r u_1 - u_2$$

$$\frac{du_3}{dt} = u_1 u_2 - b u_3 \, .$$

For the Lorenz model we find the Helmhotz-Hodge representation

$$\Phi(\mathbf{u}) = -\frac{\sigma}{2} u_1^2 - \frac{u_2^2}{2} - \frac{b}{2} u_3^2$$

and

$$W_1(\mathbf{u}) = -\frac{1}{2} u_1 u_2^2 - \frac{1}{6} u_1^3, \quad W_2(\mathbf{u}) = -\sigma u_2 u_3 + \frac{1}{6} u_2^3, \quad W_3(\mathbf{u}) = \frac{1}{2} u_1^2 u_3 - \frac{r}{2} u_1^2 + \frac{\sigma}{2} u_3^2 \, .$$

Here we have $\text{div} W = 0$. Using a Hamilton function $H(u_1, u_2)$ we find the representation

$$\Phi(\mathbf{u}) = -\sigma \frac{u_1^2}{2} - \frac{u_2^2}{2} - b \frac{u_3^2}{2} + u_1 u_2 u_3$$

where

$$\frac{du_1}{dt} = \frac{\partial \Phi}{\partial u_1} + \frac{\partial H}{\partial u_2}$$

$$\frac{du_2}{dt} = \frac{\partial \Phi}{\partial u_2} - \frac{\partial H}{\partial u_1}$$

$$\frac{du_3}{dt} = \frac{\partial \Phi}{\partial u_3}$$

and the Hamilton function is given by

$$H(u_1, u_2) = \frac{u_2^2}{2} (\sigma - u_3) + u_1^2 (u_3 - r/2) \, .$$

For this case we also find a vector field W

$$W_1(\mathbf{u}) = -u_1u_3^2 + ru_1u_3, \quad W_2(\mathbf{u}) = 0, \quad W_3(\mathbf{u}) = \sigma u_2 - u_2u_3\,.$$

Note that div$W \neq 0$ for this case. Another example is the Rikitake two disc dynamo

$$\begin{aligned}\frac{du_1}{dt} &= -\mu u_1 + u_2u_3\\ \frac{du_2}{dt} &= -\mu u_2 + (u_3 - a)u_1\\ \frac{du_3}{dt} &= 1 - u_1u_2\,.\end{aligned}$$

Here we find the Helmholtz-Hodge decomposition

$$\Phi(\mathbf{u}) = -\frac{\mu}{2}(u_1^2 + u_2^2) + u_3$$

and

$$W_1(\mathbf{u}) = \frac{1}{2}u_1u_2^2 + \frac{1}{6}u_1^3, \quad W_2(\mathbf{u}) = -\frac{1}{2}u_2u_3^2 - \frac{1}{6}u_2^3, \quad W_3(\mathbf{u}) = -\frac{1}{2}(u_3 + a)u_1^2 + \frac{1}{6}u_3^3$$

where div$W = 0$. A decomposition using the scalar potential Φ

$$\Phi(\mathbf{u}) = u_3 - \frac{\mu}{2}(u_1^2 + u_2^2)$$

the Hamilton functions

$$H_{12} = -\frac{1}{2}(u_3 - a)u_1^2, \qquad H_{13} = \frac{1}{2}(u_1^2 + u_3^2)u_2$$

is given by

$$\begin{aligned}\frac{du_1}{dt} &= \frac{\partial\Phi}{\partial u_1} + \frac{\partial H_{12}}{\partial u_2} + \frac{\partial H_{13}}{\partial u_3}\\ \frac{du_2}{dt} &= \frac{\partial\Phi}{\partial u_2} - \frac{\partial H_{12}}{\partial u_1}\\ \frac{du_3}{dt} &= \frac{\partial\Phi}{\partial u_3} - \frac{\partial H_{13}}{\partial u_1}\end{aligned}$$

where we have to use two Hamilton functions H_{12}, H_{13}.

The decompositions can be extended to higher dimensions. For example consider the Rössler system with hyperchaotic behaviour

$$\begin{aligned}\frac{du_1}{dt} &= -u_2 - u_3\\ \frac{du_2}{dt} &= u_1 + \frac{1}{4}u_2 + u_4\\ \frac{du_3}{dt} &= 3 + u_1u_3\\ \frac{du_4}{dt} &= -\frac{1}{2}u_3 + 0.05u_4\,.\end{aligned}$$

Here we find a decomposition

$$\begin{aligned}\frac{du_1}{dt} &= \frac{\partial\Phi}{\partial u_1} + \frac{\partial H}{\partial u_2} + \frac{\partial\widetilde{H}}{\partial u_3}\\ \frac{du_2}{dt} &= \frac{\partial\Phi}{\partial u_2} - \frac{\partial H}{\partial u_1} - \frac{\partial\widetilde{H}}{\partial u_4}\\ \frac{du_3}{dt} &= \frac{\partial\Phi}{\partial u_3} + \frac{\partial H}{\partial u_4} - \frac{\partial\widetilde{H}}{\partial u_1}\\ \frac{du_4}{dt} &= \frac{\partial\Phi}{\partial u_4} - \frac{\partial H}{\partial u_3} + \frac{\partial\widetilde{H}}{\partial u_2}\end{aligned}$$

where

$$\Phi(\mathbf{u}) = \frac{1}{6}u_1^3 + 0.15u_4^2$$

and

$$H = -\frac{1}{2}(u_1^2 + u_2^2) + \frac{1}{4}u_3^2 + 3u_4, \quad \widetilde{H} = -\frac{1}{2}u_1^2u_3 - \frac{1}{2}u_3^2 - \frac{1}{2}u_4^2 - \frac{1}{4}u_2u_4\,.$$

We have described two decompositions of vector fields. In both cases we have produced a gradient part, but in the two decompositions the gradient parts do not agree in general.

Chapter 7

First integrals and integral preserving iterators

Above we have looked at the decomposition of vector fields into one gradient and one or more Hamilton vector fields. Each Hamilton sub-systems has an associated Hamiltonian. This Hamiltonian is a first integral for the Hamilton system. When a vector field with a specific properties needs to be iterated numerically, care needs to be taken to preserve those properties. This chapter examines how general first integrals can be preserved in numerical integration [9].

7.1 Definition

The Poisson bracket of two smooth functions in $\mathbf{R}^{2n}$ is defined by

$$\{f, g\} := \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right)$$

A function $I(\mathbf{p}, \mathbf{q})$ is called a first integral with respect to the Hamiltonian H if we have

$$\{I, H\} = 0.$$

This is equivalent to $\frac{dI}{dt} = 0$.

Example: Consider the Hamilton function

$$H(\mathbf{p}, \mathbf{q}) = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + V(\| \mathbf{q} \|)$$

in $\mathbf{R}^6$ where

$$\| \mathbf{q} \| = \sqrt{p_1^2 + p_2^2 + p_3^2}.$$

The functions

$$h_1(\mathbf{p},\mathbf{q}) = q_2p_3 - q_3p_2$$

$$h_2(\mathbf{p},\mathbf{q}) = q_3p_1 - q_1p_3$$

$$h_3(\mathbf{p},\mathbf{q}) = q_1p_2 - q_2p_1$$

are first integrals corresponding to H.
This is shown for h_1. For this case, we have $n = 3$. Then

$$\begin{aligned}\{h_1, H\} &= \left(\frac{\partial h_1}{\partial q_1}\frac{\partial H}{\partial p_1} - \frac{\partial h_1}{\partial p_1}\frac{\partial H}{\partial q_1}\right) + \left(\frac{\partial h_1}{\partial q_2}\frac{\partial H}{\partial p_2} - \frac{\partial h_1}{\partial p_2}\frac{\partial H}{\partial q_2}\right) + \left(\frac{\partial h_1}{\partial q_3}\frac{\partial H}{\partial p_3} - \frac{\partial h_1}{\partial p_3}\frac{\partial H}{\partial q_3}\right)\\
&= p_2p_3 + q_3\left(\frac{\partial V}{\partial q_2}\right) - p_2p_3 - q_2\left(\frac{\partial V}{\partial q_3}\right)\\
&= q_3\left(\frac{\partial V}{\partial q_2}\right) - q_2\left(\frac{\partial V}{\partial q_3}\right).\end{aligned}$$

Now let $\| \,.\, \| = f$. Then have $V(\| \mathbf{q} \|) = V(f(\mathbf{q}))$. By the chain rule, we have

$$\frac{\partial V}{\partial q_i} = \frac{\partial V}{\partial f}\cdot\frac{\partial f}{\partial q_i}.$$

So,

$$\frac{\partial V}{\partial q_2} = \frac{\partial V}{\partial f}\left(p_1^2 + p_2^2 + p_3^2\right)^{-\frac{1}{2}} q_2$$

$$\frac{\partial V}{\partial q_3} = \frac{\partial V}{\partial f}\left(p_1^2 + p_2^2 + p_3^2\right)^{-\frac{1}{2}} . q_3.$$

Therefore

$$q_3\left(\frac{\partial V}{\partial q_2}\right) - q_2\left(\frac{\partial V}{\partial q_3}\right) = q_3q_2\frac{\partial V}{\partial f}\left(p_1^2 + p_2^2 + p_3^2\right)^{-\frac{1}{2}} - q_2q_3\frac{\partial V}{\partial f}\left(p_1^2 + p_2^2 + p_3^2\right)^{-\frac{1}{2}} = 0.$$

Likewise, it is found that $\{h_2, H\} = 0$ and $\{h_3, H\} = 0$. ♠

7.2 Integral preserving iterators

Geometric integration involves the numerical solution of differential equations, while as far as possible, one or more of the geometrical properties of the system. Here we are interested in preserving the first integrals of a system. First integrals of systems are important as they often represent such important concepts as the total energy, momentum, or the angular-momentum of the system. This gives rise to integral preserving iterators.

First, linear gradient methods are used to study first integrals. Then a method for finding integral-preserving-iterators is discussed. Finally, we look at first integrals explicitly dependent on time, and how these can be preserved.

Consider the autonomous system of first order ordinary differential equations

$$\frac{d\mathbf{u}}{dt} = V(\mathbf{u}) \tag{7.1}$$

where $V_j : \mathbf{R}^n \to \mathbf{R}$ are analytic functions. Many of these dynamical systems admit first integrals $I(\mathbf{u}(t))$ and explicitly time-dependent first integrals $I(\mathbf{u}(t), t)$ [1]. The later case we often find in dynamical systems with chaotic behaviour, for example the Lorenz model and the Rikitake two disc dynamo [2]. That $I(\mathbf{u}(t))$ is a first integral is expressed as $dI/dt = 0$.

We first consider first integrals which are not explicitly time dependent. Using the analytic vector field [3]

$$V = \sum_{j=1}^{n} V_j(\mathbf{u}) \frac{\partial}{\partial u_j}$$

we can express this condition as $L_V I(\mathbf{u}) = 0$, where L_V denotes the Lie derivative. Thus from $dI/dt = 0$ we can write

$$\sum_{j=1}^{n} \frac{\partial I}{\partial u_j} V_j(\mathbf{u}) = 0$$

which can be written as

$$(\nabla I(\mathbf{u}))^T V(\mathbf{u}) = 0\,. \tag{7.2}$$

It can be shown that (7.1) can be written as

$$\frac{d\mathbf{u}}{dt} = S(\mathbf{u}) \nabla I(\mathbf{u}) \tag{7.3}$$

where $S(\mathbf{u})$ is a skew-symmetric $n \times n$ matrix and ∇ denotes the gradient. Note that ∇I is considered as a column vector.
This is demonstrated as follows. We want to find matrix S such that

$$S(\mathbf{u})\nabla I(\mathbf{u}) = V(\mathbf{u}).$$

Then using (7.2), we have

$$(\nabla I(\mathbf{u}))^T S(\mathbf{u})\nabla I(\mathbf{u}) = (\nabla I(\mathbf{u}))^T V(\mathbf{u}) = 0.$$

This results in

$$\sum_{i,j=1}^{n} s_{ij}\frac{\partial I}{\partial u_i}\frac{\partial I}{\partial u_j} = 0$$

which can be written as

$$\sum_{I}(s_{ij} - s_{ji})\frac{\partial I}{\partial u_i}\frac{\partial I}{\partial u_j} = 0$$

where I is the multi-index (i, j) with $i < j$. If all the $\frac{\partial I}{\partial u_i}$ are functionally independent, then it follows that

$$s_{ij} = -s_{ji}.$$

With $S(\mathbf{u})$ anti-symmetric, equation (7.3) can be represented as

$$-(g^{\sharp}(dI))\rfloor\mathbf{S} = g^{\flat}(V)$$

where $\mathbf{S}(\mathbf{u}) = \sum_{i,j=1}^{n} s_{ij}(\mathbf{u})\, dx_i \wedge dx_j$ with $s_{ij} = 0$ for $i \geq j$.
The matrix $S(\mathbf{u})$ is given by

$$S(\mathbf{u}) = \frac{1}{|\nabla I|^2}(V(\nabla I)^T - (\nabla I)V^T) \qquad (7.4)$$

where ^{T} denotes transpose. Obviously we have to assume that $|\nabla I|$ is non-vanishing. An integral preserving discrete version of this is given by ([4], [5])

$$\frac{\mathbf{u}' - \mathbf{u}}{\tau} = \overline{S}(\mathbf{u}, \mathbf{u}', \tau)\overline{\nabla} I(\mathbf{u}, \mathbf{u}')$$

where τ is the step length, $\mathbf{u}, \mathbf{u}'$ denote $\mathbf{u}_n$ and $\mathbf{u}_{n+1}$, respectively. The matrix $\overline{S}$ is a skew symmetric matrix satisfying for consistency

$$\overline{S}(\mathbf{u}, \mathbf{u}', \tau) = S(\mathbf{u}) + O(\tau)\,.$$

The general discrete gradient $\overline{\nabla} I$ is defined by

$$(\mathbf{u}' - \mathbf{u}) \cdot \overline{\nabla} I(\mathbf{u}', \mathbf{u}) := I(\mathbf{u}') - I(\mathbf{u})\,.$$

The discrete gradient $\overline{\nabla} I(\mathbf{u}', \mathbf{u})$ may be expanded in the form

$$\overline{\nabla} I(\mathbf{u}', \mathbf{u}) = \nabla I + B(\mathbf{u})(\mathbf{u}' - \mathbf{u}) + (\mathbf{u}' - \mathbf{u})^T M(\mathbf{u})(\mathbf{u}' - \mathbf{u}) + O(||\mathbf{u}' - \mathbf{u}||).$$

From this it is found that

$$B_{ij} + B_{ji} = I_{ij}, \tag{7.5}$$

$$M_{ijk} + M_{jki} + M_{kij} = \frac{1}{2} I_{ijk}. \tag{7.6}$$

A discrete gradient $\overline{\nabla} I$ is a first order integral-preserving-iterator(IPI) if $\overline{\nabla} I(\mathbf{u}', \mathbf{u}) \neq \overline{\nabla} I(\mathbf{u}, \mathbf{u}')$, and the skew symmetric matrix $\overline{S} = S(\mathbf{u})$.

7.2.1 Higher order iterators

The method for finding iterators of higher orders of accuracy described below is given by McLaren and Quispel [4]. The method is more efficient than Yoshida's composition method [8]. Systems having a constant matrix $S = S(\mathbf{u})$ are considered. Subscripts denote the order of the iterator. In the following, subscripts are indices of tensor components, except for subscripts used with I, where $I_j := \frac{\partial I}{\partial u_j}$ and $I_{ij} := \frac{\partial^2 I}{\partial u_i \partial u_j}$.

For first order we have

$$S_1 = S(\mathbf{u}).$$

For second order we have

$$S_2 = S + \tau S Q(\mathbf{u}) S$$

where $Q(\mathbf{u}) = \frac{1}{2} H(\mathbf{u}) - B(\mathbf{u})$. Here H is the Hessian $H = \frac{\partial^2 I}{\partial u_i \partial u_j}$ and B is given by (7.5).

Then for third order

$$S_3 = S_2 + \tau^2 \overline{R} = S + \tau S Q(\mathbf{u}) S + \tau^2 \overline{R}$$

where

$$\overline{R} := SQSQS - \frac{1}{12}SHSHS + \overline{E}$$

$$\overline{E}_{rs} :=_{ri} P_{ijk}S_{jm}(\overline{\nabla}I)_m S_{kn}$$

$$P_{ijk} := \frac{1}{6}I_{ijk} - M_{ijk}.$$

As a choice for a non-symmetric gradient, we can use

$$\overline{\nabla}I(\mathbf{u},\mathbf{u}') = \begin{pmatrix} \frac{I(u_1',u_2,u_3,\ldots u_n)-I(u_1,u_2,u_3,\ldots u_n)}{u_1'-u_1} \\ \frac{I(u_1',u_2',u_3,\ldots u_n)-I(u_1',u_2,u_3,\ldots u_n)}{u_2'-u_2} \\ \cdot \\ \cdot \\ \frac{I(u_1',u_2',u_3',\ldots u_n')-I(u_1',u_2',\ldots,u_{n-1}',u_n)}{u_n'-u_n} \end{pmatrix}$$

Then

$$B_{ij} = \begin{cases} 0 & \text{if} \quad i < j; \\ \frac{1}{2}I_{ij} & \text{if} \quad i = j; \\ I_{ji} & \text{if} \quad i > j. \end{cases}$$

$$M_{kij} = \begin{cases} 0 & \text{for} \quad i,j > k \quad \text{if } k \le n-1; \\ \frac{1}{2}I_{kii} & \text{for} \quad i = 1,2\ldots,k-1; \quad j = i; \\ \frac{1}{6}I_{iii} & \text{if} \quad i = k; \quad j = i; \\ \frac{1}{2}I_{kij} & \text{for} \quad j = 2,3,\ldots,k-1; \quad i = 1,2,\ldots,j-1; \\ \frac{1}{4}I_{kik} & \text{for} \quad i = 1,2,\ldots,k-1; \quad j = k; \\ M_{kji} & \text{(symmetric)}. \end{cases}$$

7.2.2 Multiple first integrals and explicitly time dependent first integrals

As an example consider a Lotka-Volterra model with three species $(u_1, u_2, u_3 > 0)$

$$\frac{du_1}{dt} = u_1u_2 - u_1u_3$$

$$\frac{du_2}{dt} = u_2u_3 - u_1u_2$$

$$\frac{du_3}{dt} = u_3u_1 - u_2u_3\,.$$

It describes the interaction between three species, where species 1 feeds on species 2, species 2 feeds on species 3 and species 3 feeds on species 1. The model is of interest since it admits two first integrals, namely $I_1(\mathbf{u}) = u_1 + u_2 + u_3$ and $I_2(\mathbf{u}) = u_1 u_2 u_3$. The fixed points of this system is the manifold $\{(u_1, u_2, u_3) : u_1 = u_2 = u_3\}$. From the constant of motions $u_1 + u_2 + u_3 = C_1$ and $u_1 u_2 u_3 = C_2$, where $C_1 > 0$, $C_2 > 0$ and stability analysis we find that the system admits closed orbits as solutions. Using the approach given above we have to decide which of the two first integrals we use for the discretization.

For explicitly time-dependent first integrals we extended the autonomous system (1) to the autonomous system in $\mathbf{R}^{n+1}$

$$\frac{d\mathbf{u}}{d\epsilon} = V(\mathbf{u})$$

$$\frac{dt}{d\epsilon} = 1$$

where $t(\epsilon = 0) = 0$. Then we have the vector field in $\mathbf{R}^{n+1}$

$$W = V + \frac{\partial}{\partial t}$$

and the definition for the explicitly first integral $dI(\mathbf{u}(t), t)/dt = 0$ can be written as $L_W I(\mathbf{u}, t) = 0$. Thus the methods for first integral presevation described above can be extended to explicitly time-dependent first integrals. As an example with explicitly time-dependent first integrals consider the Lorenz model

$$\frac{du_1}{dt} = \sigma(u_2 - u_1)$$

$$\frac{du_2}{dt} = -u_2 - u_1(u_3 - r)$$

$$\frac{du_3}{dt} = u_1 u_2 - b u_3 \,.$$

For example for $b = 2\sigma$ and r arbitrary we find the explicitly time-dependent first integral

$$I(\mathbf{u}(t), t) = (u_1^2 - 2\sigma u_3)\exp(2\sigma t)\,.$$

Other explicitly time-dependent first integrals for the Lorenz model are given by Kus [6]. These first integrals can be found using an ansatz given by Steeb [2] and the Carleman embedding [7]. Another example of a dynamical system with explicitly time-dependent first integrals is the Rikitake-two-disc dynamo

$$\frac{du_1}{dt} = -ru_1 + u_3u_2$$
$$\frac{du_2}{dt} = -ru_2 + (u_3 - a)u_1$$
$$\frac{du_3}{dt} = 1 - u_1u_2 \, .$$

In this case we have the explicitly time-dependent first integral for $a = 0$ and r arbitrary [2]

$$I(\mathbf{u}(t), t) = (u_1^2 - u_2^2)\exp(2rt) \, .$$

There are also systems which admit two explicitly time-dependent first integrals. For example

$$\frac{du_1}{dt} = cu_1 + c_{23}u_2u_3$$
$$\frac{du_2}{dt} = cu_2 + c_{13}u_1u_3$$
$$\frac{du_3}{dt} = cu_3 + c_{12}u_1u_2$$

with the explicitly time-dependent first integrals

$$I_1(\mathbf{u}(t), t) = \frac{1}{2}(c_{13}u_1^2 - c_{23}u_2^2)e^{-2ct}, \qquad I_2(\mathbf{u}(t), t) = \frac{1}{2}(c_{12}u_1^2 - c_{23}u_3^2)e^{-2ct} \, .$$

Using the approach for the discretization given above we again have to decide which of the explicitly time-dependent first integrals we are applying.

Example. Consider the Lorenz model given above.
For $b = 2\sigma$ and r arbitrary we find the explicitly time-dependent first integral

$$I(\mathbf{u}(t), t) = (u_1^2 - 2\sigma u_3)\exp(2\sigma t) \, .$$

To find the first order integral preserving iterator, we write the system as

$$\frac{du_1}{d\epsilon} = \sigma(u_2 - u_1)$$

$$\frac{du_2}{d\epsilon} = -u_2 - u_1(u_3 - r)$$

$$\frac{du_3}{d\epsilon} = u_1 u_2 - b u_3$$

$$\frac{dt}{d\epsilon} = 1.$$

giving the vector field $W = V + \frac{\partial}{\partial t}$. Then choosing

$$S(\mathbf{u}) = \begin{pmatrix} 0 & s_{12} & s_{13} & s_{14} \\ -s_{21} & 0 & s_{23} & s_{24} \\ -s_{13} & -s_{23} & 0 & s_{34} \\ -s_{14} & -s_{24} & -s_{34} & 0 \end{pmatrix}$$

with

$$s_{12} = \frac{u_2}{2u_1 e^{2\sigma t}}$$

$$s_{13} = \frac{\sigma u_2 - \sigma u_1 - 1}{-2\sigma e^{2\sigma t}}$$

$$s_{14} = 1$$

$$s_{23} = \frac{u_1 u_3}{2\sigma e^{2\sigma t}}$$

$$s_{24} = r u_1$$

$$s_{34} = -2\sigma u_3 + u_1^2 + \frac{u_1}{\sigma}$$

we find

$$L_W I(\mathbf{u}, t) = 0.$$

It is found that $I(\mathbf{u}, t)$ has a constant value of $\frac{1}{2\sigma}$. ♠

Chapter 8

Conclusion

The full theoretical background to the work of Roels [1] and Mendes and Duarte [2] has been presented in this thesis. The decomposition theorems of Roels and Mendes have been elucidated. A theorem has been presented that allows for the decomposition of vector fields on non-compact Euclidean spaces. The decomposition theorems have been simplified into an algorithm that can be implemented using computer algebra. The Roels extended decomposition has been applied to vector fields on non-compact manifolds. The decomposition has been graphed for the van der Pol system. The decomposition of Helmholz has been investigated, and the use of Legendre polynomials in the decomposition has been demonstrated. An equivalent form of the Roels decomposition has been given [3]. The preservation of autonomous and non-autonomous first integrals during numeric integration has been investigated. A theoretical backbone for the preservation has been provided [4]. Also investigated are methods for finding integral preserving iterators of higher order [5].

The application of the vector field decomposition to geometric integration theories could prove fruitful. That is, preserving certain geometric properties of the separate vector fields may result in the preservation of these properties to a greater order of accuracy in the original vector field. A method for numerically integrating a vector field could involve the decomposition of the field according to Mendes and Duatre, and the preservation of the Hamiltonians of the separate sub-manifolds.

Another area open to examination is the way that certain geometrical properties of the original vector field may be preserved in the decomposed fields, for example chaotic behavior and curvature. The comparison of properties of the resulting vector fields to each other could also be interesting.

An interesting exercise would be to determine if the theory of chapter five could be carried over to fractional gradient and Hamiltonian systems [6]. More specifically, given any $\alpha \in \mathbf{R}$, can any vector field be decomposed into gradient and Hamiltonian vector fields of fractional order α. This would provide a more general form of the theory. It seems that this would require a fractional form of the Hodge decomposition theorem, which to my knowledge, does not at present exist.

Chapter 9

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SUMMARY

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