GAME-THEORETIC METHODS IN LOGIC

by

Michael Jacobus Dorfling

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Supervisor: Prof. V. F. Goranko

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To my wife, Samantha, with love.

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Summary

The aim of the thesis is to develop game-theoretic techniques for dealing with common problems in model theory, mainly that of showing logical equivalence between structures, and to illustrate the effectiveness of the game-theoretic approach by means of examples.

Chapter 1 gives the basic definitions regarding first-order logic and structures.

Chapter 2 introduces Ehrenfeucht's game and the associated characterization of elementary equivalence. We give some applications to definability and completeness and we show how the restrictions in Ehrenfeucht's theorem can be circumvented.

In Chapter 3 we obtain extensions of Ehrenfeucht's theorem for monadic second-order logic, infinitary logic, logics with cardinality quantifiers and first-order logic with a bounded number of variables.

Chapter 4 discusses modal logic and the game-theoretic counterparts of bisimulation and bounded bisimulation. We also obtain bisimulations as fixed points of certain operators.

In Chapter 5 we discuss a general framework in which all our games fit and we briefly mention a game-theoretic approach to forcing and game-theoretic semantics.

Opsomming

Die doel van die skripsie is om spel-teoretiese tegnieke te ontwikkel waarmee tipiese probleme in model teorie, veral logiese ekwivalensie van strukture, hanteer kan word en om die doeltreffendheid van die spel-teoretiese benadering met voorbeelde te illustreer.

In Hoofstuk 1 word die basiese definisies aangaande eerste-orde logika en strukture gegee.

Hoofstuk 2 handel oor Ehrenfeucht se spel en die gepaardgaande karakterisering van elementêre ekwivalensie van strukture. Ons gee 'n paar toepassings in definiëerbaarheid en aksiomatiserings en ons toon aan hoe die beperkings in Ehrenfeucht se stelling oorbrug kan word.

In Hoofstuk 3 verkry ons uitbreidings van Ehrenfeucht se stelling vir monadiese tweede-orde logika, logikas met oneindige disjunksies, logikas met kardinaliteitskwantore en eerste-orde logika met 'n beperkte aantal veranderlikes.

Hoofstuk 4 bespreek modale logika en die spel-teoretiese duale van bisimulasie en begrensde bisimulasie. Ons verkry ook bisimulasies as vaste punte van sekere operatore.

In Hoofstuk 5 bespreek ons 'n algemene raamwerk waarin al ons spelle pas en ons noem kortliks 'n spel-teoretiese benadering tot konstruksie en spel-teoretiese semantiek.

Chapter 0

Introduction

A typical problem in model theory is that of showing that two structures are equivalent with respect to some language. This type of problem usually arises when proving that a certain property is not definable in a certain language, or when proving that a theory is complete with respect to some class of structures. A classic example is the first-order equivalence of the orderings of the rational numbers and the real numbers, proving that continuity of a linear ordering cannot be expressed in first-order logic.

Showing directly that two structures are equivalent can be a difficult task, even for simple examples such as the one mentioned above. This task was greatly simplified with the algebraic characterization of elementary equivalence between structures given by R. Fraïssé in 1954. Seven years later A. Ehrenfeucht gave an elegant game-theoretic characterization, which is still the best technique available. Ehrenfeucht's result is essentially a restatement of Fraïssé's result, however, the game-theoretic approach has several advantages. Firstly, in specific situations it is easy to understand exactly what has to be done in order to obtain the required result. The example of the rationals and the reals above, for instance, becomes almost trivial once the problem is stated in game-theoretic terms. Secondly, the use of games prove to be an invaluable aid to the intuition, making it easier to see arguments that may otherwise be overlooked.

We will concentrate on the game-theoretic method. The interested reader is referred to [8] or [14] for a discussion on Fraïssé's theorem. References to the two original papers by Ehrenfeucht and Fraïssé can be found in [5].

Ehrenfeucht's characterization can be extended to various other languages, of which we will discuss in detail the following:

1. monadic second-order logic,

2. infinitary logic,

3. logics with cardinality quantifiers,

4. first-order logic with a bounded number of variables.

There is also a close connection between games and the bisimulations of modal logic. Bisimulations can be seen as winning strategies in certain games on Kripke models. We explore this connection further in Chapter 4, where we also define games that characterize bounded bisimulation.

The core of the thesis deals with characterizations of logical equivalence of structures. There are of course many other applications of games in logic. Topics briefly discussed in Chapter 5 include a game-theoretic version of forcing and game-theoretic semantics.

Some familiarity with the basic definitions and technicalities regarding first-order logic is assumed. The purpose of Chapter 1 is simply to fix the notation and terminology. For more information a good reference is the book *Model Theory* by W. Hodges.

As is common practise, we will deal rather informally with games. An intuitive idea gained from experience with games such as chess and draughts is sufficient for our purposes. After all, it is the intuition behind games that makes the method so appealing. One notion that may be worth defining (intuitively) is that of a *winning strategy*, which is a set of rules telling a player how to play in order to win, regardless of his opponent's moves. All the games considered will be *determined*, i.e. one of the players has a winning strategy. For most of the games this will be nearly obvious because they are so-called *finite zero-sum games*, i.e. the game always ends in finitely many steps and there are no draws. In any event, we will see in Chapter 5 how the games fit into the general framework of closed games and a proof of the determinacy of these games will be given.

Where proofs and results were taken from the literature the appropriate references are given at the beginning of the chapter. The following is a list of proofs and results that are my own. In most cases I found the statement itself in the references (without proof), otherwise I am sure that they are known anyway. I only list the non-trivial proofs and the results (which are indicated by an asterisk *).

Chapter 2: Example 2.2.11, Example 2.3.9, Theorem 2.5.1*, Theorem 2.5.4, Theorem 2.5.8, Lemma 2.6.1, Theorem 2.6.2, Theorem 2.6.3*.

Chapter 3: Theorem 3.1.2, Theorem 3.3.1, Corollary 3.3.2*, Theorem 3.4.1, Theorem 3.5.2.

Chapter 4: Theorem 4.2.4, Example 4.3.2, Lemma 4.3.5*, Theorem 4.3.6*, Example 4.3.7*, Theorem 4.4.3*.

Chapter 5: Theorem 5.1.8.

All omitted proofs are either trivial, or straightforward adaptions of other proofs, or can be found in the references mentioned at the beginning of the relevant chapter.

Chapter 1

Basic Concepts

In this chapter we introduce the basic notions regarding first-order logic and structures, in order to fix the notation and terminology. More detail can be found in e.g. [8] or [14]. The definitions for second-order logic, modal logic etc. will be given in the relevant sections. For set-theoretic concepts the reader is referred to [4].

1.1 First-order Languages

A first-order language consists of a set of a set of *individual variables*, denoted by $x_0, x_1, \ldots$, a set of *logical symbols* and a *signature*. The logical symbols are fixed for every first-order language while the signature varies. The logical symbols are:

1. the *equality symbol*: $=$,

2. *connectives*: $\neg$ and $\rightarrow$,

3. the *existential quantifier*: $\exists$,

4. *grouping symbols*: (,) and ,.

The signature consists of:

1. *relation symbols*: $\mathbf{r}_0$, $\mathbf{r}_1$, $\ldots$,

2. *function symbols*: $\mathbf{f}_0$, $\mathbf{f}_1$, $\ldots$,

3. *constant symbols (constants)*: $\{\mathbf{c}_i : i \in I\}$ for some index set I.

The equality symbol is treated like a relation symbol. The reason for including it with the logical symbols is merely to ensure that every signature contains it, since we will consider *normal structures* only, that is, structures in which $=$ is interpreted by equality.

1.2 Terms and Formulae

The *terms* of a language with signature σ are recursively defined by the clauses:

1. all variables and constants are terms,
2. if $\mathbf{f}$ is an n-ary function symbol and $t_1, \ldots, t_n$ are terms then $\mathbf{f}(t_1, \ldots, t_n)$ is a term.

A term is called *closed* if it contains no variables.

An *atomic formula (atom)* is an expression of the form $t_1 = t_2$ where t_1 and t_2 are terms, or $\mathbf{r}(t_1, \ldots . t_n)$ where $\mathbf{r}$ is an n-ary relation symbol and $t_1, \ldots, t_n$ are terms.

A *σ-formula* is an expression that can be obtained inductively from the following:

1. all atoms are formulae,
2. if α and β are formulae and x is a variable then $\neg\alpha$, $\alpha \wedge \beta$, and $\exists x\alpha$ are formulae.

Other connectives and the universal quantifier are introduced by the abbreviations:

1. $\alpha \wedge \beta = \neg(\alpha \rightarrow \neg\beta)$,
2. $\alpha \vee \beta = \neg\alpha \rightarrow \beta$,
3. $\alpha \leftrightarrow \beta = (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$,
4. $\forall x\alpha = \neg\exists x\neg\alpha$.

The set $Subf(\alpha)$ of *subformulae* of α is defined by:

1. $Subf(\alpha) = \{\alpha\}$ if α is atomic,
2. $Subf(\neg\alpha) = Subf(\alpha) \cup \{\neg\alpha\}$,
3. $Subf(\alpha \wedge \beta) = Subf(\alpha) \cup Subf(\beta) \cup \{\alpha \wedge \beta\}$,
4. $Subf(\exists x\alpha) = Subf(\alpha) \cup \{\exists x\alpha\}$.

If $\exists x\beta$ is a subformula of α we say that β is the *scope* of the occurrence of $\exists x$. If a variable y occurs in the scope of some quantifier $\exists y$ then that occurrence of y is said to be *bound* by that occurrence of the quantifier. A variable that is not bound is called *free*.

Let α be any formula and t a term. Then t is called *free for substitution for x in α* if x does not occur free in the scope of $\exists y$ where y is any variable occurring in t.

A σ-formula that contains no free variables is called a *σ-sentence*. The set of all σ-sentences will be denoted by SEN_σ.

We will write $\alpha(x_1, \ldots, x_k)$ to denote that α is a formula with free variables among $x_1, \ldots, x_k$. If $t_1, \ldots, t_k$ are terms such that t_i is free for x_i, $i = 1, \ldots, k$, we then denote the formula obtained by substituting t_i for every free occurrence of x_i by $\alpha(t_1, \ldots, t_k)$.

1.3 Structures

Let σ be a signature. A *σ-structure* is a pair $\mathcal{A} = \langle A, \nu \rangle$ where A is a non-empty set, called the *universe* of $\mathcal{A}$, and ν is a function defined on σ such that

1. $\nu(\mathbf{c}) \in A$ if $\mathbf{c}$ is a constant symbol,
2. if $\mathbf{r}$ is an n-ary relation symbol then $\nu(\mathbf{r})$ is an n-ary relation over A,
3. if $\mathbf{f}$ is an n-ary function symbol then $\nu(\mathbf{r})$ is an n-ary function over A.

The function ν is called an *interpretation in $\mathcal{A}$*. We will write $\mathbf{s}^{\mathcal{A}}$ for $\nu(\mathbf{s})$ where $\mathbf{s} \in \sigma$.

We will always assume that A is the universe of $\mathcal{A}$, B the universe of $\mathcal{B}$, A_i the universe of $\mathcal{A}_i$ etc.

If $\sigma = \{\mathbf{r}, \ldots, \mathbf{f}, \ldots, \mathbf{c}, \ldots\}$ we use the notation $\langle A, \mathbf{r}^{\mathcal{A}}, \ldots, \mathbf{f}^{\mathcal{A}}, \ldots, \mathbf{c}^{\mathcal{A}}, \ldots \rangle$ for a σ-structure.

A σ-structure $\mathcal{A}$ is called *relational* if σ contains relation symbols only.

Let $\sigma \subseteq \sigma'$. For a σ-structure $\mathcal{A} = \langle A, \nu \rangle$ and a σ'-structure $\mathcal{B} = \langle A, \nu' \rangle$, we say that $\mathcal{A}$ is a *reduct* of $\mathcal{B}$, and $\mathcal{B}$ an *expansion* of $\mathcal{A}$, and we write $\mathcal{A} = \mathcal{B}|_\sigma$, if $\nu = \nu'|\sigma$ (where $\nu'|\sigma$ denotes the restriction of ν to σ).

Given a structure $\mathcal{A}$ and $a_0, \ldots, a_n \in A$, the structure $\mathcal{B}$ obtained by adding constants $\{\mathbf{c}_0, \ldots, \mathbf{c}_n\}$ and interpreting each $\mathbf{c}_i$ by a_i is called a *simple expansion* of $\mathcal{A}$ and we write $\mathcal{B} = \langle \mathcal{A}, a_0, \ldots, a_n \rangle$. An expansion obtained by adding finitely many constants, relations and functions is denoted similarly.

Given a family $\{\mathcal{A}_i : i \in I\}$ of σ-structures their *product*, $\Pi\{\mathcal{A}_i : i \in I\}$, is the structure $\mathcal{A} = \langle A, \nu \rangle$ where:

1. $A = \Pi\{A_i : i \in I\} = \{f : f$ is a function with domain I into $\bigcup_{i \in I} A_i$ such that $f(i) \in A_i$ for every $i \in I\}$, that is, the *Cartesian product* of the family of sets $\{A_i : i \in I\}$,
2. if $\mathbf{c}$ is a constant symbol then $\mathbf{c}^{\mathcal{A}}(i) = \mathbf{c}^{\mathcal{A}_i}$ for every $i \in I$,
3. for a function symbol $\mathbf{f}$ and $a_1, \ldots, a_n \in A$, $\mathbf{f}^{\mathcal{A}}(a_1, \ldots, a_n)(i) = \mathbf{f}^{\mathcal{A}_i}(a_1(i), \ldots, a_n(i))$ for every $i \in I$,
4. if $\mathbf{r}$ is a relation symbol and $a_1, \ldots, a_n \in A$ then $\mathbf{r}^{\mathcal{A}}(a_1, \ldots, a_n)$ iff $\mathbf{r}^{\mathcal{A}_i}(a_1(i), \ldots, a_n(i))$ for every $i \in I$.

1.4 Truth, Validity and Logical Consequence

From now on we omit the prefix σ- where the context makes it clear what the signature is.

If $\mathcal{A}$ is a structure then an *$\mathcal{A}$-assignment* is a function from the set of variables into the universe of $\mathcal{A}$. An assignment v is extended to the set of all terms as follows:

1. $v(\mathbf{c}) = \mathbf{c}^{\mathcal{A}}$ for a constant $\mathbf{c}$,

2. $v(\mathbf{f}(t_1, \ldots, t_n)) = \mathbf{f}^{\mathcal{A}}(v(t_1), \ldots, v(t_n))$ if $\mathbf{f}$ is an n-ary function symbol and $t_1, \ldots, t_n$ are terms.

For a structure $\mathcal{A}$, an $\mathcal{A}$-assignment v and a formula α we define the notion "*α is satisfied by v in $\mathcal{A}$*", written $\mathcal{A} \models_v \alpha$, as follows:

1. $\mathcal{A} \models_v (s = t)$ iff $v(s) = v(t)$,

2. $\mathcal{A} \models_v \mathbf{r}(t_1, \ldots, t_n)$ iff $\mathbf{r}^{\mathcal{A}}(v(t_1), \ldots, v(t_n))$,

3. $\mathcal{A} \models_v \neg\alpha$ iff $\mathcal{A} \not\models_v \alpha$,

4. $\mathcal{A} \models_v \alpha \wedge \beta$ iff $\mathcal{A} \models_v \alpha$ and $\mathcal{A} \models_v \beta$,

5. $\mathcal{A} \models_v \exists x\alpha$ iff $\mathcal{A} \models_{v'} \alpha$ for some assignment v' such that $v'(y) = v(y)$ if $y \neq x$.

If $\mathcal{A} \models_v \alpha$ we say that α is *true* in $\mathcal{A}$ under v. If Γ is a set of formulae, we write $\mathcal{A} \models_v \Gamma$ if $\mathcal{A} \models_v \alpha$ for every $\alpha \in \Gamma$. If $\mathcal{K}$ is a class of structures we define $\mathcal{K} \models_v \alpha$ and $\mathcal{K} \models_v \Gamma$ similarly.

If $\mathcal{A} \models_v \alpha$ for every assignment v we write $\mathcal{A} \models \alpha$ and say that α is *valid in $\mathcal{A}$*. In this case we call $\mathcal{A}$ a *model* of α. Again we define $\mathcal{K} \models \alpha$, $\mathcal{K} \models \Gamma$ and $\mathcal{A} \models \Gamma$ similarly.

A formula α is *valid*, denoted by $\models \alpha$, if α is valid in any structure.

If $\alpha = \alpha(x_1, \ldots, x_k)$ and $a_1, \ldots, a_k \in A$ we will write $\mathcal{A} \models \alpha(a_1, \ldots, a_k)$ to denote that $\mathcal{A} \models_v \alpha$ for every assignment v with $v(x_i) = a_i$, $i = 1, \ldots, k$.

A formula α is called a *logical consequence* of Γ, denoted $\Gamma \models \alpha$, if $\mathcal{A} \models_v \Gamma$ implies $\mathcal{A} \models_v \alpha$ for every structure $\mathcal{A}$ and $\mathcal{A}$-assignment v.

1.5 Isomorphism and Equivalence

Let $\mathcal{A}$ and $\mathcal{B}$ be structures.

A function $h : A \to B$ is an *isomorphic embedding* if it satisfies:

1. for every n-ary relation symbol $\mathbf{r}$ and $a_1, \ldots, a_n \in A$,

$$\mathbf{r}^{\mathcal{A}}(a_1, \ldots, a_n) \quad \text{iff} \quad \mathbf{r}^{\mathcal{B}}(h(a_1), \ldots, h(a_n)),$$

2. for every n-ary function symbol $\mathbf{f}$ and $a_1, \ldots, a_n \in A$,

$$h(\mathbf{f}^{\mathcal{A}}(a_1, \ldots, a_n)) = \mathbf{f}^{\mathcal{B}}(h(a_1), \ldots, h(a_n)),$$

3. for every constant $\mathbf{c}$, $h(\mathbf{c}^{\mathcal{A}}) = \mathbf{c}^{\mathcal{B}}$.

Note that since we have equality in the language h is necessarily injective. If h is surjective it is called an *isomorphism*.

$\mathcal{A}$ and $\mathcal{B}$ are *isomorphic*, written $\mathcal{A} \cong \mathcal{B}$ if there exists an isomorphism h from A onto B.

$\mathcal{A}$ and $\mathcal{B}$ are *equivalent*, denoted by $\mathcal{A} \equiv \mathcal{B}$, if for every sentence α, $\mathcal{A} \models \alpha$ iff $\mathcal{B} \models \alpha$.

$\mathcal{A}$ is a *substructure* of $\mathcal{B}$ and $\mathcal{B}$ is an *extension* of $\mathcal{A}$, written $\mathcal{A} \subseteq \mathcal{B}$, if the identity mapping $id : \mathcal{A} \to \mathcal{B}$ is an isomorphic embedding.

If $\mathcal{A}$ is a structure and $X \subseteq A$ there is a smallest substructure of $\mathcal{A}$ containing X, the substructure of $\mathcal{A}$ *generated by* X. We denote this substructure by $[X]_{\mathcal{A}}$.

1.6 Theories

A *theory* is a set of sentences.

A theory Γ is *closed* if for every sentence α, $\Gamma \models \alpha$ implies $\alpha \in \Gamma$.

A theory Γ is *consistent* if there is no sentence α such that $\Gamma \models \alpha$ and $\Gamma \models \neg\alpha$.

The *closure* of a theory Γ is the set $Con(\Gamma) = \{\alpha : \Gamma \models \alpha\}$.

For a class of structures $\mathcal{K}$, the *theory of* $\mathcal{K}$ is $TH(\mathcal{K}) = \{\alpha : \mathcal{K} \models \alpha\}$. If $\mathcal{K} = \{\mathcal{A}\}$ we just write $TH(\mathcal{A})$.

The class of models of a theory Γ is denoted by $MOD(\Gamma)$.

1.7 Orderings and Ordinals

A (not necessarily reflexive) *partial ordering* (or simply an *ordering*) on a set A is a binary relation $\prec$ that is:

1. *transitive*: for all $a, b, c \in A$, $a \prec b$ and $b \prec c$ implies $a \prec c$, and
2. *antisymmetric*: for all $a, b \in A$, $a \prec b$ and $b \prec a$ implies $a = b$.

A partial ordering on A is *reflexive* if for all $a \in A$, $a \preceq a$.

A *strict* partial ordering on A is an ordering such that for all $a \in A$, $a \not\prec a$.

We will consistently use $\preceq$ or $\leq$ for a reflexive ordering and $\prec$ or $<$ for a strict ordering.

If $\preceq$ is a partial ordering on A we also call the structure $\langle A, \preceq\rangle$ a partial ordering.

An ordering is *linear* if it is also *connected*, i.e. for all $a, b \in A$, $a \preceq b$ or $b \preceq a$ or $a = b$.

Some examples of linear orderings are:

1. $\mathbf{n} = \langle \{0, 1, \ldots, n-1\}, < \rangle$,

2. $\omega = \langle \mathcal{N}, < \rangle$,

3. $\zeta = \langle \mathcal{Z}, < \rangle$,

4. $\eta = \langle \mathcal{Q}, < \rangle$,

5. $\lambda = \langle \mathcal{R}, < \rangle$.

We use von Neumann's ordinals, i.e. if α is an ordinal then $\alpha = \{\beta : \beta < \alpha\}$.

For an ordinal α we will use α and $\langle \alpha, < \rangle$ interchangeably, relying on the context to make it clear whether α is considered to be an ordinal or an ordering.

A *tree* is a partial ordering $\langle T, \leq \rangle$ such that for all $a, b, c \in T$, $a \leq c$ and $b \leq c$ implies $a = b$ or $a \leq b$ or $b \leq a$, i.e. for every $c \in T$ the set $\{a : a \leq c\}$ is linearly ordered by $\leq$.

If $\mathcal{T}$ is a tree and $a \in T$ then $\mathcal{T}_a$ is the subtree *rooted* at a, i.e. $\mathcal{T}_a = \langle T_a, \leq_a \rangle$ where $T_a = \{x \in T : a \leq x\}$ and $\leq_a = \leq \cap (T_a \times T_a)$.

If $\langle A, \preceq \rangle$ is an ordering then $\langle A, \preceq \rangle^*$ denotes the ordering $\langle A, \preceq^* \rangle$ where $a \preceq^* b$ iff $b \preceq a$.

If A and B are disjoint and $\langle A, \preceq \rangle$ and $\langle B, \leq \rangle$ are orderings then $\langle A, \preceq \rangle + \langle B, \leq \rangle$ is the ordering $\langle A \cup B, \preceq \cup \leq \cup (A \times B) \rangle$.

Chapter 2

The Ehrenfeucht Game

This chapter introduces the Ehrenfeucht game for first-order logic. Ehrenfeucht's game provides a useful technique for proving non-definability and completeness results. It is particularly useful in finite model theory, where the usual methods such as Compactness and Löwenheim-Skolem fail. In Section 5 we discuss unnested Ehrenfeucht games, used for signatures that include function symbols. Section 6 generalizes the ideas in Section 5 to obtain a few methods related to Ehrenfeucht's theorem.

Most of the first three sections is based on [5], section 4 is based on [1] and the final sections on [14].

2.1 Local Isomorphisms

Definition: A finite relation $h = \{\langle a_1, b_1\rangle, \dots, \langle a_n, b_n\rangle\} \subseteq A \times B$ is a *local isomorphism* between σ-structures $\mathcal{A}$ and $\mathcal{B}$ if for every atomic formula $\alpha(x_1, \dots, x_n)$,

$$\mathcal{A} \models \alpha(a_1, \dots, a_n) \text{ iff } \mathcal{B} \models \alpha(b_1, \dots, b_n).$$

This is equivalent to saying that the simple expansions $\langle \mathcal{A}, a_1, \dots, a_n\rangle$ and $\langle \mathcal{B}, b_1, \dots, b_n\rangle$ satisfy the same atomic sentences.

Equivalently, h is a local isomorphism if and only if it can be extended to an isomorphism between $[\{a_1, \dots, a_n\}]_\mathcal{A}$ and $[\{b_1, \dots, b_n\}]_\mathcal{B}$.

Examples:

1. The empty relation is a local isomorphism between any two relational structures, since there are no atomic sentences to consider.
2. $\{\langle 0, 2\rangle, \langle 3, 3\rangle, \langle 5, 4\rangle\}$ is a local isomorphism between **5** and **9**.
3. $\{\langle 0, 0\rangle, \langle 1, 2\rangle\}$ is a local isomorphism between $\langle N, <, +\rangle$ and $\langle Z, <, +\rangle$.

4. Any restriction of a local isomorphism is a local isomorphism.

A local isomorphism h is necessarily an injective function, for suppose that $\langle a_i, b_i\rangle, \langle a_j, b_j\rangle \in h$. Let α be the formula $x_i = x_j$. Then $\mathcal{A} \models \alpha(a_1, \ldots, a_n)$ iff $\mathcal{B} \models \alpha(b_1, \ldots, b_n)$, i.e. $a_i = a_j$ iff $b_i = b_j$.

2.2 Ehrenfeucht's Game

Definition: Let $\mathcal{A}$ and $\mathcal{B}$ be σ-structures and let n be a natural number. The *Ehrenfeucht game of length n on $\mathcal{A}$ and $\mathcal{B}$*, $E(\mathcal{A}, \mathcal{B}, n)$, is played as follows. There are two players, *Player I* and *Player II* (also called the *Spoiler* and the *Duplicator*). The game is played over n *rounds*, a round consisting of Player I first choosing an element of either structure and Player II then choosing an element of the other structure. The game ends after n rounds.

Each round determines an ordered pair $\langle a, b\rangle$ and therefore, after n rounds, a finite relation $h \subseteq A \times B$ is determined. The relation h is called a *play* of the game. Player II *wins* the play if h is a local isomorphism between $\mathcal{A}$ and $\mathcal{B}$, otherwise Player I wins.

A player loses if it is his turn to move and he has no element to choose (one or two of the structures could be empty).

Definition: If Player II has a winning strategy for the game $E(\mathcal{A}, \mathcal{B}, n)$ we denote this by $II(\mathcal{A}, \mathcal{B}, n)$.

The proof of the following lemma is a straightforward application of the definitions.

Lemma 2.2.1

1. *If $II(\mathcal{A}, \mathcal{B}, n)$ and $m \leq n$ then $II(\mathcal{A}, \mathcal{B}, m)$.*
2. *If $II(\mathcal{A}, \mathcal{B}, n)$ then $II(\mathcal{B}, \mathcal{A}, n)$.*
3. *If $\mathcal{A} \cong \mathcal{B}$ then $II(\mathcal{A}, \mathcal{B}, n)$ for every n.*
4. *If $II(\mathcal{A}, \mathcal{B}, n)$ and $II(\mathcal{B}, \mathcal{C}, n)$ then $II(\mathcal{A}, \mathcal{C}, n)$.* □

Example: If σ is relational then Player II wins the game $E(\mathcal{A}, \mathcal{B}, 0)$ for any σ-structures $\mathcal{A}$ and $\mathcal{B}$.

Example: Consider the game $E(\mathbf{6}, \mathbf{7}, 2)$. Player II wins easily: If, in the first round, Player I chooses the least or greatest element of either structure then Player II accordingly chooses the least or greatest element of the other structure. Otherwise Player II chooses any other element, after which his second move is obvious.

Example: Consider the game $E(\omega, \zeta, 2)$. Player I wins by first choosing the 0 in ω. Whatever element of ζ Player II responds with, Player I can choose a smaller one, and wins.

Before we get to the main result of this chapter, namely Ehrenfeucht's game-theoretic characterization of equivalence between structures, we consider several important definitions and results.

Definition: The *quantifier rank* $qr(\alpha)$ of a formula α is the maximum number of nested quantifiers in α, calculated recursively by:

1. $qr(\alpha) = 0$ if α is atomic,
2. $qr(\neg\alpha) = qr(\alpha)$,
3. $qr(\alpha \wedge \beta) = max\{qr(\alpha), qr(\beta)\}$,
4. $qr(\exists x\alpha) = qr(\alpha) + 1$.

Definition: Two structures $\mathcal{A}$ and $\mathcal{B}$ are *n-equivalent*, denoted by $\mathcal{A} \equiv_n \mathcal{B}$, if they satisfy the same sentences of quantifier rank at most n.

Lemma 2.2.2 *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are structures, $a \in A$ and $b \in B$. For a finite relation $h \subseteq A \times B$ the following are equivalent:*

1. h is a local isomorphism between $\langle \mathcal{A}, a\rangle$ and $\langle \mathcal{B}, b\rangle$.

2. $h \cup \{\langle a, b\rangle\}$ is a local isomorphism between $\mathcal{A}$ and $\mathcal{B}$.

Proof: h is a local isomorphism between $\langle \mathcal{A}, a\rangle$ and $\langle \mathcal{B}, b\rangle$ if and only if $\langle\langle \mathcal{A}, a\rangle, a_1, \ldots, a_n\rangle$ and $\langle\langle \mathcal{B}, b\rangle, b_1, \ldots, b_n\rangle$ satisfy the same atomic sentences, if and only if $\langle \mathcal{A}, a, a_1, \ldots, a_n\rangle$ and $\langle \mathcal{B}, b, b_1, \ldots, b_n\rangle$ satisfy the same atomic sentences, if and only if $h \cup \{\langle a, b\rangle\}$ is a local isomorphism between $\mathcal{A}$ and $\mathcal{B}$. $\square$

Lemma 2.2.3 $II(\mathcal{A}, \mathcal{B}, n+1)$ *iff*

1. ("forth") $\forall a \in A\ \exists b \in B\ II(\langle \mathcal{A}, a\rangle, \langle \mathcal{B}, b\rangle, n)$.

2. ("back") $\forall b \in B\ \exists a \in A\ II(\langle \mathcal{A}, a\rangle, \langle \mathcal{B}, b\rangle, n)$.

Proof: Suppose that $II(\mathcal{A}, \mathcal{B}, n+1)$. Fix a winning strategy S for Player II in the game $E(\mathcal{A}, \mathcal{B}, n+1)$. Suppose that $a \in A$. Consider a as a first move of Player I. Now S produces an answer $b \in B$. We prove that $II(\langle \mathcal{A}, a\rangle, \langle \mathcal{B}, b\rangle, n)$:
Player II can use S as a winning strategy. In order to do so he pretends to play the game $E(\mathcal{A}, \mathcal{B}, n+1)$ in which a first move $\langle a, b\rangle$ has already been made. After the play of $E(\langle \mathcal{A}, a\rangle, \langle \mathcal{B}, b\rangle, n)$ a relation h has been produced. Since S is a winning strategy for Player II in $E(\mathcal{A}, \mathcal{B}, n+1)$, $h \cup \{\langle a, b\rangle\}$ is a local isomorphism between $\mathcal{A}$ and $\mathcal{B}$. Then, by Lemma 2.2.2, h is a local isomorphism between $\langle \mathcal{A}, a\rangle$ and $\langle \mathcal{B}, b\rangle$, hence this strategy is winning for Player II.

Conversely, assume that the conditions hold and let the players play the game $E(\mathcal{A},\mathcal{B},n+1)$. Suppose Player I chooses an element of $\mathcal{A}$ in the first round. By the "forth" condition there is an element b in $\mathcal{B}$ such that $II(\langle\mathcal{A},a\rangle,\langle\mathcal{B},b\rangle,n)$. Player II then chooses b and thereafter plays according to his winning strategy S for the game $E(\langle\mathcal{A},a\rangle,\langle\mathcal{B},b\rangle,n)$. At the end of the game a relation $h\cup\{\langle a,b\rangle\}$ is produced (where h consists of all the moves from the second move on). Since S is winning for Player II in $E(\langle\mathcal{A},a\rangle,\langle\mathcal{B},b\rangle,n)$, h is a local isomorphism between $\langle\mathcal{A},a\rangle$ and $\langle\mathcal{B},b\rangle$. Then, by Lemma 2.2.2, $h\cup\{\langle a,b\rangle\}$ is a local isomorphism between $\mathcal{A}$ and $\mathcal{B}$, so Player II wins. Similarly if Player I's first choice is from $\mathcal{B}$. $\square$

Lemma 2.2.4 *If $\mathcal{A}$ and $\mathcal{B}$ satisfy the same atomic sentences, then they satisfy the same quantifier free sentences.*

Proof: Since every quantifier free sentence is a Boolean combination of atomic sentences, the result follows immediately by induction on α. $\square$

Lemma 2.2.5 *Let σ be a finite signature containing no function symbols. For every k and n there are only finitely many inequivalent σ-formulae of rank at most n that have $x_1,\ldots,x_k$ free.*

Proof: The proof is by induction on n.
$n=0$: Since the signature is finite and contains no function symbols there are only finitely many atoms in $x_1,\ldots,x_k$. Every formula of rank 0 is quantifier free and has an equivalent in disjunctive normal form. Since there are only finitely many atoms, there are, up to equivalence, only finitely many disjunctive normal forms. Induction step: let α be a formula of rank at most $n+1$ with free variables among $x_1,\ldots,x_k$. Then α is equivalent to a formula β that is in disjunctive normal form. β is built up from formulae of rank at most n and formulae of the form $\exists x_{k+1}\gamma$ where γ has rank at most n and $x_1,\ldots,x_{k+1}$ free. By the induction hypothesis there are only finitely many of those, up to equivalence. $\square$

Theorem 2.2.6 *(Ehrenfeucht) Let σ be a finite signature containing no function symbols and let $\mathcal{A}$ and $\mathcal{B}$ be σ-structures. Then, for every $n<\omega$,*

$$II(\mathcal{A},\mathcal{B},n) \text{ if and only if } \mathcal{A}\equiv_n\mathcal{B}.$$

Proof: By induction on n.
$n=0$: Note that $\mathcal{A}\equiv_0\mathcal{B}$ means that $\mathcal{A}$ and $\mathcal{B}$ satisfy the same quantifier free sentences and that $(\mathcal{A},\mathcal{B},0)$ means that $\mathcal{A}$ and $\mathcal{B}$ satisfy the same atomic sentences. Now apply Lemma 2.2.4.

Induction step: Assume the result for n.
Suppose that $II(\mathcal{A},\mathcal{B},n+1)$ and let α be a sentence of rank at most $n+1$. We prove that $\mathcal{A}\models\alpha$ iff $\mathcal{B}\models\alpha$ by induction on α. The cases for $\alpha=\beta\wedge\gamma$ and $\alpha=\neg\beta$ are immediate,

so assume that $\alpha = \exists x\beta(x)$ and that $\mathcal{A} \models \alpha$. Then there exists $a \in A$ such that $\mathcal{A} \models \beta(a)$ or, equivalently, $\langle \mathcal{A}, a\rangle \models \beta(\mathbf{c})$ where $\mathbf{c}$ is a new constant symbol interpreted by a. By Lemma 2.2.3 we obtain $b \in B$ such that $II(\langle \mathcal{A}, a\rangle, \langle \mathcal{B}, b\rangle, n)$. By the induction hypothesis, $\langle \mathcal{A}, a\rangle \equiv_n \langle \mathcal{B}, b\rangle$. Therefore, since $\beta(\mathbf{c})$ has quantifier rank at most n, $\langle \mathcal{B}, b\rangle \models \beta(\mathbf{c})$, i.e. $\mathcal{B} \models \beta(b)$, so $\mathcal{B} \models \exists x\beta(x)$.

Conversely, suppose that $\mathcal{A} \equiv_{n+1} \mathcal{B}$. We show that for every $a \in A$ there exists $b \in B$ such that $\langle \mathcal{A}, a\rangle \equiv_n \langle \mathcal{B}, b\rangle$. It then follows from the induction hypothesis that $II(\langle \mathcal{A}, a\rangle, \langle \mathcal{B}, b\rangle, n)$. Similarly for the other way around. By Lemma 2.2.3 we then have $II(\mathcal{A}, \mathcal{B}, n+1)$. So let $a \in A$ and suppose that such a b does not exist. Then for every $b \in B$ there exists a sentence $\alpha_b(\mathbf{c})$ of rank at most n satisfied by $\langle \mathcal{A}, a\rangle$ but not by $\langle \mathcal{B}, b\rangle$. By Lemma 2.2.5 the set $\{\alpha_b : b \in B\}$ is finite, up to equivalence. Then the sentence $\exists x \bigwedge_b \alpha_b(x)$ of rank at most $n+1$ is satisfied in $\mathcal{A}$ since $\langle \mathcal{A}, a\rangle \models \bigwedge_b \alpha_b(\mathbf{c})$, but not in $\mathcal{B}$ since $\langle \mathcal{B}, b'\rangle \models \bigwedge_b \alpha_b(\mathbf{c})$ implies $\langle \mathcal{B}, b'\rangle \models \alpha_{b'}(\mathbf{c})$. This contradicts our assumption that $\mathcal{A} \equiv_{n+1} \mathcal{B}$. □

Corollary 2.2.7 *Let σ be a finite signature with no function symbols and let $\mathcal{A}$ and $\mathcal{B}$ be σ-structures. Then*

$$\mathcal{A} \equiv \mathcal{B} \textit{ if and only if } E(\mathcal{A}, \mathcal{B}, n) \textit{ for every } n < \omega. \qquad \Box$$

Example: From the three examples following Lemma 2.2.1 we have:

1. If $\mathcal{A}$ and $\mathcal{B}$ are relational structures then $\mathcal{A} \equiv_0 \mathcal{B}$.
2. $\mathbf{6} \equiv_2 \mathbf{7}$. These structures are not 3-equivalent, as is shown by the sentence $\exists x(\exists y(\exists z(z < y < x) \land \exists z(y < z < x)) \land \exists y(\exists z(x < z < y) \land \exists z(x < y < z)))$.
3. $\omega \not\equiv_2 \zeta$. A sentence of rank 2 that distinguishes these structures is $\exists x \forall y(\neg y < x)$.

The example below shows that the restrictions on the signature are necessary.

Example 2.2.8

To see why finiteness of the signature is needed, consider the structures $\langle \omega, 0, 1, 2, \ldots\rangle$ (that is, the strucuture with universe the set of natural numbers and for each natural number k a constant symbol $\mathbf{c}_k$ interpreted by k) and $\langle \omega + \zeta, 0, 1, 2, \ldots\rangle$ (here we use $\omega + \zeta$ simply as the set of natural numbers combined with the set of integers, not as an ordering). In section 2.6 below we will be able to prove that they are elementarily equivalent. However, Player I wins the 1-round game by picking any element from ζ.

Similarly, the structures $\langle \omega, S, 0\rangle$ and $\langle \omega + \zeta, S, 0\rangle$ (where S is the successor function) show that function symbols have to be excluded. □

The next few examples involve linear orderings. For this the following notation will be useful. If $\langle A, <\rangle$ is a linear ordering and $a \in A$ then $a\uparrow$ denotes the substructure with universe $\{x \in A : a < x\}$. We define $a\downarrow$ similarly. Lemma 2.2.3 then implies:

Lemma 2.2.9 *If $\mathcal{A}$ and $\mathcal{B}$ are linear orderings then $H(\mathcal{A},\mathcal{B},n+1)$ iff*

1. *("forth") $\forall a \in A\ \exists b \in B\ H(a\!\downarrow, b\!\downarrow, n)$ and $H(a\!\uparrow, b\!\uparrow, n)$.*
2. *("back") $\forall b \in B\ \exists a \in A\ H(a\!\downarrow, b\!\downarrow, n)$ and $H(a\!\uparrow, b\!\uparrow, n)$.* □

Example 2.2.10 *If $k, m \geq 2^n - 1$ then $H(\mathbf{k}, \mathbf{m}, n)$.*

Proof: By induction on n, using Lemma 2.2.9.

The case $n = 0$ is immediate since the structures are relational and therefore the empty relation is a local isomorphism.

Now assume the result holds for n and that $k, m \geq 2^{n+1} - 1$. By Lemma 2.2.9 it suffices to show that for every element $i \in \{0, \ldots, k-1\}$ there exists an element $j \in \{0, \ldots, m-1\}$ such that $H(i\!\downarrow, j\!\downarrow, n)$ and $H(i\!\uparrow, j\!\uparrow, n)$ and conversely. Note that $i\!\downarrow = \mathbf{i}$ and $i\!\uparrow = \mathbf{k-i-1}$. There are three cases to consider.

1. $i, k-i-1 \geq 2^n - 1$:
   Since $m \geq 2^{n+1}$ and $2^{n+1} - 1 = (2^n - 1) + 1 + (2^n - 1)$ there exists j, $0 \leq j < m$ such that $j, m-j-1 \geq 2^n - 1$. Take such a j. By the induction hypothesis we have that $H(i\!\uparrow, j\!\uparrow, n)$ and $H(i\!\downarrow, j\!\downarrow, n)$.
2. $i < 2^n - 1$:
   Put $j = i$. Then we have that $H(i\!\downarrow, j\!\downarrow, n)$. Since $k, m \geq 2^{n+1} = (2^n - 1) + 1 + (2^n - 1)$ and $i = j < 2^n - 1$ we have that $k - i - 1, m - j - 1 \geq 2^n - 1$, hence it follows from the induction hypothesis that $H(i\!\uparrow, j\!\uparrow, n)$.
3. $k - i - 1 < 2^n$:
   Choose j such that $m - j - 1 = k - i - 1$ and argue as in case 2. □

Example 2.2.11 *If $m \geq 2^n - 1$ then $H(\omega + \omega^*, \mathbf{m}, n)$.*

Proof: By induction on n. Again the case $n = 0$ is trivial since the structures are relational.

Assume therefore that the result holds for n and let $m \geq 2^{n+1} - 1$. We have to show that for every element $i \in \omega + \omega^*$ there exists an element $j \in \{0, \ldots, m-1\}$ such that $H(i\!\downarrow, j\!\downarrow, n)$ and $H(i\!\uparrow, j\!\uparrow, n)$ and conversely. Suppose first that $i \in \omega$. There are two cases to consider:

1. $i \leq 2^n - 1$.
   Put $j = i$. Then we have that $H(i\!\downarrow, j\!\downarrow, n)$. Since $m \geq 2^{n+1} - 1$ we have $m - i - 1 \geq 2^n - 1$, hence by the induction hypothesis $H(i\!\uparrow, j\!\uparrow, n)$ since $j\!\uparrow = \mathbf{m - i - 1}$ and $i\!\uparrow = \omega + \omega^*$.

2. $i > 2^n - 1$.
   Put $j = 2^n - 1$. Then $II(i\downarrow, j\downarrow, n)$ by Example 2.2.10 and $II(i\uparrow, j\uparrow, n)$ by the induction hypothesis.

Similarly if $i \in \omega^*$ or $i \in \mathbf{m}$. $\square$

Example 2.2.12 *If α_1, α_2, β_1 and β_2 are linear orderings such that $II(\alpha_1, \beta_1, n)$ and $II(\alpha_2, \beta_2, n)$ then $II(\alpha_1 + \alpha_2, \beta_1 + \beta_2, n)$.*

Proof: If Player I chooses from α_1 or β_1 then Player II uses his winning strategy for $E(\alpha_1, \beta_1, n)$, otherwise he uses his winning strategy for $E(\alpha_2, \beta_2, n)$. $\square$

Theorem 2.2.13 $\omega \equiv \omega + \zeta$.

Proof: From Example 2.2.11 and Example 2.2.12 we have $II(\mathbf{2^n} - \mathbf{1} + \omega, (\omega + \omega^*) + \omega, n)$ for every $n < \omega$, hence $\omega \equiv_n \omega + \zeta$ for every $n < \omega$. $\square$

2.3 Definability

Definition: A class $\mathcal{K}$ of structures is *definable* (also called finitely axiomatizable) if it is the class of models of some sentence. A class $\mathcal{K}$ is *axiomatizable* if it is the class of models of some theory. A class $\mathcal{K}$ is *definable within a class $\mathcal{M}$* if it is the class of models of some sentence, intersected with $\mathcal{M}$. A *property* is just an isomorphism-closed class of structures, i.e. we identify a property with the class of all structures that have the property.

A class $\mathcal{K}$ is therefore definable (within a class $\mathcal{M}$) if and only if there is a sentence α such that for every structure $\mathcal{A}$ (in $\mathcal{M}$),

$$\mathcal{A} \models \alpha \text{ if and only if } \mathcal{A} \in \mathcal{K}.$$

Ehrenfeucht's theorem is particularly useful in obtaining negative definability results. In order to prove that a class $\mathcal{K}$ is not axiomatizable it suffices to show that there are structures $\mathcal{A}$ and $\mathcal{B}$ with $\mathcal{A} \in \mathcal{K}$, $\mathcal{B} \notin \mathcal{K}$ and $\mathcal{A} \equiv \mathcal{B}$. Of course, this also shows that $\mathcal{K}$ is not definable. However, the following condition is generally easier to use.

Lemma 2.3.1 *A class $\mathcal{K}$ of structures is not definable (within a class $\mathcal{M}$) if for every $n < \omega$ there are structures $\mathcal{A}$ and $\mathcal{B}$ (in $\mathcal{M}$) with $\mathcal{A} \in \mathcal{K}$, $\mathcal{B} \notin \mathcal{K}$ and $\mathcal{A} \equiv_n \mathcal{B}$.*

Proof: Suppose $\mathcal{K}$ is definable, say $\mathcal{K} = MOD(\alpha)$. Let $n = qr(\alpha)$. Then for any structures $\mathcal{A}$ and $\mathcal{B}$, if $\mathcal{A} \in \mathcal{K}$ and $\mathcal{B} \notin \mathcal{K}$ then $\mathcal{A} \not\equiv_n \mathcal{B}$. $\square$

If the signature is finite and contains no function symbols then the converse is also true: Suppose that there is an n such that for every $\mathcal{A} \in \mathcal{K}$ and $\mathcal{B} \notin \mathcal{K}$ there is a formula $\phi_{\mathcal{A},\mathcal{B}}$ of rank at most n such that $\mathcal{A} \models \phi_{\mathcal{A},\mathcal{B}}$ and $\mathcal{B} \not\models \phi_{\mathcal{A},\mathcal{B}}$. For every $\mathcal{A} \in \mathcal{K}$ the set $\{\phi_{\mathcal{A},\mathcal{B}} : \mathcal{B} \notin \mathcal{K}\}$ is finite, up to equivalence. Let $\phi_{\mathcal{A}} = \bigwedge\{\phi_{\mathcal{A},\mathcal{B}} : \mathcal{B} \notin \mathcal{K}\}$ for every $\mathcal{A} \in \mathcal{K}$. Then, up to equivalence, the set $\{\phi_{\mathcal{A}} : \mathcal{A} \in \mathcal{K}\}$ is finite and we can let $\phi = \bigvee\{\phi_{\mathcal{A}} : \mathcal{A} \in \mathcal{K}\}$. Then $\mathcal{K} = MOD(\phi)$.

Example 2.3.2 *The class of finite linear orderings is not definable.*

Proof: For any $n < \omega$ there is an m with $m \geq 2^n - 1$. Then $\mathbf{m}$ is a finite linear ordering and $\omega + \omega^*$ is not. Hence, by Example 2.2.11 and Lemma 2.3.1, the class of finite linear orderings is not definable. $\Box$

Note that we cannot replace "definable" by "axiomatizable" in Lemma 2.3.1, for it would follow from the proof of Example 2.3.2 that the class of infinite linear orderings is not axiomatizable. This class is however axiomatized by the theory $\{\lambda_n : n < \omega\} \cup LO$ where LO is the set of defining properties of a linear ordering and, for every $n < \omega$, λ_n is the sentence $\exists x_1 \cdots \exists x_n (\bigwedge_{i \neq j} x_i \neq x_j)$.

Example 2.3.3 *The property of having an even number of elements is not definable.*

Proof: For any $n < \omega$ the structures $\langle\{0, 1, \ldots, n-1\}\rangle$ and $\langle\{0, 1, \ldots, n\}\rangle$ are n-equivalent, while the one has an even number of elements and the other one does not. Here is a winning strategy for Player II: If Player I selects an element x that has not been selected by either player in a previous round, he selects an element from the other structure that has not been selected before. This is always possible since both structures have at least n elements. Otherwise Player II selects the other element that was selected in the round x was first selected. $\Box$

The proof of this example shows that having an even number of elements is not definable even within the class of finite structures. In fact, it is not definable within the class of finite linear orderings.

Example 2.3.4 *The property of having an even number of elements is not definable within the class of finite linear orderings.*

Proof: Immediate from Example 2.2.10. $\Box$

Example 2.3.5 *The property of being a linear ordering such that between any two elements there are only finitely many others is not axiomatizable.*

Proof: Immediate from Theorem 2.2.13. □

For our next example we need several definitions.

Definition: A *graph* is a structure $\langle V, E\rangle$ where E is a symmetric and irreflexive binary relation on V.

A graph $\langle V, E\rangle$ is *connected* if, for all $x, y \in V$, there exist $x_1, \ldots, x_n \in V$ such that $xEx_1E\cdots Ex_nEy$.

If $\mathcal{A} = \langle V, <\rangle$ is a partial order then $\mathcal{A}^{nb}$ denotes the graph $\langle V, E\rangle$ where xEy if and only if $(x < y \wedge \neg\exists z(x < z < y)) \vee (y < x \wedge \neg\exists z(y < z < x))$.

The graph $\mathbf{n}^{nb}$ is denoted by P_n. With $P_n = \langle n, E\rangle$, the graph $\langle n, E \cup \{\langle 0, n-1\rangle, \langle n-1, 0\rangle\}\rangle$ is denoted by C_n.

If $G_1 = \langle V_1, E_1\rangle$ and $G_2 = \langle V_2, E_2\rangle$ are graphs then their *union*, denoted by $G_1 \cup G_2$, is the structure $\langle V_1 \cup V_2, E_1 \cup E_2\rangle$. (We always assume that $V_1 \cap V_2 = \emptyset$.)

Lemma 2.3.6 *Let G_1, G_2, H_1 and H_2 be graphs. If $II(G_1, H_1, n)$ and $II(G_2, H_2, n)$ then $II(G_1 \cup G_2, H_1 \cup H_2, n)$.*

Proof: Whenever Player I chooses from G_1 or H_1, Player II uses his winning strategy for $E(G_1, H_1, n)$, otherwise he uses his strategy for $E(G_2, H_2, n)$. □

Lemma 2.3.7 *Let $\mathcal{A}$ and $\mathcal{B}$ be partial orders (possibly with distinguished elements). If $\mathcal{A} \equiv_{n+1} \mathcal{B}$ then $\mathcal{A}^{nb} \equiv_n \mathcal{B}^{nb}$.*

Proof: Use Theorem 2.6.3 below. □

Example 2.3.8 *Connectedness of graphs is not axiomatizable.*

Proof: Let $G_1 = \omega^{nb}$ and $G_2 = (\omega + \zeta)^{nb}$. From Theorem 2.2.13 and Lemma 2.3.7 we have $G_1 \equiv G_2$, while G_1 is connected and G_2 is not. □

Example 2.3.9 *Connectedness is not definable within the class of finite graphs.*

Proof: We show that if $m \geq 2^{n+1}$ then $II(C_m \cup P_m, P_m, n)$.

First we prove that $m \geq 2^n$ implies $C_m \equiv_n \zeta^{nb}$. A little thought shows that for Player II to win the n-round game on C_m and ζ^{nb} it suffices to win the $n-1$-round game on $\langle P_{m+1}, 0, m\rangle$ and $\langle(\omega + \omega^*)^{nb}, 0, 0^*\rangle$ (here we distinguish the endpoints of the two successor structures). Player II can win this game if he can win the game $E(\langle \mathbf{m+1}, 0, m\rangle, \langle\omega +$

$\omega^*, 0, 0^*\rangle, n)$, by Lemma 2.3.7. For this Player II only has to win the game $E(\mathbf{m-1}, \omega+\omega^*, n)$ (Example 2.2.12). Now use Example 2.2.11.

From Lemma 2.3.7 and Example 2.2.11 it also follows that $II(P_m, (\omega+\omega^*)^{nb}, n)$ if $m \geq 2^{n+1}$.

Now, by Lemma 2.3.6, $II(C_m \cup P_m, \zeta^{nb} \cup (\omega+\omega^*)^{nb}, n)$. Since $\zeta^{nb} \cup (\omega+\omega^*)^{nb} = (\omega+\zeta+\omega^*)^{nb}$ the proof will be completed if we can show that $(\omega+\zeta+\omega^*)^{nb} \equiv (\omega+\omega^*)^{nb}$. This follows immediately from Lemma 2.3.7, Theorem 2.2.13 and Example 2.2.12. □

2.4 Completeness

Definition: An *axiomatization* of a theory Δ is a theory Γ such that $Con(\Gamma) = \Delta$.

In order to prove that a theory Γ axiomatizes a theory Δ we have two things to do: show that $Con(\Gamma) \subseteq \Delta$ (soundness) and that $\Delta \subseteq Con(\Gamma)$ (completeness).

If Δ is the theory of some class $\mathcal{K}$ this is usually done as follows:

Soundness is proved by showing that every element of $\mathcal{K}$ is a model of Γ, from which it immediately follows that $\mathcal{K} \models Con(\Gamma)$.

For completeness we can prove that for every n, every model of Γ is n-equivalent to some element of $\mathcal{K}$. To see that this proves completeness, suppose that $\alpha \notin Con(\Gamma)$. Then there is a model $\mathcal{A}$ of Γ in which $\neg\alpha$ is satisfied. Let n be the quantifier rank of $\neg\alpha$. Then $\mathcal{A}$ is n-equivalent to some element of $\mathcal{K}$, hence $\neg\alpha$ is satisfied in some element of $\mathcal{K}$, so $\alpha \notin TH(\mathcal{K}) = \Delta$.

If the signature is finite and contains no function symbols then this condition is actually necessary, for suppose that for some n there is a model $\mathcal{A}$ of Γ that is not n-equivalent to any element of $\mathcal{K}$. Then, for every $\mathcal{B} \in \mathcal{K}$, there is a sentence $\phi_{\mathcal{B}}$ of rank at most n such that $\mathcal{B} \models \phi_{\mathcal{B}}$ and $\mathcal{A} \not\models \phi_{\mathcal{B}}$. Up to equivalence the set $\{\phi_{\mathcal{B}} : \mathcal{B} \in \mathcal{K}\}$ is finite. Then the sentence $\phi = \bigvee\{\phi_{\mathcal{B}} : \mathcal{B} \in \mathcal{K}\}$ is not in $Con(\Gamma)$ but is true in every element of $\mathcal{K}$.

The rest of this section is devoted to an example showing how Ehrenfeucht's game can be applied to the axiomatization of a theory. We let $\mathcal{K}$ be the class of finite trees and we will show that $TH(\mathcal{K})$ is axiomatized by Γ, the set of sentences listed below. The signature consists of a single binary relation $\leq$. We will also write $x < y$ as an abbreviation for $x \leq y \wedge x \neq y$.

A1 $\forall x \forall y \forall z (x \leq y \wedge y \leq z \rightarrow x \leq z)$

A2 $\forall x \forall y (x \leq y \wedge y \leq x \rightarrow x = y)$

A3 $\forall x \forall y \forall z (x \leq z \wedge y \leq z \rightarrow (x \leq y \vee y \leq x \vee x = y))$

A4 $\forall x \forall y (x \leq y \rightarrow \exists z (x \leq z \leq y \wedge \neg \exists t (x < t < z)))$

A5 $\forall x \exists y (y \leq x \wedge \neg \exists z (z < y))$

A6 $\forall x(\phi(x) \to \exists y(x \leq y \wedge \phi(y) \wedge \forall z(y < z \to \neg\phi(z))))$ for every formula $\phi(x)$ in one free variable x.

It is easily verified that every finite tree satisfies Γ, hence soundness.

To prove completeness we now fix an n and show that every model of Γ is n-equivalent to some finite tree. Throughout the rest of this section we let L_n be the set of all formulae of quantifier rank at most n.

First we get an idea of what a model $\mathcal{T}$ of Γ looks like. **A1**, **A2** and **A3** are just the defining properties of a tree. By **A5** there is a set V of nodes that have no predecessors and such that every node is a successor of some node in V, i.e. $\mathcal{T}$ has a first level. By **A4** every node in V that has a successor has an immediate successor. The set of all these immediate successors forms a second level. Similarly, there is a third level, a fourth level, and so on. This prompts the following definition.

Definition: If $\mathcal{T} \models \Gamma$, the i'th *level* of $\mathcal{T}$, $i \geq 1$, is defined by:

1. the 1'st level of $\mathcal{T}$ is the set of all nodes with no predecessors,
2. the $i+1$'th level of $\mathcal{T}$ is the set of all immediate successors to elements of the i'th level.

A node on level i will be called an *i-node.*

Lemma 2.4.1 *There exists a finite set of sentences $\Phi \subseteq L_n$ such that*

1. *Every structure satisfies exactly one sentence from Φ.*
2. *Structures $\mathcal{A}$ and $\mathcal{B}$ satisfy the same sentence from Φ if and only if they are n-equivalent.*

Proof: Let Φ' be a finite set of sentences from L_n such that every element of L_n is logically equivalent to an element of Φ'. Such a set exists by Lemma 2.2.5. Let Φ be the set of all conjunctions of elements of Φ' such that for every element $\phi \in \Phi'$ either ϕ or $\neg\phi$ occurs in the conjunction. The result is now immediate. $\square$

We call an element of Φ a *structure-type* and say that a structure $\mathcal{B}$ has type ϕ if $\mathcal{B} \models \phi$.

Lemma 2.4.2 *For every structure-type ϕ there exists a formula $\phi'(x)$ such that for every tree $\mathcal{T}$ and $a \in T$, $\mathcal{T}_a \models \phi$ iff $\mathcal{T} \models \phi'(a)$.*

Proof: ϕ' is obtained from ϕ by relativizing all quantification to successors of x, i.e. ϕ' is defined inductively by:

1. $\phi' = \phi$ if ϕ is atomic,

2. $(\neg\alpha)' = \neg\alpha'$,

3. $(\alpha \wedge \beta)' = \alpha' \wedge \beta'$,

4. $(\exists y\alpha)' = \exists y(x \leq y \wedge \alpha')$. (We may assume that $y \neq x$, since $\exists x\alpha(x)$ is equivalent to $\exists y\alpha(y)$ for any y that does not occur in $\alpha(x)$.) $\square$

Definition: For trees $\mathcal{T}$ and $\mathcal{T}'$ we denote the tree obtained by substituting $\mathcal{T}'$ for a subtree $\mathcal{T}_a$ by $S(\mathcal{T}, \mathcal{T}', a)$. Formally $S(\mathcal{T}, \mathcal{T}', a)$ is defined as follows:

1. $S(\mathcal{T}, \mathcal{T}', a) = \langle S, R\rangle$ where

2. $S = T \setminus T_a \cup T'$,

3. xRy iff $x, y \in T \setminus T_a$ and $x \leq y$, or $x, y \in T'$ and $x \leq y$, or $x \leq a$ and $y \in T'$.

Lemma 2.4.3 *Let $\mathcal{T}$ and $\mathcal{T}'$ be trees, $a \in T$ and $\mathcal{T}_a \equiv_n \mathcal{T}'$. Then $S(\mathcal{T}, \mathcal{T}', a) \equiv_n \mathcal{T}$.*

Proof: Player II does the following. If Player I chooses an element from $T \setminus T_a$ he chooses exactly the same element. Otherwise he uses his winning strategy for the game $E(\mathcal{T}_a, \mathcal{T}', n)$ to produce a response. That this is a winning strategy is easy to see. $\square$

Lemma 2.4.4 *Let $\mathcal{T}$ be a model of Γ. Then $\mathcal{T}$ is n-equivalent to a subtree of $\mathcal{T}$ in which there are finitely many nodes on the first level.*

Proof: For a given structure-type ϕ, let V_1^ϕ be the set of nodes of structure-type ϕ on the first level. It suffices to show that $\mathcal{T}$ is n-equivalent to a subtree of $\mathcal{T}$ in which there are finitely many nodes of structure-type ϕ on the first level.

If V_1^ϕ is finite there is nothing to prove. Otherwise, let $\mathcal{T}'$ be the tree obtained by removing all but n subtrees rooted at elements of V_1^ϕ and let V^ϕ be the remaining nodes of V_1^ϕ.

Player II uses the following strategy: Suppose Player I chooses an element a from $\mathcal{T}$. Let $a \in \mathcal{T}_r$ where r is on the first level. If $r \notin V_1^\phi$ then Player II chooses a in $\mathcal{T}'$. Otherwise, if no element of $\mathcal{T}_r$ was chosen before, Player II first selects an element $b \in V^\phi$ such that no element of $\mathcal{T}_b'$ has been chosen before (this is always possible since V^ϕ has n elements) and then uses his winning strategy for the game $E(\mathcal{T}_a, \mathcal{T}_b', n)$ to choose a reply. From then on, whenever Player I chooses an element from $\mathcal{T}_a$ or $\mathcal{T}_b'$, Player II continues using his strategy for $E(\mathcal{T}_a, \mathcal{T}_b', n)$.

Similarly if Player I chooses an element from $\mathcal{T}'$. $\square$

Lemma 2.4.5 *Let $\mathcal{T}$ be a model of Γ and $a \in T$. Then $\mathcal{T}$ is n-equivalent to a subtree $\mathcal{T}'$ of $\mathcal{T}$ in which a has finitely many immediate successors. Moreover, for every $b \in T'$, $\mathcal{T}_b$ and $\mathcal{T}_b'$ have the same structure-type.*

Proof: The argument for the first part is similar to that in the proof of the previous lemma.

For the second part, let $b \in T''$. If $b \not\leq a$ then $\mathcal{T}_b = \mathcal{T}'_b$. Otherwise, by repeating the argument in the first part with $\mathcal{T}_b$ instead of $\mathcal{T}$, it follows that $\mathcal{T}_b \equiv_n \mathcal{T}'_b$. □

Theorem 2.4.6 *Every model of* Γ *is* n*-equivalent to some finite tree.*

Proof: Let $\mathcal{T}$ be a model of Γ. We construct a sequence of trees $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_k$ with the properties:

1. $\mathcal{T}_i$ has finitely many nodes on levels $1, 2, \ldots, i$.
2. If $a \in T_i$ is a node of structure-type ϕ on level $j \leq i$ then no strict successor of a has structure-type ϕ.

By Lemma 2.4.4 $\mathcal{T}$ is n-equivalent to a tree $\mathcal{T}'_1$ that has finitely many nodes on level 1. Let a be a 1-node of $\mathcal{T}'_1$ of structure-type ϕ. By **A6** there is a greatest successor b of a with the same type. By Lemma 2.4.3 the tree obtained by substituting $(\mathcal{T}'_1)_a$ by $(\mathcal{T}'_1)_b$ is n-equivalent to $\mathcal{T}'_1$. Let $\mathcal{T}_1$ be the tree obtained by repeating this procedure with every 1-node of $\mathcal{T}'_1$.

Suppose that $\mathcal{T}_i$ has been defined. By Lemma 2.4.5, $\mathcal{T}_i$ is n-equivalent to a tree $\mathcal{T}'_{i+1}$ that has finitely many nodes on level $i+1$. We now repeat the construction above to obtain the tree $\mathcal{T}_{i+1}$.

Both properties follow immediately from the construction (and, for Property 2, the second part of Lemma 2.4.5).

By Property 2, this process terminates after finitely many steps. In fact, if there are m different structure-types then $k \leq m$. The tree $\mathcal{T}_k$ is then the required finite tree. □

2.5 Unnested Ehrenfeucht Games

As we have seen, Theorem 2.2.6 is very useful for proving definability and completeness results. However, it is restrictive in that it only applies to finite signatures without function symbols. The requirement that the signature must be finite is easily overcome. To allow for function symbols we will modify the game to obtain similar results for signatures that may include function symbols.

In order to prove that two σ-structures are equivalent it suffices to show that all reducts to finite subsets of σ are equivalent. This follows from the fact that only finitely many symbols occur in any formula. Together with Theorem 2.2.6 we have:

Theorem 2.5.1 *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be* σ*-structures, where* σ *contains no function symbols. Then, for every* $n < \omega$,

$\mathcal{A} \equiv_n \mathcal{B}$ *if and only if* $II(\mathcal{A}|_{\sigma'}, \mathcal{B}|_{\sigma'}, n)$ *for every finite subset* σ' *of* σ. □

Example: 2.2.8 (continued). Using Theorem 2.5.1 the equivalence of $\mathcal{A} = \langle \omega, 0, 1, 2, \ldots \rangle$ and $\mathcal{B} = \langle \omega + \zeta, 0, 1, 2, \ldots \rangle$ follows easily by noting that $\mathcal{A}|_{\sigma'} \cong \mathcal{B}|_{\sigma'}$ for any finite subset σ'.

Before we come to the variation of Ehrenfeucht's game that allows for function symbols we need a few preliminaries.

Definition: An *unnested* atomic formula of a signature σ is an atomic formula of one of the following forms (where each of $x_1, \ldots, x_k$, x and y is a variable or a constant):

1. $x = y$,
2. $\mathbf{f}(x_1, \ldots, x_k) = y$ for some function symbol $\mathbf{f}$,
3. $\mathbf{r}(x_1, \ldots, x_k)$ for some relation symbol $\mathbf{r}$.

A formula is called *unnested* if all its atomic subformulae are unnested.

Theorem 2.5.2 *Let σ be a signature. Then every atomic σ-formula is logically equivalent to an unnested σ-formula.*

Proof: For any terms $t_1, \ldots, t_k$ and i with $1 \leq i \leq k$ the following pairs of formulae are logically equivalent: (In each case x is a new variable.)

1. $\mathbf{r}(t_1, \ldots, t_i, \ldots, t_k)$ and $\forall x(t_i = x \rightarrow \mathbf{r}(t_1, \ldots, x, \ldots, t_k))$.
2. $t_1 = t_2$ and $\forall x(t_1 = x \rightarrow t_2 = x)$.
3. $\mathbf{f}(t_1, \ldots, t_i, \ldots, t_k) = y$ and $\forall x(t_i = x \rightarrow \mathbf{f}(t_1, \ldots, x, \ldots, t_k) = y)$.

Now any atomic formula can be reduced to an unnested formula by repeatedly applying these equivalences. □

Corollary 2.5.3 *Let σ be any signature. Then every σ-formula is logically equivalent to an unnested σ-formula.*

Proof: Use Theorem 2.5.2 to replace all atomic subformulae by unnested formulae. □

Definition: We modify the game $E(\mathcal{A}, \mathcal{B}, n)$ as follows. The game is played exactly as before, only the condition for Player II to win a play changes. We now stipulate that Player II wins the play $\{\langle a_1, b_1 \rangle, \ldots, \langle a_n, b_n \rangle\}$ if for every unnested atomic formula $\alpha(x_1, \ldots, x_n)$, $\mathcal{A} \models \alpha(a_1, \ldots, a_n)$ iff $\mathcal{B} \models \alpha(b_1, \ldots, b_n)$.

This game, denoted by $E^u(\mathcal{A}, \mathcal{B}, n)$, is called an *unnested Ehrenfeucht game.* The situation that Player II has a winning strategy is now denoted by $II^u(\mathcal{A}, \mathcal{B}, n)$ and we write $\mathcal{A} \equiv_n^u \mathcal{B}$ to denote that structures $\mathcal{A}$ and $\mathcal{B}$ satisfy the same unnested sentences of quantifier rank at most n.

Theorem 2.2.6 now takes the form:

Theorem 2.5.4 *Let σ be a finite signature and let $\mathcal{A}$ and $\mathcal{B}$ be σ-structures. Then, for every $n < \omega$,*

$$II^u(\mathcal{A}, \mathcal{B}, n) \text{ if and only if } \mathcal{A} \equiv_n^u \mathcal{B}.$$

Proof: Let σ' be the signature obtained by replacing each m-ary function symbol of σ by an $m+1$-ary relation symbol $\mathbf{r_f}$. Let $\mathcal{A}'$ be the σ' structure with

1. $\mathbf{r}^{\mathcal{A}'} = \mathbf{r}^{\mathcal{A}}$ for every relation symbol $\mathbf{r} \in \sigma$,
2. $\mathbf{c}^{\mathcal{A}'} = \mathbf{c}^{\mathcal{A}}$ for every constant symbol $\mathbf{c} \in \sigma$,
3. $\mathbf{r_f}^{\mathcal{A}'} = G_{\mathbf{f}^{\mathcal{A}}}$, the graph of $\mathbf{f}^{\mathcal{A}}$, for every function symbol $\mathbf{f} \in \sigma$.

and define $\mathcal{B}'$ similarly.

Note that for every m-ary function symbol $\mathbf{f}$ and $a_1, \ldots, a_m, a \in A$, $\mathcal{A} \models \mathbf{f}(a_1, \ldots, a_m) = a$ iff $\mathcal{A}' \models \mathbf{r_f}(a_1, \ldots, a_m, a)$ and for $b_1, \ldots, b_m, b \in B$, $\mathcal{B} \models \mathbf{f}(b_1, \ldots, b_m) = b$ iff $\mathcal{B}' \models \mathbf{r_f}(b_1, \ldots, b_m, b)$.

More generally, for an unnested σ-formula $\alpha(z_1, \ldots, z_k)$, $a_1, \ldots, a_k \in A$ and $b_1, \ldots, b_k \in B$, if we let α' be the result of replacing every atomic subformula of the form $\mathbf{f}(x_1, \ldots, x_m) = y$ by $\mathbf{r_f}(x_1, \ldots, x_m, y)$, it follows by induction on α that

$$\begin{gathered}\mathcal{A} \models \alpha(a_1, \ldots, a_k) \text{ iff } \mathcal{A}' \models \alpha'(a_1, \ldots, a_k) \text{ and} \\ \mathcal{B} \models \alpha(b_1, \ldots, b_k) \text{ iff } \mathcal{B}' \models \alpha'(b_1, \ldots, b_k).\end{gathered}$$

Note also that α and α' have the same quantifier rank. The same holds true if we start with a σ'-formula and replace subformulae of the form $\mathbf{r_f}(x_1, \ldots, x_m, y)$ by $\mathbf{f}(x_1, \ldots, x_m) = y$.

Furthermore, $II^u(\mathcal{A}, \mathcal{B}, n)$ if and only if $II(\mathcal{A}', \mathcal{B}', n)$ (in fact a winning strategy for one game is a winning strategy for the other game).

The result now follows from this and Theorem 2.2.6. □

Together with Corollary 2.5.3 we obtain:

Corollary 2.5.5 *Let σ be a finite signature and let $\mathcal{A}$ and $\mathcal{B}$ be σ-structures. Then*

$$II^u(\mathcal{A},\mathcal{B},n), \textit{ for every } n, \textit{ if and only if } \mathcal{A} \equiv \mathcal{B}. \qquad \square$$

Example 2.5.6 *Let G_1, G_2 and H be groups. If $G_1 \equiv G_2$ then $G_1 \times H \equiv G_2 \times H$.*

Proof: It suffices to show that if $G_1 \equiv G_2$ then Player II has a winning strategy for the game $E^u(G_1 \times H, G_2 \times H, n)$ for every n. Assume that $G_1 \equiv G_2$. Then $II^u(G_1, G_2, n)$. In order to win the game $E^u(G_1 \times H, G_2 \times H, n)$ Player II uses his winning strategy for the game $E^u(G_1, G_2, n)$ as follows: next to the actual play of $E^u(G_1 \times H, G_2 \times H, n)$ he also plays the game $E^u(G_1, G_2, n)$ on the side. Whenever Player I chooses an element, say the element $\langle g, h\rangle \in G_1 \times H$, Player II pretends that g was chosen in the side game and uses his winning strategy to choose an element $g' \in G_2$. His official reply is then the element $\langle g', h\rangle \in G_2 \times H$. Similarly if Player I chooses an element from $G_2 \times H$.

At the end of the game let the play be $\{\langle\langle g_1, h_1\rangle, \langle g'_1, h_1\rangle\rangle, \ldots, \langle\langle g_n, h_n\rangle, \langle g'_n, h_n\rangle\rangle\}$. Now the unnested atomic formulae in the language of groups are the formulae of the form $x = y$, $1 = y$, $x_1x_2 = y$, $x^{-1} = y$ (and $1 = 1$) where x, y, x_1 and x_2 are variables. Since Player II has won the side game we have for all $i, j, l \leq n$

1. $g_i = g_j$ iff $g'_i = g'_j$,
2. $1 = g_i$ iff $1 = g'_i$,
3. $g_ig_j = g_l$ iff $g'_ig'_j = g'_l$,
4. $g_i^{-1} = g_j$ iff $g_i'^{-1} = g'_j$.

By the definition of cartesian products this implies that for all $i, j, l \leq n$ we have

1. $\langle g_i, h_i\rangle = \langle g_j, h_j\rangle$ iff $\langle g'_i, h_i\rangle = \langle g'_j, h_j\rangle$,
2. $1 = \langle g_i, h_i\rangle$ iff $1 = \langle g'_i, h_i\rangle$,
3. $\langle g_i, h_i\rangle\langle g_j, h_j\rangle = \langle g_l, h_l\rangle$ iff $\langle g'_i, h_i\rangle\langle g'_j, h_j\rangle = \langle g'_l, h_l\rangle$,
4. $\langle g_i, h_i\rangle^{-1} = \langle g_j, h_j\rangle$ iff $\langle g'_i, h_i\rangle^{-1} = \langle g'_j, h_j\rangle$.

(For example, $\langle g_i, h_i\rangle = \langle g_j, h_j\rangle$ iff $g_i = g_j$ and $h_i = h_j$ iff $g'_i = g'_j$ and $h_i = h_j$ iff $\langle g'_i, h_i\rangle = \langle g'_j, h_j\rangle$.) So Player II wins the official game too, which proves the theorem. $\square$

Analogous to Theorem 2.5.1 we have the following.

Theorem 2.5.7 *Let $\mathcal{A}$ and $\mathcal{B}$ be σ-structures. Then, for every $n < \omega$,*

$$\mathcal{A} \equiv^u_n \mathcal{B} \textit{ if and only if } II^u(\mathcal{A}|_{\sigma'}, \mathcal{B}|_{\sigma'}, n) \textit{ for every finite subset } \sigma' \textit{ of } \sigma. \qquad \square$$

Using this result we can now generalize Example 2.5.6.

Theorem 2.5.8 *Let* $\{\mathcal{A}_i : i \in I\}$ *and* $\{\mathcal{B}_i : i \in I\}$ *be families of* σ*-structures such that* $\mathcal{A}_i \equiv \mathcal{B}_i$ *for every* $i \in I$. *Then* $\Pi\{\mathcal{A}_i : i \in I\} \equiv \Pi\{\mathcal{B}_i : i \in I\}$.

Proof: Let $\mathcal{A} = \Pi\{\mathcal{A}_i : i \in I\}$ and $\mathcal{B} = \Pi\{\mathcal{B}_i : i \in I\}$. By Theorem 2.5.7 it suffices to prove the result for finite signatures. The argument is the same as above. Suppose Player I selects the function $a \in A$. For every $i \in I$, Player II uses his winning strategy for the game $E^u(\mathcal{A}_i, \mathcal{B}_i, n)$ to select an element $b(i) \in B_i$. This determines a function $b \in \Pi\{B_i : i \in I\}$, which is then his official reply.

After n rounds, let the play be $\{\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle\}$. Further, let $a_{n+1}, \ldots, a_m$ and $b_{n+1}, \ldots, b_m$ be the interpretations, in $\mathcal{A}$ and $\mathcal{B}$ respectively, of the constant symbols in σ. Then for all $k_1, \ldots, k_{l+1} \leq m$ and $i \in I$ we have

1. $a_{k_1}(i) = a_{k_2}(i)$ iff $b_{k_1}(i) = b_{k_2}(i)$,
2. $\mathbf{f}^{\mathcal{A}_i}(a_{k_1}(i), \ldots, a_{k_l}(i)) = a_{k_{l+1}}(i)$ iff $\mathbf{f}^{\mathcal{B}_i}(b_{k_1}(i), \ldots, b_{k_l}(i)) = b_{k_{l+1}}(i)$,
3. $\mathbf{r}^{\mathcal{A}_i}(a_{k_1}(i), \ldots, a_{k_l}(i))$ iff $\mathbf{r}^{\mathcal{B}_i}(b_{k_1}(i), \ldots, b_{k_l}(i))$.

From the definition of products it now follows that

1. $a_{k_1} = a_{k_2}$ iff $b_{k_1} = b_{k_2}$, (since $a_{k_1} = a_{k_2}$ iff $a_{k_1}(i) = a_{k_2}(i)$ for every $i \in I$ iff $b_{k_1}(i) = b_{k_2}(i)$ for every $i \in I$ iff $b_{k_1} = b_{k_2}$),
2. $\mathbf{f}^{\mathcal{A}}(a_{k_1}, \ldots, a_{k_l}) = a_{k_{l+1}}$ iff $\mathbf{f}^{\mathcal{B}}(b_{k_1}, \ldots, b_{k_l}) = b_{k_{l+1}}$,
3. $\mathbf{r}^{\mathcal{A}}(a_{k_1}, \ldots, a_{k_l})$ iff $\mathbf{r}^{\mathcal{B}}(b_{k_1}, \ldots, b_{k_l})$.

Therefore Player II wins and it follows that $\mathcal{A} \equiv \mathcal{B}$. □

2.6 Definable Symbols

The idea of the previous section can be summed up as: We are given two structures $\mathcal{A}$ and $\mathcal{B}$ of a signature that includes function symbols. In order to prove that they are equivalent we cannot use Ehrenfeucht's theorem directly, so we modify the structures (by considering every n-ary function to be an $n+1$-ary relation) to obtain structures $\mathcal{A}'$ and $\mathcal{B}'$ of a finite function-free signature and, using Ehrenfeucht's theorem, prove that $\mathcal{A}' \equiv \mathcal{B}'$. Moreover, $\mathcal{A}'$ and $\mathcal{B}'$ are chosen in such a way that $\mathcal{A} \equiv \mathcal{B}$ follows immediately from $\mathcal{A}' \equiv \mathcal{B}'$. In this section we generalize this idea of proving the equivalence of two structures by proving the equivalence of two closely related structures.

Definition: Let $\sigma \subseteq \sigma'$ and $\mathbf{s} \in \sigma'$. An *explicit definition* of $\mathbf{s}$ in terms of σ is a formula ψ of the form:

1. $\forall x_1 \cdots \forall x_n(\mathbf{r}(x_1, \ldots, x_n) \leftrightarrow \phi(x_1, \ldots, x_n))$ if $\mathbf{s}$ is a relation symbol $\mathbf{r}$,

2. $\forall x(\mathbf{c} = x \leftrightarrow \phi(x))$ if $\mathbf{s}$ is a constant symbol $\mathbf{c}$,

3. $\forall x_1 \cdots \forall x_n \forall y(\mathbf{f}(x_1, \ldots, x_n) = y \leftrightarrow \phi(x_1, \ldots, x_n, y))$ if $\mathbf{s}$ is a function symbol $\mathbf{f}$,

where in each case ϕ is a σ-formula.

We say that the formula ϕ (or ψ) *defines* $\mathbf{s}^{\mathcal{A}}$ on a σ'-structure $\mathcal{A}'$ if $\mathcal{A}' \models \psi$.

If $\mathcal{A}$ is a σ-structure and S is a relation, function or constant in A then S is *definable* in $\mathcal{A}$ if there is some σ-formula ϕ that defines S in $\langle \mathcal{A}, S\rangle$. In this case we also say that ϕ defines S in $\mathcal{A}$.

Note that 2 and 3 imply respectively:

1. $\exists! y \phi(y)$,

2. $\forall x_1 \cdots \forall x_n \exists! y \phi(x_1, \ldots, x_n, y)$,

where $\exists! y$ means "there exists a *unique* y". These sentences are called the *admissability conditions* of the explicit definitions in 2 and 3. If we want to expand a structure by adding a constant or function such that some definition ψ holds, these conditions must be satisfied. We are, however, not interested in definitional expansions. We are interested in the case where we start with structures in which some symbol is defined by some formula, and then consider corresponding reducts, in which case the admissability conditions will necessarily be satisfied.

Lemma 2.6.1 *Let $\sigma \subseteq \sigma'$. Let, for every $\mathbf{s} \in \sigma'$, $\psi_{\mathbf{s}}$ be an explicit definition of $\mathbf{s}$ in terms of σ. Then for every σ'-formula α' there is a σ-formula α such that, for every σ'-structure $\mathcal{A}'$ with $\mathcal{A}' \models \{\psi_{\mathbf{s}} : \mathbf{s} \in \sigma'\}$, $\mathcal{A}' \models \alpha'$ iff $\mathcal{A}'|_\sigma \models \alpha$.*

Proof: For atomic α' the formula α is given, namely the formula ϕ that occurs in $\psi_{\mathbf{s}}$. Otherwise, by Corollary 2.5.3 α' is equivalent to an unnested σ'-formula and α is then obtained by replacing every atomic subformula by the corresponding σ-formula. The required property now follows by induction on α. □

Theorem 2.6.2 *Let $\mathcal{A} \equiv \mathcal{B}$ be σ-structures and let $\mathcal{A}'$ and $\mathcal{B}'$ be σ'-structures, where $\mathcal{A}$ and $\mathcal{A}'$ have the same universe and $\mathcal{B}$ and $\mathcal{B}'$ have the same universe. If, for every $\mathbf{s} \in \sigma'$, there is a σ-formula ϕ that defines $\mathbf{s}^{\mathcal{A}'}$ in $\mathcal{A}$ and $\mathbf{s}^{\mathcal{B}'}$ in $\mathcal{B}$ then $\mathcal{A}' \equiv \mathcal{B}'$.*

Proof: Let $\mathcal{A}''$ be the $\sigma \cup \sigma'$-structure with $\mathbf{s}^{\mathcal{A}''} = \mathbf{s}^{\mathcal{A}}$ if $\mathbf{s} \in \sigma$ and $\mathbf{s}^{\mathcal{A}''} = \mathbf{s}^{\mathcal{A}'}$ otherwise. For every σ''-formula α'' there is, by Lemma 2.6.1, a σ-formula α such that $\mathcal{A}'' \models \alpha''$ iff $\mathcal{A} \models \alpha$ and $\mathcal{B}'' \models \alpha''$ iff $\mathcal{B} \models \alpha$. It now follows immediately from $\mathcal{A} \equiv \mathcal{B}$ that $\mathcal{A}'' \equiv \mathcal{B}''$ and therefore $\mathcal{A}' \equiv \mathcal{B}'$. □

Theorem 2.6.3 *Let $\mathcal{A}$ and $\mathcal{B}$ be σ-structures and let $\mathcal{A}'$ and $\mathcal{B}'$ be σ'-structures, where $\mathcal{A}$ and $\mathcal{A}'$ have the same universe, $\mathcal{B}$ and $\mathcal{B}'$ have the same universe and σ' is finite and relational. If, for every $\mathbf{r} \in \sigma'$, there is a σ-formula $\phi_{\mathbf{r}}$ that defines $\mathbf{r}^{\mathcal{A}'}$ in $\mathcal{A}$ and $\mathbf{r}^{\mathcal{B}'}$ in $\mathcal{B}$ and $k = max\{qr(\phi_{\mathbf{r}}) : \mathbf{r} \in \sigma'\}$ then $\mathcal{A} \equiv_{n+k} \mathcal{B}$ implies $\mathcal{A}' \equiv_n \mathcal{B}'$.*

Proof: Let $\mathcal{A}''$ and $\mathcal{B}''$ be as in the proof of Theorem 2.6.2. As in the proof of Lemma 2.6.1. using the fact that σ' is relational, it follows that for every $\sigma \cup \sigma'$-formula α'' there is a σ-formula α of rank at most $n + k$ such that $\mathcal{A}'' \models \alpha''$ iff $\mathcal{A} = \mathcal{A}''|_\sigma \models \alpha$ and $\mathcal{B}'' \models \alpha''$ iff $\mathcal{B} = \mathcal{B}''|_\sigma \models \alpha$. From $\mathcal{A} \equiv_{n+k} \mathcal{B}$ it now follows that $\mathcal{A}'' \equiv_n \mathcal{B}''$, hence $\mathcal{A}' \equiv_n \mathcal{B}'$. □

We are now in a position to complete Example 2.2.8.

Example: 2.2.8(continued).

To show that $\langle \omega, S, 0\rangle$ and $\langle \omega + \zeta, S, 0\rangle$ are equivalent, note that the successor function is defined in both $\langle \omega, <\rangle$ and $\langle \omega + \zeta, <\rangle$ by the formula $x < y \wedge \neg\exists z(x < z < y)$. Also, 0 is defined by $\neg\exists y(y < x)$. From Theorem 2.2.13 and Theorem 2.6.2 we now obtain $\langle \omega, S, 0\rangle \equiv \langle \omega + \zeta, S, 0\rangle$. (Alternatively, this can be proved directly using unnested Ehrenfeucht games.)

The equivalence of $\langle \omega, 0, 1, 2, \ldots\rangle$ and $\langle \omega + \zeta, 0, 1, 2, \ldots\rangle$ can now be proved using the explicit definition $\forall x(\mathbf{c}_n = x \leftrightarrow x = \mathbf{s}^n(0))$ of n in $\langle \omega, S\rangle$ and $\langle \omega + \zeta, S\rangle$. (Here $\mathbf{c}_n$ is the constant symbol interpreted by n and $\mathbf{s}^n(0)$ denotes the formula $\mathbf{s}(\mathbf{s}(\ldots(\mathbf{s}(0))\ldots))$, where $\mathbf{s}$ is iterated n times.) □

Chapter 3

Variations of Ehrenfeucht's Game

The games of Chapter 2 characterize first-order equivalence of structures. By adapting the game we can generalize the results obtained to various extensions and restrictions of first-order logic. In this chapter we will describe games and obtain results similar to Ehrenfeucht's theorem for monadic second order logic, infinitary logic, the logic $L(Q_\alpha)$ and first order logic with a bounded number of variables.

All the games mentioned in this chapter can be found in [5], [6] or [14]. The proofs are modifications of the proof of Theorem 2.2.6, the exception being Theorem 3.2.4, which is from [5]. For more on the logics and games introduced in this chapter the reader is referred to [2], [3], [7] and [8].

3.1 Monadic Second-order Logic

Monadic second-order logic is obtained from first order logic by adding:

1. a set $\{X_0, X_1, \ldots\}$ of *unary relation variables*,
2. to the definition of a formula the clauses:
   if X is a relation variable and t a term then $X(t)$ is a formula,
   if α is a formula and X a relation variable then $\exists X\alpha$ is a formula.

A *monadic second-order assignment* is a function v that assigns to every individual variable an element from the universe and to every relation variable a subset of the universe. The semantics of monadic second-order logic is then obtained by adding the following clauses:

1. if X is a relation variable and t is a term then $\mathcal{A} \models_v X(t)$ iff $t^{\mathcal{A}} \in v(X)$,
2. $\mathcal{A} \models_v \exists X\alpha$ iff there exists $B \subseteq A$ such that $\mathcal{A} \models_{v'} \alpha$ where $v'(X) = B$ and otherwise v' agrees with v.

The version of Ehrenfeucht's game for monadic second-order logic is obtained by allowing Player I to also choose a subset from either structure. If Player I chooses a subset, Player II must then choose a subset from the other structure. At the end of the game a finite relation $h = \{\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle\}$ is determined (some of the a_i's and b_i's may be subsets). Player II wins if the expansions $\langle \mathcal{A}, a_1, \ldots, a_n\rangle$ and $\langle \mathcal{B}, b_1, \ldots, b_n\rangle$ satisfy the same atomic sentences.

This game is denoted by $E^m(\mathcal{A}, \mathcal{B}, n)$. If Player II has a winning strategy for this game we write $II^m(\mathcal{A}, \mathcal{B}, n)$ and we write $\mathcal{A} \equiv^m \mathcal{B}$ to denote that $\mathcal{A}$ and $\mathcal{B}$ have the same true monadic second-order sentences. The quantifier rank of a formula is as before, with $qr(X(t)) = 0$ for a predicate variable X and a term t, and $qr(\exists X \alpha) = qr(\alpha) + 1$. If structures $\mathcal{A}$ and $\mathcal{B}$ satisfy the same monadic second-order sentences of quantifier rank at most n we write $\mathcal{A} \equiv^m_n \mathcal{B}$.

Example: Player I wins the game $E^m(\omega, \omega + \zeta, 3)$ by selecting the subset ζ in $\omega + \zeta$. Whatever subset of ω Player II responds with, it has a least element, which is then Player I's next move. Player II then has to select an element from ζ, after which Player I wins by selecting any smaller element in ζ.

Corresponding to Lemma 2.2.3 we have:

Lemma 3.1.1 *$II^m(\mathcal{A}, \mathcal{B}, n+1)$ iff*

1. $\forall a \in A\ \exists b \in B\ II^m(\langle \mathcal{A}, a\rangle, \langle \mathcal{B}, b\rangle, n)$.
2. $\forall b \in B\ \exists a \in A\ II^m(\langle \mathcal{A}, a\rangle, \langle \mathcal{B}, b\rangle, n)$.
3. $\forall C \subseteq A\ \exists D \subseteq B\ II^m(\langle \mathcal{A}, C\rangle, \langle \mathcal{B}, D\rangle, n)$.
4. $\forall D \subseteq B\ \exists C \subseteq A\ II^m(\langle \mathcal{A}, C\rangle, \langle \mathcal{B}, D\rangle, n)$.

Proof: By modifying the definition of a local isomorphism to allow for pairs of subsets as well as pairs of elements, Lemma 2.2.2 still holds. Also, Lemma 2.2.5 still holds. The result then follows by the same arguments as in Lemma 2.2.3. □

The main result corresponding to Ehrenfeucht's theorem is:

Theorem 3.1.2 *Let $\mathcal{A}$ and $\mathcal{B}$ be σ-structures, where σ is finite and contains no function symbols. Then, for every $n < \omega$,*

$$II^m(\mathcal{A}, \mathcal{B}, n) \text{ if and only if } \mathcal{A} \equiv^m_n \mathcal{B}.$$

Proof: The proof is exactly as for Theorem 2.2.6, with the case $\alpha = \exists X \beta$ being similar to the case $\alpha = \exists x \beta$ in the proof of Theorem 2.2.6. □

Example 3.1.3 *The ordering of the rationals (η) and the ordering of the reals (λ) are not monadic second-order equivalent.*

Proof: This is of course due to the fact that λ is continuous (a property that can be expressed in monadic second-order logic) while η is not. In terms of games, Player I wins by selecting the set $A = \{x \in \mathcal{Q} : x < \sqrt{2}\}$ in η, then the least upper bound of Player II's response (which will exist unless Player II has already lost) and then an element of A greater than Player II's second move (again, this is impossible only if Player II has already lost). □

There is nothing in the proof of Theorem 3.1.2 that relies on the quantification ranging over *all* subsets. In *weak second-order logic*, for instance, quantification ranges over finite subsets. If we modify the game $E^m(\mathcal{A}, \mathcal{B}, n)$ by insisting that the players select finite subsets only Player II will have a winning strategy if and only if $\mathcal{A}$ and $\mathcal{B}$ satisfy the same weak second-order sentences of rank at most n (under the usual restrictions on the signature). We generalize this as follows:

Definition: A *generalized monadic second-order quantifier* is a (class) function that assigns to every cardinal κ a set of cardinals $\leq \kappa$.

For example, the quantifier that assigns to every cardinal κ the set of all cardinals $\leq \kappa$ is the usual monadic second-order quantifier and the quantfier Q with $Q(\kappa) = \{k < \omega : k \leq \kappa\}$ is the weak monadic second-order quantifier.

If Q is a generalized monadic second-order quantifier the logic $L(Q)$ is obtained from second-order logic by replacing the second-order quantifier with the quantifier Q. Syntactically, Q is treated just like the second-order quantifier. The semantics are as follows (with $|A|$ the cardinality of A for a set A):

$\mathcal{A} \models_v QX\alpha$ iff there exists $B \subseteq A$ such that $|B| \in Q(|A|)$ and $\mathcal{A} \models_{v'} \alpha$ where $v'(X) = B$ and otherwise v' agrees with v.

The game corresponding to this is modified from $E^m(\mathcal{A}, \mathcal{B}, n)$ as follows: If Player I chooses a subset of A, it has to be a subset with cardinality an element of $Q(|A|)$. Player II must then reply with a subset of B of cardinality some element of $Q(|B|)$. Similarly if Player I chooses a subset of B.

Analogous to Theorem 3.1.2 we then obtain that Player II has a winning strategy for this game if and only if $\mathcal{A}$ and $\mathcal{B}$ satisfy the same $L(Q)$-sentences of quantifier rank at most n, provided that the signature is finite and contains no function symbols.

3.2 Infinitary Logic

The infinitary logic $L_{\infty\omega}$ is the logic obtained from first-order logic by allowing infinite conjunctions and disjunctions, i.e. $L_{\infty\omega}$ is obtained by adding:

1. the symbol $\bigwedge$,

2. the clause: If Φ is a set of formulae then $\bigwedge \Phi$ is a formula,

3. for the semantics the clause: $\mathcal{A} \models_v \bigwedge \Phi$ iff $\mathcal{A} \models_v \phi$ for every $\phi \in \Phi$.

If structures $\mathcal{A}$ and $\mathcal{B}$ satisfy the same infinitary sentences we write $\mathcal{A} \equiv_{\infty\omega} \mathcal{B}$.

The game corresponding to $L_{\infty\omega}$ is the infinite version of the game $E(\mathcal{A}, \mathcal{B}, n)$. In other words, the game is played as before, but now there is no bound on the number of moves. The game goes on forever and Player II now wins if at every finite stage of the play, the moves made up to that point constitute a local isomorphism. This game is denoted by $E^{\infty}(\mathcal{A}, \mathcal{B})$.

Definition: If Player II has a winning strategy for the game $E^{\infty}(\mathcal{A}, \mathcal{B})$ we say that $\mathcal{A}$ and $\mathcal{B}$ are *partially isomorphic* and we write $II^{\infty}(\mathcal{A}, \mathcal{B})$.

Example 3.2.1 *No well-ordering is partially isomorphic with a non-well-ordering.*

Proof: Let Player I play an infinite descending sequence in the non-well-ordering. □

Example 3.2.2 *Any two dense linear orderings without endpoints are partially isomorphic.*

Proof: Player II must play in such a way that if the position at the end of round n is $h = \{\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle\}$ and (without loss of generality) $a_1 \leq a_2 \leq \ldots \leq a_n$ then $b_1 \leq b_2 \leq \ldots \leq b_n$.

For $n = 1$ this is trivial. Now suppose that Player II has managed to maintain this situation up to the end of round n and suppose that Player I selects a_{n+1}. If $a_{n+1} < a_1$ Player II can choose any element $< b_1$ in $\mathcal{B}$. Such an element exists since $\mathcal{B}$ has no least element. Similarly if $a_{n+1} > b_n$. Otherwise $a_i < a_{n+1} < a_{i+1}$ for some i with $1 \leq i < n$. In this case Player II can choose any b_{n+1} with $b_i < b_{n+1} < b_{i+1}$, which exists since the ordering is dense. Similarly if Player I chooses from $\mathcal{B}$. □

Example 3.2.3 *Well-orderings of different types are not partially isomorphic.*

Proof: Let Player I play the element a of the larger one such that $a\downarrow$ is isomorphic to the smaller one. Thereafter, Player I can always counter a move c with a move d such that $c\downarrow$ is isomorphic to $d\downarrow$. The moves of Player I (and hence of Player II) then form a strictly descending sequence. Therefore Player II must eventually pick the least element of one structure, after which Player I picks the least element of the other structure and Player II cannot find a smaller element to counter with. □

Theorem 3.2.4 *(Karp) For any structures $\mathcal{A}$ and $\mathcal{B}$,*

$$II^{\infty}(\mathcal{A}, \mathcal{B}) \text{ if and only if } \mathcal{A} \equiv_{\infty\omega} \mathcal{B}.$$

Proof: Suppose $II^{\infty}(\mathcal{A}, \mathcal{B})$. By induction on ϕ we show that $\mathcal{A} \models \phi$ iff $\mathcal{B} \models \phi$. For atomic sentences this follows from the fact that if $II^{\infty}(\mathcal{A}, \mathcal{B})$ then $II(\mathcal{A}, \mathcal{B}, 0)$. The induction steps for the connectives are trivial.

Assume that $\mathcal{A} \models \exists x\phi(x)$. Let $a \in A$ be such that $\langle \mathcal{A}, a\rangle \models \phi(\mathbf{c})$, where $\mathbf{c}$ is the new constant symbol interpreted by a in $\langle \mathcal{A}, a\rangle$. Consider a as a first move of Player I in the infinite game. Let Player II use his winning strategy to determine an answer $b \in B$. Since the game is infinite we have that $II^{\infty}(\langle \mathcal{A}, a\rangle, \langle \mathcal{B}, b\rangle)$. Therefore, by the induction hypothesis applied to $\phi(\mathbf{c})$, we have that $\langle \mathcal{B}, b\rangle \models \phi(\mathbf{c})$. It follows that $\mathcal{B} \models \exists x\phi$.

Conversely, assume that $\mathcal{A}$ and $\mathcal{B}$ satisfy the same infinitary sentences. Player II's strategy consists of playing in such a way that after his n-th move a position $\{\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle\}$ is obtained for which

$$\langle \mathcal{A}, a_1, \ldots, a_n\rangle \equiv_{\infty\omega} \langle \mathcal{B}, b_1, \ldots, b_n\rangle.$$

This obviously suffices for Player II to win. So suppose that Player II has succeeded in maintaining this condition up to and including the n-th move and suppose that Player I plays $a_{n+1} \in A$. We must find a b_{n+1} such that

$$\langle \mathcal{A}, a_1, \ldots, a_{n+1}\rangle \equiv_{\infty\omega} \langle \mathcal{B}, b_1, \ldots, b_{n+1}\rangle.$$

Assume that no such b_{n+1} exists. Then for every $b \in B$ there is a formula $\phi_b(x)$ such that $\langle \mathcal{A}, a_1, \ldots, a_n\rangle \models \phi_b(a_{n+1})$ and $\langle \mathcal{B}, b_1, \ldots, b_n\rangle \not\models \phi_b(b)$. Let $\Phi = \{\phi_b : b \in B\}$. Then

$$\langle \mathcal{A}, a_1, \ldots, a_n\rangle \models \bigwedge \Phi(a_{n+1})$$

and hence

$$\langle \mathcal{A}, a_1, \ldots, a_n\rangle \models \exists x_{n+1} \bigwedge \Phi.$$

But

$$\langle \mathcal{B}, b_1, \ldots, b_n\rangle \not\models \exists x_{n+1} \bigwedge \Phi,$$

a contradiction. □

Theorem 3.2.5 *Countable partially isomorphic structures are isomorphic.*

Proof: Let Player I enumerate all the elements of the two structures and let Player II play according to his winning strategy. The resulting play is the required isomorphism. □

From this and Example 3.2.2 we immediately have Cantor's characterization of the ordering of the rationals.

Corollary 3.2.6 *The ordering of the rationals is, up to isomorphism, the only countable dense linear ordering without endpoints.* □

3.3 Ordinal-bounded Games

The game $E(\mathcal{A}, \mathcal{B}, n)$ can be generalized in the following way: Let α be an ordinal. The game $E(\mathcal{A}, \mathcal{B}, \alpha)$ is played like before, except that at the beginning of the first round Player I first chooses an ordinal less than α. Thereafter, at the beginning of each round Player I chooses an ordinal less than the previous ordinal selected. The game ends when Player I has no ordinals to choose from, i.e. at the end of the round in which 0 is chosen. The condition for Player II to win stays the same and the situation that Player II has a winning strategy is denoted by $II(\mathcal{A}, \mathcal{B}, \alpha)$.

Note that when α is a finite ordinal this game is equivalent to the ordinary Ehrenfeucht game, in the sense that a player has a winning strategy for one game if and only if he has a winning strategy for the other game. Note also that the game always ends in finitely many rounds, by the well-ordering of the ordinals. However, while the number of rounds is known from the start in the ordinary game, Player I can make it last as long as he wants without telling Player II (at least for a while). For example, if $\alpha = \omega + 2$ Player I has two moves before he has to let Player II know how long the game will last; if $\alpha = \omega \cdot 2$ Player I can keep Player II in the dark for an arbitrary number of rounds, etc.

Example: If $\mathcal{A}$ and $\mathcal{B}$ are structures of a finite signature with no function symbols then $II(\mathcal{A}, \mathcal{B}, \omega)$ if and only if $\mathcal{A} \equiv \mathcal{B}$.

Example: 3.2.3(continued) If α and β are ordinals with $\alpha < \beta$ then Player I has a winning strategy for $E(\alpha, \beta, \alpha + 1)$.

Proof: Player I plays as before, with the ordinal he chooses at the beginning of each round the same as the element he chooses in that round. (So, for example, Player I starts by choosing α as the first ordinal and then α as an element from β.) □

The unbounded version of this game $E(\mathcal{A}, \mathcal{B}, \infty)$ is obtained by allowing Player I to choose any ordinal in the first round. This game, although it always ends in finitely many rounds, turns out to be equivalent to the game $E^{\infty}(\mathcal{A}, \mathcal{B})$. This fact will become clear once we have established the meaning of the game.

Definition: The quantifier rank of an infinitary formula ϕ is defined as before, with $qr(\bigwedge \Phi) = sup\{qr(\phi) : \phi \in \Phi\}$. If structures $\mathcal{A}$ and $\mathcal{B}$ satisfy the same infinitary sentences of quantifier rank at most α we write $\mathcal{A} \equiv^{\alpha}_{\infty\omega} \mathcal{B}$.

Theorem 3.3.1 *For any structures $\mathcal{A}$ and $\mathcal{B}$ and an ordinal α,*

$$II(\mathcal{A}, \mathcal{B}, \alpha) \text{ if and only if } \mathcal{A} \equiv^{\alpha}_{\infty\omega} \mathcal{B}.$$

Proof: By induction on α. The case for α a limit ordinal is trivial, while the case for α a successor ordinal follows exactly as in the proof of Theorem 2.2.6. Now, however, we do not need Lemma 2.2.5 since infinite conjunctions are allowed. □

Corollary 3.3.2 *For any structures $\mathcal{A}$ and $\mathcal{B}$,*

$$II(\mathcal{A}, \mathcal{B}, \infty) \text{ if and only if } II^{\infty}(\mathcal{A}, \mathcal{B}).$$

Proof: Note that $II(\mathcal{A}, \mathcal{B}, \infty)$ if and only if $II(\mathcal{A}, \mathcal{B}, \alpha)$ for every ordinal α. Also, $\mathcal{A} \equiv_{\infty\omega} \mathcal{B}$ if and only if $\mathcal{A} \equiv^{\alpha}_{\infty\omega} \mathcal{B}$ for every ordinal α. The result now follows from Theorem 3.2.4 and Theorem 3.3.1. $\square$

3.4 The Logic $L(Q_\alpha)$

Let ω_α be an infinite cardinal. The logic $L(Q_\alpha)$ extends first-order logic by allowing quantification of the form "there exist at least ω_α elements satisfying ...". Formally, $L(Q_\alpha)$ is obtained from first-order logic by adding:

1. the symbol Q_α,
2. the rule: if β is a formula and x a variable then $Q_\alpha x\beta$ is a formula,
3. the semantics: $\mathcal{A} \models_v Q_\alpha x\beta$ iff there exists a set $B \subseteq A$ of cardinality at least ω_α such that $\mathcal{A} \models_{v'} \beta$ for every assignment v' such that $v'(x) \in B$ and otherwise v' agrees with v.

The game that characterizes equivalence with respect to this logic is the game $E^{Q_\alpha}(\mathcal{A}, \mathcal{B}, n)$, played as follows: In each round Player I chooses either an element c or a set X of at least ω_α elements of one structure. If Player I chooses a single element, Player II must reply with an element from the other structure. Otherwise Player II must choose a set Y of at least ω_α elements of the other structure, Player I chooses an element d of Y and Player II then chooses an element c of X. The pair $\langle c, d\rangle$ (or $\langle d, c\rangle$) is then recorded as the move. The condition for Player II to win is then the same as for $E(\mathcal{A}, \mathcal{B}, n)$.

The situation that Player II has a winning strategy is of course denoted by $II^{Q_\alpha}(\mathcal{A}, \mathcal{B}, n)$. We also write $\mathcal{A} \equiv^{Q_\alpha} \mathcal{B}$ if $\mathcal{A}$ and $\mathcal{B}$ have the same true $L(Q_\alpha)$-sentences and $\mathcal{A} \equiv^{Q_\alpha}_n \mathcal{B}$ if $\mathcal{A}$ and $\mathcal{B}$ have the same true $L(Q_\alpha)$-sentences of quantifier rank at most n, where we add the following clause to the definition of quantifier rank:

$qr(Q_\alpha x\beta) = qr(\beta) + 1$.

Example: Consider the game $E^{Q_0}(\omega + \omega^*, \omega + \zeta + \omega^*, 3)$. Player I wins by first choosing an element x of ζ in $\omega + \zeta + \omega^*$. If Player II responds with an element x of ω, then Player I chooses ω in $\omega + \zeta + \omega^*$ and Player II cannot find a countable set of elements in $\omega + \omega^*$ all of which are less than x. Similarly if Player II's first choice is in ω^*.

The formula corresponding to Player I's winning strategy is $\exists x(Q_0 y(y < x) \wedge Q_0 y(x < y))$, stating that there exists an element with at least countably many predecessors and at least countably many successors. $\square$

Theorem 3.4.1 *Let $\mathcal{A}$ and $\mathcal{B}$ be any structures of a finite signature containing no function symbols. Then*

$$II^{Q_\alpha}(\mathcal{A},\mathcal{B},n) \textit{ if and only if } \mathcal{A} \equiv_n^{Q_\alpha} \mathcal{B}.$$

Proof: For a set X and a cardinal κ we write $\mathcal{P}_{\geq\kappa}(X)$ for the set of subsets of X of cardinality at least κ. Again, the proof is similar to the proof of Ehrenfeucht's theorem, with Lemma 2.2.3 becoming:

$II^{Q_\alpha}(\mathcal{A},\mathcal{B},n+1)$ iff

1. $\forall a \in A\ \exists b \in B\ II^{Q_\alpha}(\langle\mathcal{A},a\rangle,\langle\mathcal{B},b\rangle,n)$.
2. $\forall b \in B\ \exists a \in A\ II^{Q_\alpha}(\langle\mathcal{A},a\rangle,\langle\mathcal{B},b\rangle,n)$.
3. $\forall C \in \mathcal{P}_{\geq\omega_\alpha}(A)\ \exists D \in \mathcal{P}_{\geq\omega_\alpha}(B)\ \forall b \in D\ \exists a \in C\ II^{Q_\alpha}(\langle\mathcal{A},a\rangle,\langle\mathcal{B},b\rangle,n)$.
4. $\forall D \in \mathcal{P}_{\geq\omega_\alpha}(B)\ \exists C \in \mathcal{P}_{\geq\omega_\alpha}(A)\ \forall a \in C\ \exists b \in D\ II^{Q_\alpha}(\langle\mathcal{A},a\rangle,\langle\mathcal{B},b\rangle,n)$.

and the addition of the case $\phi = Q_\alpha x\beta$ which we treat as follows:

Assume the result for n. Suppose that $II^{Q_\alpha}(\mathcal{A},\mathcal{B},n+1)$ and that $\mathcal{A} \models \phi'$ iff $\mathcal{B} \models \phi'$ for all sentences ϕ' of quantifier rank at most n. Let $\mathcal{A} \models Q_\alpha x\phi(x)$ with $qr(\phi(x)) \leq n$. Then there is a set $C \in \mathcal{P}_{\geq\omega_\alpha}(A)$ such that $\langle\mathcal{A},a\rangle \models \phi(\mathbf{c})$ for every $a \in C$, where $\mathbf{c}$ is the new constant symbol interpreted by a. By the variation of Lemma 2.2.3 above there is $D \in \mathcal{P}_{\geq\omega_\alpha}(B)$ such that $\forall b \in D\ \exists a \in C\ II^{Q_\alpha}(\langle\mathcal{A},a\rangle,\langle\mathcal{B},b\rangle,n)$. By the induction hypothesis this is equivalent to $\forall b \in D\ \exists a \in C\ \langle\mathcal{A},a\rangle \equiv_n^{Q_\alpha} \langle\mathcal{B},b\rangle$, hence for every $b \in D$ $\langle\mathcal{B},b\rangle \models \phi(\mathbf{c})$ since $\langle\mathcal{A},a\rangle \models \phi(\mathbf{c})$ for every $a \in C$. Therefore $\mathcal{B} \models Q_\alpha x\phi(x)$.

Next, assume that $\mathcal{A} \equiv_{n+1}^{Q_\alpha} \mathcal{B}$. Let $C \in \mathcal{P}_{\geq\omega_\alpha}(A)$ and suppose that there is no $D \in \mathcal{P}_{\geq\omega_\alpha}(B)$ such that $\forall b \in D\ \exists a \in C\ II^{Q_\alpha}(\langle\mathcal{A},a\rangle,\langle\mathcal{B},b\rangle,n)$. Then for every $D \in \mathcal{P}_{\geq\omega_\alpha}(B)$ there exists $b \in D$ such that for every $a \in C$, not $II^{Q_\alpha}(\langle\mathcal{A},a\rangle,\langle\mathcal{B},b\rangle,n)$. Using the induction hypothesis, for every $D \in \mathcal{P}_{\geq\omega_\alpha}(B)$ there exists $b \in D$ such that for every $a \in C$ there is a sentence $\phi_{D,a}(\mathbf{c})$ with $\langle\mathcal{A},a\rangle \models \phi_{D,a}(\mathbf{c})$ and $\langle\mathcal{B},b\rangle \not\models \phi_{D,a}(\mathbf{c})$. Again, up to equivalence, the set $\{\phi_{D,a}(\mathbf{c}) : D \in \mathcal{P}_{\geq\omega_\alpha}(B), a \in C\}$ is finite. Then the quantifier rank at most $n+1$ sentence $Q_\alpha x \bigwedge_{D\subseteq B, a\in C} \phi_{D,a}(x)$ is satisfied in $\mathcal{A}$ but not in $\mathcal{B}$.

Therefore the last two conditions of the back-and-forth lemma hold. The first two follow exactly as in the proof of Theorem 2.2.6, hence $II^{Q_\alpha}(\mathcal{A},\mathcal{B},n+1)$. □

3.5 k-Pebble Games

For $k < \omega$ the logic L_k is the restriction of first-order logic obtained by allowing only k variables $x_1,\ldots,x_k$.

The version of Ehrenfeucht's game corresponding to L_k is Immerman's k-pebble game, which is played as follows: Players I and II are each given k pebbles, marked $1,\ldots,k$. Player I now moves by placing one of his pebbles on an element of either structure. Player II then has to place his corresponding pebble on an element of the other structure. The players are allowed to move their own pebbles from one element to another. The game ends after n rounds and Player II wins if at every stage of the game the relation determined by the pebbles is a local isomorphism.

This game will be denoted by $E^k(\mathcal{A}, \mathcal{B}, n)$, L_k-equivalence by $\mathcal{A} \equiv^k \mathcal{B}$ and n-equivalence by $\mathcal{A} \equiv^k_n \mathcal{B}$.

Definition: A *position* in the game $E^k(\mathcal{A}, \mathcal{B}, n)$ is a relation $g = \{\langle a_1, b_1\rangle, \ldots, \langle a_j, b_j\rangle\} \subseteq A \times B$ with $j \leq k$. The n-round k-pebble game on $\mathcal{A}$ and $\mathcal{B}$ *starting from position* g is played like $E^k(\mathcal{A}, \mathcal{B}, n)$, with the two pebbles labelled i placed on a_i and b_i before the start of the game, for all i with $1 \leq i \leq j$. If Player II has a winning strategy for this game we write $II^k(\mathcal{A}, \mathcal{B}, g, n)$.

Lemma 3.5.1 *$II^k(\mathcal{A}, \mathcal{B}, g, n+1)$ iff for every i, $1 \leq i \leq k$,*

1. $\forall a \in A \ \exists b \in B \ II^k(\mathcal{A}, \mathcal{B}, (g \setminus \langle a_i, b_i\rangle) \cup \langle a, b\rangle, n)$.
2. $\forall b \in B \ \exists a \in A \ II^k(\mathcal{A}, \mathcal{B}, (g \setminus \langle a_i, b_i\rangle) \cup \langle a, b\rangle, n)$. □

Theorem 3.5.2 *Let $\mathcal{A}$ and $\mathcal{B}$ be σ-structures with σ finite and with no function symbols. Then, for any $n, k < \omega$*

$$II^k(\mathcal{A}, \mathcal{B}, n) \text{ if and only if } \mathcal{A} \equiv^k_n \mathcal{B}.$$

Proof: We prove that $II^k(\mathcal{A}, \mathcal{B}, \{\langle a_1, b_1\rangle, \ldots, \langle a_j, b_j\rangle\}, n)$ if and only if for every L_k-formula $\phi(x_1, \ldots, x_j)$ of rank at most n, $\mathcal{A} \models \phi(a_1, \ldots, a_j)$ iff $\mathcal{B} \models \phi(b_1, \ldots, b_j)$. The result is then the special case with $j = 0$.

The proof is by induction on n, for fixed $\mathcal{A}$, $\mathcal{B}$ and k. For $n = 0$ this follows directly from the definitions.

Assume that the statement holds for n and suppose first that $II^k(\mathcal{A}, \mathcal{B}, g, n+1)$. We prove that $\mathcal{A} \models \phi(a_1, \ldots, a_j)$ iff $\mathcal{B} \models \phi(b_1, \ldots, b_j)$ by induction on ϕ. For atomic ϕ, negations and conjunctions this is trivial.

Assume therefore that $\phi = \exists x_i \psi$, with $qr(\psi) \leq n$. Then ϕ has fewer than k free variables, since x_i does not occur free in ϕ. Hence we must have $j < k$ and, up to equivalence, we may assume that $i = j+1$ and $\psi = \psi(x_1, \ldots, x_j, x_{j+1})$. Now suppose that $\mathcal{A} \models \phi(a_1, \ldots, a_j)$. Then $\mathcal{A} \models \psi(a_1, \ldots, a_j, a)$ for some $a \in A$. By Lemma 3.5.1 there is a $b \in B$ such that $II^k(\mathcal{A}, \mathcal{B}, g \cup \langle a, b\rangle, n)$. By the induction hypothesis, $\mathcal{A} \models \psi(a_1, \ldots, a_j, a)$ if and only if $\mathcal{B} \models \psi(b_1, \ldots, b_j, b)$, hence $\mathcal{B} \models \phi(b_1, \ldots, b_j)$.

For the converse, suppose that for every L_k-formula $\phi(x_1, \ldots, x_j)$ of rank at most n, $\mathcal{A} \models \phi(a_1, \ldots, a_j)$ iff $\mathcal{B} \models \phi(b_1, \ldots, b_j)$. We show that the two conditions in Lemma 3.5.1 hold. Suppose condition 1 fails. We may assume that $j < k$. Then, using the induction hypothesis, there is an $a \in A$ such that for every $b \in B$ there is a formula $\phi_b(x_1, \ldots, x_j, x_{j+1})$ such that $\mathcal{A} \models \phi_b(a_1, \ldots, a_j, a)$ and $\mathcal{B} \not\models \phi_b(b_1, \ldots, b_j, b)$. By Lemma 2.2.5 the set $\{\phi_b : b \in B\}$ is finite, up to equivalence. Let ψ be the L_k-formula $\exists x_{j+1} \bigwedge\{\phi_b : b \in B\}$. Now $\mathcal{A} \models \psi(a_1, \ldots, a_j)$ and $\mathcal{B} \not\models \psi(b_1, \ldots, b_j)$, contradicting the assumption. The proof of the second condition is analogous. $\square$

Example 3.5.3 *If $\mathcal{A}$ and $\mathcal{B}$ are linear orderings with the same valid 3-variable sentences of quantifier rank at most n then $\mathcal{A} \equiv_n \mathcal{B}$.*

Proof: We show, using induction on n, that if Player II has a winning strategy in the 3-pebble game at position h then he has a winning strategy in the ordinary game at position h.

The case $n = 0$ is trivial. Assume therefore the result for n and suppose that Player II has a winning strategy in the game $E^k(\mathcal{A}, \mathcal{B}, n)$ at position h. There are two cases.

1. At position h at most four pebbles have been placed: Then each player has at least one free pebble. Thus, for every $a \in A$ there exists $b \in B$ and for every $b \in B$ there exists $a \in A$ such that Player II has a winning strategy in the 3-pebble game of length n at position $h \cup \{\langle a, b\rangle\}$. By the induction hypothesis, for every $a \in A$ there exists $b \in B$ and for every $b \in B$ there exists $a \in A$ such that Player II has a winning strategy in the ordinary game of length n at position $h \cup \{\langle a, b\rangle\}$. This means that Player II has a winning strategy in the ordinary game of length $n + 1$ at position h.

2. At position h all six pebbles have been used: Let $h = \{\langle a_0, b_0\rangle, \langle a_1, b_1\rangle, \langle a_2, b_2\rangle\}$ with $a_0 < a_1 < a_2$ and $b_0 < b_1 < b_2$. Then Player II has winning strategies for the two 3-pebble games of length $n + 1$ at the 2-pebble positions $\{\langle a_0, b_0\rangle, \langle a_1, b_1\rangle\}$ and $\{\langle a_1, b_1\rangle, \langle a_2, b_2\rangle\}$. The argument in case 1 shows that Player II has winning strategies S and T, respectively, in the ordinary games of length $n+1$ at positions $\{\langle a_0, b_0\rangle, \langle a_1, b_1\rangle\}$ and $\{\langle a_1, b_1\rangle, \langle a_2, b_2\rangle\}$. But then Player II has a winning strategy in the ordinary game of length $n + 1$ at position h: moves $< a_1$ or $< b_1$ are countered using S, while moves $> a_1$ or $> b_1$ are countered using T. $\square$

3.6 Other Games

All the games discussed so far have their unnested analogues, with corresponding results similar to Theorem 2.5.4. There are also numerous other games, which we will not discuss in detail. A few examples are given below.

1. In (full) second-order logic quantification over arbitrary relations is allowed. In the corresponding version of Ehrenfeucht's game Player I is also allowed to pick any relation on either structure. Player II must then respond with a relation of the same arity on the other structure.

2. The game $E^h(\mathcal{A}, \mathcal{B}, n)$ is defined exactly like $E(\mathcal{A}, \mathcal{B}, n)$ except that Player II wins if the play constitutes a *local homomorphism*, which is a relation $\{\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle\} \subseteq A \times B$ such that every atomic sentence satisfied by $\langle \mathcal{A}, a_1, \ldots, a_n\rangle$ is also satisfied by $\langle \mathcal{B}, b_1, \ldots, b_n\rangle$.

   This game relates to *positive* formulae, which are constructed from atomic formulae using only $\wedge$,$\vee$, $\forall$ and $\exists$. Player II has a winning strategy for the game $E^h(\mathcal{A}, \mathcal{B}, n)$ if and only if $\mathcal{B}$ satisfies every positive sentence of quantifier rank at most n satisfied by $\mathcal{A}$.

3. Another version of the game is obtained from $E^\infty(\mathcal{A}, \mathcal{B})$ by insisting that Player I always choose from $\mathcal{A}$ (and Player II always choose from $\mathcal{B}$). If $\mathcal{A}$ is at most countable then Player II has a winning strategy for this game if and only if $\mathcal{A}$ is embeddable in $\mathcal{B}$.

4. Hausdorff's game on a linear ordering $\mathcal{A}$ is played as follows. First Player I chooses an element a_1 of $\mathcal{A}$, so that $\mathcal{A} = \mathcal{B}_1 + a_1 + \mathcal{B}_2$. Player II then chooses one of $\mathcal{B}_1$ and $\mathcal{B}_2$, call it $\mathcal{B}$. Player I then chooses an element a_2 from $\mathcal{B}$, so that $\mathcal{B} = \mathcal{C}_1 + a_2 + \mathcal{C}_2$. Player II then chooses one of $\mathcal{C}_1$ and $\mathcal{C}_2$, etc. If Player II chooses an empty interval the game ends and he wins. It turns out that Player II has a winning strategy for this game if and only if $\mathcal{A}$ is *scattered*, that is $\mathcal{A}$ does not embed the ordering of the rationals.

Chapter 4

Modal Logic, Bisimulation and Games

This chapter introduces modal logic and the well-known characterization of logical equivalence in terms of bisimulations. In terms of processes, bisimulations characterize operational equivalence. Two processes are considered equivalent if any action that the one can carry out can be simulated by the other, i.e. the other process can perform some action with an equivalent outcome. As many authors have noted, bisimulations are essentially game-theoretic. In Section 3 we make this correspondence explicit. We also introduce bounded bisimulations and corresponding games, with a result similar to Theorem 3.3.1. In Section 4 we see how bisimulations can be obtained as fixed points of operators and how bounded bisimulations can be seen as approximations to these fixed points.

Sections 2 and 3 are based on [10] and Section 4 on [11]. For more background on modal logic see also [10].

4.1 Modal Logic

Definition: Let Φ be a set of propositional variables and I an index set. The formulae of *infinitary modal logic*, L_∞, are defined by the clauses:

1. p is a formula for every $p \in \Phi$,
2. if A is a formula then $\neg A$ is a formula and, for each $i \in I$, $\Diamond_i A$ is a formula,
3. if Δ is a set of formulae then $\bigwedge \Delta$ is a formula.

Definition: A *Kripke frame* is a pair $\langle W, \{R_i : i \in I\}\rangle$ where W is a nonempty set and R_i is a binary relation on W for every $i \in I$. A *valuation* is a function V that assigns a subset of W to every $p \in \Phi$. A *Kripke model* is a pair $\langle \mathcal{F}, V\rangle$ where $\mathcal{F}$ is a Kripke frame and V is a valuation. A *pointed Kripke model* is a pair $\langle \mathcal{M}, w\rangle$ where $\mathcal{M}$ is a Kripke model and $w \in \mathcal{M}$.

Definition: The definitions of truth in a model are:

1. $\langle \mathcal{M}, w\rangle \models p$ iff $w \in V(p)$,

2. $\langle \mathcal{M}, w\rangle \models \neg A$ iff $\langle \mathcal{M}, w\rangle \not\models A$,

3. $\langle \mathcal{M}, w\rangle \models \bigwedge \Delta$ iff $\langle \mathcal{M}, w\rangle \models A$ for all $A \in \Delta$,

4. $\langle \mathcal{M}, w\rangle \models \Diamond_i A$ iff there is a v such that wR_iv and $\langle \mathcal{M}, w\rangle \models A$.

We will sometimes abuse the notation and write $V(w)$ for the set $\{p \in \Phi : \langle \mathcal{M}, w\rangle \models p\}$.

Two pointed models $\langle \mathcal{M}, w\rangle$ and $\langle \mathcal{M}', w'\rangle$ are *equivalent*, denoted $\langle \mathcal{M}, w\rangle \equiv_\infty \langle \mathcal{M}', w'\rangle$, if for any L_∞-formula A, $\langle \mathcal{M}, w\rangle \models A$ if and only if $\langle \mathcal{M}', w'\rangle \models A$.

Definition: The *depth* of a modal formula A, denoted $d(A)$, is calculated as follows:

1. $d(p) = 0$ for $p \in \Phi$,

2. $d(\neg A) = d(A)$,

3. $d(\bigwedge \Delta) = sup\{d(A) : A \in \Delta\}$,

4. $d(\Diamond_i A) = d(A) + 1$.

For each ordinal α we denote by L_α the class of all L_∞-formulae of depth at most α. If $\langle \mathcal{M}, w\rangle$ and $\langle \mathcal{M}', w'\rangle$ satisfy the same L_α-formulae we write $\langle \mathcal{M}, w\rangle \equiv_\alpha \langle \mathcal{M}', w'\rangle$.

The classical modal logic is obtained by allowing finite conjunctions only. We denote this logic by L and L-equivalence between pointed models by $\equiv$.

In order to simplify notation we will consider languages with a single modal operator throughout the rest of this chapter. All results can, however, be generalized easily.

We let $V_\emptyset$ be the valuation with $V(p) = \emptyset$ for all $p \in \Phi$ and, throughout, $\mathcal{M}$ and $\mathcal{M}'$ will be the models $\langle W, R, V\rangle$ and $\langle W', R', V'\rangle$ respectively.

4.2 Bisimulation

Definition: A relation $C \subseteq W \times W'$ is a *bisimulation* between models $\langle W, R, V\rangle$ and $\langle W', R', V'\rangle$ if for all $w \in W$ and $w' \in W'$, if wCw' then:

1. $w \in V(p)$ iff $w' \in V'(p)$ for all $p \in \Phi$,

2. if wRv then there is a v' such that $w'R'v'$ and vCv',

3. if $w'R'v'$ then there is a v such that wRv and vCv'.

Two pointed models $\langle \mathcal{M}, w\rangle$ and $\langle \mathcal{M}', w'\rangle$ are *bisimilar*, $\langle \mathcal{M}, w\rangle \sim \langle \mathcal{M}', w'\rangle$, if there is a bisimulation C between $\mathcal{M}$ and $\mathcal{M}'$ such that wCw'.

Example: Let $\mathcal{M} = \langle 2, R, V_\emptyset\rangle$ and $\mathcal{M}' = \langle 1, R', V_\emptyset\rangle$ where R and R' are universal. Then $C = \{\langle 0,0\rangle, \langle 1,0\rangle\}$ is a bisimulation between $\mathcal{M}$ and $\mathcal{M}'$ hence $\langle \mathcal{M}, 0\rangle \sim \langle \mathcal{M}', 0\rangle$ and $\langle \mathcal{M}, 1\rangle \sim \langle \mathcal{M}', 0\rangle$.

Example: Let $\mathcal{M} = \langle \omega, <, V_\emptyset\rangle$ and $\mathcal{M}' = \langle 2, R', V_\emptyset\rangle$ where R' is the universal relation. Then the function f with $f(x) = 0$ if x is even and $f(x) = 1$ otherwise, viewed as a relation, is a bisimulation between $\mathcal{M}$ and $\mathcal{M}'$. In particular, $\langle \mathcal{M}, 0\rangle \sim \langle \mathcal{M}', 0\rangle$.

Example: Let $\mathcal{M}_1 = \langle W_1, R_1, V_1\rangle$ and $\mathcal{M}_2 = \langle W_2, R_2, V_2\rangle$ with $W_1 \cap W_2 = \emptyset$ and let $\mathcal{M}$ be their disjoint union, that is, $\mathcal{M} = \langle W_1 \cup W_2, R_1 \cup R_2, V\rangle$ where $V(p) = V_1(p) \cup V_2(p)$ for every $p \in \Phi$. Then $C = \{\langle x, x\rangle : x \in W_i\}$ is a bisimulation between $\mathcal{M}$ and $\mathcal{M}_i$, $i = 1, 2$.

Definition: Let α be an ordinal. We define an α- *bisimulation* between pointed models by induction on α. A relation $C \subseteq W \times W'$ is an α-bisimulation between $\langle \mathcal{M}, w\rangle$ and $\langle \mathcal{M}', w'\rangle$ if

1. wCw',
2. if vCv' then $v \in V(p)$ iff $v' \in V'(p)$ for all $p \in \Phi$,
3. for all $\beta < \alpha$, if wRv then there is a v' such that $w'R'v'$ and C is a β-bisimulation between $\langle \mathcal{M}, v\rangle$ and $\langle \mathcal{M}', v'\rangle$,
4. for all $\beta < \alpha$, if $w'R'v'$ then there is a v such that wRv and C is a β-bisimulation between $\langle \mathcal{M}, v\rangle$ and $\langle \mathcal{M}', v'\rangle$.

$\langle \mathcal{M}, w\rangle$ and $\langle \mathcal{M}', w'\rangle$ are called *α-bisimilar* if there is an α-bisimulation between them. We denote this by $\langle \mathcal{M}, w\rangle \sim_\alpha \langle \mathcal{M}', w'\rangle$.

Example: Let $\mathcal{M} = \langle \omega, S, V\rangle$ and $\mathcal{M}' = \langle n, S, V'\rangle$ with V' the restriction of V to n and S the successor relation, i.e. xSy iff $y = x + 1$. Then $\langle \mathcal{M}, 0\rangle \not\sim \langle \mathcal{M}', 0\rangle$ while $\{\langle x, x\rangle : x < n\}$ is an $n - 1$-bisimulation between $\langle \mathcal{M}, 0\rangle$ and $\langle \mathcal{M}', 0\rangle$. □

From the definitions we immediately obtain the following two lemmas.

Lemma 4.2.1 *Two pointed models $\langle \mathcal{M}, w\rangle$ and $\langle \mathcal{M}', w'\rangle$ are α-bisimilar if and only if*

1. *$w \in V(p)$ iff $w' \in V'(p)$ for all $p \in \Phi$,*
2. *for all $\beta < \alpha$, if wRv then there is a v' such that $w'R'v'$ and $\langle \mathcal{M}, v\rangle \sim_\beta \langle \mathcal{M}', v'\rangle$,*
3. *for all $\beta < \alpha$, if $w'R'v'$ then there is a v such that wRv and $\langle \mathcal{M}, v\rangle \sim_\beta \langle \mathcal{M}', v'\rangle$.* □

Lemma 4.2.2

1. $\langle \mathcal{M}, w\rangle \sim \langle \mathcal{M}', w'\rangle$ *if and only if* $\langle \mathcal{M}, w\rangle \sim_\alpha \langle \mathcal{M}', w'\rangle$ *for all ordinals* α.
2. *If* $\alpha \leq \beta$ *and* C *is a* β*-bisimulation between* $\langle \mathcal{M}, w\rangle$ *and* $\langle \mathcal{M}', w'\rangle$, *then* C *is an* α*-bisimulation between* $\langle \mathcal{M}, w\rangle$ *and* $\langle \mathcal{M}', w'\rangle$.
3. *If* α *is a limit ordinal then* $\langle \mathcal{M}, w\rangle \sim_\alpha \langle \mathcal{M}', w'\rangle$ *if and only if* $\langle \mathcal{M}, w\rangle \sim_\beta \langle \mathcal{M}', w'\rangle$ *for all* $\beta < \alpha$. □

Theorem 4.2.3 $\langle \mathcal{M}, w\rangle \equiv_\alpha \langle \mathcal{M}', w'\rangle$ *if and only if* $\langle \mathcal{M}, w\rangle \sim_\alpha \langle \mathcal{M}', w'\rangle$.

Proof: First we prove, by induction on A, that if $d(A) = \alpha$, $\langle \mathcal{M}, w\rangle \models A$ and $\langle \mathcal{M}, w\rangle \sim_\alpha \langle \mathcal{M}', w'\rangle$ then $\langle \mathcal{M}', w'\rangle \models A$.

The only nontrivial case is $\Diamond A$. Let $d(A) = \alpha$. Then $d(\Diamond A) = \alpha + 1$. Assume $\langle \mathcal{M}, w\rangle \sim_{\alpha+1} \langle \mathcal{M}', w'\rangle$ and $\langle \mathcal{M}, w\rangle \models \Diamond A$. Then there is a v such that wRv and $\langle \mathcal{M}, v\rangle \models A$. Since $\langle \mathcal{M}, w\rangle \sim_{\alpha+1} \langle \mathcal{M}', w'\rangle$ there is a v' such that $\langle \mathcal{M}, v\rangle \sim_\alpha \langle \mathcal{M}', v'\rangle$ and $w'R'v'$. By the induction hypothesis, $\langle \mathcal{M}', v'\rangle \models A$, hence $\langle \mathcal{M}', w'\rangle \models \Diamond A$.

We prove the converse by induction on α, using Lemma 4.2.1. The case $\alpha = 0$ is immediate.

Assume that $\langle \mathcal{M}, w\rangle \equiv_{\alpha+1} \langle \mathcal{M}', w'\rangle$ and suppose that wRv. Suppose that there is no v' such that $w'R'v'$ and $\langle \mathcal{M}, v\rangle \sim_\alpha \langle \mathcal{M}', v'\rangle$. Then, by the induction hypothesis, for every v' such that $w'R'v'$ there is a formula $A_{v'}$ such that $\langle \mathcal{M}, v\rangle \models A_{v'}$ and $\langle \mathcal{M}', v'\rangle \not\models A_{v'}$. Then $\langle \mathcal{M}, w\rangle \models \Diamond \bigwedge\{A_{v'} : w'R'v'\}$ but $\langle \mathcal{M}', w'\rangle \not\models \Diamond \bigwedge\{A_{v'} : w'R'v'\}$, contradicting our assumption.

Assume $\langle \mathcal{M}, w\rangle \equiv_\alpha \langle \mathcal{M}', w'\rangle$ with α a limit ordinal. Then $\langle \mathcal{M}, w\rangle \equiv_\beta \langle \mathcal{M}', w'\rangle$ for all $\beta < \alpha$ so by Lemma 4.2.2, $\langle \mathcal{M}, w\rangle \sim_\alpha \langle \mathcal{M}', w'\rangle$. □

Definition: The binary relation R of a model $\mathcal{M}$ is called *image-finite* if for every $w \in W$ the set $R(w) = \{v \in W : wRv\}$ is finite.

For image-finite models it turns out that bisimilarity is equivalent to ω-bisimilarity.

Theorem 4.2.4 *Let* $\langle \mathcal{M}, w\rangle$ *and* $\langle \mathcal{M}', w'\rangle$ *be image-finite. Then* $\langle \mathcal{M}, w\rangle \sim \langle \mathcal{M}', w'\rangle$ *if and only if* $\langle \mathcal{M}, w\rangle \sim_\omega \langle \mathcal{M}', w'\rangle$.

Proof: For the non-trivial direction we prove that $\langle \mathcal{M}, w\rangle \equiv \langle \mathcal{M}', w'\rangle$ implies $\langle \mathcal{M}, w\rangle \sim \langle \mathcal{M}', w'\rangle$. The result then follows from Theorem 4.2.3 and the fact that $L \subseteq L_\omega$. To do this we prove that $\equiv$ is a bisimulation. (Here we consider $\equiv$ to be a binary relation between W and W' in a natural way, i.e. $w \sim w'$ iff $\langle \mathcal{M}, w\rangle$ and $\langle \mathcal{M}', w'\rangle$ are bisimilar.)

Assume therefore that $w \equiv w'$. The first condition for bisimulation is immediate. Now let $w'R'v'$ and suppose there is no v such that wRv and $v \equiv v'$. Then, for every v such that wRv there is an L-formula A_v such that $\langle \mathcal{M}, v\rangle \models A_v$ and $\langle \mathcal{M}', v'\rangle \not\models A_v$. Then the formula $\Diamond \bigwedge\{a_v : v \in R(w)\}$ is an L-formula (since $R(w)$ is finite) true at w but not at v, contradicting the assumption.

The other condition follows similarly. $\square$

4.3 Ehrenfeucht Games

Bisimulation and equivalence between pointed models can also be characterized by Ehrenfeucht games. For every ordinal α we define the *Ehrenfeucht game bounded by* α by induction on α as follows:

The game is played by two players on two pointed models $\langle \mathcal{M}, w\rangle$ and $\langle \mathcal{M}', w'\rangle$. In the 0-game the players do nothing and Player II wins if and only if $V(w) = V(w')$.

In the $\alpha + 1$-game Player I must choose a v from $\langle \mathcal{M}, w\rangle$ or a v' from $\langle \mathcal{M}', w'\rangle$ such that wRv or $w'R'v'$ and Player II must reply with a v' or v from the other model such that wRv or $w'R'v'$ (depending on Player I's choice). The players then play the α-game on $\langle \mathcal{M}, v\rangle$ and $\langle \mathcal{M}', v'\rangle$. Player II wins if and only if $V(v) = V'(v')$ and he wins the α-game on $\langle \mathcal{M}, v\rangle$ and $\langle \mathcal{M}', v'\rangle$.

For α a limit ordinal Player I chooses an ordinal β less than α and the players play the β-game. Player II wins if and only if he wins this game.

Theorem 4.3.1 *Player II has a winning strategy for the α-game on $\langle \mathcal{M}, w\rangle$ and $\langle \mathcal{M}', w'\rangle$ if and only if $\langle \mathcal{M}, w\rangle \sim_\alpha \langle \mathcal{M}', w'\rangle$.*

Proof: The proof is by induction on α. For $\alpha = 0$ this is immediate.

Assume that Player II has a winning strategy for the $\alpha + 1$-game. Then, for each v with wRv there is a v' with $w'R'v'$ such that Player II has a winning strategy for the α-game on $\langle \mathcal{M}, v\rangle$ and $\langle \mathcal{M}', v'\rangle$, and the other way around. Then, by the induction hypothesis and the definition of $\alpha + 1$-bisimulation, $\langle \mathcal{M}, w\rangle \sim_{\alpha+1} \langle \mathcal{M}', w'\rangle$.

If α is a limit ordinal then Player II has a winning strategy for the α-game precisely when, for each $\beta < \alpha$, he has a winning strategy for the β-game. Then, by the induction hypothesis, $\langle \mathcal{M}, w\rangle \sim_\beta \langle \mathcal{M}', w'\rangle$ for every $\beta < \alpha$, hence $\langle \mathcal{M}, w\rangle \sim_\alpha \langle \mathcal{M}', w'\rangle$.

Now assume that $\langle \mathcal{M}, w\rangle \sim_{\alpha+1} \langle \mathcal{M}', w'\rangle$ and suppose Player I chooses v from $\langle \mathcal{M}, w\rangle$. Then wRv and, since $\langle \mathcal{M}, w\rangle \sim_{\alpha+1} \langle \mathcal{M}', w'\rangle$, Player II can find a v' such that $w'R'v'$ and $\langle \mathcal{M}, v\rangle \sim_\alpha \langle \mathcal{M}', v'\rangle$. By the induction hypothesis Player II has a winning strategy for the α-game on $\langle \mathcal{M}, v\rangle$ and $\langle \mathcal{M}', v'\rangle$, and we are finished.

If α is a limit ordinal and $\langle \mathcal{M}, w\rangle \sim_\alpha \langle \mathcal{M}', w'\rangle$ then $\langle \mathcal{M}, w\rangle \sim_\beta \langle \mathcal{M}', w'\rangle$ for every $\beta < \alpha$. By the induction hypothesis Player II has a winning strategy for each β-game with $\beta < \alpha$. Hence, whatever β Player I chooses, Player II wins the β-game and therefore wins the α-game. $\square$

Definition: For an ordinal α we let α^+ be the pointed model $\langle\langle\alpha+1, >, V_\emptyset\rangle, \alpha\rangle$.

Example 4.3.2 *Let α and β be ordinals with $\alpha < \beta$. Then Player II wins the α-game on α^+ and β^+ while Player I wins the $\alpha+1$-game.*

Proof: For the α-game, note that Player II wins as soon as he is able to copy Player I's moves, i.e. as soon as Player I chooses an element that is less than every element of the other model previously selected. The only way for Player I to avoid this is to select elements from β^+ only. However, Player II then wins by letting the ordinal selected by Player I be his choice of element in α^+ (in those rounds where Player I selects an ordinal, i.e. the rounds when the players play a δ-game for some limit ordinal δ, otherwise Player II selects the predecessor of his previous move).

For the $\alpha+1$-game Player I uses exactly the same strategy as for the game $E(\alpha, \beta, \alpha+1)$ (see Example 3.3). □

From this example we also see that the relations $\sim_\alpha$ are proper approximations to $\sim$.

Corollary 4.3.3 *For every ordinal α there exist models $\langle\mathcal{M}, w\rangle$ and $\langle\mathcal{M}', w'\rangle$ such that $\langle\mathcal{M}, w\rangle \sim_\alpha \langle\mathcal{M}', w'\rangle$ but $\langle\mathcal{M}, w\rangle \not\sim \langle\mathcal{M}', w'\rangle$.* □

We define the *unbounded game on $\langle\mathcal{M}, w\rangle$ and $\langle\mathcal{M}', w'\rangle$* as follows: Player I selects any ordinal α and the Players play the α-game.

Theorem 4.3.4 *Player II has a winning strategy for the unbounded game on $\langle\mathcal{M}, w\rangle$ and $\langle\mathcal{M}', w'\rangle$ iff $\langle\mathcal{M}, w\rangle \sim \langle\mathcal{M}', w'\rangle$ iff $\langle\mathcal{M}, w\rangle \equiv_\infty \langle\mathcal{M}', w'\rangle$.*

Proof: Immediate from Lemma 4.2.2, Theorem 4.3.1 and Theorem 4.2.3. □

Since, in general, L_ω is more expressive than L (see Example 4.3.7 below) we do not have a game-theoretic characterization of L-equivalence yet. We do however have the following.

Lemma 4.3.5 *If Φ is finite then, for any $\langle\mathcal{M}, w\rangle$ and $\langle\mathcal{M}', w'\rangle$, $\langle\mathcal{M}, w\rangle \equiv_\omega \langle\mathcal{M}', w'\rangle$ iff $\langle\mathcal{M}, w\rangle \equiv \langle\mathcal{M}', w'\rangle$.*

Proof: One direction is trivial. For the other direction assume that $\langle\mathcal{M}, w\rangle \equiv \langle\mathcal{M}', w'\rangle$. We prove by induction on A that if $A \in L_\omega$ and $\langle\mathcal{M}, w\rangle \models A$ then $\langle\mathcal{M}', w'\rangle \models A$.

The only nontrivial case is $A = \Diamond B$. In this case $d(A) < \omega$ hence it suffices to show that every L_ω-formula A of finite depth is equivalent to an L-formula. This follows by induction on A, using the fact that since Φ is finite there are, up to equivalence, only finitely many L_ω-formulae of depth n for every $n < \omega$. □

Using this lemma we can obtain a version of the game that characterizes L-equivalence. First we need a definition.

Definition: For a model $\mathcal{M}$ and a subset Ψ of Φ we denote by $\mathcal{M}|_\Psi$ the model $\langle W, R, V|_\Psi \rangle$ where $V|_\Psi(p) = V(p)$ if $p \in \Psi$ and $V|_\Psi(p) = \emptyset$ otherwise.

The *L-game on $\langle \mathcal{M}, w \rangle$ and $\langle \mathcal{M}', w' \rangle$* is played as follows: Player I selects a finite subset Ψ of Φ and the players play the ω-game on $\langle \mathcal{M}|_\Psi, w \rangle$ and $\langle \mathcal{M}'|_\Psi, w' \rangle$.

Theorem 4.3.6 *Player II has a winning strategy for the L-game on $\langle \mathcal{M}, w \rangle$ and $\langle \mathcal{M}', w' \rangle$ if and only if $\langle \mathcal{M}, w \rangle \equiv \langle \mathcal{M}', w' \rangle$.*

Proof: Suppose Player II has a winning strategy and let $A \in L$. Let Ψ be the set of atomic formulae occurring in A. If Player I chooses Ψ at the start of the game then Player II wins the ω-game on $\langle \mathcal{M}|_\Psi, w \rangle$ and $\langle \mathcal{M}'|_\Psi, w' \rangle$, hence $\langle \mathcal{M}|_\Psi, w \rangle \equiv \langle \mathcal{M}'|_\Psi, w' \rangle$. From the definition of Ψ it follows that $\langle \mathcal{M}, w \rangle \models A$ iff $\langle \mathcal{M}', w' \rangle \models A$.

Conversely, assume that $\langle \mathcal{M}, w \rangle \equiv \langle \mathcal{M}', w' \rangle$. Let Player I choose $\Psi \subseteq \Phi$. By the definition of $\langle \mathcal{M}|_\Psi, w \rangle$ and $\langle \mathcal{M}'|_\Psi, w' \rangle$ we can conclude from Lemma 4.3.5 that $\langle \mathcal{M}|_\Psi, w \rangle \equiv_\omega \langle \mathcal{M}'|_\Psi, w' \rangle$, so Player II wins. □

Example 4.3.7 *If Φ is infinite then the relations $\equiv$ and $\equiv_\omega$ are distinct.*

Proof: Set $\Phi = \{p_0, p_1, \ldots\}$. Let $\mathcal{M} = \langle \{-1\} \cup \omega, R, V \rangle$ and $\mathcal{M}' = \langle \{-2, -1\} \cup \omega, R', V' \rangle$ where $R = \{-1\} \times \omega$, $R' = \{-1\} \times (\{-2\} \cup \omega)$ and for every $p_i \in \Phi$, $V(p_i) = \{i, i+1, \ldots\}$ and $V'(p_i) = \{-2, i, i+1, \ldots\}$.

Player II wins the L-game on $\langle \mathcal{M}, -1 \rangle$ and $\langle \mathcal{M}', -1 \rangle$ since $\langle \mathcal{M}|_\Psi, -1 \rangle \sim \langle \mathcal{M}'|_\Psi, -1 \rangle$ for any finite subset Ψ of Φ. Therefore $\langle \mathcal{M}, -1 \rangle \equiv \langle \mathcal{M}', -1 \rangle$. However, $\langle \mathcal{M}, -1 \rangle \not\equiv_\omega \langle \mathcal{M}', -1 \rangle$. Player I wins by selecting -2. (The formula $\Diamond \bigwedge \{p_i : i < \omega\}$ distinguishes the two pointed models.) □

4.4 Bisimulations as Fixed Points

Definition: A mapping $F : 2^S \to 2^S$ defined on the powerset of some set S is called an *operator on S*. A subset T of S is a

1. *pre-fixed point* if $F(T) \subseteq T$,
2. *post-fixed point* if $T \subseteq F(T)$,
3. *fixed point* if $F(T) = T$.

If $F(T) \subseteq F(T')$ whenever $T \subseteq T'$ we call F *monotone*.

Theorem 4.4.1 *(Knaster-Tarski) Let F be a monotone operator on S. Then F has a least fixed point, namely $\mu(F) = \bigcap\{T \subseteq S : F(T) \subseteq T\}$ and a greatest fixed point, namely $\nu(F) = \bigcup\{T \subseteq S : T \subseteq F(T)\}$.* □

We now fix two models $\mathcal{M}$ and $\mathcal{M}'$. The relation $\sim$ (on the class of pointed models) naturally induces a binary relation between W and W'. Throughout this section we consider $\sim$ to be this binary relation, i.e. for all $w \in W$ and $w' \in W'$, $w \sim w'$ iff $\langle \mathcal{M}, w\rangle$ and $\langle \mathcal{M}', w'\rangle$ are bisimilar. (This viewpoint was exploited in the proof of Theorem 4.2.4.) Similarly we consider $\sim_\alpha$ to be a binary relation between W and W'.

For any binary relation $\rho \subseteq W \times W'$ we let $F(\rho)$ be the set of all pairs $\langle w, w'\rangle \in \rho$ that satisfy the three conditions in the definition of bisimulation. Then F is a monotone operator on $W \times W'$.

For any $\rho \subseteq W \times W'$ we have that ρ is a bisimulation if and only if $\rho \subseteq F(\rho)$. Hence $\nu(F) = \bigcup\{\rho \subseteq W \times W' : \rho \subseteq F(\rho)\}$ is a bisimulation between $\mathcal{M}$ and $\mathcal{M}'$ and so is the greatest bisimulation between $\mathcal{M}$ and $\mathcal{M}'$. Clearly, $\nu(F) = \sim$.

Definition: We define a downwards hierarchy of approximations to the greatest fixed point of F as follows:

1. $F\downarrow 0 = \{\langle w, w'\rangle \in W \times W' : V(w) = V'(w')\}$,
2. $F\downarrow \alpha + 1 = F(F\downarrow \alpha)$,
3. $F\downarrow \delta = \bigcap_{\alpha<\delta} F\downarrow \alpha$ for δ a limit ordinal.

(The standard definition for an operator F on a set S is $F\downarrow 0 = S$, but for our purposes the definition given here is more convenient.)

It turns out that these approximations are precisely the relations $\sim_\alpha$.

Theorem 4.4.2 *For every ordinal α, $F\downarrow \alpha = \sim_\alpha$.*

Proof: By induction on α, using Lemma 4.2.1. □

4.5 Related Topics

We conclude this chapter with references to a few related topics in modal logic:

1. In [16] C. Stirling defines games that characterize equivalence with respect to *modal μ-calculus*.

2. A well-known result of Kamp states that the temporal operators *Since* and *Until* are as expressive as first-order logic on linear time-frames. Hodkinson [15] gives another proof of this using games.

3. Bisimulations play an important role in characterizing modal definability of first-order formulae. Let σ be a signature consisting of a binary relation and countably many unary predicates. Note that a Kripke $\mathcal{M}$ model can be seen as a σ-structure by letting $\mathbf{p}_i^{\mathcal{M}} = V(p_i)$ for every unary predicate $\mathbf{p}_i$. For every modal formula ψ there is a first-order formula $\phi(x)$, called the *translation* of ψ, such that for any model $\langle \mathcal{M}, w \rangle$, $\langle \mathcal{M}, w \rangle \models \psi$ iff $\mathcal{M} \models \phi(w)$. We say that a first-order formula $\phi(x)$ is *invariant for bisimulation* if, for any bisimulation between pointed models $\langle \mathcal{M}, w \rangle$ and $\langle \mathcal{M}', w' \rangle$, $\mathcal{M} \models \phi(w)$ iff $\mathcal{M}' \models \phi(w')$. The following result is due to Van Benthem [17].

Theorem 4.5.1 *A first-order formula ϕ in one free variable is equivalent to the translation of a modal formula if and only if ϕ is invariant for bisimulation.* $\square$

Chapter 5

Some Other Topics

Our main interest was game-theoretic characterizations of logical equivalence in various languages. We now turn our attention to a more general view of games and some other applications of game theory in logic.

Section 1 is from [14], Section 2 is based on [13] and Section 3 on [12].

5.1 Closed Games

All the games we have dealt with thus far belong to a class of games known as closed games. In this section we consider the general notion of a closed game and we show how Ehrenfeucht games fit into this category. We also give a proof that every closed game is determined.

Definition: A two-player *closed game* of perfect information and of length ω has the following form.

The game consists of:

1. a partitioning of ω into two sets M_I and M_{II},
2. a set X,
3. for each $i < \omega$, a set W_i of ordered i-tuples of elements of X.

At the ith step ($i < \omega$), one of the players must choose an element of X. If $i \in M_I$ then Player I makes the choice, otherwise Player II chooses. After ω moves, a sequence $\bar{x} = \langle x_0, x_1, \ldots \rangle$ of elements of X has been chosen. This sequence is called a *play* of the game G. Player II *wins* the play $\bar{x}$ if

$$\text{for every } i < \omega \text{ the } i\text{-tuple } \bar{x}|i \text{ is in } W_i.$$

Otherwise Player I wins the play. There may also be further restrictions on which elements a player is allowed to choose.

Example 5.1.1 *Well-founded partial orders.*

Put $M_{II} = \omega$, let X be a partially ordered set and let W_i be the set of all i-tuples $\langle x_0, \ldots, x_{i-1}\rangle$ such that $x_0 > x_1 \ldots > x_{i-1}$. This defines a closed game G. The game consists of Player II choosing an infinite sequence of elements of X and he wins if and only if the sequence is strictly decreasing. Player I never moves, which is a winning strategy if and only if X is well-founded.

Example 5.1.2 *Svenonius games.*

Let σ be a signature, and for each $n < \omega$ let $\phi_n(x_0, \ldots, x_{n-1})$ be a first-order σ-formula. Let ϕ be the expression

$$\forall x_0 \exists x_1 \forall x_2 \exists x_3 \ldots \textstyle\bigwedge_{n<\omega} \phi_n.$$

For every σ-structure $\mathcal{A}$, ϕ defines a closed game $Sv(\phi, \mathcal{A})$ (called a *Svenonius game*) by setting

$M_I = \{0, 2, 4, \ldots\}$,

$X = A$,

$W_n = \{(a_0, \ldots, a_{n-1}) \in A^n : \mathcal{A} \models \phi_n(a_0, \ldots, a_{n-1})\}$.

We write $\mathcal{A} \models \phi$ if Player II has a winning strategy for the game $Sv(\phi, \mathcal{A})$. The expression ϕ is called a *Svenonius sentence.*

Example 5.1.3 *Ehrenfeucht games.*

The game $E^\infty(\mathcal{A}, \mathcal{B})$ is a closed game. We could give it in the exact format above, but it is more convenient to count the two moves making up a round as a single step. Each W_n is the set of all $2n$-tuples $\langle x_0, \ldots, x_{2n-1}\rangle$ such that for each $m < n$, if x_{2m} is from A then x_{2m+1} is from B and vice versa, and if we list the choices from A in order as $a_0, \ldots, a_{n-1}$ and those from B as $b_0, \ldots, b_{n-1}$ then $\langle \mathcal{A}, a_0, \ldots, a_{n-1}\rangle \equiv_0 \langle \mathcal{B}, b_0, \ldots, b_{n-1}\rangle$.

The game $E(\mathcal{A}, \mathcal{B}, n)$ and variations thereof can similarly be seen to be closed.

Definition: If $\bar{x}$ is a play of a closed game G then for every $n < \omega$ $\bar{x}|n$ is a *position* of G. In other words a position is a tuple of possible moves, starting at the beginning of the game.

A *strategy* for a player is a function S from the set of all positions in which it is that player's move, into X. A strategy is *winning* for a player if he always wins using that strategy, regardless of his opponent's moves.

A position $\bar{a}$ is *winning* for a player if he has a strategy that guarantees him to win once the play has reached $\bar{a}$. For example the initial position $\langle\rangle$ is winning for Player II if and only if Player II has a winning strategy for the game G.

We rank the positions of a game G according to how near Player II is to winning. The rank of a position is either -1, an ordinal or ∞. ∞ is considered to be greater than every ordinal. The *rank*, $rk(\bar{x})$, of a position $\bar{x}$ of length n is defined as follows:

1. $rk(\bar{x}) \geq 0$ iff $\bar{x}|m \in W_m$ for all $m \leq n$,
2. $rk(\bar{x}) \geq \alpha + 1$ iff Player II can ensure that after the next choice y, $rk(\bar{x}y) \geq \alpha$,
3. for a limit ordinal δ, $rk(\bar{x}) \geq \delta$ iff for all $\alpha < \delta$, $rk(\bar{x}) \geq \alpha$.

Condition 2 states that $rk(\bar{x}y) \geq \alpha$ for all y if the choice of y belongs to Player I (i.e. $n \in M_I$) and that $rk(\bar{x}y) \geq \alpha$ for some y otherwise.

Lemma 5.1.4 *If α is an ordinal such that every position of rank $\geq \alpha$ is also of rank $\geq \alpha+1$, then every position of rank $\geq \alpha$ is winning for Player II.*

Proof: Suppose $\bar{x} = \langle x_0, \ldots, x_{n-1}\rangle$ is a position of rank $\geq \alpha$. By assumption $\bar{x}$ also has rank $\geq \alpha + 1$ and so Player II can ensure that $\bar{x}x_n$ has rank $\geq \alpha$ too. Repeating the argument, Player II can ensure that $\langle x, \ldots, x_{i-1}\rangle$ has rank $\geq \alpha$ for each i with $n \leq i < \omega$. Therefore, if $\bar{z}$ is the resulting play, $rk(\bar{z}|i) \geq 0$ for each i, $n \leq i < \omega$. It then follows that $\bar{z}|i \in W_i$ for each i, so Player II wins. □

Theorem 5.1.5 *A position $\bar{x}$ has rank ∞ if and only if it is winning for Player II.*

Proof: Suppose $\bar{x}$ has rank ∞. Put $\alpha = sup\{rk(\bar{y}) + 1 : \bar{y}$ is a position of rank $< \infty\}$. Such an ordinal exists since the collection of positions is a set. By Lemma 5.1.4 every position of rank $\geq \alpha$ is winning for Player II.

Conversely we show by induction on β that if $\bar{x}$ is a winning position for Player II and β is an ordinal, then $rk(\bar{x}) \geq \beta$. The case when β is a limit ordinal follows directly from the definition of rank, so assume that $\beta = \gamma + 1$. Let S be a winning strategy for Player II which is winning from $\bar{x}$ onwards and imagine that the game continues according to S. Then the position $\bar{x}x_n$ is still winning for Player II, regardless of Player I's choice. By the induction hypothesis, Player II can ensure that $rk(\bar{x}x_n) \geq \gamma$. Then $rk(\bar{x}) \geq \beta$ by 2 of the definition of rank. □

Corollary 5.1.6 *(Gale-Stewart theorem) Every closed game is determined.*

Proof: If the initial position has rank ∞ then, by Theorem 5.1.5, Player II has a winning strategy. Suppose therefore that $rk(\langle\rangle) < \infty$. From 2 of the definition, Player II can never increase the rank, while Player I can decrease the rank so long as it is ≥ 0 (condition 2 implies that if $rk(\bar{x}) < \infty$ then $rk(\bar{x}y) < rk(\bar{x})$ for every y). Since the ordinals are well-ordered, Player I can decrease the rank to -1 in finitely many steps, and so he wins. □

There is an interesting axiom called the *Axiom of Determinacy* which states that every game is determined. As the next result shows this contradicts the Axiom of Choice.

Theorem 5.1.7 *(Assuming the Axiom of Choice.) Not every game is determined.*

Proof: The two-player game $G(X)$ where X is a set of infinite binary sequences (that is, $X \subseteq {}^{\omega}2$) is played as follows: Player I chooses an element x_0 of 2, then Player II chooses x_1, then Player I chooses x_2, etc. After ω steps this yields a sequence $\bar{x} = \langle x_0, x_1, \ldots \rangle$. Player II wins if and only if $\bar{x} \in X$. We show that X can be chosen in such a way that neither player has a winning strategy.

List all possible strategies for either player as $\langle S_i : i < 2^{\omega} \rangle$ (this is where we use the Axiom of Choice). This is possible since the set of all strategies is a subset of the set ${}^{<\omega}2 \times 2$, which has cardinality 2^{ω} (where ${}^{<\omega}2$ is the set of finite binary sequences).

If S is a strategy for one of the players we denote by $D(S)$ the set of all infinite sequences $\bar{y}$ such that there is a strategy T for the other player that will result in the play being $\bar{y}$ if the players play according to S and T. Note that if S is a strategy for Player I and $\bar{y} \in X$ then S is not winning. If S is a strategy for Player II and $\bar{y} \notin X$ then S is not winning.

Given a sequence $\langle x_1, x_3, x_5, \ldots \rangle$ in ${}^{\omega}2$ we define a sequence $\bar{y} = \langle y_0, y_1, \ldots \rangle$ inductively by setting $y_i = x_i$ if i is odd and $y_i = S(\langle y_0, y_1, \ldots, y_{i-1} \rangle)$ if i is even. Then $\bar{y} \in D(S)$. This shows that $D(S)$ has cardinality 2^{ω} if S is a strategy for Player I. Similarly if S is a strategy for Player II.

By induction on i we construct a sequence $\langle \bar{y}_i : i \in 2^{\omega} \rangle$ in ${}^{\omega}2$ such that no element occurs more than once.

Once $\bar{y}_j$ has been defined for every $j < i$ we let $\bar{y}_i$ be any element of $D(S_i) \setminus \{\bar{y}_j : j < i\}$. Such an element exists since $D(S_i)$ has cardinality 2^{ω} while $\{\bar{y}_j : j < i\}$ has cardinality less than 2^{ω} (since $i < 2^{\omega}$ and 2^{ω} is a cardinal).

Now set $X = \{\bar{y}_i : S_i \text{ is a strategy for Player I}\}$. Then, for every strategy S, if S is a strategy for Player I then X contains an element of $D(S)$ and if S is a strategy for Player II then some element of $D(S)$ is not in X. Therefore the game $G(X)$ is not determined. □

5.2 Forcing

Forcing is a method of building structures with prescribed properties introduced by Cohen when in 1963 he proved his famous result on the independence of the continuum hypothesis. We now give a brief introduction to Hodges' game-theoretic approach to forcing. To give an idea of what the method is about we will outline a proof of the well-known Compactness theorem, for countable signatures. (See [13] for more.)

We now fix a countable signature σ', a set $W = \{\mathbf{c}_i : i < \omega\}$ of new constant symbols, called *witnesses*, and we let $L(W)$ be the first order language with signature $\sigma = \sigma' \cup W$.

Definition: A *notion of forcing for* $L(W)$ is a set N of sets of σ-sentences such that the following hold for every $p \in N$:

1. for every closed term t, $p \cup \{t = t\} \in N$,
2. for every atomic formula $\phi(x)$ and all closed terms s and t, if $\phi(s) \in p$ and p contains at least one of $s = t$ and $t = s$ then $p \cup \{\phi(t)\} \in N$,
3. for every atomic sentence ϕ, ϕ and $\neg\phi$ are not both in p,
4. for every sentence ϕ, if $\neg\neg\phi \in p$ then $p \cup \{\phi\} \in N$.

   For all sentences ϕ and ψ,
5. if $\phi \wedge \psi \in p$ then $p \cup \{\phi, \psi\} \in N$,
6. if $\neg(\phi \wedge \psi) \in p$ then either $p \cup \{\neg\phi\} \in N$ or $p \cup \{\neg\psi\} \in N$,
7. if $\phi \vee \psi \in p$ then either $p \cup \{\phi\} \in N$ or $p \cup \{\psi\} \in N$,
8. if $\neg(\phi \vee \psi) \in p$ then $p \cup \{\neg\phi, \neg\psi\} \in N$.

   For all variables x and formulae $\psi(x)$,
9. if $\forall x\psi \in p$ then for every closed term t, $p \cup \{\psi(t)\} \in N$,
10. if $\neg\forall x\psi \in p$ then for some closed term t, $p \cup \{\neg\psi(t)\} \in N$,
11. if $\exists x\psi \in p$ then for some closed term t, $p \cup \{\psi(t)\} \in N$,
12. if $\neg\exists x\psi \in p$ then for every closed term t, $p \cup \{\neg\psi(t)\} \in N$,
13. if $p \in N$, t is a closed term of $L(W)$ and c is a witness which occurs nowhere in p or t, then $p \cup \{t = c\} \in N$,
14. at most finitely many witnesses occur in any $p \in N$.

The elements of N are called N-*conditions*. A chain $\bar{p} = \langle p_i : i < \omega \rangle$ of N-conditions is called an N-*construction sequence*. We denote $\bigcup_{i<\omega} p_i$ by $\bigcup \bar{p}$. There is a least $=$-closed set U which contains all the atomic sentences in $\bigcup \bar{p}$, where $=$-*closed* means that for every closed term t, $t = t$ is in U, and for every atomic formula $\phi(x)$ and all closed terms s and t, if U contains $\phi(s)$ and at least one of the sentences $s = t$ and $t = s$ then $\phi(t) \in U$. It can be shown that there is a unique model $\mathcal{A}$ of U with the properties:

1. every element of $\mathcal{A}$ is of the form $t^{\mathcal{A}}$ for some closed term t,

2. for every atomic sentence ϕ, $\mathcal{A} \models \phi$ iff $\phi \in U$.

We write $\mathcal{A}^+(\bar{p})$ for $\mathcal{A}$ and call it the *compiled structure.*

Let P be a property which the set $\bigcup \bar{p}$ can have or not have. This includes properties of the compiled structure. The game $G_N(P)$ is played as follows: Players I and II take turns to choose between them the elements p_i of an N-construction sequence $\bar{p}$. Player I chooses p_0, Player II chooses p_1 and so on. Player II wins if and only if at the end of the game, $\bigcup \bar{p}$ has the property P.

A property P is called N*-enforceable* if Player II has a winning strategy for the game $G_N(P)$.

The next theorem, which we will not prove here, is the key result.

Theorem 5.2.1 *Let N be a notion of forcing. The following property is N-enforceable: "The compiled structure $\mathcal{A}^+(\bar{p})$ is a model of $\bigcup(\bar{p})$".* □

Using this we can now prove the Compactness theorem for countable signatures.

Theorem 5.2.2 *Let Γ be a first-order theory in a countable signature such that every finite subset of Γ has a model. Then Γ has a model.*

Proof: Let N be the following set of sets of sentences: $p \in N$ iff

1. finitely many witnesses occur in sentences in p,
2. every finite subset of p has a model.

It is easy to check that N is a notion of forcing for $L(W)$ and that $\Gamma \in N$, so by Theorem 5.2.1 Γ has a model. □

5.3 Game Quantification

We have seen how games can be used to characterize logical equivalence in various languages. In this section we consider how the situation can be reversed, in other words, how we can use games to define semantic notions such as truth of a formula in a structure or equivalence of structures. For instance, we may just as well define two structures $\mathcal{A}$ and $\mathcal{B}$ of a finite function-free signature to be first-order equivalent if Player II has a winning strategy for the game $E(\mathcal{A}, \mathcal{B}, \omega)$, without any reference to the syntactic notion of a formula.

To get an idea of how games can serve to define truth of formulae we next give a game-theoretic approach to defining the semantics of the universal and existential quantifiers in first-order logic (assuming that truth of quantifier-free formulae is defined).

Definition: A first-order formula α is in *prenex* form if $\alpha = Q_1x_1Q_2x_2\cdots Q_nx_n\beta$ where every Q_i is either $\exists$ or $\forall$ and β is quantifier-free.

Let α be a σ-sentence in prenex form, say $\alpha = Q_1x_1Q_2x_2\cdots Q_nx_n\beta(x_1,\ldots,x_n)$ with β quantifier-free and let $\mathcal{A}$ be a σ-structure. We define a game $E(\mathcal{A},\alpha)$ as follows: The game is played over n rounds. In round i, if Q_i is $\forall$ then Player I selects an element a_i from $\mathcal{A}$. otherwise Player II selects a_i. Player II wins if at the end of the play, $\mathcal{A} \models \beta(a_1,\ldots,a_n)$. We can now define α to be true in $\mathcal{A}$ if and only if Player II has a winning strategy for $E(\mathcal{A},\alpha)$. It is not difficult to see that this agrees with the usual definition.

In some cases, game-theory is essentially the only way of defining truth. An example is *independence friendly* first-order logic, obtained by adding the following rules, to be applied to formulae in negation normal form after the usual formation rules:

If $\exists x$ occurs within the scope of the quantifiers $\forall y_1,\ldots,\forall y_n$, among others, then it may be replaced by $(\exists x/\forall y_1,\ldots,\forall y_n)$.

If $\vee$ occurs within the scope of the quantifiers $\forall y_1,\ldots,\forall y_n$, among others, then it may be replaced by $(\vee/\forall y_1,\ldots,\forall y_n)$.

If $\forall x$ occurs within the scope of the quantifiers $\exists y_1,\ldots,\exists y_n$, among others, then it may be replaced by $(\forall x/\exists y_1,\ldots,\exists y_n)$.

If $\wedge$ occurs within the scope of the quantifiers $\exists y_1,\ldots,\exists y_n$, among others, then it may be replaced by $(\wedge/\exists y_1,\ldots,\exists y_n)$.

The intended meaning of a sentence such as

$$(\forall x)(\forall z)(\exists y/\forall z)(\exists u/\forall x)\mathbf{r}(x,z,y,u)$$

is that $\exists y$ depends on $\forall x$ but not on $\forall z$, while $\exists u$ depends on $\forall z$ but not on $\forall x$.

Tarskian truth-definitions do not work here, because it cannot avoid interpreting at least one existential quantifier as being dependent on both the universal quantifiers. However, a game-theoretical interpretation is available. In fact, we simply modify the game we defined above for first-order logic to a game of *imperfect information*. For the sentence above, on Player II's first move he is not allowed to know Player I's second move, and on Player II's second move, he has no access to Player I's first move. For more on this see Hintikka and Sandu [12].

Bibliography

[1] R. Backofen, J. Rogers and K. Vijay-Shanker, *A First-order Axiomatization of the Theory of Finite Trees*, Journal of Logic, Language and Information, **4**, 5-39, 1995.

[2] J. Barwise, *Handbook of Mathematical Logic*, North-Holland, 1977.

[3] J. Barwise and S. Feferman, *Model-theoretic Logics*, Springer-Verlag, 1985.

[4] K. Devlin, *The Joy of Sets*, Springer-Verlag, 2nd ed., 1993.

[5] K. Doets, *Basic Model Theory*, CSLI publications, Stanford, California, 1996.

[6] K. Doets, *Completeness and Definability*, Ph. D. Thesis, Univ. of Amsterdam, 1987.

[7] H.-D. Ebbinghaus and J. Flum, *Finite Model Theory*, Springer-Verlag, 1995.

[8] H.-D. Ebbinghaus, J. Flum and W. Thomas, *Mathematical Logic*, Springer-Verlag, 2nd ed., 1994.

[9] J. Gerbrandy, *Bisimulation and Bounded Bisimulation*, Dept. of Philosophy/ILLC, Univ. of Amsterdam, preprint.

[10] R. Goldblatt, *Logics of Time and Computation* 2nd ed., CSLI lecture notes no. 7, 1992.

[11] R. Goldblatt, *Modal Logics of Programs*, Lecture Notes for the Workshop on Formal Aspects of Programming, Univ. of Cape Town, 1994.

[12] J. Hintikka and G. Sandu, *Game-theoretical Semantics*, in: Handbook of Logic and Language, Edited by J. van Benthem and A. ter Meulen, Elsevier Science, 1997.

[13] W. Hodges, *Building Models by Games*, Cambridge Univ. Press, 1985.

[14] W. Hodges, *Model Theory*, Cambridge Univ. Press, 1993.

[15] I. Hodkinson, *Expressive Completeness of Until and Since over Dedekind Complete Linear Time*, in: Modal Logic and Process Algebra, CSLI Publications, 1995.

[16] C. Stirling, *Games and Modal Mu-Calculus*, Dept. of Computer Science, Univ. of Edinburgh, preprint.

[17] J. van Benthem, *Classical Logic and Modal Logic*, Bibliopolis, Napoli, 1985.