REGULARITIES AND THE INDEX FOR FREDHOLM ELEMENTS IN A BANACH ALGEBRA VIA A TRACE

by

ANDRÉ MICHAEL SWARTZ

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SUPERVISOR: PROF H RAUBENHEIMER

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Dedication

I dedicate this work to the memory of my parents.

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Introduction

Fredholm theory in Banach algebras had its start with the two papers by Barnes, [5, 6]. In [6] the author defines Fredholm elements of a ring as those elements which are invertible modulo the socle. This set of Fredholm elements forms an open semigroup on which the author defines a generalized index function. The index function is based on the order of an ideal. The index function is then shown to satisfy a multiplicative property and to be continuous on the set of Fredholm elements. In [10], the authors generalize the definition of a Fredholm element to elements which are invertible modulo an ideal. On such an ideal a *trace* supplies the structure required to define an index function. The content of [10] and [12] describe the development of this theory and shows that the index defined in this way has the same desirable attributes as the index for Fredholm operators defined on a Banach space relative to the ideal of compact operators. In addition to the papers by Barnes, see also [7], Chapter F. In this thesis we are going to focus on an index function defined on Fredholm elements in a Banach algebra relative to an ideal on which a trace is defined. The basic properties of this index function is developed in [10]. However, one fact that eluded the authors of [10] was to prove that an index zero Fredholm element can be decomposed as the sum of an invertible and a socle element. One of the goals of this thesis is to prove that an index zero Fredholm element has the property that it can be written as a sum of an invertible element and a socle element. We prove this in Chapter 2, section 2.4 and we also refer the reader to [12].

In the context of spectral theory, a regularity is a generalization of the group of invertible elements in a Banach algebra. Just as the spectrum of an element is defined relative to the group of invertible elements, we define the R - spectrum of an element relative to a regularity R. A component concept is that of a semiregularity, and the properties of regularities and semiregularities have been extensively investigated by Müller and others and documented in [23], and [24]. Some regularities and semiregularities have an interesting property of being spectral radius preserving. A second goal in this thesis is to identify spectral radius preserving regularities and semiregularities. In [17] the authors present us with a way of identifying spectral radius preserving regularities and semiregularities. A theme of Chapter 3 is the use of this theorem in investigating the spectral radius preserving properties of certain regularities and semiregularities.

The thesis is organized as follows. In chapter 1 we gather basic notions and facts that will be used in later chapters. New results in the thesis appear in chapters 2 and 3. When we refer to Theorem x.y.z in the thesis, it is Theorem z in section y of chapter x.

Chapter 1

Banach algebras and spectral theory

1.1 Overview

In this chapter we set up all the tools we need for our discussions to come. This includes the definitions of all the basic concepts we will require. Having these tools readily available will allow us to follow a train of thought without being sidetracked by the definition of basic concepts. For ease of reference, we also state Jacobson's Density Theorem.

1.2 Banach algebras

An *algebra* is a vector space A over a field K, with a multiplication operation such that for all $x, y, z \in A$ and $\lambda \in K$:

1. $x(yz) = (xy)z$
2. $(x + y)z = xz + yz$
3. $x(y + z) = xy + xz$
4. $\lambda(xy) = (\lambda x)y = x(\lambda y)$.

If, in addition, A is a Banach space for a norm $||\cdot||$ and satisfies the norm inequality $||xy|| \leq ||x|| \cdot ||y||$, for all $x, y \in A$, we say that A is a *Banach algebra.*

If the field K is either $\mathbb{R}$ or $\mathbb{C}$, the Banach algebra is called a *real* or *complex* Banach algebra.

An element $1 \in A$ such that for all $x \in A$

$$1x = x1 = x$$

is called an *identity* of A. In this thesis 1 will always denote the identity of a Banach algebra. When we wish to emphasize the Banach algebra, we will use 1_A to represent the identity.

If a Banach algebra has an identity, it is unique and the Banach algebra is called *unital*. If a Banach algebra A has an identity 1, we can assume that $||1|| = 1$, otherwise we can replace $||\cdot||$ with an equivalent norm $|||\cdot|||$, satisfying $|||1||| = 1$ and $|||xy||| \leq |||x||| \cdot |||y|||$ for all $x, y \in A$.

If a Banach algebra A does not have an identity, it is always possible to embed it isometrically in the Banach algebra with identity $\tilde{A} = A \times \mathbb{C}$. Here we define the operations and the norm as:

1. $(x, \alpha) + (y, \beta) = (x + y, \alpha + \beta)$,
2. $\lambda(x, \alpha) = (\lambda x, \lambda \alpha)$,
3. $(x, \alpha) \cdot (y, \beta) = (xy + \beta x + \alpha y, \alpha\beta)$,
4. $||(\alpha, x)|| = ||x|| + |\alpha|$

In this document, A will always represent a complex, unital Banach algebra.

A set B is called a *subalgebra* of A if B is a subspace of A and B is algebraically closed under the operation of multiplication.

A Banach algebra A is called *commutative* if $xy = yx$ for all $x, y \in A$.

If X is a (complex) Banach space, denote by $\mathcal{L}(X)$ the Banach space of bounded linear operators defined on X. If we define multiplication on $\mathcal{L}(X)$ as composition of operators, then $\mathcal{L}(X)$ becomes a Banach algebra under the operator norm $||T|| = \sup\{||Tx|| : ||x|| = 1\}$, because $||ST|| \leq ||S|| \cdot ||T||$ for all $S, T \in \mathcal{L}(X)$. The identity in $\mathcal{L}(X)$ is the operator I in $\mathcal{L}(X)$. $\mathcal{L}(X)$ is not commutative, unless $dim X = 1$.

An element $x \in A$, is called *left invertible* (respectively *right invertible*) if there exists $y \in A$ such that $yx = 1$ (respectively $xy = 1$). The element y is called a *left inverse* (*right inverse*) of x. If an element $x \in A$ is both left and right invertible, the left and right inverses coincide and x is called *invertible*. In that case the inverse of x is denoted by x^{-1}.

The sets of left invertible, right invertible and invertible elements in A are denoted by A_l^{-1}, A_r^{-1} and A^{-1} respectively.

An element $a \in A$ is called *nilpotent* if there exists $n \in \mathbb{N}$ such that $a^n = 0$.

An element $p \in A$ such that $p^2 = p$ is called an *idempotent*.

A nonzero idempotent p is called *minimal* if pAp is a division algebra.

A set W of idempotents of A is *orthogonal* if $ef = fe = 0$ for $e, f \in W$ and $e \neq f$.

Two elements a and b in a Banach algebra A are called *similar* if there exists an element $u \in A^{-1}$ such that $a = u^{-1}bu$.

An element a of a Banach algebra A is called a *left topological divisor of zero* if

$$\inf\{||ax|| : x \in A, ||x|| = 1\} = 0.$$

Similarly, a is a *right topological divisor of zero* if

$$\inf\{||xa|| : x \in A, ||x|| = 1\} = 0.$$

Let A be a Banach algebra. A set $I \subset A$ is called a *left (right) ideal* in A if I is a linear subspace of A and $ax \in I$ ($xa \in I$) for every $a \in A, x \in I$. If I is both a left and a right ideal of A we will call it a *two-sided ideal.* An ideal (left, right or two-sided) I of A is called *proper* if $I \neq A$.

A (left, right, two-sided) ideal I of A is *minimal* if $I \neq \{0\}$ and if for any (left, right, two-sided) ideal $I' \subset I$ either $I' = \{0\}$ or $I' = I$.

A (left, right, two-sided) ideal I of A is called *maximal* if I is a proper ideal and if the only proper ideal (left, right, two-sided) that contains I is I itself.

Let I be a closed two-sided ideal of A. Then the quotient A/I is a Banach space for the norm

$$||x + I|| = \inf\{||x + u|| : u \in I\}$$

where

$$A/I = \{x + I : x \in A\}$$

and addition and scalar multiplication are defined by

$$(x+I)+(y+I)=(x+y)+I \text{ and}$$
$$\alpha(x+I)=\alpha x+I$$

for all $x, y \in A$ and $\alpha \in \mathbb{C}$.
With the above norm, the quotient Banach space A/I becomes a Banach algebra because

$$||(x+I)\cdot(y+I)|| \leq ||x+I||\cdot||y+I||$$

for all $x, y \in A$ if we define multiplication in A/I by

$$(x+I)\cdot(y+I)=xy+I.$$

1.3 Spectral theory

Let A be a Banach algebra. For $x \in A$ we define the *spectrum* of x as

$$\sigma(x)=\{\lambda \in \mathbb{C} \;:\; \lambda - x \notin A^{-1}\}.$$

The following two results are stated without proof and demonstrate an interesting property of the spectrum function.

Theorem 1.3.1 *([24], Theorem 1.1.29) Let $a, b \in A$ and let λ be a non-zero complex number. Then $ab-\lambda$ is left (right) invertible if and only if $ba-\lambda$ is left (right) invertible.*

Corollary 1.3.2 *([24], Corollary 1.1.30) Let x, y be elements of a Banach algebra A. Then*

$$\sigma(xy)\setminus\{0\}=\sigma(yx)\setminus\{0\}.$$

We define the *spectral radius* of $x \in A$ as

$$r(x)=max\{|\lambda| \;:\; \lambda \in \sigma(x)\}.$$

Let A be a Banach algebra. An element $a \in A$ is *quasinilpotent* if $\sigma(x)=\{0\}$. The set of all such elements is denoted by $\mathrm{QN}(A)$.

Let $K \subset \mathbb{C}$, K compact. Then ηK denotes the *connected hull of* K, i.e., ηK is the union of K with the bounded components of the complement of K ([20], p 342).

1.4 The Radical and the Socle of a Banach algebra A

Let A be a Banach algebra. Two important ideals for A is the radical of A and the socle of A. We describe these here. We state the following result without proof:

Theorem 1.4.1 *([24], Theorem 1.1.41) Let A be a Banach algebra. The following sets are identical:*

1. *the intersection of all maximal left ideals in A.*
2. *the intersection of all maximal right ideals in A.*
3. *the set of all $x \in A$ such that $1 - ax$ is invertible for every $a \in A$.*
4. *the set of all $x \in A$ such that $1 - xa$ is invertible for every $a \in A$.*

The set of all x with the properties 1 - 4 in the previous theorem is called the *radical* of A and we will denote it by $\operatorname{Rad} A$.

A is said to be *semisimple* if $\operatorname{Rad} A = \{0\}$. A is called *semiprime* if $0 \neq u \in A$ implies there exists $x \in A$ such that $uxu \neq 0$. All semisimple Banach algebras are semiprime ([8], Proposition VI.30.5).

Let $\{I_\lambda : \lambda \in \Lambda\}$ be a family of left (right) ideals in an algebra A. Then the smallest left (right) ideal that contains each I_λ is called the *sum* of the ideals I_λ. The sum of all minimal left (right) ideals in A is called the *left (right) socle*. The left (right) socle exists if and only if A contains minimal left (right) ideals. If the left and right socles exist and are equal, then the resulting two-sided ideal is simply called the *socle* of A. We denote the socle of A by $\operatorname{Soc} A$.

If I is a closed ideal in A then $b \in A$ is called *Riesz* relative to I if $b+I \in \operatorname{QN}(A/I)$. We denote by $R(A, I)$ the set of elements in A which are Riesz relative to I. The set $\mathsf{kh}(I)$ is defined by $\mathsf{kh}(I) := \{b \in A \ : \ b + I \in \operatorname{Rad}(A/I)\}$. The following inclusions are clear:

$$I \subset \mathsf{kh}(I) \subset R(A, I).$$

An element $a \neq 0$ in a semiprime Banach algebra A is called of *rank one* if there exists a linear functional f_a on A such that $axa = f_a(x)a$ for all $x \in A$.

Minimal idempotents are examples of rank one elements ([8], Proposition VI.31.3), and conversely, if a is a rank one element, then $p = f_a(1)^{-1}a$ is easily seen to be a minimal idempotent.

The *finite rank elements* of A, denoted by $F(A)$, is the set of all $a \in A$ of the form $a = \sum_{i=1}^{n} a_i$ with each a_i a rank one element. By [26], Lemma 2.7, $F(A)$ is an ideal in A. In the case of a semiprime Banach algebra, $F(A) = \operatorname{Soc} A$.

If X is a Banach space, then two well known ideals in the Banach algebra $\mathcal{L}(X)$ are the closed ideal $\mathcal{K}(X)$ of compact operators defined on X and the ideal $\mathcal{F}(X)$ of finite rank operators defined on X.

1.5 Regularities and semiregularities

Within the spectral theory of Banach algebras lie a vast number of spectra, defined for a single element of a Banach algebra. These spectra have been investigated extensively. It is an interesting question as to whether all spectra can be described axiomatically. Such a theory would attempt to formulate properties characteristic of spectra in general and provide a means of classification. Potentially, the theory would define spectra in a simple and general manner. To date, significant progress has been made in this regard, and the topic remains an active and widely discussed issue for researchers in the field.

Research in this direction began in 1974 when Zbigniew Słodkowski and Wieslaw Z̊elazko ([30]) examined the *joint spectrum* of an n-tuple of elements in a commutative, complex, unital Banach algebra. Their work examined the basic properties of a number of familiar spectra in the literature, and focused on establishing the spectral mapping theorem for these spectra. In 1979 Z̊elazko followed up on earlier progress made in [31]. However, there arise a number of spectra, usually defined for a single element of a Banach algebra, that are not covered by the axiomatic theory of Z̊elazko.

In 1996, Vladimír Kordulla, Vladimir Müller and Mostafa Mbekhta addressed this issue and created the theory of *regularities* in [15] and [31]. Their unique idea was not to describe a spectrum axiomatically, but rather to describe the underlying set of elements in a Banach algebra which gives rise to the spectrum. Their work was initiated by a simple axiomatic definition of a *regularity* chosen in such a way that there were many natural classes of elements in a Banach algebra satisfying it. These classes included the collection of invertible elements, left and right invertible elements, the collection of all elements that are not left (right) topological divisors of zero, and various classes of operators in the Banach algebra $\mathcal{L}(X)$. The principal consequence of the theory of regularities was that every regularity R gave rise to a corresponding spectrum σ_R defined by

$$\sigma_R(a) = \{\lambda \in \mathbb{C} : \lambda - a \notin R\}$$

for every $a \in A$.

The definition of a regularity provided a starting point from which to develop the theory further. In 2000, Müller introduced the theory of *semiregularities* ([23]). By dividing the axioms of the definition of a regularity into two halves, it gave rise to what Müller termed *lower semiregularity* and *upper semiregularity*. As with regularities, semiregularities gave rise to spectra which only satisfied a one-way Spectral Mapping Theorem. The spectra arising from regularities satisfy the Spectral Mapping Theorem. Semiregularities were defined in such a way that every regularity was both a lower semiregularity and an upper semiregularity. Because of this there existed a multitude of examples of semiregularities. However, the discovery of lower semiregularities that were not upper semiregularities and vice versa demonstrated that these theories were by no means symmetric, and gave merit to Müller's work. Furthermore, it provided a means with which to classify familiar spectra like the exponential spectrum and the Weyl spectrum which did not satisfy the definition of a regularity. In essence, semiregularities completed the theory of regularities.

Regularities have been a research interest since 1996. The theory has been examined in connection with various classes of bounded linear operators (defined by means of kernels and ranges) ([19]), Fredholm theory ([17]), commutative Banach algebras and recently, generalized invertibility ([18]). The notions defined in this section will be investigated in Chapter 3. For all undefined concepts concerning regularities and semiregularities we refer the reader to [15, 19, 23, 24].

Definition 1.5.1 ([23], Definition 1) Let R be a non-empty subset of a Banach algebra A. Then R is called a *lower semiregularity* if

1. $a \in A, n \in \mathbb{N}, a^n \in R \implies a \in R$,
2. if a, b, c, d are mutually commuting elements of A satisfying $ac + bd = 1_A$ and $ab \in R$ then $a, b \in R$.

Müller gives us the following simplified way of showing that a nonempty subset of a Banach algebra is a lower semiregularity.

Lemma 1.5.2 *([23], Remark 3) Let A be a Banach algebra and suppose that $R \subset A$ is a nonempty subset satisfying*

$$a, b \in A, ab = ba, ab \in R \implies a, b \in R$$

Then clearly R is a lower semiregularity.

Definition 1.5.3 ([23], Definition 10) A subset R of a Banach algebra A is called an *upper semiregularity* if

1. $a \in R, n \in \mathbb{N} \implies a^n \in R$,
2. if a, b, c, d are mutually commuting elements of A satisfying $ac + bd = 1_A$ and $a, b \in R$ then $ab \in R$,
3. R contains a neighbourhood of the unit element 1_A.

Again, Müller gives us the following simplified way of showing that a nonempty subset of a Banach algebra is an upper semiregularity.

Lemma 1.5.4 *([23], Remark 11) Let R be a nonempty subset of a Banach algebra A. Suppose R is a semigroup that contains a neighbourhood of 1_A. Then R is an upper semiregularity.*

A regularity R is both an upper and a lower semiregularity.

Definition 1.5.5 ([24], Definition 1.6.1) A subset R of a Banach algebra A is called a *regularity* if

1. if $a \in A, n \in \mathbb{N}$ then $a \in R \iff a^n \in R$,
2. if a, b, c, d are mutually commuting elements of A satisfying $ac + bd = 1_A$ then $a, b \in R \iff ab \in R$.

The following theorem provides us with a simplified way to show that certain sets are regularities.

Theorem 1.5.6 *([24], Theorem 1.6.4) Let R be a non-empty subset of a Banach algebra A satisfying*

$$ab \in R \iff a \in R \text{ and } b \in R$$

for all commuting elements $a, b \in A$. Then R is a regularity.

The property in Theorem 1.5.6 is referred to as the P1 property and we call a regularity that satisfies this property a P1 regularity.

We now generalize the property mentioned in Theorem 1.3.1 and Corollary 1.3.2. Let A be a Banach algebra and $M \subset A$. Let $a, b \in A$, $\lambda \in \mathbb{C}$, $\lambda \neq 0$. If M satisfies

$$\lambda - ab \in M \iff \lambda - ba \in M$$

then we will say that M satisfies the *Jacobson property*.

A regularity (semiregularity) R in a Banach algebra A assigns to each $a \in A$ a subset $\sigma_R(a)$ of $\mathbb{C}$ called the *spectrum of a relative to R* with

$$\sigma_R(a) = \{\lambda \in \mathbb{C} : \lambda - a \notin R\}.$$

In Chapter 3 we are going to compare regularities (semiregularities) R and S in A with $R \subset S$ and also investigate the spectra $\sigma_R(a)$ and $\sigma_S(a)$, $a \in A$.

1.6 Jacobson's Density Theorem

Let A and B be Banach algebras. We call a linear map $T : A \to B$ a *homomorphism* if $T(ab) = (Ta)(Tb)$ and $T1_A = 1_B$ for all $a, b \in A$.

Let A be a complex Banach algebra and let X be a complex vector space of dimension greater than or equal to 1. Let $\mathcal{L}(X)$ be the algebra of linear operators on X. Let $\pi : A \to \mathcal{L}(X)$ be a nontrivial homomorphism. Then we call π a *representation of* A *on* X.

If a linear subspace Y of X satisfies $\pi(x)Y \subset Y$ for all $x \in A$, we say that Y is *invariant under* π. A representation π is said to be *irreducible* if the only linear subspaces invariant under π are $\{0\}$ and X.

The representation π is said to be *bounded* if X is a Banach space and if $\pi(x)$ is a bounded linear operator on X for all $x \in A$. It is said to be *continuous* if it is bounded and if there exists a constant $C > 0$ such that $||\pi(x)|| \leq C||x||$ for all $x \in A$.

Theorem 1.6.1 *(Jacobson's Density Theorem) ([1], Theorem 4.2.5) Let π be a continuous irreducible representation of a Banach algebra A on a Banach space X. If $\xi_1, ..., \xi_n$ are linearly independent in X and if $\eta_1, ..., \eta_n$ are in X there exists $a \in A$ such that $\pi(a)\xi_i = \eta_i$, for $i = 1, ..., n$.*

Corollary 1.6.2 *(A. Sinclair) ([1], Corollary 4.2.6) With the hypotheses of Theorem 1.6.1, we suppose further that $\eta_1, ..., \eta_n$ are linearly independent. Then there exists a, invertible in A, such that $\pi(a)\xi_i = \eta_i$ for $i = 1, ..., n$.*

Chapter 2

The index for Fredholm elements in a Banach algebra via a trace

2.1 Introduction

The existence of a continuous trace on an operator ideal of operators on a Banach space has long been known to provide a useful tool for developing the Fredholm theory of operators (see for instance the monograph of A. Pietsch, [25] and the paper [13]). The problem of defining traces on ideals in a Banach algebra attracted the attention of many authors, (see the papers by Puhl, [26] and Aupetit and Mouton, [2]). The main thrust of these papers was to show that a trace and a Fredholm determinant exist on the socle of a semisimple Banach algebra ([2], [26]) and on the question of extending the trace or determinant to larger ideals ([26] and [3]). Our aim in this chapter, on the one hand, is to present an axiomatic approach by assuming a trace to exist and then to show how useful such a trace is by applying it firstly to develop the index theory for abstract Fredholm elements in a semisimple Banach algebra (see the research notes of Barnes, Murphy, Smyth and West, [7] and of Caradus, Pfaffenberger and Yood, [9]). Here, the commutativity property of the trace provides some elegant proofs of the properties of the index.

Next, we define a concept which is central for the rest of the chapter.

Definition 2.1.1 ([10], Definition 2.1) Let I be an ideal in a Banach algebra A. A function $\tau : I \to \mathbb{C}$ is called a trace on I if:
(TN) $\tau(p) = 1$ for every rank one idempotent $p \in I$.
(TA) $\tau(a + b) = \tau(a) + \tau(b)$ for all $a, b \in I$.
(TH) $\tau(\alpha a) = \alpha\tau(a)$ for all $\alpha \in \mathbb{C}$ and $a \in I$.
(TC) $\tau(ba) = \tau(ab)$ for all $a \in I$ and $b \in A$.

We shall refer to an ideal on which a trace is defined as a *trace ideal*. A trace τ on I is called *nilpotent* if $\tau(a) = 0$ for all nilpotent elements $a \in I$.

In the following discussion we present three examples of traces in Banach algebras.

Example 2.1.2 Let X be a Banach space and let T be a continuous linear operator on X such that its range $Y := R(T)$ is finite-dimensional. Let $\{x_1, ..., x_n\}$ be a basis for Y. Then there exist continuous linear functionals $\{x'_1, ..., x'_n\}$ such that

$$Tx = \sum_{i=1}^{n} x'_i(x)x_i = (\sum_{i=1}^{n} x'_i \otimes x_i)(x).$$

The trace $\tau(T)$ of T is defined as

$$\tau(T) = \sum_{i=1}^{n} x'_i(x_i).$$

Note that this is the trace of the matrix $(x'_i(x_j))_{i,j=1}^n$, which, as can easily be checked, is the matrix representing the operator T restricted to Y. Since this trace is independent of the choice of the basis, it is a well defined number. From the well known properties of the matrix trace it follows that the conditions (TN), (TA), (TH) and (TC) above are satisfied for this trace on the ideal consisting of all operators with finite-dimensional range.

An example of a trace on the ideal of finite rank elements in a Banach algebra A was provided by Puhl in [26]. Let A be a semiprime Banach algebra and let $0 \neq u \in A$ be a rank one element. Puhl defined a trace $tr(u)$ of u to be given by:

$$u^2 = tr(u)u$$

(see [26], Section 2). It follows from the definition of rank one elements that $tr(u) = f_u(1)$, and that $tr(p) = 1$ for every every rank one idempotent p.
If $u \in \operatorname{Soc} A$ and $u = \sum_{i=1}^{n} u_i$ with all the u_i rank one elements then $tr(u) := \sum_{i=1}^{n} tr(u_i)$ ([26], Definition 4.4) is well defined ([26], Section 4). It easily follows from the properties of a general trace τ on an inessential ideal I that τ coincides with Puhl's trace on $I \cap \operatorname{Soc} A$. In fact, if a is a rank one element, then, as we mentioned earlier, $p = f_a(1)^{-1}a$ is a minimal idempotent. Consequently, $1 = \tau(p) = f_a(1)^{-1}\tau(a)$. Hence, $\tau(a) = f_a(1) = tr(a)$. By the linearity of τ it follows that $\tau(a) = tr(a)$ for all $a \in \operatorname{Soc} A$. Puhl also proves that his trace is nilpotent on $\operatorname{Soc} A$, and hence by what we have shown, any trace restricted to $\operatorname{Soc} A$ is also nilpotent.

Finally, we remark that B. Aupetit and H. du T. Mouton in [2] also defined a trace on $\operatorname{Soc} A$ for A a semiprime Banach algebra, by putting

$$Tr(a) = \sum_{\lambda \in \sigma(a)} m(\lambda, a)\lambda,$$

where the number $m(\lambda, a)$ turns out to be the algebraic multiplicity of λ for a. This trace is of course nilpotent on $\mathrm{Soc}\, A$ and it has the properties of an abstract trace. It is therefore equal to the Puhl trace and consequently coincides with any abstract trace defined on $\mathrm{Soc}\, A$.

As shown by the definition of Aupetit and Mouton, the trace on finite rank elements is *spectral*, i.e., if $\lambda_1, \cdots, \lambda_n$ are the eigenvalues of a, each repeated according to its algebraic multiplicity, then

$$\tau(a) = \sum_{i=1}^{n} \lambda_i.$$

Since for every idempotent $p \in \mathrm{Soc}\, A$ we have $\sigma(p) \subset \{0, 1\}$, we immediately see that $\tau(p) \in \mathbb{N}$ for every such p.

The fact that any abstract trace τ defined on I corresponds to the Puhl trace on $I \cap \mathrm{Soc}\, A$ shows that τ can be extended to elements of $\mathrm{Soc}\, A$. It can thus be extended to the ideal $I + \mathrm{Soc}\, A$. For this reason we will assume that if a trace is defined on an ideal I, then I includes $\mathrm{Soc}\, A$.

2.2 Fredholm elements and index theory

We start off in a very general setting in which the assumptions on the trace are minimal (in particular, we do not assume the trace ideal to be closed).

Let I be a proper ideal of a Banach algebra A and suppose that $\tau : I \to \mathbb{C}$ is a trace defined on I. The following definition is a generalization of Definitions F.2.5 and F.3.12 in [7] (pp. 31, 42).

Definition 2.2.1 ([10], Definition 3.1) Let A be a Banach algebra and let I be an ideal in A. We call the element $a \in A$ a Fredholm element relative to I if there exists an element $a_0 \in A$ such that

1. $aa_0 - 1 \in I$;
2. $a_0 a - 1 \in I$.

The set of all Fredholm elements relative to I is denoted by $\Phi(A, I)$.

Clearly, $a \in \Phi(A, I)$ if and only if $\bar{a} = a + I$ is invertible in the quotient algebra A/I. Also, $A^{-1} \subset \Phi(A, I)$ and $\Phi(A, I)$ is a multiplicative semi-group. Note that since we do not assume I to be closed, A/I will not be a Banach algebra in general. The next example motivates the definition (see Definition 2.2.3) of an index via a trace.

Example 2.2.2 ([10], Example 3.2) Let X be a Banach space and let T be a Fredholm operator defined on X. The index of T, $i(T)$ is defined to be

$$i(T) = dimN(T) - dimN(T') = \alpha - \beta.$$

For a Fredholm operator T on X there exists an operator T_0 on X such that

$$T_0 T = I - F_1 \text{ and } TT_0 = I - F_2$$

where F_1 is a projection of X onto $N(T)$ and F_2 the projection of X onto a finite dimensional subspace $X_0 \subset X$ satisfying $X = R(T) \oplus X_0$ and $dimX_0 = \beta$ ([28], Theorem V.1.4). Since F_1 has finite dimensional range, it is of the form $\sum_{i=1}^{\alpha} x_i' \otimes x_i$, and since it is a projection, $x_i'(x_j) = \delta_{ij}$. It follows that its trace $\tau(F_1)$ satisfies $\tau(F_1) = \alpha$. Similarly, $\tau(F_2) = dim(X_0) = \beta$. Hence,

$$i(T) = \alpha - \beta = \tau(F_1 - F_2) = \tau(TT_0 - T_0T).$$

Following the example, we define an index on $\Phi(A, I)$ with the aid of a trace as follows.

Definition 2.2.3 ([10], Definition 3.3) Let A be a Banach algebra and let τ be a trace on the ideal I in A. We define the index function $\iota : \Phi(A, I) \to \mathbb{C}$ by

$$\iota(a) = \tau(aa_0 - a_0a) \text{ for all } a \in \Phi(A, I)$$

where $a_0 \in A$ satisfies $aa_0 - 1 \in I$ and $a_0a - 1 \in I$.

This definition was suggested by J.J. Grobler in [10]. Let $a \in \Phi(A, I)$. Suppose there exist $a_0, a_0' \in A$ such that $aa_0 - 1 \in I, a_0a - 1 \in I$ and $a_0'a - 1 \in I, aa_0' - 1 \in I$. We show that the value of the index function is independent of a_0 (and a_0').

Proposition 2.2.4 *([10], Proposition 3.4) Let A be a Banach algebra and let I be a trace ideal in A. The index function is well defined on $\Phi(A, I)$.*

Proof. We note firstly that $aa_0 - a_0a = aa_0 - 1 - (a_0a - 1) \in I$, and so $\tau(aa_0 - a_0a)$ exists. Next, let $a_0' \in A$ also satisfy $aa_0' - 1 \in I$ and $a_0'a - 1 \in I$. Then we have $\tau(aa_0' - a_0'a) - \tau(aa_0 - a_0a) = \tau(a(a_0' - a_0) - (a_0' - a_0)a)$, and since $a_0' \equiv a_0 \ mod \ I$, both terms in the argument are in I. By the properties of a trace we therefore get

$$\tau(aa_0' - a_0'a) - \tau(aa_0 - a_0a) = \tau(a(a_0' - a_0)) - \tau((a_0' - a_0)a) = 0$$

and so ι is well defined. □

We note that the definition of Fredholm elements is symmetric. If $a \in \Phi(A, I)$ and a_0 satisfies $a_0a - 1 \in I$ and $aa_0 - 1 \in I$ then clearly $a_0 \in \Phi(A, I)$ as well and its index satisfies $\iota(a_0) = -\iota(a)$. Also, if a is invertible we can take $a_0 = a^{-1}$ and get $\iota(a) = 0$. The following four results explore some of the properties of the index function.

Proposition 2.2.5 *([10], Proposition 3.5) Let A be a Banach algebra and I a trace ideal in A. If $a, b \in \Phi(A, I)$, then $\iota(ab) = \iota(a) + \iota(b)$.*

Proof. As observed above, if $a, b \in \Phi(A, I)$ then $ab \in \Phi(A, I)$. Let $a_0, b_0 \in A$ and satisfy $aa_0 - 1, a_0 a - 1 \in I$ and $bb_0 - 1, b_0 b - 1 \in I$. It is easy to see that

$$(ab)(b_0 a_0) - 1 \in I, \quad (b_0 a_0)(ab) - 1 \in I, \quad b(b_0 a_0)a - 1 \in I.$$

Therefore,

$$\begin{aligned}
\iota(ab) &= \tau((ab)(b_0 a_0) - (b_0 a_0)(ab)) \\
&= \tau((ab)(b_0 a_0) - b(b_0 a_0)a) + \tau(b(b_0 a_0)a - (b_0 a_0)(ab)) \\
&= \tau(a(bb_0 a_0) - (bb_0 a_0)a) + \tau(b(b_0 a_0 a) - (b_0 a_0 a)b) = \iota(a) + \iota(b)
\end{aligned}$$

□

Proposition 2.2.6 *([10], Proposition 3.7) Let A be a Banach algebra and let I be a trace ideal in A. If $a \in \Phi(A, I)$ then the following holds:*

(i) For every $q \in I$ we have $\iota(a + q) = \iota(a)$.

(ii) For every $0 \neq \lambda \in \mathbb{C}$ we have $\lambda a \in \Phi(A, I)$ and $\iota(\lambda a) = \iota(a)$.

(iii) For every $0 \neq \lambda \in \mathbb{C}$ and every $q \in I$ we have $\iota(\lambda - q) = 0$.

(iv) The set $\Phi(A, I)$ is open in A.

(v) The index function ι is constant on every component of $\Phi(A, I)$.

(vi) The index function $\iota : \Phi(A, I) \to \mathbb{C}$ is continuous.

Proof.

(i) With a_0 as before and for $q \in I$ we have $(a + q)a_0 - 1 = (aa_0 - 1) + qa_0 \in I$ and likewise $a_0(a + q) - 1 \in I$. Hence, $a + q \in \Phi(A, I)$ and

$$\iota(a + q) = \tau((a + q)a_0 - a_0(a + q)) = \tau(aa_0 - a_0 a) + \tau(qa_0 - a_0 q) = \iota(a).$$

(ii) For $\lambda \neq 0$ we have $(\lambda a)(\lambda^{-1} a_0) - 1 \in I$ and $(\lambda^{-1} a_0)(\lambda a) - 1 \in I$. It follows immediately that $\iota(\lambda a) = \iota(a)$.

(iii) This follows immediately from (i) and (ii) since $\iota(1) = 0$.

(iv) Let $a \in \Phi(A, I)$ and let $aa_0 - 1 = p \in I$ and $a_0 a - 1 = q \in I$. Let $b \in A$ satisfy $||a - b|| < ||a_0||^{-1}$. It follows from $ba_0 = 1 + p - (a - b)a_0 = p + (1 - (a - b)a_0)$, and the fact that $u := 1 - (a - b)a_0$ is invertible, that $ba_0 u^{-1} - 1 = pu^{-1} \in I$. Similarly, we get $v^{-1}a_0 b - 1 = v^{-1}q \in I$ where $v := 1 - a_0(a - b)$. By the series expansion for u^{-1} and v^{-1} we find that $a_0 u^{-1} = v^{-1}a_0$ and this shows by definition that $b \in \Phi(A, I)$. Hence, $\Phi(A, I)$ is open.

(v) Using the argument and notation in (iv) we find that for all b in a neighbourhood of a that $bb_0 - 1 \in I$ and $b_0 b - 1 \in I$ with $b_0 = a_0 u^{-1}$. It follows that

$$\iota(b) = -\iota(b_0) = -\iota(a_0) - \iota(u^{-1}) = -\iota(a_0) = \iota(a)$$

(vi) This follows immediately from (iv) and (v).

□

To continue the development of the theory of an index via a trace, we make use of Barnes idempotents and annihilators as described below:

Definition 2.2.7 ([10], Definition 3.9) Let A be a Banach algebra and $a \in A$. An idempotent p is called a *left Barnes* idempotent for $a \in A$ if

$$aA = (1 - p)A \tag{2.1}$$

Similarly, q is called a *right Barnes* idempotent for a if:

$$Aa = A(1 - q) \tag{2.2}$$

Also, for $a \in A$ we define the *right* and *left annihilators* respectively of a as the sets:

$$N_r(a) := \{x \in A \ : \ ax = 0\}, \quad N_\ell(a) := \{x \in A \ : \ xa = 0\}$$

We shall illustrate that a Barnes idempotent belonging to $a \in A$ need not be unique. We note the following facts from [10].

Proposition 2.2.8 *([10], Proposition 3.10) Let A be a unital Banach algebra, $a \in A$. If p is a left Barnes idempotent for a then $Ap = N_\ell(a)$, and in particular, $p \in N_\ell(a)$. Similarly, if q is a right Barnes idempotent for $a \in A$ then $qA = N_r(a)$ and so $q \in N_r(a)$.*

Proof. Let p be a left Barnes idempotent for a. By (2.1) there exists some $b \in A$ such that $a = (1 - p)b$. Hence, $pa = p(1 - p)b = 0$ and so $p \in N_\ell(a)$. It follows that $Ap \subset N_\ell(a)$. On the other hand, again by (2.1) we have $1 - p = ab$ for some $b \in A$. So, if $x \in N_\ell(a)$, we have $x(1 - p) = xab = 0$. Thus, $x = xp \in Ap$. This establishes $Ap = N_\ell(a)$. □

Theorem 2.2.9 *([10], Theorem 2.2.12) Let A be a semisimple unital Banach algebra and let the trace ideal I satisfy* $\operatorname{Soc} A \subset I \subset \mathsf{kh}(\operatorname{Soc} A)$. *Then*

(i) $a \in \Phi(A, I)$ *if and only if there exist left and right Barnes idempotents p and q, respectively, in* $\operatorname{Soc} A$ *and an element $a_0 \in A$ such that*

$$aa_0 = 1 - p \quad \textit{and} \quad a_0 a = 1 - q. \tag{2.3}$$

(ii) $\iota(a) \in \mathbb{Z}$.

Proof.

(i) Since A is a semisimple unital Banach algebra it follows that $\operatorname{Soc} A$ is non-trivial. As we remarked earlier, we may assume that the trace ideal I contains $\operatorname{Soc} A$. Since an element a is invertible modulo $\operatorname{Soc} A$ if and only if it is invertible modulo $\mathsf{kh}(\operatorname{Soc} A)$ (see [7], Theorem BA.2.4, p 103 or [1], Theorem 5.7.2), we have $\Phi(A, I) = \Phi(A, \operatorname{Soc} A)$. It then follows from [7], Theorem F.1.10 that $a \in \Phi(A, I)$ if and only if there exist left and right Barnes idempotents in the socle for a. Let $p \in \operatorname{Soc} A \subset I$ be a left Barnes idempotent for a; then there exist by (2.1) an element $a_0 \in A$ such that $aa_0 = 1 - p$ and also an element $b \in A$ such that $a = (1 - p)b$. The latter relation implies that $(1 - p)a = a$. Hence, $a_0 a$ is an idempotent, because

$$(a_0 a)^2 = a_0(aa_0)a = a_0(1 - p)a = a_0 a. \tag{2.4}$$

Let $q := 1 - a_0 a$. Since $a \in \Phi(A, I)$ and the equivalence class containing a_0 is a right inverse of a modulo $I = \operatorname{Soc} A$, it is also a left inverse of a modulo I. It follows that $q \in I = \operatorname{Soc} A$ and $aq = a(1 - q)$. The two equalities

$$1 - q = a_0 a \quad \text{and} \quad a = a(1 - q)$$

together imply that $Aa = A(1 - q)$ and so q is a right Barnes idempotent for a. Hence, (i) holds.

(ii) This follows immediately from (2.3) because

$$\iota(a) = \tau(aa_0 - a_0 a) = \tau(1 - p - (1 - q)) = \tau(q - p) = \tau(q) - \tau(p).$$

But, as remarked earlier, τ is a spectral trace on $\operatorname{Soc} A$ and consequently $\tau(p), \tau(q) \in \mathbb{N}$. Thus, $\iota(a) \in \mathbb{Z}$.

□

We call the Barnes idempotents p and q in Theorem 2.2.9 *associated* Barnes idempotents.

Corollary 2.2.10 *([10], Corollary 3.13) Let A be a semisimple Banach algebra and let the trace ideal I satisfy $\operatorname{Soc} A \subset I \subset kh \operatorname{Soc} A$. If $a \in \Phi(A, I)$, then there exist associated Barnes idempotents p and q with $\iota(a) = \tau(q) - \tau(p)$.*

Next, we show that the traces of all left (respectively right) Barnes idempotents for a given element $a \in A$ are equal.

Theorem 2.2.11 *([10], Theorem 3.14) Let A be a semisimple Banach algebra and let the trace ideal I in A satisfy $\operatorname{Soc} A \subset I \subset kh \operatorname{Soc} A$. Let p and q be respectively left and right Barnes idempotents of the Fredholm element $a \in A$. Then $\tau(p)$ is equal to the cardinality of a maximal set of orthogonal minimal idempotents in $N_\ell(a) = Ap$. Similarly, $\tau(q)$ is equal to the cardinality of a maximal set of orthogonal minimal idempotents in $N_r(a) = qA$.*

Proof. Since $N_\ell = N_\ell(a) = Ap$ is a left ideal contained in the socle of A, we deduce by [7], Lemma F.1.7 that every orthogonal subset of minimal subsets of N_ℓ is finite. Let $\{e_1, ..., e_k\}$ be a maximal subset of minimal idempotents. Then $N_\ell = \sum_{i=1}^k Ae_i$, and we can write:

$$p = x_1 e_1 + \cdots + x_k e_k.$$

Again, by [7], Lemma F.1.7, $\bar{p} = e_1 + \cdots + e_k$ is an idempotent in $N_\ell(a)$ and $N_\ell(a) = A\bar{p}$. Note now that $p\bar{p} = p$. We then have

$$aa_0 = (1 - \bar{p})aa_0 = (1 - \bar{p})(1 - p) = 1 - \bar{p} - p + \bar{p}p$$

and it follows from the commutative property of the trace that

$$\begin{aligned} \tau(p) = \tau(1 - aa_0) &= \tau(\bar{p} + p - \bar{p}p) = \tau(\bar{p}) + \tau(p) - \tau(\bar{p}p) \\ &= \tau(\bar{p}) + \tau(p) - \tau(p) = \tau(e_1) + \cdots + \tau(e_k) = k. \end{aligned}$$

The proof for a right Barnes idempotent is similar. □

Remark 2.2.12 We note two interesting facts which follow from the proof. In the first place, since the e_i are minimal idempotents, we have $e_i x_i e_i = \lambda_i e_i$ for some $\lambda_i \in \mathbb{C}$. So by the commutativity of the trace,

$$\tau(p) = \sum_{i=1}^k \tau(x_i e_i) = \sum_{i=1}^k \tau(e_i x_i e_i) = \sum_{i=1}^k \tau(\lambda_i e_i) = \sum_{i=1}^k \lambda_i$$

Secondly, for every left Barnes idempotent p for a, we see that $p + \bar{p} - \bar{p}p$ (with the notation of the proof) is again a left Barnes idempotent for a. Similarly, if q is the right Barnes idempotent for a associated with p, then $q + \bar{q} - q\bar{q}$ is again a right Barnes idempotent for a and it is associated with $p + \bar{p} - \bar{p}p$.

Corollary 2.2.13 *([10], Corollary 3.15) Suppose A is a semisimple Banach algebra and suppose the trace ideal I satisfies $\operatorname{Soc} A \subset I \subset \mathsf{kh}\operatorname{Soc} A$. If $a \in \Phi(A, I)$ with left Barnes idempotent p and right Barnes idempotent q, then $\tau(p) = \operatorname{rank}(p)$ and $\tau(q) = \operatorname{rank}(q)$.*

Proof. This follows immediately from [2], Corollary 2.18. □

The fact that for any left Barnes idempotent p for $a \in \Phi(A, I)$ the number $\tau(p)$ is uniquely defined, even though the Barnes idempotent is not, enables us to define the nullity and deficiency of an element $a \in \Phi(A, I)$ as follows.

Definition 2.2.14 *([10], Definition 3.16) The nullity $n(a)$ and deficiency $d(a)$ of $a \in \Phi(A, I)$ are defined as*

$$n(a) := \tau(q) \text{ and } d(a) := \tau(p)$$

where q is a right Barnes idempotent and p is a left Barnes idempotent of a.

We have

Theorem 2.2.15 *([10], Theorem 3.17) Let A be a unital Banach algebra and let the trace ideal I satisfy $\operatorname{Soc} A \subset I \subset \mathsf{kh}\operatorname{Soc} A$. Then for every $a \in \Phi(A, I)$ we have*

$$\iota(a) = n(a) - d(a)$$

and

$$A^{-1} = \{a \in \Phi(A, I) : n(a) = d(a) = 0\}$$

Proof. If $a \in A$ is invertible, then 0 is a left as well as a right Barnes idempotent for a and so $n(a) = d(a) = 0$. On the other hand, $\tau(p) \geq 1$ for every nonzero left Barnes idempotent of a. If, therefore, the condition $n(a) = d(a) = 0$ holds, we see that a has a right and left inverse and hence $a \in A^{-1}$. □

Let X and Y be Banach spaces and let $B : X \to Y$ and $A : Y \to X$ be bounded linear operators. There is a classical result that if $I - AB$ is a Fredholm operator, then $\iota(I - AB) = \iota(I - BA)$, see ([4], Theorem 6). We prove the analogue of this result in general.

Theorem 2.2.16 *([10], Theorem 3.20) Let A be a semisimple Banach algebra and let I be a closed trace ideal such that $\operatorname{Soc} A \subset I \subset \mathsf{kh}\operatorname{Soc} A$. If $a, b \in A$ with $1 - ab \in \Phi(A, I)$ then $\iota(1 - ab) = \iota(1 - ba)$.*

Proof. Let p, q be respectively right and left Barnes idempotents of $1-ab \in \Phi(A, I)$ such that $\iota(1-ab) = \tau(q) - \tau(p)$ (see Corollary 2.2.10). From the definition of a right Barnes idempotent there exists an element $x \in A$ such that

$$(1-ab)x = 1-p \text{ and } x(1-ab) = 1-q \tag{2.5}$$

If $x_0 = bxa + 1$ then, using (2.5), we get

$$\begin{aligned}(1-ba)x_0 &= bxa + 1 - b(abx)a - ba \\ &= bxa + 1 - b(x-1+p)a - ba = 1 - bpa,\end{aligned} \tag{2.6}$$

$$\begin{aligned}x_0(1-ba) &= bxa - b(xab)a + 1 - ba \\ &= bxa - b(x-1+q)a + 1 - ba = 1 - bqa,\end{aligned} \tag{2.7}$$

This shows that $1 - ba \in \Phi(A, I)$ and applying (2.5) - (2.7), we have

$$\begin{aligned}\iota(1-ba) &= \tau((1-ba)x_0 - x_0(1-ba)) \\ &= \tau(1 - bpa - (1 - bqa)) \\ &= \tau(bqa) - \tau(bpa) \quad \text{(by TA)}.\end{aligned}$$

But

$$\begin{aligned}\tau(bqa) = \tau(abq) &= \tau(abq - q + q) \\ &= \tau((ab-1)q + q) = \tau(q),\end{aligned}$$

since $q \in N_r(1-ab)$, and similarly, because $p \in N_\ell(1-ab)$ we have $\tau(bpa) = \tau(p)$. It follows that $\iota(1-ba) = \tau(q) - \tau(p) = \iota(1-ab)$. $\square$

Note that if $0 \neq \lambda \in \mathbb{C}$ is such that $\lambda - ab \in \Phi(A, I)$ then $\iota(\lambda - ab) = \iota(\lambda - ba)$ because

$$\begin{aligned}\iota(\lambda - ab) &= \iota(\lambda(1 - \frac{1}{\lambda}ab)) \\ &= \iota(\lambda(1 - \frac{ab}{\lambda})) \\ &= \iota(\lambda(1 - \frac{ba}{\lambda})) \\ &= \iota(\lambda - ba),\end{aligned}$$

see Proposition 2.2.5.

2.3 Another index function

In [16], H. Kraljević, S. Suljagić and K. Veselić also defined an index function for Fredholm elements in a semisimple Banach algebra. Their construction is briefly as follows: Let Γ be the set of similarity classes of minimal idempotents in $\operatorname{Soc} A$. The authors then proved that $\operatorname{Soc} A = \sum_{\gamma\in\Gamma} A_\gamma$, where A_γ is the two-sided ideal generated by $p \in \gamma \in \Gamma$, and the sum is an algebraic direct sum ([16], Theorem 2.12). So, for every $a \in \operatorname{Soc} A$ there exists a finite set $\Gamma_a \subset \Gamma$ such that $a = \sum_{\gamma\in\Gamma_a} a_\gamma$. If we put $a_\gamma = 0$ if $\gamma \in \Gamma \setminus \Gamma_a$, we have $a = \sum_{\gamma\in\Gamma} a_\gamma$. Let p and q be left and right Barnes idempotents of a respectively and suppose that

$$p = \sum_{\gamma\in\Gamma} p_\gamma \quad q = \sum_{\gamma\in\Gamma} q_\gamma.$$

Let $a \in \Phi(A, \operatorname{Soc} A)$. The authors above defined their index function as follows:

$$\operatorname{Ind}_\gamma(a) = (\operatorname{Ind}_\gamma(a))_{\gamma\in\Gamma} \in \mathbb{Z}^\Gamma$$

where $\operatorname{Ind}_\gamma(a) = \operatorname{rank}(q_\gamma) - \operatorname{rank}(p_\gamma) = \tau(q_\gamma) - \tau(p_\gamma)$. It is clear from Lemma 3.18 in [10] that

$$\iota(a) = \tau(q) - \tau(p) = \sum_{\gamma\in\Gamma}(\tau(q_\gamma) - \tau(p_\gamma)) = \sum_{\gamma\in\Gamma} \operatorname{Ind}_\gamma(a).$$

A disadvantage of this index function is that the index of a Fredholm element is a sequence. Also, one first has to find Barnes idempotents in order to define $\operatorname{Ind}_\gamma$. An advantage of the index function $\operatorname{Ind}(a) \in \mathbb{Z}^\Gamma$ is that it is not difficult to prove that if $\operatorname{Ind}(a) = 0$, $a \in \Phi(A, \operatorname{Soc} A)$, then $a = b + c$ with $b \in A^{-1}$ and $c \in \operatorname{Soc} A$, see [10], Theorem 3.19.

2.4 Index zero Fredholm elements

In [10] the authors defined an index function for Fredholm elements relative to a trace ideal, and they developed the basic properties of this index function. One fact that eluded them was to show that an index zero Fredholm element can be written as a sum of an invertible and socle element. This section describes how to prove this.

New results in this section are Theorems 2.3.4 and 2.3.6. Their proofs show similarities with the proof of Theorem 3.19 in [10].

Let A be a semisimple Banach algebra and let I be a closed trace ideal with $\operatorname{Soc} A \subset I \subset \mathsf{kh} \operatorname{Soc} A$. We denote the set of index zero Fredholm elements by

$\Phi_0(A, I)$. Note that with the aid of the index function ι we can decompose the set $\Phi(A, I)$ of Fredholm elements into equivalence classes as follows:

$$\Phi(A, I) = \bigcup_{n=-\infty}^{\infty} \iota^{-1}(n),$$

where $\iota^{-1}(0) = \Phi_0(A, I)$.

Theorem 2.4.1 *([12], Theorem 3.1) Let A be a semisimple Banach algebra and let I be a trace ideal in A satisfying $\operatorname{Soc} A \subset I \subset kh \operatorname{Soc} A$. Let $a \in \Phi(A, I)$ and p and q be left and right Barnes idempotents for a respectively. If p and q are similar, then $a = x + y$ with $x \in A^{-1}$ and $y \in \operatorname{Soc} A$.*

Proof. Let $a \in \Phi(A, I)$. Since p and q are Barnes idempotents for a there exist $a_0 \in A$ such that $aa_0 = 1 - p$ and $a_0 a = 1 - q$ ([10], Theorem 3.11). If p and q are similar then there exists $u \in A^{-1}$ with $up = qu$. Put $\bar{u} = up = qu$ and $\bar{v} = pu^{-1} = u^{-1}q$. Then $\bar{v}\bar{u} = p$ and $\bar{u}\bar{v} = q$. In view of [10], Proposition 3.10, $\bar{u} \in N_\ell(a) \cap N_r(a)$ and $\bar{v} \in N_\ell(a_0) \cap N_r(a_0)$. If we combine these arguments it follows that

$$(a + \bar{v})(a_0 + \bar{u}) = 1 - p + p = 1 \quad \text{and} \quad (a_0 + \bar{u})(a + \bar{v}) = 1 - q + q = 1.$$

Then $a = (a + \bar{v}) - \bar{v}$ with $a + \bar{v} \in A^{-1}$ and $\bar{v} = pu^{-1} \in \operatorname{Soc} A$, since $\operatorname{Soc} A$ is an ideal and $p \in \operatorname{Soc} A$. This completes the proof. □

Corollary 2.4.2 *([12], Corollary 3.2) Under the assumption of the theorem if $a \in \Phi(A, I)$ is such that it has similar associated Barnes idempotents, then $a \in \Phi_0(A, I)$.*

Proof. Let $a = x + y \in A^{-1} + \operatorname{Soc} A$. Then

$$\iota(a) = \iota(x + y) = \iota(x) = 0$$

□

Note that in the above Theorem if $\|aa_0 - a_0 a\| < 1$, then $\|p - q\| < 1$ and so in view of a result of Zemánek ([2], Lemma 2.5) p and q are similar.

Let I be a closed ideal in a Banach algebra A. We will say that $x \in A$ is *Weyl* with respect to I if $x = a + b$ with $a \in A^{-1}$ and $b \in I$. Denote the collection of Weyl elements in A relative to I by $\mathcal{W}(A, I)$. We also define a spectrum relative to $\mathcal{W}(A, I)$ as

$$\sigma_{\mathcal{W}(A,I)}(a) = \{\lambda \in \mathbb{C} : \lambda - a \notin \mathcal{W}(A, I)\} \text{ for all } a \in A.$$

Corollary 2.4.3 *([12], Corollary 3.3) Let I be a closed trace ideal in a semisimple Banach algebra A such that $\operatorname{Soc} A \subset I \subset kh \operatorname{Soc} A$. Then $\mathcal{W}(A,I) \subset \Phi_0(A,I)$.*

Our aim is to reverse these inclusions. This brings us to our main theorem. The main tool used in the proof of this theorem is the Sinclair version of Jacobson's density theorem, see (Chapter 1).

Theorem 2.4.4 *([12], Theorem 3.4) Let A be a semisimple Banach algebra and let I be a trace ideal in A with $\operatorname{Soc} A \subset I \subset kh \operatorname{Soc} A$. If $a \in \Phi(A,I)$ with $\iota(a) = 0$, then $a = x + y$ with $x \in A^{-1}$ and $y \in \operatorname{Soc} A$, that is, $\Phi_0(A,I) = A^{-1} + \operatorname{Soc} A$.*

Proof. Let $a \in \Phi(A,I)$ with $\iota(a) = 0$. By [10], Corollary 3.13, Theorem 3.14, $\iota(a) = \tau(p) - \tau(q)$ where $\tau(q)$ is equal to the cardinality of a maximal set of orthogonal minimal idempotents in $N_\ell(a) = Ap$ and similarly, $\tau(q)$ is equal to the cardinality of a maximal set of minimal idempotents in $N_r(a) = qA$. Let k be the common cardinality and let $\{e_1, e_2, ..., e_k\}$ be a maximal set of orthogonal minimal idempotents in $N_\ell(a)$. Then $N_\ell(a) = \sum_1^k Ae_i$ and we can write $p = x_1e_1 + x_2e_2 + ... + x_ke_k$. Likewise, let $\{f_1, f_2, ..., f_k\}$ be a maximal subset of orthogonal minimal idempotents in $N_r(a)$. Again, $N_r(a) = f_iA$ and we can write $q = f_1y_1 + f_2y_2 + ... + f_ky_k$. Since the sets $\{x_1e_1, x_2e_2, ..., x_ke_k\}$ and $\{f_1y_1, f_2y_2, ..., f_ky_k\}$ are linearly independent, the Sinclair version of the Jacobson's density theorem (Corollary 1.6.2) implies that there exists an element $u \in A^{-1}$ such that $ux_ie_i = f_iy_i$ for $i = 1, ..., k$ and consequently, $up = q$.
Put $\bar{u} = up = q$ and $\bar{v} = p = u^{-1}q$. Then $\bar{u}\bar{v} = up^2 = up = q$ and $\bar{v}\bar{u} = u^{-1}q^2 = u^{-1}q = p$. In view of [10], Proposition 3.10 we have $\bar{u} \in N_\ell(a) \cap N_r(a)$ and $\bar{v} \in N_\ell(a_0) \cap N_r(a_0)$. It follows that

$$(a + \bar{v})(a_0 + \bar{u}) = 1 - p + p = 1 \text{ and } (a_0 + \bar{u})(a + \bar{v}) = 1 - q + q = 1$$

Hence, $a = a + \bar{v} - \bar{v}$ with $a + \bar{v} \in A^{-1}$ and $\bar{v} = p \in \operatorname{Soc} A$.
This completes the proof. □

Corollary 2.4.5 *([12], Corollary 3.5) Let I be a closed trace ideal in a semisimple Banach algebra A such that $\operatorname{Soc} A \subset I \subset kh \operatorname{Soc} A$. Then $\mathcal{W}(A,I) = \Phi_0(A,I)$.*

We now link our result to spectral theory.

Theorem 2.4.6 *([12], Theorem 3.6) Let I be a closed trace ideal in a semisimple Banach algebra A such that $\operatorname{Soc} A \subset I \subset kh \operatorname{Soc} A$. Then $\mathcal{W}(A,I) = \Phi_0(A,I)$ is an upper semiregularity with the Jacobson property.*

Proof. Since $\Phi_0(A,I)$ is an open semigroup containing 1 (see [10], Proposition 3.5, Proposition 3.7), it is an upper semiregularity (Lemma 1.5.4). Let $0 \neq \lambda \in \mathbb{C}$

and $a, b \in A$ with $\lambda - ab \in \Phi_0(A, I)$. By Theorem 2.2.16, we have $0 = \iota(\lambda - ab) = \iota(\lambda - ba)$ and so $\lambda - ba \in \Phi_0(A, I)$. The converse follows by symmetry and we are done. $\square$

We are now able to characterize the Weyl spectrum in terms of the index.

Corollary 2.4.7 *([12], Corollary 3.7) Let I be a closed trace ideal in a semisimple Banach algebra A such that $\operatorname{Soc} A \subset I \subset kh \operatorname{Soc} A$. Then, for every $a \in A$*

$$\sigma_{\mathcal{W}(A,I)}(a) = \{\lambda \in \mathbb{C} : \lambda - a \text{ is not Fredholm or } \iota(\lambda - \mathrm{a}) \neq 0\}.$$

Chapter 3

Regularities and semiregularities

3.1 Introduction

In this chapter we are going to compare regularities (semiregularities) S and R that satisfy $S \subset R$. In this case the spectra satisfy $\sigma_R(a) \subset \sigma_S(a)$ for all $a \in A$. For such spectra we will be interested in the case $\partial\sigma_S(a) \subset \sigma_R(a) \subset \sigma_S(a)$ for all $a \in A$. In many cases in the literature [1, 11, 17, 20, 21, 24] this condition was verified separately for many regularities (semiregularities). We are going to follow a simple uniform approach to prove this condition between the regularities (semiregularities) R and S that would guarantee $\partial\sigma_S(a) \subset \sigma_R(a) \subset \sigma_S(a)$ for all $a \in A$. If we let $B = A$ in ([17], Theorem 3.1) we get:

Theorem 3.1.1 *Let A be a Banach algebra and let R and S be regularities (semiregularities) in A such that $S \subset R$. Then:*

1. *$\sigma_R(a) \subset \sigma_S(a)$ for all $a \in A$.*
2. *If $\partial S \cap R = \emptyset$ then $\partial\sigma_S(a) \subset \sigma_R(a) \subset \sigma_S(a)$ for all $a \in A$ provided that $\sigma_S(a) \neq \emptyset$.*

Proof.

1. Let $a \in A$. If $\lambda \notin \sigma_S(a)$ then $a - \lambda \in S \subset R$, so $\lambda \notin \sigma_R(a)$.
2. Let $a \in A$ and $\lambda \in \partial\sigma_S(a)$. Then there is a sequence (λ_n) in $\mathbb{C} \setminus \sigma_S(a)$ such that $\lambda_n \to \lambda$ and a sequence (μ_n) in $\sigma_S(a)$ such that $\mu_n \to \lambda$. Then $(a - \lambda_n)$ is a sequence in S such that $a - \lambda_n \to a - \lambda$ and $(a - \mu_n)$ is a sequence in $A \setminus S$ such that $a - \mu_n \to a - \lambda$. Consequently, $a - \lambda \in \partial S$ and since $\partial S \cap R = \emptyset$ it follows that $a - \lambda \notin R$ and so $\lambda \in \sigma_R(a)$.

$\square$

Note that in Theorem 3.1.1, the statement will hold for more general R, see Proposition 3.1.8.

As a first step we let S be equal to the regularity A^{-1} and compare A^{-1} with other regularities (semiregularities). We need the following notion:

Definition 3.1.2 ([24], p. 57) Let R be a regularity in a Banach algebra A with σ_R the corresponding spectrum. We say that σ_R is *spectral radius preserving* if $\sigma_R(a)$ is closed and

$$\sup\{|\lambda| \ : \ \lambda \in \sigma_R(a)\} = \sup\{|\lambda| \ : \ \lambda \in \sigma(a)\} \text{ for all } a \in A.$$

Müller then proved:

Theorem 3.1.3 *([24], Theorem 1.6.13) Let R be a regularity in a Banach algebra A such that the corresponding spectrum σ_R is spectral radius preserving. Then $\partial\sigma(a) \subset \sigma_R(a) \subset \sigma(a)$ for all $a \in A$.*

If we compare a regularity R in a Banach algebra A with the regularity A^{-1} and $\partial\sigma(a) \subset \sigma_R(a) \subset \sigma(a)$ for all $a \in A$, then the spectrum σ_R is spectral radius preserving. We now exhibit regularities R for which $\partial A^{-1} \cap R = \emptyset$. Let A be a Banach algebra and let:

1. $R_1 = A$;
2. $R_2 = A^{-1}$;
3. $R_3 = A_l^{-1}$;
4. $R_4 = A_r^{-1}$;
5. $R_5 = \{x \in A \ : \ x$ is not a left topological zero divisor$\}$;
6. $R_6 = \{x \in A \ : \ x$ is not a right topological zero divisor$\}$.

The sets R_i $(i = 1, 2, ..., 6)$ are examples of regularities since they satisfy the P1 condition, see Theorem 1.5.6.

Proposition 3.1.4 *Let A be a Banach algebra and let $R = R_3$ or $R = R_4$. Then $\partial\sigma(a) \subset \sigma_R(a) \subset \sigma(a)$ for all $a \in A$.*

Proof. We prove the statement for $R = R_3$. The proof for $R = R_4$ is similar and is omitted. Suppose that $a \in \partial A^{-1}$. By ([24], Theorem 1.1.14), a is a topological divisor of zero, and from the same theorem it follows that $a \notin R$. Hence, $\partial A^{-1} \cap R = \emptyset$. In view of Theorem 3.1.1 it follows that $\partial\sigma(a) \subset \sigma_R(a) \subset \sigma(a)$ for all $a \in A$. □

Corollary 3.1.5 *If A is a Banach algebra then the regularities R_3 and R_4 are spectral radius preserving.*

Proof. Since R_i (i = 3, 4) are open subsets in A, see ([9], Theorem 2.1.3) it follows from ([24], Proposition 1.6.9) that $\sigma_{R_i}(a)$ is closed for every $a \in A$. The statement now follows from the proposition above. $\square$

Proposition 3.1.6 *Let A be a Banach algebra and let $R = R_5$ or $R = R_6$. Then $\partial\sigma(a) \subset \sigma_R(a) \subset \sigma(a)$ for all $a \in A$.*

Proof. We prove the statement for $R = R_5$. The proof for $R = R_6$ is similar and is omitted. Suppose that $a \in \partial A^{-1}$. By ([24], Theorem 1.1.14), a is a topological divisor of zero, and from the same theorem it follows that $a \notin R$. Hence, $\partial A^{-1} \cap R = \emptyset$. In view of Theorem 3.1.1 it follows that $\partial\sigma(a) \subset \sigma_R(a) \subset \sigma(a)$ for all $a \in A$. $\square$

For a different proof of this fact, see [14].

Corollary 3.1.7 *If A is a Banach algebra then the regularities R_5 and R_6 are spectral radius preserving.*

Proof. Since R_i (i = 5, 6) are open sets in A, see ([9], Theorem 2.4.3), it follows from ([24], Proposition 1.6.9), that $\sigma_{R_i}(a)$ is closed for every $a \in A$. The statement now follows from the proposition above. $\square$

Next we investigate a set R in a Banach algebra A which is neither an upper semiregularity nor a lower semiregularity but the spectrum σ_R is spectral radius preserving. In [21], the author defines the *boundary spectrum* of an element in a Banach algebra as follows: Let S be the set of noninvertible elements in a Banach algebra A. The set

$$S_\partial(a, A) = \{\lambda \in \mathbb{C} \ : \ \lambda - a \in \partial S\}$$

is called the *boundary spectrum* of $a \in A$. To work with this spectrum in our context let $R = A \setminus \partial A^{-1}$. Then R is an open set in A with $A^{-1} \subset R$ and $\sigma_R(a)$, $a \in A$, is the boundary spectrum of a. The reason for this is that in a topological space a set and its complement has the same boundary. So, if $S = A \setminus A^{-1}$, then $\partial A^{-1} = \partial S$.

Proposition 3.1.8 *Let A be a Banach algebra and let $R = A \setminus \partial A^{-1}$. Then $\partial\sigma(a) \subset \sigma_R(a) \subset \sigma(a)$ for all $a \in A$.*

Proof. Since $A^{-1} \subset R$, $\sigma_R(a) \subset \sigma(a)$ for every $a \in A$. Also, since $\partial A^{-1} \cap (A \setminus \partial A^{-1}) = \emptyset$ it follows from Theorem 3.1.1 that $\partial\sigma(a) \subset \sigma_R(a) \subset \sigma(a)$ for all $a \in A$. $\square$

For a different proof of this fact, see ([21], Proposition 2.1).

Corollary 3.1.9 *Let A be a Banach algebra and let $R = A \setminus \partial A^{-1}$. Then the boundary spectrum σ_R is spectral radius preserving.*

Proof. Since R is open, σ_R is closed. By Proposition 3.1.8 above, σ_R is spectral radius preserving. $\square$

We illustrate next that if A is a Banach algebra and $R = A \setminus \partial A^{-1}$, that in general R is neither an upper semiregularity nor a lower semiregularity.

Example 3.1.10 Let A be the real Banach algebra $C([-1,1])$ of real valued continuous functions defined on $[-1,1]$ with the supremum norm and $R = A \setminus \partial A^{-1}$. Let $f(x) = x$, for $x \in [-1,1]$. Then $f \in R$, but $f^2 \notin R$. Hence R is not an upper semiregularity.

Next, we use the previous example to construct a complex Banach algebra in which the given set is not an upper semiregularity.

Example 3.1.11 Let A be as in Example 3.1.10 and let $A_{\mathbb{C}}$ be the complexification of A. Let $R_{\mathbb{C}} = A_{\mathbb{C}} \setminus \partial A_{\mathbb{C}}^{-1}$. As in Example 3.1.10, let $f(x) = x$ for $x \in [-1,1]$ and $R = A \setminus \partial A^{-1}$. Then $f \in R$, hence there exists a neighbourhood, N, of f that contains no elements from A^{-1}. First we note that $(f,f) \notin A_{\mathbb{C}}^{-1}$. Suppose this is not the case, i.e. suppose that $(f,f) \in A_{\mathbb{C}}^{-1}$. Then there is $(g,h) \in A_{\mathbb{C}}$ such that $(f,f) \cdot (g,h) = (1,0)$. Then $fg - fh = 1$ and $fg + fh = 0$. This yields $f \cdot (2g) = 1$ which contradicts $f \notin A^{-1}$. Hence $(f,f) \notin A_{\mathbb{C}}^{-1}$, and $0 \in \sigma_{A_{\mathbb{C}}^{-1}}((f,f))$. Next, we show that the neighbourhood $N \times N$ contains no elements from $A_{\mathbb{C}}^{-1}$. Suppose $(g,h) \in A_{\mathbb{C}}^{-1}$ and $(g,h) \in N \times N$. Then $0 \notin \sigma_{A_{\mathbb{C}}^{-1}}((g,h))$. Since $A_{\mathbb{C}}$ is a commutative Banach algebra the spectrum function is uniformly continuous. In particular, it is continuous at (f,f). Let $K_1 = \sigma_{A_{\mathbb{C}}^{-1}}((f,f))$ and $K_2 = \sigma_{A_{\mathbb{C}}^{-1}}((g,h))$. Then $\Delta(K_1,K_2) = \max\left(\sup_{z \in K_2} \operatorname{dist}(z,K_1), \sup_{z \in K_1} \operatorname{dist}(z,K_2)\right)$ denotes the Hausdorff distance between K_1 and K_2. Since $0 \notin K_2$ and $0 \in K_1$, $\Delta(K_1,K_2) = \epsilon$ for some $\epsilon > 0$. Since the spectrum function is continuous at (f,f), there is $\delta > 0$ such that if $||(g,h) - (f,f)|| < \delta$ then $\Delta(K_1,K_2) < \frac{\epsilon}{2}$. This contradicts $0 \notin \sigma_{A_{\mathbb{C}}^{-1}}((g,h))$, and so $(f,f) \in R_{\mathbb{C}}$. Finally, $(f,f)^2 = (0,2f^2) \notin R_{\mathbb{C}}$ because $\sigma_{A_{\mathbb{C}}^{-1}}((0,2f^2)) = \{ix : 0 \leq x \leq 2\}$. This proves that $R_{\mathbb{C}}$ is not an upper semiregularity in $A_{\mathbb{C}}$.

Example 3.1.12 Let A be the real Banach algebra $C([0,1])$ of real valued continuous functions defined on $[0,1]$ with the supremum norm and $R = A \setminus \partial A^{-1}$. For $x \in [0,1]$, let

$$f(x) = \begin{cases} -3x + \dfrac{3}{2} & \text{if } x \in [0, \frac{1}{2}) \\ 3x - \dfrac{3}{2} & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$$

and

$$
\begin{aligned}
g(x) &= 1 - f(x) \\
&= \begin{cases} 3x - \dfrac{1}{2} & \text{if } x \in [0, \frac{1}{2}) \\ -3x + \dfrac{5}{2} & \text{if } x \in [\frac{1}{2}, 1] \end{cases}
\end{aligned}
$$

Note that if $s(x) = t(x) = 1$, then

$$f \cdot s + g \cdot t = 1.$$

Also,

$$
f \cdot g(x) = \begin{cases} -9x^2 + 6x - \dfrac{3}{4} & \text{if } x \in [0, \frac{1}{2}) \\ -9x^2 + 12x - \dfrac{15}{4} & \text{if } x \in [\frac{1}{2}, 1] \end{cases}
$$

Then $f \cdot g \in R$, but $f \notin R$ and $g \in R$. By definition, R is not a lower semiregularity.

At this point Example 3.1.12 is the best we can do in terms of proving that the set $R = A \setminus \partial A^{-1}$ is in general not a lower semiregularity. The search for a counterexample in a complex Banach algebra will continue. See also [22], section 4.

Proposition 3.1.13 *Let A be a Banach algebra and let $R = A_l^{-1} \cup A_r^{-1}$. Then R is an open lower semiregularity.*

Proof. We use Lemma 1.5.2, to prove this as follows. Since $1_A \in R$, we know that $R \neq \emptyset$. Next, let $a, b \in R$, and $ab = ba$. Suppose that $ab \in R$ and suppose that $ab \in A_l^{-1}$. Then there is $c \in A$ such that $c(ab) = 1_A$. hence $(ca)b = 1_A$, hence $b \in R$. Similarly, using $ab = ba$, we have $a \in R$. Both A_l^{-1} and A_r^{-1} are open, hence R is open. $\square$

Proposition 3.1.14 *Let A be a Banach algebra and let $R = A_l^{-1} \cup A_r^{-1}$. Then $\partial\sigma(a) \subset \sigma_R(a) \subset \sigma(a)$ for all $a \in A$.*

Proof. From Proposition 3.1.4 we know that $\partial A^{-1} \cap A_l^{-1} = \emptyset$ and $\partial A^{-1} \cap A_r^{-1} = \emptyset$. Hence $\partial A^{-1} \cap R = (\partial A^{-1} \cap A_l^{-1}) \cup (\partial A^{-1} \cap A_r^{-1}) = \emptyset$. Using Theorem 3.1.1, the result follows. $\square$

Corollary 3.1.15 *Let A be a Banach algebra and let $R = A_l^{-1} \cup A_r^{-1}$. Then σ_R is spectral radius preserving.*

Proof. Since R is an open lower semiregularity we know from [24], Proposition 1.6.9 that $\sigma_R(a)$ is closed for all $a \in A$. Using Proposition 3.1.14 the result follows. □

We are going to give an example of an upper semiregularity R such that $\partial A^{-1} \cap R \neq \emptyset$ and σ_R is not spectral radius preserving.

Proposition 3.1.16 *Let A be a Banach algebra and let $R = A^{-1} \cup \partial A^{-1}$. Then R is an upper semiregularity.*

Proof. Note that $R = \overline{A^{-1}}$. [In a topological space the closure of a set is the union of the set with its boundary]. Since $1_A \in A^{-1} \subset R$, R contains a neighbourhood of 1_A. Let $a, b \in R$. Then there are sequences (a_n) and (b_n) in A^{-1} such that $a_n \to a$ and $b_n \to b$. Hence, $a_n b_n \to ab$ with $(a_n b_n)$ a sequence in A^{-1}. Consequently, $ab \in R$. □

For a related result, see ([21], Lemma 2.6).

Proposition 3.1.17 *Let A be a Banach algebra and let $R = A^{-1} \cup \partial A^{-1}$. Then $\sigma_R(a) \subset \sigma(a)$ and $\partial\sigma(a) \cap \sigma_R(a) = \emptyset$ for all $a \in A$.*

Proof. Since $A^{-1} \subset R$, we have that $\sigma_R(a) \subset \sigma(a)$ for all $a \in A$. Note that $\partial A^{-1} \cap R \neq \emptyset$. If $a \in A$ and $\lambda \in \partial\sigma(a)$, then $\lambda - a \in \partial A^{-1} \subset R$ and so $\lambda \notin \sigma_R(a)$. This completes the proof. □

Corollary 3.1.18 *Let A be a Banach algebra and let $R = A^{-1} \cup \partial A^{-1}$. Then the spectrum σ_R is not spectral radius preserving.*

Proof. The result follows from Proposition 3.1.17 above. □

3.2 ExpA

Let A be a Banach algebra. We define the *generalized exponentials*, $ExpA$ by

$$ExpA = \{e^{c_1} \cdots e^{c_k} : k \in \mathbb{N}, c_1, ..., c_k \in A\}.$$

It can be shown ([14] and [1], Theorem 3.3.7) that $ExpA$ forms the connected component of A^{-1} containing 1. Furthermore, $ExpA$ is an open subset of A and a closed normal subgroup of A^{-1}:

$$a \in A^{-1} \implies a^{-1} \cdot ExpA \cdot a = ExpA.$$

If $R = ExpA$, then the spectrum σ_R defined by

$$\sigma_R(a) = \{\sigma \in \mathbb{C} : \lambda - a \notin R\},$$

is known in the literature as the *exponential spectrum* of $a \in A$, see [14]. In this section we are going to compare the exponential spectrum σ_R with the ordinary spectrum.

Proposition 3.2.1 *Let A be a Banach algebra and let $R = ExpA$. Then R is an upper semiregularity.*

Proof. We prove this fact by proving that $ExpA$ is an open semigroup that contains the identity element (Lemma 1.5.4). From ([1], Theorem 3.3.7), we know that $ExpA$ is the component of A^{-1} that contains 1. Since its a component of A^{-1} it is closed in A^{-1}. By the same theorem, we know that $ExpA$ is also open in A^{-1}. So there must be U, open in A such that $ExpA = U \cap A^{-1}$. We also know that A^{-1} is open, hence $ExpA$ is open. What remains is to show that $ExpA$ is closed with respect to the binary operation of the algebra. Suppose that $a_1, a_2 \in ExpA$ and that $a_1 = e^{x_1} \cdots e^{x_n}$, $a_2 = e^{y_1} \cdots e^{y_m}$. Then $a_1 a_2 = e^{x_1} \cdots e^{x_n} e^{y_1} \cdots e^{y_m} \in ExpA$. □

Theorem 3.2.2 *Let A be a Banach algebra and let $R = ExpA$. Then $\partial\sigma_R(a) \subset \sigma(a) \subset \sigma_R(a)$ for all $a \in A$.*

Proof. Since $R \subset A^{-1}$, $\sigma(a) \subset \sigma_R(a)$ for all $a \in A$. We claim that $\partial R \cap A^{-1} = \emptyset$: Let $a \in \partial ExpA$. So every neighbourhood of a contains points of $ExpA$ as well as points of $A \backslash ExpA$. Either $a \in A^{-1}$ or $a \notin A^{-1}$. If $a \notin A^{-1}$ then $\partial ExpA \cap A^{-1} = \emptyset$ as required. Alternatively, suppose that $a \in A^{-1}$. Then a belongs to a component of A^{-1}. If $a \in ExpA$, then $ExpA$ is a neighbourhood of a because $ExpA$ is open in A^{-1}. But then this neighbourhood avoids $A \setminus ExpA$, hence contradicts $a \in \partial ExpA$. So next, suppose that $a \in U$, where U is any other component of A^{-1}. Then again, U is open in A^{-1}, hence a neighbourhood of a. Since components are disjoint, we have that $U \cap ExpA = \emptyset$, which again contradicts $a \in \partial ExpA$. Hence $\partial ExpA \cap A^{-1} = \emptyset$ and in view of Theorem 3.1.1 we are done. □

For a different proof of this fact see ([14], Theorem 1). To compare the upper semiregularity $R = ExpA$ with the regularity A^{-1} we need to generalize the notion of spectral radius preserving as follows.

Definition 3.2.3 Let R and S be regularities (semiregularities) such that $R \subset S$ and let σ_R and σ_S be the corresponding spectra. We will say that the spectrum σ_S is R - radius preserving if σ_S is closed and $\sup\{|\lambda| \mid \lambda \in \sigma_S(a)\} = \sup\{|\lambda| : \lambda \in \sigma_R(a)\}$ for all $a \in A$.

Corollary 3.2.4 *Let A be a Banach algebra and let $R = ExpA$. Then the usual spectrum σ is R - radius preserving.*

Proof. It is well known that the usual spectrum, $\sigma(a)$ is a closed set for all $a \in A$. It follows from Theorem 3.2.2 above that the spectrum σ is R - radius preserving. $\square$

3.3 expA

Let A be a Banach algebra and $a \in A$. We define the set $expA$ by

$$expA = \{e^x : x \in A\}.$$

If $R = expA$, then we will denote the spectrum $\sigma_R(a) = \{\lambda \in \mathbb{C} : \lambda - a \notin R\}$ by $e(a, A)$. For properties of the spectrum $e(\cdot, A)$, see [20], section 3. The set $expA$ is neither an upper semiregularity nor a lower semiregularity. In fact, from [23], Lemma 2 we know that every lower semiregularity contains the set A^{-1}. Since $expA \subset A^{-1}$ (and in general this inclusion is strict), $expA$ is not a lower semiregularity. Although $expA$ contains a neighbourhood of 1 ([8], Corollary I.8.4), $expA$ is not an upper semiregularity: If $a, b \in A$ then it is well-known that $ab = ba \implies e^a e^b = e^b e^a = e^{a+b}$. The converse of this statement is not true. In [29] the author proves that $e^a e^b = e^b e^a \implies ab = ba$ provided that the spectra $\sigma(a)$ and $\sigma(b)$ are $2\pi i - congruence - free$. In view of the definition of an upper semiregularity ([23], Definition 10), $expA$ is not an upper semiregularity. However, if A is a commutative Banach algebra, then $expA = ExpA$, see the remarks above. In this case $expA$ is an upper semiregularity, see Proposition 3.2.1. Despite these remarks we show in our next result that σ_{ExpA} is $expA$-radius preserving.

Theorem 3.3.1 *Let A be a Banach algebra. Then σ_{ExpA} is $expA$-radius preserving.*

Proof. Let $S = expA$ and $R = ExpA$. Since $S \subset R$, we have that $\sigma_R(a) \subset \sigma_S(a)$ for all $a \in A$. Note that $\sigma_R(a)$ is a closed set for all $a \in A$ ([14], Theorem 1). Since $\eta\sigma_R(a) = \eta\sigma_S(a)$ for all $a \in A$ ([20], Theorem 3.2), it follows that σ_{ExpA} is $expA-$ radius preserving. $\square$

In sections 1 and 2 we compared various spectra with the usual spectrum σ and in section 3 we compared the *exponential spectrum* with the spectrum generated by $expA$. Recall that, for A a Banach algebra:
$R_1 = A$;
$R_2 = A^{-1}$;

$R_3 = A_l^{-1}$;
$R_4 = A_r^{-1}$;
$R_5 = \{x \in A \ : \ x$ is not a left topological divisor of zero$\}$;
$R_6 = \{x \in A \ : \ x$ is not a right topological divisor of zero$\}$;
Let
$R_7 = A \setminus \partial A^{-1}$;
$R_8 = A_l^{-1} \cup A_r^{-1}$;
$R_9 = ExpA$;
$R_{10} = expA$.

Corollary 3.3.2 *Let A be a Banach algebra and $a \in A$. Then*

$$\sup\{|\lambda| \ : \lambda \in \sigma(a)\} = \sup\{|\lambda| \ : \lambda \in \sigma_{R_i}(a)(i = 2, ..., 10)\}$$

Proof. This follows from Corollaries 3.1.5, 3.1.7, 3.1.9, 3.1.15, 3.2.4 and Theorem 3.3.1. □

3.4 Subalgebras

In this section we continue our investigation of comparing regularities (semiregularities) by involving subalgebras of a Banach algebra. We state three results from literature that will guide our discussions.

Proposition 3.4.1 *([17], Proposition 3.2) Let A and B be Banach algebras such that $1 \in B \subset A$. If R_A is a regularity in A and R_B is a regularity in B, then $R_A \cap R_B$ is a regularity in B.*

Corollary 3.4.2 *([17], Corollary 3.3) Let A and B be Banach algebras such that $1 \in B \subset A$. If R_A is a regularity in A then $R_A \cap B$ is a regularity in B.*

Proposition 3.4.3 *([17], Proposition 3.4) Let A and B be Banach algebras such that $1 \in B \subset A$. Suppose R_A is a regularity in A. Then $\sigma_{R_A}(b) = \sigma_{R_A \cap B}(b)$ for every $b \in B$.*

We will generalize the above statements and show that they are valid for both upper and lower semiregularities.

Proposition 3.4.4 *Let A and B be Banach algebras such that $1 \in B \subset A$. If R_A is an upper (lower) semiregularity in A and R_B is an upper (lower) semiregularity in B, then $R_A \cap R_B$ is an upper (lower) semiregularity in B.*

Proof. Suppose that R_A and R_B are upper semiregularities. Let $a \in R_A \cap R_B$, $n \in \mathbb{N}$. Then $a \in R_A$ and $a \in R_B$, hence $a^n \in R_A$ and $a^n \in R_B$ since R_A and R_B are both upper semiregularities. Hence $a^n \in R_A \cap R_B$. Next, let a, b, c, d be mutually commuting elements of B satisfying $ac + bd = 1$ and $a, b \in R_A \cap R_B$. Then a, b, c, d are mutually commuting elements of A satisfying $ac + bd = 1$ and $a, b \in R_A$. Since R_A is an upper semiregularity, we have $ab \in R_A$. Also, a, b, c, d are mutually commuting elements of B satisfying $ac+bd = 1$ and $a, b \in R_B$. Since R_B is an upper semiregularity, we know that $ab \in R_B$. Hence $ab \in R_A \cap R_B$. as required. Finally, since R_A and R_B are upper semiregularities, they each contain a neighbourhood of 1. Hence $R_A \cap R_B$ contains a neighbourhood of 1. In view of Definition 1.5.3, the above shows that $R_A \cap R_B$ is an upper semiregularity. It is clear that $R_A \cap R_B \subset B$.
Next, suppose that R_A and R_B are lower semiregularities. Suppose that $a^n \in R_A \cap R_B$, $n \in \mathbb{N}$. Then $a^n \in R_A$ and $a^n \in R_B$, $n \in \mathbb{N}$. Hence, $a \in R_A$ and $a \in R_B$, since R_A and R_B are both lower semiregularities, hence $a \in R_A \cap R_B$ as required. Next, suppose that a, b, c, d are mutually commuting elements of B and that $ac + bd = 1$ and $ab \in R_A \cap R_B$. Then $ab \in R_A$ and $ab \in R_B$. Since both R_A and R_B are lower semiregularities, we have that $a, b \in R_A$ and $a, b \in R_B$. Hence $a, b \in R_A \cap R_B$. In view of Definition 1.5.1, the above shows that $R_A \cap R_B$ is a lower semiregularity. Again, it is clear that $R_A \cap R_B \subset B$. □

Corollary 3.4.5 *Let A and B be Banach algebras such that $1 \in B \subset A$. If R_A is an upper (lower) semiregularity in A then $R_A \cap B$ is an upper (lower) semiregularity in B.*

Proof. This follows from Proposition 3.4.4 and the fact that B is an upper (lower) semiregularity. □

Proposition 3.4.6 *Let A and B be Banach algebras such that $1 \in B \subset A$. Suppose R_A is an upper (lower) semiregularity in A. Then $\sigma_{R_A}(b) = \sigma_{R_A \cap B}(b)$ for every $b \in B$.*

Proof. The proof follows from Corollary 3.4.5 and Theorem 3.1.1. □

Proposition 3.4.7 *Let A and B be Banach algebras such that $1 \in B \subset A$. If R_i is a regularity in A, then $R_i \cap B$ is a regularity in B for $i \in \{2, 3, 4, 5, 6\}$.*

Proof. For each $i \in \{2, 3, 4, 5, 6\}$, R_i is a regularity in A. In view of Corollary 3.4.2 the result follows. □

Proposition 3.4.8 *Let A and B be Banach algebras such that $1 \in B \subset A$ and $b \in B$. If R_i is a regularity in A, then $\sigma_{R_i}(b) = \sigma_{R_i \cap B}(b)$ for all $i \in \{2, 3, 4, 5, 6\}$.*

Proof. R_i is a regularity in A, for all $i \in \{2, 3, 4, 5, 6\}$. In view of Proposition 3.4.3 the result follows. $\square$

Proposition 3.4.9 *Let A and B be Banach algebras such that $1 \in B \subset A$ and let $b \in B$. Then $\sigma_{R_7}(b) = \sigma_{R_7 \cap B}(b)$ for all $b \in B$.*

Proof. R_7 is an upper semiregularity in A. The result follows from Proposition 3.4.6. $\square$

Next, we let $R_{7,A} = \overline{A^{-1}}$ and $R_{7,B} = \overline{B^{-1}}$. Then we have

Proposition 3.4.10 *Let A and B be Banach algebras such that $1 \in B \subset A$. Then $R_{7,A} \cap R_{7,B}$ is an upper semiregularity in B.*

Proof. $R_{7,A}$ is an upper semiregularity in A and $R_{7,B}$ is an upper semiregularity in B. The result follows from application of Proposition 3.4.4. $\square$

Proposition 3.4.11 *Let A and B be Banach algebras such that $1 \in B \subset A$. Then $R_8 \cap B$ is a lower semiregularity in B.*

Proof. R_8 is a lower semiregularity in A. In view of Corollary 3.4.5 the result follows. $\square$

Proposition 3.4.12 *Let A and B be Banach algebras such that $1 \in B \subset A$. Then $\sigma_{R_8}(b) = \sigma_{R_8 \cap B}(b)$ for all $b \in B$.*

Proof. R_8 is a lower semiregularity in A. In view of Proposition 3.4.6 the result follows. $\square$

Next, we let $R_{3,B} = B_l^{-1}$ and $R_{3,A} = A_l^{-1}$. Similarly, $R_{4,B} = B_r^{-1}$ and $R_{4,A} = A_r^{-1}$. Then we have

Proposition 3.4.13 *Let A and B be Banach algebras such that $1 \in B \subset A$. Then $R_{3,B} \cap R_8$ and $R_{4,B} \cap R_8$ are both lower semiregularities in B.*

Proof. $R_{8,A}$ is a lower semiregularity in A, and each of $R_{3,B}$ and $R_{4,B}$ are regularities, hence lower semiregularities in B. The result follows from application of Proposition 3.4.4. $\square$

Proposition 3.4.14 *Let A and B be Banach algebras such that $1 \in B \subset A$. Then $R_9 \cap B$ is an upper semiregularity in B.*

Proof. R_9 is an upper semiregularity in A. In view of Corollary 3.4.5 the result follows. □

Proposition 3.4.15 *Let A and B be Banach algebras such that $1 \in B \subset A$. Then $\sigma_{R_9}(b) = \sigma_{R_9 \cap B}(b)$ for all $b \in B$.*

Proof. R_9 is an upper semiregularity in A. In view of Proposition 3.4.6 the result follows. □

To complete this section we consider the two sets that were neither semiregularities nor regularities, namely R_7 and R_{10}. We cannot apply Proposition 3.4.1 or Corollary 3.4.2 or any of their generalizations to either of the two sets. However, we can ask whether they satisfy the condition stated in Proposition 3.4.3. We have the following final result for this section.

Proposition 3.4.16 *Let A and B be Banach algebras such that $1 \in B \subset A$. Let $R \in \{R_7, R_{10}\}$. Then $\sigma_R(b) = \sigma_{R \cap B}(b)$ for every $b \in B$.*

Proof. Let A and B be Banach algebras such that $1 \in B \subset A$. Let $b \in B$. If $\lambda \in \sigma_R(b, A)$ then $\lambda - b \notin R$. Hence, $\lambda - b \notin R \cap B$, and so $\lambda \in \sigma_{R \cap B}(b, B)$. This shows that $\sigma_R(b, A) \subset \sigma_{R \cap B}(b, B)$ for every $b \in B$. Conversely, let $b \in B$ and suppose $\lambda \in \sigma_{R \cap B}(b, B)$. Then $\lambda - b \notin R \cap B$. Since B is a Banach algebra, and $\lambda, b \in B$, we know that $\lambda - b \in B$, so $\lambda - b \notin R$, hence $\lambda \in \sigma_R(b, A)$. This proves that $\sigma_{R \cap B}(b, B) \subset \sigma_R(b, A)$ for all $b \in B$, hence the result follows. □

3.5 Subalgebras and superalgebras

In the previous section we investigated the intersection of a regularity or semiregularity in a Banach algebra A with a subalgebra B. In this section we look at the intersection of a regularity or semiregularity in A with the corresponding structure (or set) in a subalgebra B. For ease of reference we state the following result, which will be used often in this section:

Theorem 3.5.1 *([1], Theorem 3.2.13) Let A be a Banach algebra and let B be a closed subalgebra of A containing the unit 1. We have:*

1. *B^{-1} is the union of some components of $B \cap A^{-1}$, and the set $\partial B^{-1} \cap (B \cap A^{-1})$ is empty.*

2. *if $x \in B$, then $\sigma(x, B)$ is the union of $\sigma(x, A)$ and a (possibly empty) collection of bounded components of $\mathbb{C} \setminus \sigma(x, A)$, in particular $\partial\sigma(x, B) \subset \partial\sigma(x, A)$.*

We give an alternative proof of Theorem 3.5.1 using topological divisors of zero.

Proof. Let $b \in B$. Since $B \subset A$, we have $B^{-1} \subset A^{-1}$ and from part 1 of Theorem 3.1.1, we have that $\sigma_{A^{-1}}(b) \subset \sigma_{B^{-1}}(b)$. Next let $a \in \partial B^{-1}$. Then a is a topological divisor of zero in B ([24], Theorem 1.1.14), so there is a sequence (a_n) in B with $||a_n|| = 1$ for all $n \in \mathbb{N}$ and $aa_n \to 0$. Then (a_n) is a sequence in A with $||a_n|| = 1$ for all $n \in \mathbb{N}$ and $aa_n \to 0$, hence a is a topological divisor of zero in A, which means $a \notin A^{-1}$. Hence, $\partial B^{-1} \cap (B \cap A^{-1}) = \emptyset$, and so by Theorem 3.1.1, $\partial\sigma_{B^{-1}}(b) \subset \sigma_{B\cap A^{-1}}(b) \subset \sigma_{B^{-1}}(b)$ because $B^{-1} \subset B \cap A^{-1}$. If we employ Proposition 3.4.6 and combine our arguments we get $\partial\sigma_{B^{-1}}(b) \subset \sigma_{B\cap A^{-1}}(b) = \sigma_{A^{-1}}(b) \subset \sigma_{B^{-1}}(b)$. □

Corollary 3.5.2 *Let B be a closed subalgebra of a Banach algebra A, with $1 \in B \subset A$. Then $\sigma_{A^{-1}}$ is B^{-1} radius preserving.*

Proof. Since for every $b \in B$, $\partial\sigma_{B^{-1}}(b) \subset \sigma_{A^{-1}}(b) \subset \sigma_{B^{-1}}(b)$ and since $\sigma_{A^{-1}}(b)$ is closed, it follows that the spectrum $\sigma_{A^{-1}}$ is B^{-1} radius preserving. □

Next we consider the upper semiregularity R_9. We compare the generalized exponentials in the subalgebra B with those in the superalgebra A. We denote $ExpB$ by $R_{9,B}$ and $ExpA$ by $R_{9,A}$.

Theorem 3.5.3 *Let A be a Banach algebra and let B be a closed subalgebra of A containing the unit 1. Then $\partial\sigma_{R_{9,B}}(b, B) \subset \sigma_{R_{9,A}}(b, A) \subset \sigma_{R_{9,B}}(b, B)$ for every $b \in B$.*

Proof. We have $R_{9,B} \subset R_{9,A}$ so that $\sigma_{R_{9,A}}(b) \subset \sigma_{R_{9,B}}(b)$ for every $b \in B$ is clear. We also have that $R_{9,A} \subset A^{-1}$ and since $R_{9,B}$ is a component of B^{-1} it follows that $\partial R_{9,B} \subset \partial B^{-1}$. Hence, $\partial R_{9,B} \cap (B \cap R_{9,A}) \subset \partial B^{-1} \cap (B \cap A^{-1}) = \emptyset$. The last step follows from Theorem 3.5.1. In view of Theorem 3.1.1, we have that $\partial\sigma_{R_{9,B}}(b) \subset \sigma_{R_{9,A}\cap B}(b) \subset \sigma_{R_{9,B}}(b)$ for every $b \in B$, because $R_{9,B} \subset B \cap R_{9,A}$. In view of Proposition 3.4.15, the result follows. □

For a different proof of Theorem 3.5.3, see ([11], Theorem 4.1).

Corollary 3.5.4 *Let B be a closed subalgebra of a Banach algebra A with $1 \in B \subset A$. Then the spectrum $\sigma_{R_{9,A}}$ is $R_{9,B}$ radius preserving.*

Proof. Since $R_{9,A}$ is an open set in A, $\sigma_{R_{9,A}}$ is a closed set in the complex plane. It follows from Theorem 3.5.3 that the spectrum $\sigma_{R_{9,A}}$ is $R_{9,B}$ radius preserving. $\square$

Next, we consider the left and right invertibles, $R_3 = A_l^{-1}$, and $R_4 = A_r^{-1}$.

Proposition 3.5.5 *Let B be a closed subalgebra of a Banach algebra A with $1 \in B \subset A$. Then the spectrum $\sigma_{R_{i,A}}$ is $R_{i,B}$ radius preserving for $i \in \{3, 4\}$.*

Proof. Recall that $R_{3,A} = A_l^{-1}$, $R_{3,B} = B_l^{-1}$, $R_{4,A} = A_r^{-1}$, $R_{4,B} = B_r^{-1}$. Let $i \in \{3, 4\}$. The result follows from the facts that $R_{i,A}$ is spectral radius preserving in A (Corollary 3.1.5), and Corollary 3.5.2. $\square$

In the above cases we had $S \subset R$ and we asked whether $\partial S \cap R = \emptyset$. That being highlighted, we next consider the topological divisors of zero and the boundary spectrum.

As before we assume that A is a Banach algebra and that B is a closed Banach algebra such that $1 \in B \subset A$. To distinguish the structures in A and B we let $R_{5,A} = \{x \in A : x \text{ is not a left topological divisor of zero}\}$ and $R_{5,B} = \{x \in B : x \text{ is not a left topological divisor of zero}\}$. We know that if x is a left (right) topological divisor of zero in B, then x must be a left (right) topological divisor of zero in A. Hence we have:

Theorem 3.5.6 *([11], Theorem 3.1) Suppose B is a closed subalgebra of a Banach algebra A with $1 \in B \subset A$. Then $\sigma_{R_{5,B}}(b) \subset \sigma_{R_{5,A}}(b)$ and $\sigma_{R_{6,B}}(b) \subset \sigma_{R_{6,A}}(b)$ for all $b \in B$.*

Proof. Since $R_{5,A} \subset R_{5,B}$ it follows that $\sigma_{R_{5,B}}(b) \subset \sigma_{R_{5,A}}(b)$ for all $b \in B$. It follows similarly that $\sigma_{R_{6,B}}(b) \subset \sigma_{R_{6,A}}(b)$ for all $b \in B$. $\square$

By Example 3.6 in [11], the inclusions from Theorem 3.5.6 may be strict. However, our next result shows that these spectra are also radius preserving.

Theorem 3.5.7 *Suppose B is a closed subalgebra of a Banach algebra A with $1 \in B \subset A$. The spectrum $\sigma_{R_{i,B}}$ is $R_{i,A}$ radius preserving for $i \in \{5, 6\}$.*

Proof. We prove the theorem for $i = 5$. The proof for $i = 6$ is similar and omitted. By Theorem 3.5.6, $\sigma_{R_{5,B}}(b) \subset \sigma_{R_{5,A}}(b)$ for all $b \in B$. In view of Proposition 3.1.6, for all $b \in B$

$$\partial\sigma(b, B) \subset \sigma_{R_{5,B}}(b) \subset \sigma(b, B)$$

Hence, by Corollary 3.1.7

$$\sup\{|\lambda| \ : \ \lambda \in \sigma(b,B)\} = \sup\{|\lambda| \ : \lambda \in \sigma_{R_{5,B}}(b)\} \tag{3.1}$$

Also, again by Proposition 3.1.6 and Corollary 3.1.7, for all $b \in B$

$$\partial\sigma(b,B) \subset \sigma_{R_{5,A}}(b) \subset \sigma(b,A)$$

and so

$$\sup\{|\lambda| \ : \ \lambda \in \sigma(b,A)\} = \sup\{|\lambda| \ : \lambda \in \sigma_{R_{5,A}}(b)\} \tag{3.2}$$

But, by Corollary 3.5.2

$$\sup\{|\lambda| \ : \ \lambda \in \sigma(b,B)\} = \sup\{|\lambda| \ : \ \lambda \in \sigma(b,A)\} \tag{3.3}$$

If we combine (3.1), (3.2), and (3.3) we get

$$\sup\{|\lambda| \ : \lambda \in \sigma_{R_{5,B}}(b)\} = \sup\{|\lambda| \ : \lambda \in \sigma_{R_{5,A}}(b)\}$$

Since $R_{5,B}$ is an open set in B, the spectrum $\sigma_{R_{5,B}}$ is $R_{5,A}$ radius preserving. $\square$

Let B be a closed subalgebra of a Banach algebra A with $1 \in B \subset A$. In the following results we are going to compare the boundary spectrum σ_{R_7} in B with the boundary spectrum in A. Recall that the set R_7 is neither an upper nor a lower semiregularity, see Examples 3.1.10 and 3.1.12. Denote the boundary spectrum in B by $\sigma_{R_{7,B}}$ and the boundary spectrum in A by $\sigma_{R_{7,A}}$.

Theorem 3.5.8 *([21], Theorem 2.11) Let B be a closed subalgebra of a Banach algebra A with $1 \in B \subset A$. Then $\sigma_{R_{7,B}}(b) \subset \sigma_{R_{7,A}}(b)$ for all $b \in B$.*

Proof. Let $b \in B$. Since $\partial_B B^{-1} \subset \partial_A A^{-1}$, ([21], Theorem 2.11), it follows that

$$\begin{aligned}\sigma_{R_{7,B}}(b) &= \{\lambda \in \mathbb{C} \ : \ \lambda - b \in \partial_B B^{-1}\} \\ &\subset \{\lambda \in \mathbb{C} \ : \ \lambda - b \in \partial_A A^{-1}\} \\ &= \sigma_{R_{7,A}}(b).\end{aligned}$$

$\square$

The next example shows that the inclusion from Theorem 3.5.8 is in general strict.

Example 3.5.9 Let $\Gamma = \{z \in \mathbb{C} : |z| = 1\}$ and $D = \{z \in \mathbb{C} : |z| \leq 1\}$. Let $C(\Gamma)$ be the Banach algebra of continuous, complex valued functions defined on Γ, with the supremum norm. Let $A(D)$ be the Banach algebra of complex valued functions which are continuous on Γ and analytic on the interior of D.

Let $A = C(\Gamma) \times C(\Gamma)$ and $B = A(D) \times A(D)$. Let $f(z) = z$ for $z \in D$. Then $f \in A(D) \subset C(\Gamma)$. It is easy to see that $\sigma(f, C(\Gamma)) = \Gamma$ and $\sigma(f, A(D)) = D$. We show that $(f, 0) \in \partial A^{-1}$ but $(f, 0) \notin \partial B^{-1}$. To see that $(f, 0) \in \partial A^{-1}$, let (λ_n) be any sequence in $\mathbb{C}$ with $0 < |\lambda_n| < 1$ and $\lambda_n \to 0$. Then $((f - \lambda_n, -\lambda_n)) \to (f, 0)$ and $(f - \lambda_n, -\lambda_n) \in A^{-1}$ for $n \in \mathbb{N}$. Also, $((f, 0)) \to (f, 0)$, and note that $(f, 0) \notin A^{-1}$ so $0 \in \sigma_{R_{7,A}}((f, 0))$.

Next we show that $0 \notin \sigma_{R_{7,B}}((f, 0))$. From [11], Example 3.6 we know that f is not a topological divisor of zero in $A(D)$, hence $f \notin \partial[A(D)]^{-1}$. Since $f \notin [A(D)]^{-1}$ as well, there must be a neighbourhood of f in $A(D)$ that contains no elements from $[A(D)]^{-1}$. Hence there is a neighbourhood of $(f, 0)$ in B which contains no elements from B^{-1}. Hence $(f, 0) \notin \partial B^{-1}$, or $0 \notin \sigma_{R_{7,B}}((f, 0))$. Hence $\sigma_{R_{7,B}}((f, 0)) \subsetneq \sigma_{R_{7,A}}((f, 0))$.

Theorem 3.5.10 *Let B be a closed subalgebra of a Banach algebra A with $1 \in B \subset A$. Then the spectrum $\sigma_{R_{7,B}}$ is $R_{7,A}$ radius preserving.*

Proof. Let $b \in B$. Since $R_{7,B}$ is an open set in B, the spectrum $\sigma_{R_{7,B}}$ is a closed set in $\mathbb{C}$. From Theorem 3.5.8, we have that $\sigma_{R_{7,B}}(b) \subset \sigma_{R_{7,A}}(b)$. In view of Proposition 3.1.8, $\partial\sigma(b, B) \subset \sigma_{R_{7,B}}(b) \subset \sigma(b, B)$ and so

$$\sup\{|\lambda| \ : \lambda \in \sigma(b, B)\} = \sup\{|\lambda| \ : \ \lambda \in \sigma_{R_{7,B}}(b)\} \tag{3.4}$$

Again, by Proposition 3.1.8, $\partial\sigma(b, A) \subset \sigma_{R_{7,A}}(b) \subset \sigma(b, A)$, and so

$$\sup\{|\lambda| \ : \lambda \in \sigma(b, A)\} = \sup\{|\lambda| \ : \ \lambda \in \sigma_{R_{7,A}}(b)\} \tag{3.5}$$

But by Corollary 3.5.2

$$\sup\{|\lambda| \ : \lambda \in \sigma(b, B)\} = \sup\{|\lambda| \ : \lambda \in \sigma(b, A)\} \tag{3.6}$$

If we combine (3.4), (3.5) and (3.6) we get

$$\sup\{|\lambda| \ : \ \lambda \in \sigma_{R_{7,B}}(b)\} = \sup\{|\lambda| \ : \ \lambda \in \sigma_{R_{7,A}}(b)\}$$

□

Next we consider the lower semiregularity $R_8 = A_l^{-1} \cup A_r^{-1}$. We need the following theorem:

Theorem 3.5.11 *Let A be a Banach algebra and let B be a closed subalgebra of A containing the unit 1 of A. Then $\partial B_l^{-1} \cap (B \cap A_r^{-1}) = \emptyset$ and $\partial B_r^{-1} \cap (B \cap A_l^{-1}) = \emptyset$.*

Proof. We prove the statement for $\partial B_l^{-1} \cap (B \cap A_r^{-1}) = \emptyset$. The second part is similar and omitted. Suppose $x \in \partial B_l^{-1}$. Then, using [24], Theorem 1.1.14, we have that x is a right topological divisor of zero in B. Hence x is a right topological divisor of zero in A. Using the same theorem, $x \notin A_r^{-1}$. Hence the result follows. $\square$

We let $R_{8,A} = A_l^{-1} \cup A_r^{-1}$ and $R_{8,B} = B_l^{-1} \cup B_r^{-1}$.

Proposition 3.5.12 *Let B be a closed subalgebra of a Banach algebra A such that $1 \in B \subset A$. Then the spectrum $\sigma_{R_{8,A}}$ is $R_{8,B}$ radius preserving.*

Proof. Let B be a closed subalgebra of a Banach algebra A such that $1 \in B \subset A$. Then using Corollaries 3.1.15, 3.5.2, and 3.1.15 for $b \in B$, we get

$$\begin{aligned}\sup\{|\lambda| : \lambda \in \sigma_{R_{8,A}}(b)\} &= \sup\{|\lambda| : \lambda \in \sigma_A(b)\} \\ &= \sup\{|\lambda| : \lambda \in \sigma_B(b)\} \\ &= \sup\{|\lambda| : \lambda \in \sigma_{R_{8,B}}(b)\}\end{aligned}$$

Also, $R_{8,A}$ is open, hence $\sigma_{R_{8,A}}$ is closed. The result follows. $\square$

Theorem 3.5.13 *Let A be a Banach algebra. Then $\partial A_l^{-1} \cap A_r^{-1} = \emptyset$ and $\partial A_r^{-1} \cap A_l^{-1} = \emptyset$. Also, $\partial A_l^{-1} \cap A_l^{-1} = \emptyset$ and $\partial A_r^{-1} \cap A_r^{-1} = \emptyset$.*

Proof. We prove that $\partial A_l^{-1} \cap A_r^{-1} = \emptyset$ and $\partial A_l^{-1} \cap A_l^{-1} = \emptyset$. The second pair of relationships is similar and omitted. Suppose $x \in \partial A_l^{-1}$. Then, using ([24], Theorem 1.1.14), we have that x is a right topological divisor of zero in A. Using the same theorem, $x \notin A_r^{-1}$. We have shown that $\partial A_l^{-1} \cap A_r^{-1} = \emptyset$. Next, we note that A_l^{-1} is an open set in A. Hence $A \setminus A_l^{-1}$ is a closed set and so $\partial A_l^{-1} \subset (A \setminus A_l^{-1})$, which means that $\partial A_l^{-1} \cap A_l^{-1} = \emptyset$. This completes the proof. $\square$

Theorem 3.5.14 *Let A be a Banach algebra. Then $\partial \sigma_{A_l^{-1}}(a) \subset \sigma_{A_l^{-1} \cup A_r^{-1}}(a) \subset \sigma_{A_l^{-1}}(a)$ for all $a \in A$. Also, $\partial \sigma_{A_r^{-1}}(a) \subset \sigma_{A_r^{-1} \cup A_l^{-1}}(a) \subset \sigma_{A_r^{-1}}(a)$ for all $a \in A$.*

Proof. We show that $\partial \sigma_{A_l^{-1}}(a) \subset \sigma_{A_l^{-1} \cup A_r^{-1}}(a) \subset \sigma_{A_l^{-1}}(a)$ for all $a \in A$. The proof of the second relationship is similar and omitted.

$$\begin{aligned}&\partial A_l^{-1} \cap (A_l^{-1} \cup A_r^{-1}) \\ &= (\partial A_l^{-1} \cap (A_l^{-1}) \cup (\partial A_l^{-1} \cap A_r^{-1})) \\ &= \emptyset,\end{aligned}$$

using Theorem 3.5.13 $\square$

Corollary 3.5.15 *Let A be a Banach algebra. Then the spectrum $\sigma_{R_{8,A}}$ is $R_{3,A}$ radius preserving and $R_{4,A}$ radius preserving.*

Proof. Since $R_{8,A}$ is an open set in A, the spectrum $\sigma_{R_{8,A}}$ is a closed set in $\mathbb{C}$. It follows from Theorem 3.5.13 that the spectrum $\sigma_{R_{8,A}}$ is $R_{3,A}$ radius preserving and $R_{4,A}$ radius preserving. $\square$

3.6 Fredholm Theory

In this final section we tie together two themes, the one from Chapter 2 and the one from Chapter 3.

Let A be a Banach algebra and let I be a closed ideal in A. We say that an element $a \in A$ is *Fredholm relative to I*, if the coset $a + I$ is invertible in the quotient algebra A/I, i.e., $a + I \in (A/I)^{-1}$. We denote the collection of Fredholm elements in A relative to I by $\Phi(A, I)$. See also Definition 2.2.1. Let $\pi : A \to A/I$ be the canonical homomorphism, i.e., $\pi(a) = a + I$ $(a \in A)$. Since $(A/I)^{-1}$ is a regularity in the quotient algebra A/I, $\Phi(A, I) = \pi^{-1}((A/I)^{-1})$ is a regularity in A, see [24], Theorem 1.6.3 (iii). Since π is continuous and $(A/I)^{-1}$ is an open set in A/I, $\Phi(A, I)$ is an open regularity in A. The spectrum relative to this regularity is called the *Fredholm spectrum* (relative to I) and is denoted by $\sigma_{\Phi(A,I)}$ Note that for $a \in A$

$$\begin{aligned}\sigma_{\Phi(A,I)}(a) &= \{\lambda : \lambda - a \notin \Phi(A, I)\}\\ &= \{\lambda : (\lambda - a) + I \notin (A/I)^{-1}\}\\ &= \sigma(a + I, A/I)\end{aligned}$$

An element $a \in A$ is called *Weyl* relative to I if $a = b+c$ with $b \in A^{-1}$ and $c \in I$. The collection of Weyl elements in A relative to I will be denoted by $\mathcal{W}(A, I)$. Since $\mathcal{W}(A, I)$ is a semigroup containing A^{-1}, it is an upper semiregularity, see Lemma 1.5.4. The spectrum $\sigma_{W(A,I)}(a)$ is called the *Weyl spectrum* of $a \in A$ relative to I. It is easy to see that

$$A^{-1} \subset \mathcal{W}(A, I) \subset \Phi(A, I)$$

and so

$$\sigma_{\Phi(A,I)}(a) \subset \sigma_{\mathcal{W}(A,I)}(a) \subset \sigma(a)$$

for all $a \in A$. In Fredholm theory the spectra $\sigma_{\Phi(A,I)}$ and $\sigma_{\mathcal{W}(A,I)}$ play an important role, see [7]. If we restrict ourselves to a closed trace ideal and invoke the theory in Chapter 2, we can say more.

Theorem 3.6.1 *Let A be a semisimple Banach algebra and let I be a closed trace ideal in A such that* $\operatorname{Soc} A \subset I \subset kh(\operatorname{Soc} A)$. *Then* $\partial\sigma_{\mathcal{W}(A,I)}(a) \subset \sigma_{\Phi(A,I)}(a) \subset \sigma_{\mathcal{W}(A,I)}(a)$ *for all* $a \in A$.

Proof. From the assumptions and Corollary 2.4.5 it is clear that $\mathcal{W}(A, I) = \Phi_0(A, I) \subset \Phi(A, I)$. We claim that $\partial\mathcal{W}(A, I) \cap \Phi(A, I) = \emptyset$. Let $a \in \partial\mathcal{W}(A, I)$. So every neighbourhood of a contains points from $\mathcal{W}(A, I)$ and points from $A \setminus \mathcal{W}(A, I)$. Either $a \in \Phi(A, I)$ or $a \notin \Phi(A, I)$. If $a \notin \Phi(A, I)$ then $\partial\mathcal{W}(A, I) \cap \Phi(A, I) = \emptyset$ as required. Alternatively, suppose that $a \in \Phi(A, I)$. Then a must be in a component of $\Phi(A, I)$. If $a \in \mathcal{W}(A, I)$ then $\mathcal{W}(A, I)$ is a neighbourhood of a since $\mathcal{W}(A, I) = \Phi_0(A, I)$ which is open in $\Phi(A, I)$. But this neighbourhood avoids $A \setminus \mathcal{W}(A, I)$, hence contradicts $a \in \partial\mathcal{W}(A, I)$. So next, suppose that $a \in U$ where U is any component of $\Phi(A, I)$. Then again, U is open in $\Phi(A, I)$, hence a neighbourhood of a. Since components are disjoint, we have that $U \cap \mathcal{W}(A, I) = \emptyset$, which again contradicts $a \in \partial\mathcal{W}(A, I)$. Hence $\partial\mathcal{W}(A, I) \cap \Phi(A, I) = \emptyset$. The result follows from applying Theorem 3.1.1. □

Corollary 3.6.2 *Let A be a semisimple Banach algebra and let I be a closed trace ideal in A such that* $\operatorname{Soc} A \subset I \subset kh(\operatorname{Soc} A)$. *Then the Fredholm spectrum* $\sigma_{\Phi(A,I)}$ *is* $\mathcal{W}(A, I)$ *radius preserving.*

Proof. From the discussion preceding Theorem 3.6.1 we have that $\Phi(A, I)$ is an open set in A. (see also Proposition 2.2.6 (iv)). Hence, for $a \in A$, $\sigma_{\Phi(A,I)}(a)$ is closed in $\mathbb{C}$. It follows from Theorem 3.6.1 that the Fredholm spectrum $\sigma_{\Phi(A,I)}$ is $\mathcal{W}(A, I)$ radius preserving. □

If X is a Banach space and one wants to develop Fredholm theory in the algebra $\mathcal{L}(X)$ relative to the closed ideal $\overline{\mathcal{F}(X)}$, one can apply Theorem 3.6.1 because $\overline{\mathcal{F}(X)}$ is a closed trace ideal in the Banach algebra $\mathcal{L}(X)$. See Example 2.1.2.

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