An Analytic Approach To The Spectral Characterization Of The Radical

by

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Introduction

In Zemánek's proof of his well known characterization of the radical in a Banach algebra ([6], [7], [8]), an important step was to prove the following theorem.

Theorem 0.0.1. *Let A be a Banach algebra and let $a \in A$ have the property that $[a, x]$ is quasi-nilpotent for every x in A. Then $[a, x]$ belongs to the radical of A for every x in A.*

Zemánek proved the above Theorem using the Jacobson's Density Theorem, in other words representation theory. The above Theorem was also independently proved by Le Page [5] using representation theory. Although representation theory in the theory of Banach algebras is a very powerful tool, it has the disadvantage that it involves another level of abstraction.

Since Zemánek's characterisation of the radical in 1977, many authors ([2], [3], [4]) proved spectral characterizations of the radical using analytic techniques, avoiding representation theory altogether. In this dissertation we intend to illustrate, following [1], that it is possible to give a simple proof of the above Theorem using only elementary complex analysis.

In the first chapter we provide introductory definitions and theory around Banach algebras, representation theory and complex analysis. This will give the supporting information for the remaining chapters, as well as some background to important concepts and definitions. Some of the relevant notation will also be discussed here and assumed for the rest of the dissertation.

In chapter two we focus on the Radical in a Banach algebra. Our aim in this dissertation is to show that one can prove Theorem 0.0.1 using representation theory and that one can prove Theorem 0.0.1 using analytic methods. We start by defining the notion of a Radical and give some relevant results that will shed more light on the radical in a Banach algebra.

In chapter three and four we will use different approaches in proving our main result, i.e. that if $[a, x]$ is quasi-nilpotent for every $x \in A$, then $[a, x]$ belongs to the radical of A for every $x \in A$. In chapter three we prove this fact using representation theory. In chapter four we use an analytic approach, see [1], to prove the above fact. In both chapters we start with elementary theorems, and work our way up to the final result.

Finally, we will summarize the benefits of each approach and highlight some of the challenges. We will also give our subjective preference based on our results, however noting that both of these approaches are very useful and relevant in their own way. Some future work and development will also be highlighted for the sake of interest.

Throughout this dissertation, if we write Theorem x.y.z we mean Theorem z in Section y of Chapter x.

Chapter 1

Preliminaries

In this chapter we give some preliminary definitions and results that will help us throughout this dissertation. These concepts provide the necessary background for our investigation. Some of the theorems in the subsequent chapters will also reference back to these definitions. It will also provide a platform for notation and conventions used throughout. If more detail is required than provided below, the reader is encouraged to read through some of the mentioned references.

1.1 Banach algebra

A *vector space* over a field K is a non-empty set X of elements (called *vectors*) together with two algebraic operations. These operations are called *vector addition* and *multiplication of vectors by scalars*, that is, by elements of K [9, Definition 2.1-1].

Vector addition [9] associates with every ordered pair (x, y) of vectors a vector $x+y$, called the *sum* of x and y, in such a way that the following properties hold. Vector addition is commutative and associative, that is, for all vectors we have

$$x + y = y + x$$
$$x + (y + z) = (x + y) + z$$

furthermore, there exists a vector 0, called the *zero vector*, and for every vector x there exists a vector $-x$, such that for all vectors we have

$$\begin{aligned} x + 0 &= x \\ x + (-x) &= 0. \end{aligned}$$

Multiplication by scalars [9] associates with every vector x and scalar α a vector αx (also written $x\alpha$), called the *product* of α and x, in such a way that for all vectors x, y and scalars α, β we have

$$\begin{aligned} \alpha(\beta x) &= (\alpha\beta)x \\ 1x &= x \end{aligned}$$

and the distributive laws

$$\begin{aligned} \alpha(x + y) &= \alpha x + \alpha y \\ (\alpha + \beta)x &= \alpha x + \beta x. \end{aligned}$$

A *normed space* X [9, Definition 2.2-1] is a vector space with a norm defined on it. Here a *norm* on a vector space X is a real-valued function on X whose value at an $x \in X$ is denoted by

$$\|x\|$$

and which has the properties

$$\begin{aligned} \|x\| &\geq 0 \\ \|x\| &= 0 \text{ if and only if } x = 0 \\ \|\alpha x\| &= |\alpha|\|x\| \\ \|x + y\| &\leq \|x\| + \|y\| \quad \text{(Triangle inequality)}; \end{aligned}$$

where x and y are arbitrary vectors in X and α is any scalar.

A *subspace* [9] of a vector space X is a non-empty subset Y of X such that for all $y_1, y_2 \in Y$ and all scalars α, β we have $\alpha y_1 + \beta y_2 \in Y$. A subspace Y [9] of a normed space X is a subspace of X considered as a vector space, with the norm obtained by restricting the norm on X to the subset Y.

An *Algebra* A [9] over a field K is a vector space A over K such that for each ordered pair of elements $x, y \in A$ a unique product $xy \in A$ is defined with the properties

$$\begin{aligned}(xy)z &= x(yz)\\ x(y+z) &= xy + xz\\ (x+y)z &= xz + yz\\ \alpha(xy) &= (\alpha x)y = x(\alpha y)\end{aligned}$$

for all $x, y, z \in A$ and scalars α.

In our dissertation we consider only the case where K is the set of complex numbers, denoted by $\mathbb{C}$. In this case we will say that A is a complex algebra. Furthermore, we consider the case where A is an algebra with identity (denoted by 1). In other words [9], A contains an element 1 such that for all $x \in A$,

$$1x = x1 = x$$

and we note that this identity is unique.

A *normed algebra* A [9, Definition 7.6-1] is a normed space which is an algebra such that for all $x, y \in A$,

$$\begin{aligned}\|xy\| &\leq \|x\|\|y\|\\ \|1\| &= 1\end{aligned}$$

A *Banach algebra* is a normed algebra which is complete, considered as a normed space [9, Definition 7.6-1]. Throughout this dissertation, A will denote a complex Banach algebra with unit, 1.

1.2 Ideals

A subspace I of a Banach algebra A is called a *two-sided ideal* (or just an ideal) if $AI \subset I$ and $IA \subset I$ where

$$\begin{aligned} AI &= \{ab \mid a \in A, b \in I\} \text{ and} \\ IA &= \{ba \mid b \in I, a \in A\}. \end{aligned}$$

If we have that $AI \subset I$ then I is called a *left ideal* and if $IA \subset I$ then I is called a *right ideal*.

For any proper ideal I of a Banach algebra A we have that the unit 1 does not belong to I. Also, I is not empty since $0 \in I$. Furthermore, if I and J are left ideals, then $I + J$ is also a left ideal. Similarly, if I and J are right ideals, then $I + J$ is also a right ideal.

A left(right) ideal I of a Banach algebra A is called a *maximal left(right) ideal* of A, if

- I is a proper left(right) ideal of A;
- There exists no other proper left(right) ideal J of A so that $I \subset J$;
- For any ideal J with $I \subseteq J$ either $J = I$ or $J = A$.

If I is a closed ideal in a Banach algebra A, then $A/I = \{a+I \mid a \in A\}$ is a Banach space under the norm $\|a+I\| = inf\{\|a-x\| \mid x \in I\}$. One can verify that A/I is a Banach algebra, called the *quotient algebra* of A by I, because

$$\|a + I + b + I\| \leq \|a + I\| + \|b + I\| \text{ and}$$

$$\|(a + I) \cdot (b + I)\| \leq \|a + I\| \cdot \|b + I\| \text{ where}$$

$(a + I) + (b + I) = (a + b) + I$ and $(a + I) \cdot (b + I) = ab + I$ for all $a, b \in A$. If A has unit 1, then $1 + I$ is the unit in A/I.

Theorem 1.2.1. *[3, Lemma 3.1.1] Let A be a ring with unit 1. Every left ideal of A is contained in a maximal left ideal. Similarly, every right ideal of A is contained in a maximal right ideal.*

Since every Banach algebra is a ring, every ideal in a Banach algebra is contained in a maximal ideal.

1.3 Spectral theory

In a Banach algebra A, we denote the set of invertible elements of A with A^{-1}. We note that A^{-1} is a group which contains 1. From [3, Theorem 3.2.1, Theorem 3.2.3] we get the following two theorems.

Theorem 1.3.1. *Suppose that A is a Banach algebra, $x \in A$ and $\|x\| < 1$. Then $1 - x$ is invertible and*

$$(1-x)^{-1} = \sum_{k=0}^{\infty} x^k.$$

Theorem 1.3.2. *Suppose that A is a Banach algebra and that a is invertible. If $\|x - a\| < \frac{1}{\|a^{-1}\|}$, then x is invertible. Moreover, the mapping $x \mapsto x^{-1}$ is a homeomorphism from A^{-1} onto A^{-1}.*

Next, we define the notion of the *spectrum* of x. For any $x \in A$, the spectrum of x, denoted by $\sigma(x)$, is the set of $\lambda \in \mathbb{C}$ such that $\lambda - x$ is not invertible in A. Hence we have that

$$\sigma(x) = \{\lambda \in \mathbb{C} \mid \lambda - x \notin A^{-1}\}.$$

In the definition of the spectrum we write $\lambda - x$ for $\lambda 1 - x$.

The *spectral radius* of an element x in a Banach algebra is denoted by $r(x)$ and it is defined by

$$r(x) = sup\{|\lambda| \mid \lambda \in \sigma(x)\}.$$

For the spectrum $\sigma(x)$ of x we have the following basic facts:

- If $0 \notin \sigma(x)$ then it follows that $x \in A^{-1}$;
- If $0 \in \sigma(x)$ then it follows that $x \notin A^{-1}$;
- $\sigma(1) = \{\lambda \in \mathbb{C} \mid \lambda - 1 \notin A^{-1}\} = \{1\}$ and also that $r(1) = 1$;
- $\sigma(0) = \{\lambda \in \mathbb{C} \mid \lambda - 0 \notin A^{-1}\} = \{0\}$ and also that $r(0) = 0$;
- If x is a non-trivial idempotent, i.e. $x = x^2$ and $0 \neq x \neq 1$, then $\sigma(x) = \{0, 1\}$.

Now, from our definition of $\sigma(x)$, note that:

$$\lambda - x = \lambda(1 - \frac{x}{\lambda}) \text{ with } \lambda \neq 0.$$

Then, by Theorem 1.3.1, $\lambda - x$ is invertible for $\|x\| < |\lambda|$. So $\sigma(x)$ is a bounded subset of $\mathbb{C}$ and $r(x) \leq \|x\|$.

The following interesting results [3, Theorem 3.2.8, Corollary 3.2.10] have powerful consequences.

Theorem 1.3.3 (I.M. Gelfand)**.** *Let A be a Banach algebra and $x \in A$. Then*

1. $\lambda \mapsto (\lambda - x)^{-1}$ *is analytic on* $\mathbb{C} \backslash \sigma(x)$ *and goes to* 0 *at infinity;*
2. $\sigma(x)$ *is compact and non-empty;*
3. $r(x) = \lim_{n \to \infty} \|x^n\|^{\frac{1}{n}}$

Theorem 1.3.4. *Let A be a Banach algebra. Suppose that $x, y \in A$ satisfy $xy = yx$. Then $r(x + y) \leq r(x) + r(y)$ and $r(xy) \leq r(x)r(y)$.*

Lastly we define the notion of *quasi-nilpotent elements*. When the spectrum of an element is trivial, i.e. $\sigma(x) = \{0\}$, then x is said to be quasi-nilpotent. The set of all quasi-nilpotent elements of A will be denoted by $QN(A)$, and is defined as follows:

$$QN(A) = \{x \in A \mid \sigma(x) = \{0\}\},$$

and similarly it can be defined as follows:

$$QN(A) = \{x \in A \mid r(x) = 0\}.$$

Throughout, if A is a Banach algebra and $a, x \in A$, we write

$$[a, x] = ax - xa.$$

1.4 Representation theory

We are going to need the theory of representations in chapter three. To this end, let A be a Banach algebra. We say π is a *representation of* A on a complex vector space X, with $dimX \geq 1$, if π is a non-trivial homomorphism from A into the algebra of linear operators on X. If a linear subspace Y of X satisfies $\pi(x)Y \subset Y$ for all $x \in A$, then we say that Y is *invariant under* $\pi(x)$.

A representation π is said to be *irreducible* if the only linear subspaces of X invariant under $\pi(x)$, with $x \in A$, are $\{0\}$ and X. A representation π is said to be *bounded* if X is a Banach space and if $\pi(x)$ is a bounded linear operator on X for all $x \in A$. Moreover, a representation π is said to be *continuous* if it is bounded and if there exists a constant $C > 0$ such that $\|\pi(x)\| \leq C\|x\|$ for all $x \in A$. It can be shown that the notions of bounded irreducible representations and continuous irreducible representations are the same.

Let L be a maximal left ideal of a Banach algebra A. Since maximal ideals in a Banach algebra are closed, A/L is a Banach algebra. The *standard representation* π of A on A/L is defined by

$$\pi(x)(a+L) = xa + L \qquad (x, a \in A),$$

so that $\pi : A \mapsto \mathcal{L}(A/L)$ is a homomorphism. We define the *quotient* $L : A$ of A by

$$(L : A) = \{a \in A \mid aA \subset L\}.$$

Thus, $(L : A)$ is the kernel of the above representation π of A on A/L.

In order to prove our main result in chapter three we need the following results:

Theorem 1.4.1. *[3, Theorem 4.2.1] Let A be a Banach algebra.*

1. *For every irreducible representation π of A, there exists a maximal left ideal L such that $Ker(\pi) = (L : A)$.*

2. *The radical of A is the intersection of the kernels of all continuous irreducible representations of A.*

3. *For every $x \in A$ the spectrum of x is the union of all the spectra of the $\pi(x)$ in the corresponding algebras $\pi(A)$ for all continuous irreducible representations π, i.e.*

$$\sigma(x, A) = \cup\{\sigma(\pi(x), \pi(A)) \mid \pi \text{ is a continuous irreducible representation of } A\}.$$

Theorem 1.4.2. *[3, Theorem 4.2.2 (I. Schur)] Let A be a Banach algebra and let π be a continuous irreducible representation of A on a Banach space X. Then $C = \{T \mid T \in \mathcal{L}(X), T\pi(x) = \pi(x)T,$ for all $x \in A\}$ is isomorphic to $\mathbb{C}$.*

Theorem 1.4.3. *[3, Theorem 4.2.3] Let* π *be a continuous irreducible representation of a Banach algebra* A *on a Banach space* X*. If* ξ_1, ξ_2 *are linearly independent in* X*, then there exists* $a \in A$ *such that* $\pi(a)\xi_1 = 0$ *and* $\pi(a)\xi_2 \neq 0$*.*

Theorem 1.4.4. *[3, Theorem 4.2.4] Let* π *be a continuous irreducible representation of a Banach algebra* A *on a Banach space* X*. If* $\xi_1, ..., \xi_n$ *are linearly independent in* X*, then there exists* $a \in A$ *such that* $\pi(a)\xi_i = 0$ *for* $1 \leq i \leq n-1$ *and* $\pi(a)\xi_n \neq 0$*.*

Theorem 1.4.5. *[3, Theorem 4.2.5 (Jacobson Density Theorem)] Let* π *be a continuous irreducible representation of a Banach algebra* A *on a Banach space* X*. If* $\xi_1, ..., \xi_n$ *are linearly independent in* X *and if* $\eta_1, ..., \eta_n$ *are in* X*, then there exists* $a \in A$ *such that* $\pi(a)\xi_i = \eta_i$ *for* $i = 1, ..., n$*.*

Theorem 1.4.6. *[3, Theorem 4.2.7 (I. Kaplansky)] Let* X *be a complex vector space and let* T *be a linear operator from* X *into* X*. Suppose there exists an integer* $n \geq 1$ *such that* $\xi, T\xi, ..., T^n\xi$ *are linearly dependent for all* $\xi \in X$*. Then* T *is algebraic of degree less than or equal to* n*.*

Chapter 2

The Radical In A Banach Algebra

In this chapter we focus on the radical in a Banach algebra. We start by defining the notion of a radical and give some relevant results. This is required before we can continue with the two different approaches used in this dissertation to prove Theorem 0.0.1. Some equivalent definitions are also given that will facilitate our investigation.

2.1 Preliminary definition

In the literature, the following result [3, Lemma 3.1.2] is known as Jacobson's Lemma.

Theorem 2.1.1. *Let A be an algebra with unit 1. If $x, y \in A$ and $\lambda \in \mathbb{C}$ with $\lambda \neq 0$, then $\lambda - xy$ is invertible in A if and only if $\lambda - yx$ is invertible in A.*

Proof.
Let $x, y \in A$. Suppose that $\lambda - xy$ has an inverse u in A. Then it means there exists $u \in A$ with

$$(\lambda - xy)u = u(\lambda - xy) = 1.$$

So we first note that since $(\lambda - xy)u = 1$ we have that $\lambda u - xyu = 1$. From this we get that $\lambda u - 1 = xyu$. Similarly, since $u(\lambda - xy) = 1$ we have that $u\lambda - 1 = uxy$. Using this, with our assumption that $\lambda - xy$ has an inverse, we get the following

$$\begin{aligned}(\lambda - yx)(yux + 1) &= \lambda yux + \lambda - yxyux - yx \\ &= \lambda yux - y(xyu)x - yx + \lambda \text{ (since addition is associative)} \\ &= \lambda yux - y(\lambda u - 1)x - yx + \lambda \text{ (from the above)} \\ &= \lambda yux - y\lambda ux + yx - yx + \lambda \\ &= \lambda.\end{aligned}$$

Similarly, we also get the following

$$\begin{aligned}(yux + 1)(\lambda - yx) &= yux\lambda - y(uxy)x + \lambda - yx \text{ (since addition is associative)} \\ &= yux\lambda - y(u\lambda - 1)x + \lambda - yx \text{ (from the above)} \\ &= yux\lambda - yux\lambda + yx + \lambda - yx \\ &= \lambda.\end{aligned}$$

But then we have that $(\lambda - yx)(yux + 1) = (yux + 1)(\lambda - yx) = \lambda$ with $\lambda \neq 0$. And so we get the following

$$(\lambda - yx)(\frac{yux + 1}{\lambda}) = (\frac{yux + 1}{\lambda})(\lambda - yx) = 1.$$

Hence we have shown that $\lambda - yx$ is invertible in A. The proof is similar for the converse. Hence our result follows.

□

Using our definitions of the spectrum and spectral radius, the above theorem can be reformulated as follows: for any $x, y \in A$, where A is a Banach algebra,

$$\sigma(xy) \cup \{0\} = \sigma(yx) \cup \{0\}$$

and

$$r(xy) = r(yx).$$

2.2 The Radical

We are now ready to define the notion of a radical of a Banach algebra. Our next result [3, Lemma 3.1.3] motivates the definition of the radical in a Banach algebra.

Theorem 2.2.1. *Let A be a ring with unit 1. Then the following sets are identical:*

1. *The intersection of all maximal left ideals of A.*
2. *The intersection of all maximal right ideals of A.*
3. *The set of x such that $1 - zx$ is invertible in A, for all $z \in A$.*
4. *The set of x such that $1 - xz$ is invertible in A, for all $z \in A$.*

A two-sided ideal in a Banach algebra A having any of the equivalent properties in Theorem 2.2.1 is called the *Jacobson radical* of A and it is denoted by $Rad(A)$. If $Rad(A) = \{0\}$ we say that A is *semi-simple*.

From now on, when we say radical we mean the Jacobson radical. The above theorem is very important for this dissertation, and will provide much of the groundwork required in subsequent results. Additionally, we are able to illustrate some relationships that $Rad(A)$ has with A^{-1}, $\sigma(x)$ and $QN(A)$. Since maximal left(right) ideals are closed, it follows from Theorem 2.2.1 that $Rad(A)$ is a closed two-sided ideal in A.

2.3 Properties of the Radical

The results in this section are elementary and one does not need any representation theory or analytic techniques to prove them. To discuss the relationship between $Rad(A)$ and A^{-1}, we need the following Lemma.

Lemma 2.3.1. *If A is a Banach algebra, then $A = A^{-1} + A^{-1}$.*

Proof.
Since A is a Banach algebra it is closed under addition, hence it follows that $A^{-1} + A^{-1} \subset A$.
Now, let $x \in A$. Then $x = x - \lambda + \lambda$ for all $\lambda \in \mathbb{C}$. In particular, if $|\lambda| > \|x\|$, then $x = (x - \lambda) + \lambda$ with $x - \lambda \in A^{-1}$ and $\lambda = \lambda 1 \in A^{-1}$. Hence it follows that $A \subset A^{-1} + A^{-1}$, and so our result follows. □

Theorem 2.3.2. *Let A be a Banach algebra and $a \in A$. Then $a \in Rad(A)$ if and only if $a + A^{-1} \subset A^{-1}$.*

Proof.
$\Rightarrow$:
Let a be any element in $Rad(A)$. Then we have from Theorem 2.2.1(*4*) that $za - 1 \in A^{-1}$ for all $z \in A$. Now, let x be an element in A^{-1}. Then $x(za - 1) \in A^{-1}$ for all $z \in A$ since A^{-1} is closed under multiplication. So, in particular, with $z = x^{-1} \in A$ we have that $x(x^{-1}a - 1) = a - x \in A^{-1}$. Since x was arbitrarily chosen, we have that $a + A^{-1} \subset A^{-1}$.

$\Leftarrow$:
For the converse we suppose that $a + x \in A^{-1}$ for all $x \in A^{-1}$. Then $a + x = x(x^{-1}a + 1) \in A^{-1}$ for all $x \in A^{-1}$. But since $x \in A^{-1}$ it follows that

$$x^{-1}a + 1 \in A^{-1} \quad \text{for all } x \in A^{-1}. \qquad (\alpha)$$

Next, let x to be an element of A. By Lemma 2.3.1, $x = x_1 + x_2$ with $x_1, x_2 \in A^{-1}$. Then $xa + 1 = x_1a + x_2a + 1 = (x_1a + 1)((x_1a + 1)^{-1}x_2a + 1)$ since $x_1a + 1 \in A^{-1}$ by (α). Since $x_2 \in A^{-1}$, $(x_1a + 1)^{-1}x_2 \in A^{-1}$ because A^{-1} is closed under multiplication. Again, by (α), $(x_1a + 1)^{-1}x_2a + 1 \in A^{-1}$. So we have shown that $xa + 1 \in A^{-1}$ for all $x \in A$. By Theorem 2.2.1(*3*) it follows that $a \in Rad(A)$. □

Our next result is a characterization of the radical in terms of the spectrum function.

Theorem 2.3.3. *Let A be a Banach algebra and $a \in A$. Then $a \in Rad(A)$ if and only if $\sigma(a + x) = \sigma(x)$ for all $x \in A$.*

Proof.
$\Rightarrow$:
Suppose $a \in Rad(A)$. If $\lambda \notin \sigma(x)$ we have that $\lambda - x \in A^{-1}$ for all $x \in A$. In view of Theorem 2.3.2

$$\lambda - (x + a) = (\lambda - x) - a \in A^{-1}$$

because $-a \in Rad(A)$ since $Rad(A)$ is an ideal. Hence, $\lambda \notin \sigma(x + a)$. So we have shown that $\sigma(x + a) \subset \sigma(x)$ for all $x \in A$. To prove the reverse containment, note that

$$\sigma(x) = \sigma(x + a - a) \subset \sigma(x + a)$$

by the first part of the proof, because $-a \in Rad(A)$. If we combine our arguments we get $\sigma(x) = \sigma(x + a)$ for all $x \in A$.

$\Leftarrow$:
Suppose that $\sigma(a + x) = \sigma(x)$ for all $x \in A$. Hence, if $x \in A^{-1}$, then $0 \notin \sigma(x)$ and so we have that $0 \notin \sigma(x + a)$. Hence, $x + a \in A^{-1}$. Since $x \in A^{-1}$ was arbitrarily chosen we conclude that $a + A^{-1} \subset A^{-1}$. In view of Theorem 2.3.2, $a \in Rad(A)$. $\square$

Lastly, in the following results we investigate how $QN(A)$ is related to $Rad(A)$. This relationship is quite important for our dissertation, and we remind ourselves that we want to show that *if $a \in A$ and if $[a, x] \in QN(A)$ for every $x \in A$, then $[a, x] \in Rad(A)$ for every $x \in A$.*

Theorem 2.3.4. *Let A be a Banach algebra with unit 1. Then*

$$Rad(A) = \{a \in A \mid aA \subset QN(A)\}.$$

Proof.
If $a \in Rad(A)$, then we have that $\lambda - az \in A^{-1}$ for all $z \in A$ with $\lambda \neq 0$. This implies that $aA \subset QN(A)$, since $\sigma(az) \neq \varnothing$ (Theorem 1.3.3(*2*)), so that $\sigma(az) = \{0\}$ for all $z \in A$. Hence we have shown that $Rad(A) \subset \{a \in A \mid aA \subset QN(A)\}$.

To prove the reverse inclusion, let $aA \subset QN(A)$. Hence, $\lambda - az \in A^{-1}$ for all $z \in A$ since $\sigma(az) = \{0\}$. By Theorem 2.2.1(*4*), we have that $a \in Rad(A)$. So we have shown that $\{a \in A \mid aA \subset QN(A)\} \subset Rad(A)$ and hence we have proved the Theorem. □

Proposition 2.3.5. *If A is a Banach algebra, then $Rad(A) \subset QN(A)$.*

Proof.
If $x \in Rad(A)$, then by Theorem 2.2.1, $\lambda - zx \in A^{-1}$ for all $z \in A$. So in particular, $\lambda - x \in A^{-1}$ for all $0 \neq \lambda \in \mathbb{C}$. Hence, $\sigma(x) = \{0\}$ because $\sigma(x) \neq \varnothing$ (Theorem 1.3.3(*2*)). Hence we have that $x \in QN(A)$. □

Proposition 2.3.6. *Let A be a Banach algebra and $x \in A$ with $x \in QN(A)$. If $xy = yx$ for all $y \in A$, then $x \in Rad(A)$.*

Proof.
Let $x \in A$ with $x \in QN(A)$ and $xy = yx$ for all $y \in A$. This implies that $xy \subset QN(A)$ for all $y \in A$, since $r(xy) \leq r(x)r(y)$ by Theorem 1.3.4 and since $x \in QN(A)$ we have that $r(x) = 0$, so it follows that $r(xy) = 0$ for all $y \in A$. Hence $1 - xy \in A^{-1}$ for all $y \in A$. By Theorem 2.2.1, we have that $x \in Rad(A)$. □

Theorem 2.3.7. *Let A be a Banach algebra and $a \in A$. Then $a \in Rad(A)$ if and only if $a + QN(A) \subset QN(A)$.*

Proof.
$\Rightarrow$:
Let $a \in Rad(A)$. Then from Theorem 2.3.3, $\sigma(a + x) = \sigma(x)$ for all $x \in A$. In particular, if $x \in QN(A)$ then we have that $\sigma(x) = \{0\}$. But then it follows that $\sigma(a + x) = \{0\}$. Hence $a + x \in QN(A)$ for all $x \in QN(A)$ and it follows that $a + QN(A) \subset QN(A)$.

$\Leftarrow$:
To prove that if $a + QN(A) \subset QN(A)$ then it follows that $a \in Rad(A)$, we will need to make use of representation theory. This will be done in the next chapter, and we will revisit this result. □

Chapter 3

Representation Theory

Our goal in this chapter is to prove our main result using representation theory. Our main result is:

Theorem 0.0.1. *Let A be a Banach algebra and let $a \in A$ have the property that $[a, x]$ is quasi-nilpotent for every x in A. Then $[a, x]$ belongs to the radical of A for every x in A.*

3.1 Proof of our main result

In order to prove our main result, we first need some preparation.

Let A be a Banach algebra. We have that the *centre modulo the radical of A* is the set

$$Z(A) = \{a \in A \mid ax - xa \in Rad(A) \text{ for all } x \in A\}.$$

If K is a set, then $\#K$ denotes the number of elements in K.

Theorem 3.1.1. *[3, Theorem 5.2.1] Let A be a Banach algebra and suppose $a \in A$ has the property that $\#\sigma(ax - xa) = 1$ for all $x \in A$. Then $a \in Z(A)$.*

Proof.
Let $a \in A$ have the property mentioned in the formulation of the theorem. Let π be a continuous irreducible representation of A on a Banach space X. Suppose that there exists $\xi \in X$ such that $\xi, \eta = \pi(a)\xi$ and $\pi(a)\eta$ are linearly independent. Then by Theorem 1.4.5, there exists $x \in A$ such that:

- $\pi(x)\xi = 0$;
- $\pi(x)\eta = -\xi$;
- $\pi(x)\pi(a)\eta = -\eta$.

Since π is a homomorphism, we have

$$\begin{aligned}
\pi(ax - xa)\xi &= \pi(ax)\xi - \pi(xa)\xi \\
&= \pi(a)\pi(x)\xi - \pi(x)\pi(a)\xi \\
&= \pi(a)0 - \pi(x)\eta \\
&= 0 - (-\xi) \\
&= \xi.
\end{aligned}$$

Similarly, we also have

$$\begin{aligned}
\pi(ax - xa)\eta &= \pi(a)\pi(x)\eta - \pi(x)\pi(a)\eta \\
&= -\pi(a)\xi - (-\eta) \\
&= -\eta + \eta \\
&= 0.
\end{aligned}$$

Hence, $\pi(ax - xa)\xi = \xi = 1\xi$ and $\pi(ax - xa)\eta = 0 = 0\eta$. This means that 1 and 0 are eigenvalues of the operator $\pi(ax - xa)$ and so $\{0, 1\} \subset \sigma(\pi(ax - xa)) \subset \sigma(ax - xa)$ since π is a homomorphism. But this is a contradiction, since $\#\sigma(ax - xa) = 1$ by assumption. Consequently, for $\xi \in X$, the vectors $\xi, \eta = \pi(a)\xi$ and $\pi(a)\eta$ are linearly dependent. Hence by Theorem 1.4.6, $\pi(a)$ is algebraic of degree ≤ 2.

So if we suppose that $\pi(a)^2 = \alpha\pi(a) + \beta 1$ and taking $a' = a - \frac{\alpha}{2}1$, we get

$$\begin{aligned} a'x - xa' &= (a - \frac{\alpha}{2})x - x(a - \frac{\alpha}{2}) \\ &= ax - \frac{\alpha x}{2} - xa + \frac{\alpha x}{2} \\ &= ax - xa. \end{aligned}$$

Hence we have that $\sigma(a'x - xa') = \sigma(ax - xa)$ for all $x \in A$. Since π is a homomorphism, we also have

$$\begin{aligned} \pi(a')^2 &= \pi(a - \frac{\alpha}{2})\pi(a - \frac{\alpha}{2}) \\ &= (\pi(a) - \frac{\alpha}{2})(\pi(a) - \frac{\alpha}{2}) \quad \text{(because } \pi(1) = 1\text{)} \\ &= \pi(a)^2 - \frac{\alpha}{2}\pi(a) - \frac{\alpha}{2}\pi(a) + \frac{\alpha^2}{4} \\ &= \alpha\pi(a) + \beta 1 - \alpha\pi(a) + \frac{\alpha^2}{4} \\ &= (\beta + \frac{\alpha^2}{4})1. \end{aligned}$$

We may therefore assume without loss of generality that $\pi(a)^2 = \gamma 1$ for some $\gamma \in \mathbb{C}$. Now, if $\pi(a)$ is not algebraic of degree 1, there exists $\xi \in X$ such that ξ and $\eta = \pi(a)\xi$ are linearly independent. In view of Theorem 1.4.5, there exists $x \in A$ such that

- $\pi(x)\xi = \xi$;
- $\pi(x)\eta = \xi + \eta$.

This, together with π being a homomorphism, implies

$$\begin{aligned}\pi(ax - xa)\xi &= \pi(ax)\xi - \pi(xa)\xi \\ &= \pi(a)\pi(x)\xi - \pi(x)\pi(a)\xi \\ &= \pi(a)\xi - (\xi + \eta) \\ &= \eta - \xi - \eta \\ &= -\xi.\end{aligned}$$

And similarly we also have that

$$\begin{aligned}\pi(ax - xa)\eta &= \pi(a)\pi(x)\eta - \pi(x)\pi(a)\eta \\ &= \pi(a)(\xi + \eta) - \pi(x)\pi(a)^2\xi \\ &= \pi(a)\xi + \pi(a)\eta - \gamma\pi(x)\xi \quad (\pi(a)^2 = \gamma 1) \\ &= \eta + \pi(a)^2\xi - \gamma\xi \\ &= \eta + \gamma\xi - \gamma\xi \\ &= \eta.\end{aligned}$$

Hence $\pi(ax - xa)\xi = -1\xi$ and $\pi(ax - xa)\eta = 1\eta$. This means that -1 and 1 are eigenvalues of the operator $\pi(ax - xa)$ and so $\{-1, 1\} \subset \sigma(\pi(ax - xa)) \subset \sigma(ax - xa)$ since π is a homomorphism. But this is a contradiction, because $\#\sigma(ax - xa) = 1$ by assumption. Hence $\pi(a)$ is algebraic of degree 1.

In view of $\pi(a)$ being algebraic of degree 1, there exists a polynomial $p(\lambda)$ of degree 1 such that $p(\pi(a)) = 0$. Say $p(\lambda) = \lambda - \alpha 1$. This implies that $p(\pi(a)) = \pi(a) - \alpha 1 = 0$ and hence we get that $\pi(a) = \alpha 1$ for some $\alpha \in \mathbb{C}$. This, together with π being a homomorphism, implies

$$\begin{aligned}\pi(ax - xa) &= \pi(a)\pi(x) - \pi(x)\pi(a) \\ &= \alpha\pi(x) - \alpha\pi(x) \\ &= 0\end{aligned}$$

for all continuous irreducible representations π of A.

Hence, $ax - xa \in Ker(\pi)$. By Theorem 1.4.1(ii) it follows that $ax - xa \in Rad(A)$ for all $x \in A$. Hence we get our result, i.e. that $a \in Z(A)$. □

We are now ready to prove our main result.

Theorem 3.1.2. *Let A be a Banach algebra and let $a \in A$ have the property that $[a, x] \in QN(A)$ for every $x \in A$. Then $[a, x] \in Rad(A)$ for every $x \in A$.*

Proof.
Let an element a in a Banach algebra A have the property that $[a, x] \in QN(A)$ for every $x \in A$. Then we have that $\sigma([a, x]) = \{0\}$ and so it follows that $\#\sigma([a, x]) = 1$ for all $x \in A$. By Theorem 3.1.1, $a \in Z(A)$. In view of the definition of $Z(A)$, $[a, x] = ax - xa \in Rad(A)$ for every $x \in A$. □

3.2 J. Zemánek's theorem

The following theorem is Zemánek's famous characterization of the radical in a Banach algebra:

Theorem 3.2.1. *[3, Theorem 5.3.1 (J. Zemánek)] Let A be a Banach algebra. Then the following properties are equivalent:*

1. *a is in the Jacobson radical of A,*
2. *$\sigma(a + x) = \sigma(x)$ for all $x \in A$,*
3. *$r(a + x) = 0$ for all quasi-nilpotent elements x in A,*
4. *$r(a+x) = 0$ for all quasi-nilpotent elements x in a neighbourhood of 0 in A,*
5. *there exists $C > 0$ such that $r(x) \le C\|x - a\|$ for all x in a neighbourhood of a in A.*

In Theorem 2.3.7, we were able to prove that if A is a Banach algebra and $a \in A$, then if $a \in Rad(A)$ it implies that $a + QN(A) \subset QN(A)$. We did this by using only elementary techniques. However, we were not able to prove the reverse implication without the use of Representation theory. So, to this end, let A be a Banach algebra, $a \in A$ and suppose $a + QN(A) \subset QN(A)$. But since we have that $a+QN(A) \subset QN(A)$, it implies that $a+x \in QN(A)$ for all $x \in QN(A)$. In particular this means that $r(a + x) = 0$ for all $x \in QN(A)$. Now, from Theorem 3.2.1, it follows that $a \in Rad(A)$. Hence we have proven the reverse implication of Theorem 2.3.7.

Chapter 4

An Analytic Approach

In this chapter we make use of an analytic approach (as used in [1]) to prove our main result:

Theorem 0.0.1. *Let A be a Banach algebra and let $x \in A$ have the property that $[x, a]$ is quasi-nilpotent for every a in A. Then $[x, a]$ belongs to the radical of A for every a in A.*

We start of by proving some initial theorems, and work our way up to the final result in Theorem 4.2.1. We will also provide our own alternative proof to that used in [1] to prove Theorem 4.2.1.

4.1 Main result

To prove our main result (Theorem 0.0.1) by analytic techniques, we first need to prove Theorem 4.1.1.

Theorem 4.1.1. *[1, Theorem 2],[10] Let A be a complex Banach algebra with 1, let L be a maximal left ideal of A and let x be an element of A such that $Lx \subseteq L$. Then there is some $\lambda \in \mathbb{C}$ such that $x - \lambda \in L$.*

Before we start with the proof of the above theorem, we mention some easy facts:

- We firstly note that there exists at least one $x \in A$ such that $Lx \subseteq L$. This is true for any maximal left ideal L, since we have that 1 is in A and so, with $x = 1$ we have that $Lx = L1 \subseteq L$.

- The λ that exists in Theorem 4.1.1 belongs to $\sigma(x)$, because an ideal does not contain invertible elements.

- It is easy to see that Lx is a left ideal in A.

- The converse to Theorem 4.1.1 is also true. To show this we let L be a maximal left ideal of A, and $\lambda \in \mathbb{C}$ such that $x - \lambda \in L$. If $y \in L$ then $y(x - \lambda) \in L$ since L is a left ideal. Hence we have that $yx - \lambda y \in L$. But we have from our assumption that $y \in L$ so that $\lambda y \in L$ since L is a left ideal. Hence we have that $yx \in L$ because $yx = (yx - \lambda y) + \lambda y$. Since y was arbitrarily chosen, it follows that $Lx \subseteq L$.

- Lastly, we have that for a given L and x, the complex number λ is uniquely determined. To show this, we assume the converse. So suppose that the complex number λ is not uniquely determined for a given L and x. Hence, there exists $\lambda, \mu \in \mathbb{C}$ such that $x - \lambda \in L$ and $x - \mu \in L$. But since L is a left ideal, it is closed under addition and we have that $(x - \lambda) - (x - \mu) \in L$. Again, since L is a left ideal we also have that $(\frac{1}{(\mu - \lambda)})((x - \lambda) - (x - \mu)) \in L$. From this we get that $\frac{x}{\mu - \lambda} - \frac{\lambda}{\mu - \lambda} - \frac{x}{\mu - \lambda} + \frac{\mu}{\mu - \lambda} \in L$ so that $\frac{\mu - \lambda}{\mu - \lambda} \in L$ which means that $1 \in L$. But this means that $L = A$, which is a contradiction, because maximal ideals are proper and in general ideals do not contain invertible elements.

To prove Theorem 4.1.1 we need the following Lemma.

Lemma 4.1.2. *Let A be a Banach algebra and let L be a maximal left ideal of A. Suppose $x \in A$ has the property that $x \notin L$, but $yx \in L$ for some $y \in A$. Then $y \in L$.*

Proof. Let x, y and L satisfy the hypothesis of the Lemma. To obtain a contradiction, suppose $y \notin L$. We note that L and Ay are left ideals. Hence $L + Ay$ is a left ideal, since $y \notin L$. Now, we let $u \in L$, and we note that u can be expressed as $u = u + 0y$ because $0 \in A$. Then we have that $u = u + 0y \in L + Ay$ which implies that $L \subset L + Ay$. But L is a maximal left ideal, hence $L + Ay = A$ so that $L = A - Ay$. Hence, for some $z \in A$ and $1 \in A$ we have that $1 - zy \in L$. From our assumption we have that $Lx \subseteq L$. Hence $x - zyx \in L$. But we assumed $yx \in L$, hence $zyx \in L$ since L a left ideal. Hence $x \in L$ because L is closed under addition; which contradicts our assumption. Thus, if $x \notin L$ and $yx \in L$ for some $y \in A$, then $y \in L$. □

We can now start with the proof of Theorem 4.1.1:

Proof of Theorem 4.1.1.
To obtain a contradiction, suppose that A is a complex Banach algebra with 1, L a maximal left ideal of A and x an element of A such that $Lx \subseteq L$. Then for every $\lambda \in \mathbb{C}$ we have that $x - \lambda \notin L$.

Now, from this we note that L and $A(x - \lambda)$ are left ideals (and they are not the same, because $x - \lambda \notin L$). Hence $L + A(x - \lambda)$ is a left ideal. Let $u \in L$ and we note that $0 \in A$, then $u = u + 0(x - \lambda) \in L + A(x - \lambda)$ which implies that $L \subset L + A(x - \lambda)$. But L is a maximal left ideal, hence $L + A(x - \lambda) = A$ so that $L = A - A(x - \lambda)$. Now, we let $y(\lambda) \in A$ and we note that $1 \in A$. Then we have that $y(\lambda)(x - \lambda) - 1 \in L$.

Now, suppose that there exist y_1 and y_2 such that $y_1(x - \lambda) - 1 \in L$ and $y_2(x - \lambda) - 1 \in L$. Then, since L is closed under addition, we have that $y_1(x - \lambda) - 1 - y_2(x - \lambda) + 1 \in L$. Hence we have that $y_1(x - \lambda) - y_2(x - \lambda) \in L$ which implies that $(y_1 - y_2)(x - \lambda) \in L$. Note that $(x - \lambda) \notin L$ from our assumption, and we have just shown that $(y_1 - y_2)(x - \lambda) \in L$. Then by Lemma 4.1.2 we have that $(y_1 - y_2) \in L$. Hence we have that $y_1 + L = y_2 + L$ and so we have shown that our $y(\lambda)$ is uniquely determined modulo L.

Next we let $\pi : A \mapsto A/L$ be the quotient map, i.e. $\pi(x) = x + L$ ($x \in A$). We define $f(\lambda) = \pi(y(\lambda))$ with $\lambda \in \mathbb{C}$, such that $y(\lambda)$ is any element of A such that $y(\lambda)(x - \lambda 1) - 1 \in L$. Note that since $y(\lambda)$ is uniquely determined modulo L, it follows that $f(\lambda)$ is a well-defined A/L-valued function on $\mathbb{C}$. That is, if $f(\lambda) = \pi(y(\lambda)) = \pi(z(\lambda))$ then $y(\lambda) + L = z(\lambda) + L$.

Now, let $\mu \in \mathbb{C}$ and choose $y \in A$ so that $y(x - \mu) - 1 \in L$. We note that the element $1 + (\mu - \lambda)y$ is invertible for $|\lambda - \mu| < \|y\|^{-1}$. This is the case, since $1 + (\mu - \lambda)y = 1 - (\lambda - \mu)y$ and $1 - (\lambda - \mu)y$ is invertible for $\|(\lambda - \mu)y\| < 1$ by Theorem 1.3.2. Now, since $\|(\lambda - \mu)y\| < 1$ we have that $|\lambda - \mu|\|y\| < 1$ and hence that $|\lambda - \mu| < \|y\|^{-1}$.

Now, we consider $(1 + (\mu - \lambda)y)^{-1}y(x - \lambda) - 1$:

$$
\begin{aligned}
&(1 + (\mu - \lambda)y)^{-1}y(x - \lambda) - 1 \\
=&(1 + (\mu - \lambda)y)^{-1}y(x - \lambda) - (1 + (\mu - \lambda)y)^{-1}(1 + (\mu - \lambda)y) \\
=&(1 + (\mu - \lambda)y)^{-1}(y(x - \lambda) - 1 - (\mu - \lambda)y) \\
=&(1 + (\mu - \lambda)y)^{-1}(yx - \lambda y - 1 - \mu y + \lambda y) \\
=&(1 + (\mu - \lambda)y)^{-1}(y(x - \mu) - 1)
\end{aligned}
$$

But we have that $y(x - \mu) - 1 \in L$ and L is a left ideal. Hence $(1 + (\mu - \lambda)y)^{-1}(y(x - \mu) - 1) \in L$ which implies that $(1 + (\mu - \lambda)y)^{-1}y(x - \lambda) - 1 \in L$. Hence, for $|\lambda - \mu| < \|y\|^{-1}$ we can take $y(\lambda) = (1 + (\mu - \lambda)y)^{-1}y$, which is analytic in λ by Theorem 1.3.3. We also have that $(\lambda 1 - x)^{-1}$ is analytic for $|\lambda| > \|x\|$ again by Theorem 1.3.3. Hence we have shown that f is an analytic A/L-valued function for both small and large λ and so we have that f is entire.

For $|\lambda| > \|x\|$ we can take $y(\lambda) = (x - \lambda)^{-1}$ since $(x - \lambda)^{-1}(x - \lambda) - 1 = 1 - 1 = 0 \in L$. But $(x - \lambda)^{-1} \to 0$ as $|\lambda| \to \infty$, Theorem 1.3.3. Hence $f(\lambda) = \pi((x - \lambda)^{-1}) \to \pi(0)$ since π is continuous. Hence we have that $f(\lambda) \to L$ as $|\lambda| \to \infty$. Since π is continuous, we have that f is continuous, so that f is bounded.

Now, Liouville's theorem states that any bounded entire function must be constant. Hence it follows that $f(\lambda) = L$. But then we have that $f(\lambda) = \pi(y(\lambda)) = y(\lambda) + L = L$ for each $y(\lambda)$ so that $y(\lambda) \in L$. But then $(x-\lambda)^{-1} \in L$ for $|\lambda| > \|x\|$. But this means that $(x-\lambda)(x-\lambda)^{-1} \in L$ since L a left ideal, which implies that $1 \in L$, which is a contradiction.

Hence, there is some $\lambda \in \mathbb{C}$ such that $x - \lambda \in L$. □

Lemma 4.1.3. *[1, Lemma 3] Let L be a maximal left ideal of a Banach algebra A and let $x \in A$. Suppose that $[l, x] \in QN(A)$ for every $l \in L$; then $Lx \subseteq L$.*

Proof.
We assume that the statement is not true. Hence, suppose that $[l, x] \in QN(A)$ for every $l \in L$, but that $Lx \nsubseteq L$. Hence, there is some $m \in L$ with $mx \notin L$.

We note that L and $A(mx)$ are left ideals (and they are not the same since $mx \notin L$). Hence $L + A(mx)$ is a left ideal. Let $u \in L$ and we note that $0 \in A$, then $u = u + 0(mx) \in L + A(mx)$. Hence we have that $L \subset L + A(mx)$, but L is a maximal left ideal, which implies that $L + A(mx) = A$ so that $L = A - A(mx)$. Since $1 \in A$ there exists $b \in A$ with $bmx - 1 \in L$. Also, since $m \in L$, by assumption we have that $xbm \in L$ since L is a left ideal. Hence, $(bmx - 1) - xbm \in L$ since L is closed under addition. Hence we have that $[bm, x] - 1 \in L$.

Further, since $m \in L$ (from our assumption) we have that $bm \in L$, since L is a left ideal. From our hypothesis we have that $[l, x] \in QN(A)$ for every $l \in L$, hence we get that $[bm, x] \in QN(A)$. We recall that $QN(A) = \{x \in A \mid \sigma(x) = \{0\}\}$. Hence it follows that

$$\sigma([bm, x]) = \{0\}$$

This means that $[bm, x] - 1 \in A^{-1}$ because $1 \notin \{0\}$. But we have also shown that $[bm, x] - 1 \in L$. This contradicts the fact that L is a proper left ideal. Hence we have that $Lx \subseteq L$. □

Corollary 4.1.4. *[1, Corollary 4] Let an element x in a Banach algebra A have the property that $[x, a] \in QN(A)$ for every $a \in A$ and let L be a maximal left ideal of A. Then there is a unique $\lambda \in \mathbb{C}$ such that $x - \lambda \in L$.*

Proof.
From our hypothesis, and Lemma 4.1.3 we have that $Lx \subseteq L$. But then we have from Theorem 4.1.1 that there exists a $\lambda \in \mathbb{C}$ such that $x - \lambda \in L$. We have also shown in the proof of Theorem 4.1.1 that this λ is uniquely determined. Hence our result follows. $\square$

Lemma 4.1.5. *[1, Lemma 5] Let L be a maximal left ideal of a Banach algebra A and let $x \in A$. Suppose that for every $u \in A^{-1}$ there is some $\lambda = \lambda(u) \in \mathbb{C}$ with $(x - \lambda)u \in L$. Then there is a unique $\lambda \in \mathbb{C}$ with $(x - \lambda)A \subseteq L$.*

Proof.
Suppose first that for each $u \in A^{-1}$, the $\lambda = \lambda(u) \in \mathbb{C}$ with $(x-\lambda)u \in L$ is not unique. Then we have that there exists a λ_1 and λ_2 such that $(x - \lambda_1)u \in L$ and $(x - \lambda_2)u \in L$. Since L is a left ideal it is closed under addition, and we have that $(x - \lambda_1)u - (x - \lambda_2)u \in L$. Hence it follows that $xu - \lambda_1 u - xu + \lambda_2 u \in L$, and so $(\lambda_2 - \lambda_1)u \in L$. But since we have that $u \in A^{-1}$ it follows that $u \notin L$ since L is a left ideal. Hence we must have that $(\lambda_2 - \lambda_1) = 0$ so that $\lambda_2 = \lambda_1$. Hence, for each $u \in A^{-1}$, the $\lambda = \lambda(u) \in \mathbb{C}$ with $(x - \lambda 1)u \in L$ is unique.

From our hypothesis, with $u = 1$, there is a unique $\mu \in \mathbb{C}$ with $(x - \mu)1 = x - \mu \in L$. We want to show that $\lambda(u) = \mu$ for every $u \in A^{-1}$. Thus we let $u \in A^{-1}$, and we consider the following two cases:

Case i:
The set $\{1, u\}$ is linearly dependent modulo L, hence there are $\alpha, \beta \in \mathbb{C}$, not both zero, with $\alpha + \beta u = \alpha 1 + \beta u \in L$. Then we have that $(x - \mu)(\alpha + \beta u) \in L$ since L is a left ideal. Hence we have that $\alpha(x - \mu) + \beta(x - \mu)u \in L$. But we note that $\alpha(x - \mu) \in L$ since we have from our assumption that $(x - \mu) \in L$ and that L is a left ideal. Hence it follows that $\alpha(x-\mu)+\beta(x-\mu)u-\alpha(x-\mu) \in L$ since L is a left ideal, which is closed under addition. So we have $\beta(x - \mu)u \in L$. But then, since L is a left ideal we have that $\beta^{-1}\beta(x - \mu)u \in L$ $(\alpha\beta \neq 0)$, which implies that $(x - \mu)u \in L$ and that $\lambda(u) = \mu$.

Case ii:
The set $\{1, u\}$ is linearly independent modulo L. We choose $\alpha \in \mathbb{C}\backslash\{0\}$ with $u+\alpha \in A^{-1}$ (e.g. any α with $|\alpha| > \|u\|$). Hence, by our hypothesis there are $\gamma, \delta \in \mathbb{C}$ with $(x-\gamma)u \in L$ and $(x-\delta)(u+\alpha) \in L$ and we have from our assumption that $x-\mu \in L$ so that $\alpha(x-\mu) \in L$ since L is a left ideal. Hence $(x-\delta)(u+\alpha)-(x-\gamma)u-\alpha(x-\mu) \in L$. This implies that $xu+\alpha x-\delta u-\delta\alpha-xu+\gamma u-\alpha x+\alpha\mu \in L$, which means we have that $(\gamma-\delta)u+\alpha(\mu-\delta) \in L$. But, since this is the case and $\{1, u\}$ are linearly independent modulo L, it follows that $\gamma = \delta$ and $\mu = \delta$. Hence we have that $\gamma = \mu$ which implies that $(x-\mu)u \in L$ and that $\lambda(u) = \mu$.

Since u was arbitrarily chosen in both cases, we have that:

$$(x - \mu)A^{-1} \subseteq L.$$

By Lemma 2.3.1, $A = A^{-1} + A^{-1}$. This together with L being a left ideal, implies

$$(x - \mu)A \subseteq L.$$

□

Lemma 4.1.6. *[1, Lemma 6] Let A be a Banach algebra and let $x \in A$ have the property that $[x, a] \in QN(A)$ for every $a \in A$ and let L be a maximal left ideal of A. Then there is a unique $\lambda \in \mathbb{C}$ with $(x - \lambda)A \subseteq L$.*

Proof. From our hypothesis, we have that $[x, a] \in QN(A)$ for all $a \in A$. In particular we then have that $[x, uau^{-1}] \in QN(A)$. We recall that $QN(A) = \{x \in A \mid \sigma(x) = \{0\}\}$. To assist with making this proof and notation a little bit easier, we let $b = [x, uau^{-1}]$ so that $b \in QN(A)$ from the above. This means that $\lambda - b \in A^{-1}$ for all $\lambda \neq 0$. Now from this, we have that $u^{-1}(\lambda - b)u \in A^{-1}$ since $u \in A^{-1}$ and A^{-1} is closed under multiplication. Hence $u^{-1}\lambda u - u^{-1}bu \in A^{-1}$ which implies that $\lambda - u^{-1}bu \in A^{-1}$ for all $\lambda \neq 0$. This means that $\sigma(u^{-1}bu) = \{0\}$ and we have that $u^{-1}bu \in QN(A)$. So it follows that $u^{-1}[x, uau^{-1}]u \in QN(A)$ from our assumption of $b = [x, uau^{-1}]$, so we get the following:

$$\begin{aligned} u^{-1}(xuau^{-1} - uau^{-1}x)u &\in QN(A), \\ u^{-1}xua - au^{-1}xu &\in QN(A), \\ [u^{-1}xu, a] &\in QN(A). \end{aligned}$$

Hence, from Corollary 4.1.4, there is a unique $\lambda \in \mathbb{C}$ such that

$$\begin{aligned} u^{-1}xu - \lambda &\in L, \\ u^{-1}xu - u^{-1}\lambda u &\in L, \\ u^{-1}(x - \lambda)u &\in L, \\ uu^{-1}(x - \lambda)u &\in L \text{ (since } L \text{ a left ideal)}, \\ (x - \lambda)u &\in L. \end{aligned}$$

Since u was arbitrarily chosen in A^{-1}, it follows from Lemma 4.1.5 that there is a unique $\lambda \in \mathbb{C}$ with $(x - \lambda)A \subseteq L$. □

4.2 Analytic proof

We are now in a position to prove our main result.

Theorem 4.2.1. *[1, Theorem 1] Let A be a Banach algebra and let $x \in A$ have the property that $[x, a] \in QN(A)$ for every $a \in A$. Then $[x, a] \in Rad(A)$ for every $a \in A$.*

Proof.
From our hypothesis, we have that x has the property that $[x, a] \in QN(A)$ for every $a \in A$. Hence, from Lemma 4.1.6, we have that for a maximal left ideal L, there is a unique $\lambda \in \mathbb{C}$ with $(x-\lambda)A \subseteq L$. Hence $(x-\lambda)a \in L$ for all $a \in A$. So in particular we have that $(x-\lambda)1 \in L$. But since L is a left ideal, we have that $a(x-\lambda) \in L$.

So, since L is closed under addition, we have that $(x-\lambda)a-a(x-\lambda) \in L$ and hence it follows that $[x-\lambda, a] \in L$ for all $a \in A$. We note that

$$\begin{aligned}[x-\lambda, a] &= (x-\lambda)a - a(x-\lambda)\\ &= xa - \lambda a - ax + \lambda a\\ &= [x, a]\end{aligned}$$

Hence $[x, a] \in L$ for every $a \in A$.

Now, since L was arbitrarily chosen, we have that $[x, a]$ is in the intersection of all maximal left ideals of A. So for every $a \in A$ (see Theorem 2.2.1 and the remarks following it), it follows that

$$[x, a] \in Rad(A).$$

Note that the above Theorem was proved in [1, Theorem 1] by using the notion of a primitive ideal. Clearly, our proof indicates that this is not necessary. □

We conclude with a final corollary.

Corollary 4.2.2. *[1, Corollary 7] Let A be a Banach algebra and let $x \in A$. Then $x \in Rad(A)$ if and only if both $x \in QN(A)$ and $[x, a] \in QN(A)$ for all $a \in A$.*

Proof.
$\Rightarrow$:

1. Let $x \in Rad(A)$. By Proposition 2.3.5, $x \in QN(A)$.

2. Let $x \in Rad(A)$ and $a \in A$. Then both ax and $xa \in Rad(A)$, since the radical is a two-sided ideal. Hence we have that $xa - ax \in Rad(A)$, since the radical is closed under addition. So it follows that $[x, a] \in Rad(A)$. Hence from (1.), for every $a \in A$ it follows that
$$[x, a] \in QN(A).$$

$\Leftarrow$:
Now, suppose that $x \in QN(A)$ and $[x, a] \in QN(A)$ for all $a \in A$. We define $\pi : A \mapsto A/Rad(A)$, which implies that $\pi(x) = x + Rad(A)$. Hence we have that $\sigma(\pi(x)) \subset \sigma(x) = \{0\}$ since $x \in QN(A)$. This implies that $\sigma(\pi(x)) = \{0\}$, so that $\sigma(x + Rad(A)) = \{0\}$. Hence we have that $x + Rad(A) \in QN(A/Rad(A))$.

Also, $[x, a] \in QN(A)$ which means that $[x, a] \in Rad(A)$ for all $a \in A$ (by Theorem 4.2.1). This mean that $[x, a] + Rad(A) = Rad(A)$, or alternatively that $(xa - ax) + Rad(A) = Rad(A)$. So it follows that $xa + Rad(A) = ax + Rad(A)$. This implies that $(x + Rad(A))(a + Rad(A)) = (a + Rad(A))(x + Rad(A))$ for all $a \in A$.

Now, since $x + Rad(A) \in QN(A/Rad(A))$ and $(x + Rad(A))(a + Rad(A)) = (a + Rad(A))(x + Rad(A))$ for all $a \in A$, it follows from Proposition 2.3.6 that $x + Rad(A) \in Rad(A/Rad(A))$. But $Rad(A/Rad(A)) = \{Rad(A)\}$ from [3, Theorem 3.1.5], hence $x + Rad(A) = Rad(A)$. So it follows that $x \in Rad(A)$. $\square$

Conclusion

In our dissertation, our aim was to prove the following theorem:

Theorem 0.0.1. *Let A be a Banach algebra and let $a \in A$ have the property that $[a, x]$ is quasi-nilpotent for every x in A. Then $[a, x]$ belongs to the radical of A for every x in A.*

Many authors have made use of Representation theory in the proof of the above theorem. Although Representation theory in the theory of Banach algebras is a very powerful tool, there are also different techniques that can be used to achieve desired results. In this dissertation we started in Chapter 3 with the use of Representation theory to prove our main result. However, in Chapter 4 we made use of elementary complex analysis to prove this main result.

As we have demonstrated in this dissertation, our result can be achieved by using either of these techniques and we note that both of these approaches are very useful and relevant in their own way. Representation theory is widely used and recognised. It is very useful and gets straight to the point, however, it has the disadvantage that it involves another level of abstraction. This sometimes makes it difficult to understand. In this dissertation we have made use of analytic techniques as an alternative to representation theory to prove Theorem 0.0.1. This technique has the advantage that it is easy to understand and to follow. It also doesn't require full background knowledge of complex theorems and definitions. However, the disadvantage of this technique is that it can become laborious and lengthy.

We have gone through both of these techniques in detail to get to our result. Both have got their own benefits and constraints, however they are very useful irrespective of which one the reader prefers, as we have demonstrated that we get the same result with both. However, my personal preference leans more on the use of complex analysis. It is much easier to work with, and as a result creates opportunity to investigate alternative avenues and results. It is also easier to explain to others, which opens the door to integration with other fields of study. However, I reiterate that this does not lessen the importance of Representation theory.

As per the *editorial note* in [1], further results related and analogous to Theorem 0.0.1 (our main result in this dissertation) have been subsequently obtained in [2], [3], [4], [11], [12], [13] and [14]. Some interesting future work can be done in each of these cases, with the similar use of analytic techniques. It would also be interesting to determine if one singular approach can be found or modified for all of these cases.

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