TRACE CHARACTERIZATIONS AND SOCLE IDENTIFICATIONS IN BANACH ALGEBRAS

by

FRANCOIS PETRUS SCHULZ

submitted in fulfilment of the requirements for the degree

PHILOSOPHIAE DOCTOR

in

MATHEMATICS

in the

FACULTY OF SCIENCE

at the

UNIVERSITY OF JOHANNESBURG

SUPERVISOR: DR R BRITS
CO-SUPERVISOR: DR G BRAATVEDT

September 2015

Contents

Acknowledgements

1. Rudi Brits and Gareth Braatvedt. Thank you for your unparalleled guidance, overflowing knowledge and for being adept mentors. I am truly grateful beyond measure for your support and encouragement, without which, this would not have been possible.

2. My wife to be, Vanessa Cronje. Thank you for all the inspiration, love and belief in me. Words cannot express what you mean to me.

3. My family, Magda, Hannes and Jolandi Schulz. Thank you for all the motivation, love and unwavering support. I will always be indebted to you for all the sacrifices that you have made for me throughout my life.

4. The referee of the paper, entitled "Commutators, Commutativity and Dimension in the Socle of a Banach Algebra", which resulted from this research – thank you for several very useful suggestions.

5. The Faculty of Science and the Department of Pure and Applied Mathematics at the University of Johannesburg, for the research opportunities afforded to me.

Introduction

The theory of rank and trace for square complex matrices and, to a lesser degree, finite-rank operators, has been developed quite extensively. Pioneering work done rather recently by B. Aupetit and H. Du T. Mouton in [3] has led to purely spectral and analytic definitions of the aforementioned notions, and, in fact, to an extension of these definitions to a broader setting. In particular, Aupetit and Mouton managed to prove that the socle of a semisimple Banach algebra, A, coincides with the set of finite-rank elements of A. The socle, first introduced by J. Diuedonné over 70 years ago, is a well-known object in Banach algebras and in more algebraic structures (rings etc.). However, in spite of this, the aforementioned connection, the spectral definition of rank and trace, and the underlying theory thereof, has not yet been exploited to its full potential. In the discussion that follows we will produce a large number of nontrivial applications of the work done in [3] to solve various conjectures in abstract Banach algebra theory. However, it should be emphasized that we will not merely build on [3]; indeed, we shall develop an entirely new and striking theory pertaining to the structure of the socle of a semisimple Banach algebra.

Chapter 1 contains some notations and definitions relating to rank, trace, tensor products and the socle that will be used throughout this thesis. Specifically, in the first section of this chapter, we briefly summarize the important results of Aupetit and Mouton in [3]. However, we also prove some preliminary new results. In particular, we show that the trace is a linear functional on the socle (Lemma 1.1.1) and that the elements in a semisimple Banach algebra A of the form $ab - ba$, where a belongs to the socle and $b \in A$, are all traceless (Corollary 1.1.4). Furthermore, we characterize those socles which are finite-dimensional (Theorem 1.1.5). These results have all been published in [16]. In the second section we obtain important insights to the structure of the socle that will lead to some of the main results in this thesis. These insights include the fact that there exists a collection of pairwise orthogonal minimal two-sided ideals such that every socle element can be written as a finite sum of members of these ideals (Lemma 1.2.1 and Lemma 1.2.2). Moreover, we show that each minimal two-sided ideal of this form is isomorphic as an algebra to the tensor product of a corresponding minimal left and minimal right ideal (Corollary 1.2.5).

Chapter 2 can be viewed as the main chapter of this thesis. In the first section, as a follow-up to a paper by D. Petz and J. Zemánek [12], we obtain a number of equivalent conditions which characterize the trace among linear functionals on matrix algebras, finite-rank operators and the socle of a semisimple Banach algebra (Theorem 2.1.3, Theorem 2.1.4, Theorem 2.1.10 and Theorem 2.1.12). Moreover, we address the converse problem to the above; that is, given the equivalence of certain conditions which characterize the trace, what can be said about the structure of the socle? In particular, we characterize those socles which are isomorphic to full matrix algebras in this manner (Theorem 2.1.3), as well as those socles which are minimal two-sided ideals (Theorem 2.1.7 and Theorem 2.1.8). These results have also been published in [16]. By building on the theory developed in Section 1.2, in Section 2.2 we obtain that the socle is isomorphic as an algebra to the direct sum of tensor products of corresponding minimal left and right ideals (Theorem 2.2.1). Remarkably, the finite-dimensional case here reduces to the classical *Wedderburn-Artin Theorem* [1, Theorem 2.1.2], and this approach does not use any continuous irreducible representations on the algebra in question. Section 2.2 is then concluded by showing that the socle contains many "structurally friendly" subalgebras; specifically, that any finite collection of socle elements is contained in some subalgebra of the socle which has the Wedderburn-Artin structure (Theorem 2.2.13). In the third section of this chapter, we precisely characterize the structure of the socles in which *Shoda's Theorem* (see [17]) holds true as those socles which are minimal two-sided ideals (Theorem 2.3.2). After deriving a result which suggests a correlation between commutators and the dimension of certain subalgebras in the socle (Theorem 2.3.3), we proceed to show that the commutators in the socle, i.e. the set $\{xy - yx : x, y \in \mathrm{Soc}(A)\}$, is a vector subspace of the socle (Theorem 2.3.7). Furthermore, it is shown that all socle elements with zero spectrum are commutators in the socle (Lemma 2.3.10), and that the dimension of a subalgebra involving the Riesz projection associated to a socle element has a bearing to whether a traceless element is a commutator in the socle (Theorem 2.3.11). In Section 2.4 we provide a number of necessary and sufficient conditions for the socle of a semisimple Banach algebra to be contained in the center of the algebra. Noteworthy is the characterizations of central socles which appear in Theorem 2.4.4 and Theorem 2.4.6. This section is then concluded by two very attractive characterizations in Theorem 2.4.8, which together confirm a correlation between the dimension of certain subalgebras in the socle and commutativity. The results which appear in Section 2.2, Section 2.3 and Section 2.4 have all been published in [15]. In the fifth and final section of this chapter, following a problem which is identified by M. Brešar and Š. Špenko in [7], we explore the relationship between elements in a semisimple Banach algebra that satisfy certain spectral containment conditions. In particular, we use these conditions to spectrally characterize prime Banach algebras amongst the class of Banach algebras with nonzero socles (Theorem 2.5.11 and Theorem 2.5.21), as well as to obtain spectral characterizations of socles which are minimal two-sided ideals (Theorem 2.5.7 and Theorem 2.5.20). In addition, Section 2.5 contains numerous related statements which are of independent interest as well.

In the third and final chapter, the central theme is the new notion of Shoda-completeness. A semisimple Banach algebra with nonzero socle is called Shoda-complete if every traceless element in its socle can be expressed as a commutator of two elements belonging to the socle in question. The work in Chapter 3 therefore is a continuation of the work done in Chapter 2, for Chapter 2 is littered with a number of characteristic properties of Shoda-complete Banach algebras. However, the main aim of this chapter is to address the following natural question: Given a Banach algebra A which is not Shoda-complete, is it possible to isometrically embed an algebraic isomorphic copy of A in a Banach algebra which is Shoda-complete? Using the classical Wedderburn-Artin Theorem, it is almost trivial to see how this can be done in the finite-dimensional case. The infinite-dimensional case, on the other hand, is highly nontrivial. However, we show that it can be done. Section 3.1 contains a brief motivation of the idea. In Section 3.2, we further investigate minimal ideals and obtain a number of results that are not only useful in the current work, but of independent interest as well. The algebraic extension is obtained in Section 3.3. We then introduce a submultiplicative norm in Section 3.4 which extends the original norm on the Banach algebra in question. It is very unlikely that the subsequent norm completion is semisimple. However, because of the location of the radical, the quotient algebra turns out to be a semisimple Banach algebra extension which is simultaneously Shoda-complete. Finally, in the last section of this chapter, we show how the construction of the particular extension is independent of the choice of rank one projection representatives generating the socle.

Chapter 1

Rank, Trace and the Socle of a Banach Algebra

1.1 Some reminders about Rank and Trace in Banach Algebras

The determinants of infinite matrices were first investigated by astronomers, over a century ago. Nowadays, the notions of rank, trace and determinant are well-established for operator theory. Rather recently, in their paper entitled *Trace and determinant in Banach algebras* [3], Aupetit and Mouton managed to show that these notions can be developed, without the use of operators, in a purely spectral and analytic manner. This paper is fundamental to our discussion here, for this alternative point of view not only permits the possibility to consider rank and trace related problems in a more general setting, but also allows the consideration of a converse problem to [12, Theorem 2]. Indeed, we are able to gain some new insights to the structure of the socle of a semisimple Banach algebra in this manner. As in [16] we briefly summarize some of the theory in [3] before we proceed.

By A we denote a complex Banach algebra with identity element $\mathbf{1}$, norm $\|\cdot\|_A$ and *invertible group* $G(A)$. Moreover, it will be assumed throughout that A is *semisimple* (i.e. the *Jacobson radical* of A, denoted $\mathrm{Rad}(A)$, only contains 0). By $Z(A)$ we denote the *center* of A, that is, the set of all $x \in A$ such that $xy = yx$ for all $y \in A$. For $x \in A$ we denote by $\sigma_A(x) = \{\lambda \in \mathbb{C} : \lambda\mathbf{1} - x \notin G(A)\}$, $\rho_A(x) = \sup\{|\lambda| : \lambda \in \sigma_A(x)\}$ and $\sigma'_A(x) = \sigma_A(x) - \{0\}$ the *spectrum*, *spectral radius* and *nonzero spectrum* of x, respectively. If the underlying algebra is clear from the context, then we shall agree to omit the subscript A in the notation $\sigma_A(x)$, $\rho_A(x)$ and $\sigma'_A(x)$.

This convention will also be followed in the forthcoming definitions of rank, trace, and so forth. We shall also agree to reserve the notation $\cong$ exclusively for algebra isomorphisms. Any other type of isomorphism will explicitly be referred to. Moreover, we recall that an element x of A is called *nilpotent* if $x^n = 0$ for some positive integer n, and *quasinilpotent* if $\sigma(x) = \{0\}$.

For each nonnegative integer m, let

$$\mathcal{F}_m = \{a \in A : \#\sigma'(xa) \leq m \text{ for all } x \in A\},$$

where the symbol $\#K$ denotes the number of distinct elements in a set $K \subseteq \mathbb{C}$. Following Aupetit and Mouton in [3], we define the *rank* of an element a of A as the smallest integer m such that $a \in \mathcal{F}_m$, if it exists; otherwise the rank is infinite. In other words,

$$\operatorname{rank}(a) = \sup_{x \in A} \#\sigma'(xa).$$

If $a \in A$ is a finite-rank element, then

$$E(a) = \{x \in A : \#\sigma'(xa) = \operatorname{rank}(a)\}$$

is a dense open subset of A [3, Theorem 2.2]. A finite-rank element a of A is said to be a *maximal finite-rank element* if $\operatorname{rank}(a) = \#\sigma'(a)$. With respect to rank it is further useful to know that $\sigma'(xa) = \sigma'(ax)$ for all $x, a \in A$ (Jacobson's Lemma, [1, Lemma 3.1.2]) and that $\sigma(f(x)) = f(\sigma(x))$, where f is holomorphic on an open set containing $\sigma(x)$ (Spectral Mapping Theorem, [1, Theorem 3.3.3]). It can be shown [3, Corollary 2.9] that the *socle*, written $\operatorname{Soc}(A)$, of a semisimple Banach algebra A coincides with the collection $\bigcup_{m=0}^{\infty} \mathcal{F}_m$ of finite rank elements. We mention a few elementary properties of the rank of an element [3, p. 117]. Firstly, $\#\sigma'(a) \leq \operatorname{rank}(a)$ for all $a \in A$. Furthermore, $\operatorname{rank}(xa) \leq \operatorname{rank}(a)$ and $\operatorname{rank}(ax) \leq \operatorname{rank}(a)$ for all $x, a \in A$, with equality if $x \in G(A)$. Moreover, the rank is lower semicontinuous on $\operatorname{Soc}(A)$. It is also subadditive, i.e. $\operatorname{rank}(a+b) \leq \operatorname{rank}(a) + \operatorname{rank}(b)$ for all $a, b \in A$ [3, Theorem 2.14]. Finally, if p is a projection of A, then p has rank one if and only if p is a *minimal projection*, that is $pAp = \mathbb{C}p$. It is also worth mentioning here that a projection p is minimal if and only if Ap is a nontrivial left ideal which does not contain any left ideals other than $\{0\}$ and itself, that is, if and only if Ap is a nontrivial *minimal left ideal* [4, Lemma 30.2]. A similar result holds true for the right ideal pA. We will also define a *minimal two-sided ideal* in this manner, that is, as a two-sided ideal which does not contain any two-sided ideals other than $\{0\}$ and itself.

The following two results are fundamental to the theory developed in [3] and are mentioned here for convenient referencing later on:

Scarcity Theorem for Rank [3, Theorem 2.3]: Let f be an analytic function from a domain D of $\mathbb{C}$ into A. Then either the set of λ for which the rank of $f(\lambda)$ is finite has zero capacity or there exist an integer N and a closed discrete subset E of D such that $\operatorname{rank}(f(\lambda)) = N$ on $D - E$ and $\operatorname{rank}(f(\lambda)) < N$ on E.

Diagonalization Theorem [3, Theorem 2.8]: Let $a \in A$ be a nonzero maximal finite-rank element and denote by $\lambda_1, \ldots, \lambda_n$ its nonzero distinct spectral values. Then there exists n orthogonal minimal projections $p_1, \ldots, p_n \in Aa \cap aA$ such that

$$a = \lambda_1 p_1 + \cdots + \lambda_n p_n.$$

In particular, the Diagonalization Theorem easily implies the well-known result that every element of the socle is *Von Neumann regular*, that is, for each $a \in \operatorname{Soc}(A)$, there exists an $x \in \operatorname{Soc}(A) \subseteq A$ such that $a = axa$ [3, Corollary 2.10]. Another useful result is [3, Theorem 2.16], which states that for any set of nonzero orthogonal finite-rank projections $\{p_1, \ldots, p_n\}$ we have

$$\operatorname{rank}\,(\alpha_1 p_1 + \cdots + \alpha_n p_n) = \operatorname{rank}\,(p_1) + \cdots + \operatorname{rank}\,(p_n)$$

for all $\alpha_1, \ldots, \alpha_n \in \mathbb{C} - \{0\}$.

If $a \in \operatorname{Soc}(A)$ we define the *trace* of a as in [3] by

$$\operatorname{Tr}(a) = \sum_{\lambda \in \sigma(a)} \lambda m(\lambda, a),$$

where $m(\lambda, a)$ is the *multiplicity of a at λ*. A brief description of the notion of multiplicity in the abstract case goes as follows (for particular details one should consult [3]): Let $a \in \operatorname{Soc}(A)$, $\lambda \in \sigma(a)$ and let $B(\lambda, r)$ be an open disk centered at λ such that $B(\lambda, r)$ contains no other points of $\sigma(a)$. It can be shown [3, Theorem 2.4] that there exists an open ball, say $U \subseteq A$, centered at $\mathbf{1}$ such that $\#\left[\sigma(xa) \cap B(\lambda, r)\right]$ is constant as x runs through $E(a) \cap U$. This constant integer is the multiplicity of a at λ. It can also be shown that $m(\lambda, a) \geq 1$ and

$$\sum_{\alpha \in \sigma(a)} m(\alpha, a) = \begin{cases} 1 + \operatorname{rank}(a) & \text{if } 0 \in \sigma(a) \\ \operatorname{rank}(a) & \text{if } 0 \notin \sigma(a). \end{cases} \qquad (1.1.1)$$

Let $a \in \mathrm{Soc}(A)$. From (1.1.1), it readily follows that

$$\sum_{\lambda \in \sigma'(a)} m(\lambda, a) \leq \mathrm{rank}(a).$$

Consequently, as observed in [3], we have that

$$\begin{aligned} |\mathrm{Tr}(a)| &= \left| \sum_{\lambda \in \sigma(a)} \lambda m(\lambda, a) \right| = \left| \sum_{\lambda \in \sigma'(a)} \lambda m(\lambda, a) \right| \\ &\leq \sum_{\lambda \in \sigma'(a)} |\lambda| \cdot m(\lambda, a) \leq \sum_{\lambda \in \sigma'(a)} \rho(a) \cdot m(\lambda, a) \\ &= \rho(a) \cdot \sum_{\lambda \in \sigma'(a)} m(\lambda, a) \leq \rho(a) \cdot \mathrm{rank}(a). \end{aligned}$$

Furthermore, by [3, Theorem 3.3(a)] it follows that $\mathrm{Tr}(x+y) = \mathrm{Tr}(x) + \mathrm{Tr}(y)$ for each $x, y \in \mathrm{Soc}(A)$. The next lemma shows that the trace is in fact a linear functional:

Lemma 1.1.1. *Let a be a finite-rank element of A and let $\alpha \in \mathbb{C} - \{0\}$. Then $m(\lambda, a) = m(\alpha\lambda, \alpha a)$ for each $\lambda \in \sigma'(a)$. Consequently,* $\mathrm{Tr}(\alpha a) = \alpha \mathrm{Tr}(a)$ *for each $\alpha \in \mathbb{C}$.*

Proof. Set $\lambda_0 = 0$ and denote by $\lambda_1, \dots, \lambda_n$ the distinct nonzero spectral values of a. Choose $r > 0$ so that the open disks $B(\lambda_0, r), B(\lambda_1, r), \dots, B(\lambda_n, r)$ are all disjoint and $\alpha \in \mathbb{C} - \{0\}$. By the Spectral Mapping Theorem it follows that $\sigma'(\alpha a) = \{\alpha\lambda_1, \dots, \alpha\lambda_n\}$. Notice that

$$B(\alpha\lambda_0, |\alpha| r), \dots, B(\alpha\lambda_n, |\alpha| r)$$

are also all disjoint, and that, for each $i \in \{0, 1, \dots, n\}$, we have $\beta \in B(\lambda_i, r)$ if and only if $\alpha\beta \in B(\alpha\lambda_i, |\alpha| r)$. Let $i \in \{1, \dots, n\}$ be arbitrary but fixed. For $j \in \{1, \alpha\}$, let U_j be an open disk centered at $\mathbf{1}$ so that $x \in U_j \cap E(ja)$ implies that

$$\#\left(\sigma(jxa) \cap \Delta_0^j\right) = m(j\lambda_i, ja),$$

where Δ_0^j is the interior of $\partial B(j\lambda_i, jr)$. Since $\#\sigma'(xa) = \#\sigma'(\alpha xa)$ by the Spectral Mapping Theorem, and since $\mathrm{rank}(a) = \mathrm{rank}(\alpha a)$, it follows that $E(a) \subseteq E(\alpha a)$. Moreover, since $U_1 \cap U_\alpha \cap E(a) \neq \emptyset$ by the density of $E(a)$, there exists an $x_0 \in U_1 \cap U_\alpha \cap E(a)$. But then $x_0 \in U_1 \cap E(a)$ and $x_0 \in U_\alpha \cap E(\alpha a)$, so by the Spectral Mapping Theorem and our choice of contours we obtain that

$$\begin{aligned} m(\lambda_i, a) &= \#\left(\sigma(x_0 a) \cap \Delta_0^1\right) \\ &= \#\left(\sigma(\alpha x_0 a) \cap \Delta_0^\alpha\right) \\ &= m(\alpha\lambda_i, \alpha a). \end{aligned}$$

Since i was arbitrary, this completes the proof. □

In order to prove some of our main results in the next chapter, a further property of the trace is derived below:

Lemma 1.1.2. *Let f be a linear functional on $\mathrm{Soc}(A)$, and let $c > 0$ be a real number. Suppose that $|f(a)| \leq c \cdot \mathrm{rank}\,(a) \cdot \rho(a)$ for all $a \in \mathrm{Soc}(A)$. Then f is continuous on $\mathcal{F}_k$ for each nonnegative integer k.*

Proof. Let $x \in \mathcal{F}_k$ be arbitrary, and suppose that $(x_n) \subseteq \mathcal{F}_k - \{x\}$ converges to x. We must show that $f(x_n) \to f(x)$ as $n \to \infty$: By the subadditivity of the rank it follows that

$$\mathrm{rank}\ (x_n - x) \leq \mathrm{rank}\ (x_n) + \mathrm{rank}\,(x) \leq 2k$$

for each integer $n \geq 1$. Thus, by linearity and the hypothesis on f it follows that

$$\begin{aligned} |f(x_n) - f(x)| = |f(x_n - x)| &\leq c \cdot \mathrm{rank}\ (x_n - x) \cdot \rho(x_n - x) \\ &\leq c \cdot 2k \cdot \|x_n - x\|. \end{aligned}$$

Since $x_n \to x$ as $n \to \infty$, we have the desired result. Hence, since $x \in \mathcal{F}_k$ was arbitrary, the lemma is proved. □

Theorem 1.1.3. *Let f be a linear functional on $\mathrm{Soc}(A)$, and let $c > 0$ be a real number. Suppose that $|f(a)| \leq c \cdot \mathrm{rank}\,(a) \cdot \rho(a)$ for all $a \in \mathrm{Soc}(A)$. Then $f(ab) = f(ba)$ for all $a \in \mathrm{Soc}(A)$ and $b \in A$.*

Proof. Let $a \in \mathrm{Soc}(A)$ and $b \in A$ be arbitrary. Consider the analytic function $g : \mathbb{C} \to \mathrm{Soc}(A)$ given by $g(\lambda) = e^{\lambda b} a e^{-\lambda b}$. We claim that $\lambda \mapsto f(g(\lambda))$ is an entire function: To prove our claim it will suffice to show that

$$\lim_{\lambda \to \lambda_0} \frac{f(g(\lambda)) - f(g(\lambda_0))}{\lambda - \lambda_0}$$

exists for all $\lambda_0 \in \mathbb{C}$. Let $\lambda_0 \in \mathbb{C}$ be arbitrary, and let $(\alpha_n) \subseteq \mathbb{C} - \{\lambda_0\}$ be any sequence which converges to λ_0. Observe that

$$\frac{f(g(\alpha_n)) - f(g(\lambda_0))}{\alpha_n - \lambda_0} = f\left(\frac{g(\alpha_n) - g(\lambda_0)}{\alpha_n - \lambda_0}\right)$$

for each integer $n \geq 1$, and that

$$\operatorname{rank}\left(\frac{g\left(\alpha_n\right)-g\left(\lambda_0\right)}{\alpha_n-\lambda_0}\right) \leq 2 \cdot \operatorname{rank}(a)$$

by the subadditivity and the properties of the rank mentioned earlier in this section. Moreover, it can be shown that

$$g'\left(\lambda_0\right)=e^{\lambda_0 b} b a e^{-\lambda_0 b}-e^{\lambda_0 b} a b e^{-\lambda_0 b},$$

so $g'\left(\lambda_0\right) \in \operatorname{Soc}(A)$ and $\operatorname{rank}\left(g'\left(\lambda_0\right)\right) \leq 2 \cdot \operatorname{rank}(a)$ as before. Hence, by Lemma 1.1.2 it follows that

$$\lim_{n \rightarrow \infty} \frac{f\left(g\left(\alpha_n\right)\right)-f\left(g\left(\lambda_0\right)\right)}{\alpha_n-\lambda_0}=f\left(g'\left(\lambda_0\right)\right).$$

Thus, since $\lambda_0 \in \mathbb{C}$ and $\left(\alpha_n\right)$ were arbitrary, this proves our claim. However, by hypothesis,

$$\begin{aligned}
|f(g(\lambda))| & \leq c \cdot \operatorname{rank}(g(\lambda)) \cdot \rho(g(\lambda)) \\
& = c \cdot \operatorname{rank}(a) \cdot \rho(a)
\end{aligned}$$

for each $\lambda \in \mathbb{C}$, where the equality sign follows from the properties of the rank and Jacobson's Lemma. So, since $\lambda \mapsto f(g(\lambda))$ is entire and bounded, we may conclude by Liouville's Theorem that it must be constant. Hence, $0=f\left(g'(0)\right)=f(ba-ab)$, so $f(ab)=f(ba)$ as desired. □

Since the trace is a linear functional on $\operatorname{Soc}(A)$ which satisfies

$$|\operatorname{Tr}(a)| \leq \operatorname{rank}(a) \cdot \rho(a) \text { for all } a \in \operatorname{Soc}(A),$$

Theorem 1.1.3 readily gives the following:

Corollary 1.1.4. *Let* $a \in \operatorname{Soc}(A)$. *Then* $\operatorname{Tr}(ab)=\operatorname{Tr}(ba)$ *for all* $b \in A$.

Finally, we note that the trace has the following useful properties:

(i) For any $a \in A$, if $\operatorname{Tr}(ax)=0$ for each $x \in \operatorname{Soc}(A)$, then $a\operatorname{Soc}(A)=\{0\}$. Moreover, if $a \in \operatorname{Soc}(A)$, then $a=0$ [3, Corollary 3.6].

(ii) If f is an analytic function from a domain D of $\mathbb{C}$ into $\operatorname{Soc}(A)$, then $\lambda \mapsto \operatorname{Tr}(f(\lambda))$ is holomorphic on D [3, Theorem 3.1].

Let $\lambda \in \sigma(a)$ and suppose that $B(\lambda, 2r)$ separates λ from the remaining spectrum of a. Let f_λ be the holomorphic function which takes the value 1 on $B(\lambda, r)$ and the value 0 on $\mathbb{C} - \overline{B}(\lambda, r)$. If we now let Γ_0 be a smooth contour which surrounds $\sigma(a)$ and is contained in the domain of f_λ, then

$$p(\lambda, a) = f_\lambda(a) = \frac{1}{2\pi i}\int_{\Gamma_0} f_\lambda(\alpha)\,(\alpha\mathbf{1} - a)^{-1}\,d\alpha$$

is referred to as the *Riesz projection* associated with a and λ [3, p. 120]. By the Holomorphic Functional Calculus, Riesz projections associated with a and distinct spectral values are orthogonal and for $\lambda \neq 0$,

$$p(\lambda, a) = \frac{a}{2\pi i}\int_{\Gamma_0} \frac{f_\lambda(\alpha)}{\alpha}\,(\alpha\mathbf{1} - a)^{-1}\,d\alpha \in Aa \cap aA. \tag{1.1.2}$$

The following results will also be useful: Let $a \in A$ have finite rank and let $\lambda_1, \ldots, \lambda_n$ be nonzero distinct elements of its spectrum. If

$$p = p(\lambda_1, a) + \cdots + p(\lambda_n, a),$$

then by [3, Theorem 2.6] we have

$$\operatorname{rank}(p) = m(\lambda_1, a) + \cdots + m(\lambda_n, a).$$

Moreover, $\operatorname{rank}(p) = m(1, p)$ [3, Corollary 2.7]. It is customary to refer to p here as the Riesz projection associated with a and $\lambda_1, \ldots, \lambda_n$. It is also worth mentioning that the orthogonal minimal projections obtained in the conclusion of the Diagonalization Theorem are in fact the Riesz projections of the maximal finite-rank element associated with each of its corresponding nonzero spectral values.

From results in [13] it is easy to deduce that $A = \operatorname{Soc}(A)$ if and only if A is finite-dimensional. Also, it is well-known that $\operatorname{Soc}(A)$ is a proper two-sided ideal whenever A is infinite-dimensional. However, the following observation can be made concerning the dimension of $\operatorname{Soc}(A)$:

Theorem 1.1.5. *The following are equivalent:*

(a) $\operatorname{Soc}(A)$ *is finite-dimensional.*

(b) $\operatorname{Soc}(A)$ *is closed.*

(c) *There exists a real number $c > 0$ such that $|\operatorname{Tr}(a)| \leq c \cdot \rho(a)$ for all $a \in \operatorname{Soc}(A)$.*

Proof. The implication (a) $\Rightarrow$ (b) is clear. So assume that $\mathrm{Soc}(A)$ is closed. Recall that $\mathrm{Soc}(A) = \bigcup_{m=0}^{\infty} \mathcal{F}_m$. Moreover, from the lower semicontinuity of the rank on $\mathrm{Soc}(A)$ and the fact that $\mathrm{Soc}(A)$ is closed, it now follows that each $\mathcal{F}_m$ is closed. Thus, from Baire's Category Theorem, it follows that there is a smallest integer n for which there is an open set $U \neq \emptyset$ in $\mathrm{Soc}(A)$ such that $U \subseteq \mathcal{F}_n$. Let $x \in U$ be arbitrary but fixed. Let $y \in \mathrm{Soc}(A)$ be arbitrary and consider the analytic function $f : \mathbb{C} \to A$ defined by $f(\lambda) = x - \lambda(x-y)$ for each $\lambda \in \mathbb{C}$. Since U is open in $\mathrm{Soc}(A)$, and since $f(\lambda) \in \mathrm{Soc}(A)$ for each $\lambda \in \mathbb{C}$, it readily follows that there exists an $\epsilon > 0$ such that $\lambda \in B(0,\epsilon)$ implies that $f(\lambda) = x - \lambda(x-y) \in U$. Consequently, rank $(f(\lambda)) \leq n$ for all λ in a set with nonzero capacity. Thus, by the Scarcity Theorem for Rank, it follows that rank $(f(\lambda)) \leq n$ for all $\lambda \in \mathbb{C}$. So, in particular,

$$\mathrm{rank}\ (f(1)) = \mathrm{rank}\ (x - (x-y)) = \mathrm{rank}\ (y) \leq n.$$

Since $y \in \mathrm{Soc}(A)$ was arbitrary, it follows that rank $(y) \leq n$ for all $y \in \mathrm{Soc}(A)$. Consequently, it follows that

$$|\mathrm{Tr}\,(a)| \leq \mathrm{rank}\,(a) \cdot \rho(a) \leq n \cdot \rho(a)$$

for all $a \in \mathrm{Soc}(A)$. This shows that (b) $\Rightarrow$ (c). Suppose now that there exists a real number $c > 0$ such that $|\mathrm{Tr}\,(a)| \leq c \cdot \rho(a)$ for all $a \in \mathrm{Soc}(A)$. Let k be any integer such that $k \geq c$. We claim that $\#\sigma'(y) \leq k$ for all $y \in \mathrm{Soc}(A)$: Suppose this is false. Then there exists an $x \in \mathrm{Soc}(A)$ such that $\#\sigma'(x) \geq k+1$. Let $\lambda_1, \ldots, \lambda_{k+1}$ be $k+1$ distinct nonzero spectral values of x and let $p_j = p\,(\lambda_j, x)$ for each $j \in \{1, \ldots, k+1\}$. Then $p = p_1 + \cdots + p_{k+1}$ is a projection. Moreover, by our remarks earlier in this section it follows that

$$\mathrm{Tr}\,(p) = m\,(1,p) = \mathrm{rank}\ (p) = m\,(\lambda_1, x) + \cdots + m\,(\lambda_{k+1}, x) \geq k+1.$$

But then

$$|\mathrm{Tr}\,(p)| \geq k+1 > c = c \cdot \rho(p).$$

This contradiction now proves our claim. Consequently, there exists a least integer n such that rank $(y) \leq n$ for all $y \in \mathrm{Soc}(A)$. If $n = 0$, then $\mathrm{Soc}(A) = \{0\}$ and we are done. So assume that $n \geq 1$. Since n was the smallest integer with this property, it must be the case that rank $(a) = n$ for some $a \in \mathrm{Soc}(A)$. Moreover, without loss of generality, we may assume that a is a maximal finite-rank element. By the Diagonalization Theorem, there are n orthogonal rank one projections $q_1, \ldots, q_n$ such that $a = \alpha_1 q_1 + \cdots + \alpha_n q_n$, where $\alpha_1, \ldots, \alpha_n \in \mathbb{C} - \{0\}$. We claim that $\mathrm{Soc}(A) = (q_1 + \cdots + q_n)\,A$: Suppose not. Then there exists a $b \in \mathrm{Soc}(A)$ such that $b \notin (q_1 + \cdots + q_n)\,A$.

Necessarily, $b \neq 0$. Let $u = (1 - (q_1 + \ldots + q_n))\, b$. Then $u \in \mathrm{Soc}(A)$. Moreover, $u \neq 0$, for if $u = 0$, then $b = q_1 b + \cdots + q_n b$, which contradicts our choice of b. Finally, note that $q_i u = 0$ for each $i \in \{1, \ldots, n\}$. Since $u \neq 0$, it follows that $\mathrm{rank}\,(u) = r$ for some integer r with $1 \leq r \leq n$. Let $x \in A$ so that $\#\sigma'(ux) = r$ and let $\beta_1, \ldots, \beta_r$ be the distinct nonzero spectral values of ux. Now let q be the Riesz projection associated with ux and $\beta_1, \ldots, \beta_r$. Then $q \in \mathrm{Soc}(A)$. Moreover, by formula (1.1.2) we have that $q_i q = 0$ for each $i \in \{1, \ldots, n\}$. Consider the element

$$v = 1q_1 + 2q_2 + \cdots + nq_n + (n+1)q,$$

and note that $v \in \mathrm{Soc}(A)$. Since $q_i\,(i\mathbf{1} - v) = 0$ for each $i \in \{1, \ldots, n\}$, and since $((n+1)\mathbf{1} - v)\, q = 0$, it follows that $\{1, \ldots, n+1\} \subseteq \sigma(v)$. But this contradicts the fact that $\mathrm{rank}\,(v) \leq n$. Hence, $\mathrm{Soc}(A) = (q_1 + \cdots + q_n)\, A$ as claimed. By using a symmetric argument it can be shown that $\mathrm{Soc}(A) = A\,(q_1 + \cdots + q_n)$ as well. Hence, since every element of the socle is Von Neumann regular, we may infer that

$$\mathrm{Soc}(A) = \sum_{i=1}^{n} \sum_{j=1}^{n} q_i A q_j.$$

Thus, since dim $(q_i A q_j) \leq 1$ for each $i, j \in \{1, \ldots, n\}$ (see [13, Lemma 4.2]), we have shown that (c) $\Rightarrow$ (a). This establishes the result. □

If p is a projection of A, then pAp is a closed semisimple subalgebra of A with identity p [2, Lemma 2.5]. The subalgebra pAp is very useful in the theory of rank, trace and determinant, primarily because of the following reasons:

$$\sigma'_{pAp}\,(pxp) = \sigma'_A\,(pxp) \tag{1.1.3}$$

and

$$\mathrm{rank}_{pAp}(pxp) = \mathrm{rank}_A(pxp) \tag{1.1.4}$$

for each $x \in A$. The proof of (1.1.3) is not hard and (1.1.4) is a consequence of (1.1.3) and Jacobson's Lemma.

Remark. From the last part of the argument in the proof of Theorem 1.1.5 it is actually possible to deduce a bit more. If $\mathrm{Soc}(A)$ is finite-dimensional, then it is possible to find a projection p, which is the finite sum of orthogonal rank one projections, such that $\mathrm{Soc}(A) = Ap = pA$. So, since every element of the socle is von Neumann regular, it follows that $\mathrm{Soc}(A) = pAp$. But pAp is a closed semisimple subalgebra of A with identity element p. Hence, $\mathrm{Soc}(A)$ is a finite-dimensional semisimple Banach algebra with an identity element. Consequently, by the Wedderburn-Artin Theorem, if $\operatorname{Soc}(A)$ is finite-dimensional, then $\operatorname{Soc}(A) \cong M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$, where $\oplus$ denotes the direct sum.

As a final observation for this section we note that in the operator case, $A = B(X)$ (bounded linear operators on a Banach space X), the "spectral" rank and trace both coincide with the respective classical operator definitions.

1.2 Tensor Products and the Socle

Let p be a projection of A with $\operatorname{rank}(p) \leq 1$. By J_p we denote the two-sided ideal generated by p, that is, we let

$$J_p := \left\{ \sum_{j=1}^{n} x_j p y_j : x_j, y_j \in A, n \geq 1 \text{ an integer} \right\}.$$

The next two lemmas are well-known results in Ring Theory. However, since these results will play a central role in the development of the subsequent theory, we provide short proofs for both of them.

Lemma 1.2.1. *There exists a collection of two-sided ideals $\{J_p : p \in \mathcal{P}\}$ such that every element of* $\operatorname{Soc}(A)$ *can be written as a finite sum of members of the J_p. Moreover, the two-sided ideals are pairwise orthogonal, that is, if $p, q \in \mathcal{P}$ with $p \neq q$, then*

$$J_p J_q = J_q J_p = \{0\}.$$

Proof. By definition a set J_p as defined above is a two-sided ideal contained in $\operatorname{Soc}(A)$. Let $\mathcal{S}$ consist of all collections of these ideals such that the members in a collection are pairwise orthogonal. Then $\mathcal{S} \neq 0$ because (with $p = 0$) $\{\{0\}\} \in \mathcal{S}$. Partially order $\mathcal{S}$ by set containment. By Zorn's Lemma $\mathcal{S}$ has a maximal element, say $\mathcal{M}$. We show that every element of $\operatorname{Soc}(A)$ can be written as a finite sum of elements each of which belongs to a member of $\mathcal{M}$: By the density of the set $E(a)$ for $a \in \operatorname{Soc}(A)$ and the Diagonalization Theorem it suffices to show that we can do this for any minimal projection, say q. We claim that in fact q belongs to some $J_p \in \mathcal{M}$. Suppose this is not the case. Then for any particular $J_p \in \mathcal{M}$, $q \notin J_p$. This implies that for each $x, y \in A$ we must have $xpyq = 0$, for otherwise, if there exists $x_1, y_1 \in A$ such that $x_1 p y_1 q \neq 0$, then by the minimality of q we have $A(x_1 p y_1 q) = Aq$, and so $z x_1 p y_1 q = q$ for some $z \in A$. But then $q \in J_p$ which contradicts our assumption on q. So $xpyq = 0$ for all $x, y \in A$. Similarly, $qxpy = 0$ for all $x, y \in A$. If we now consider the ideal J_q, then we see that $J_p J_q = J_q J_p = \{0\}$. But if this is true for each J_p in $\mathcal{M}$ then we have a contradiction with the maximality of $\mathcal{M}$. This completes the proof. □

Lemma 1.2.2. *Let p be a projection of A with* $\operatorname{rank}(p) \leq 1$. *Then J_p is a minimal two-sided ideal.*

Proof. If $p = 0$, then the result is obviously true. So assume that $p \neq 0$. Let $\{0\} \neq J \subseteq J_p$ be a two-sided ideal. We claim that $J = J_p$: It will suffice to show that $p \in J$. By hypothesis, there exists an $a \in J$ such that $a \neq 0$ and $a = \sum_{j=1}^{n} x_j p y_j$ for some $x_1, \ldots, x_n, y_1, \ldots, y_n \in A$. Assume that $pxayp = 0$ for all $x, y \in A$. Then, by Jacobson's Lemma and the semisimplicity of A it follows that $ayp = 0$ for all $y \in A$. However, then $awa = \sum_{j=1}^{n} awx_j p y_j = 0$ for all $w \in A$. But a is Von Neumann regular, so this implies that $a = 0$ which is absurd. Therefore, $px_0 a y_0 p \neq 0$ for some $x_0, y_0 \in A$. Since p has rank one, it is minimal. Hence, $px_0 a y_0 p = \lambda p$ for some $\lambda \in \mathbb{C} - \{0\}$. Thus, $p \in J$. This proves our claim which gives the result. $\square$

Some of the theory, which appear next and is used in Chapter 2, will require elementary properties of *tensor products*. This can be found in a standard textbook such as [4, pp. 230–232]. We shall briefly introduce the notation here, after which we will proceed to develop the relevant material:

Let X, Y, Z be normed vector spaces over the same field $\mathbb{C}$. A mapping ϕ from $X \oplus Y$ into Z is said to be *bilinear* if

(i) for each $y \in Y$ the mapping $x \mapsto \phi(x, y)$ is linear,

(ii) for each $x \in X$ the mapping $y \mapsto \phi(x, y)$ is linear.

Let X, Y be normed spaces over $\mathbb{C}$ with respective dual spaces X' and Y'. Given $x \in X$ and $y \in Y$, let $x \otimes y$ be the bilinear mapping from $X' \oplus Y'$ into $\mathbb{C}$ defined by

$$(x \otimes y)(f, g) = f(x)g(y) \quad (f \in X', g \in Y').$$

The *algebraic tensor product* of X and Y, denoted $X \otimes Y$, is defined as

$$\operatorname{span}\{x \otimes y : x \in X, y \in Y\}.$$

Theorem 1.2.3. *Let p be a projection of A with* $\operatorname{rank}(p) \leq 1$. *Then there exists a linear isomorphism from $Ap \otimes pA$ onto J_p.*

Proof. If $p = 0$ then the result is trivially true. So assume that p is a rank one projection. Let $\tau : Ap \oplus pA \to Ap \otimes pA$ be the *tensor map*, that is, let

$$\tau(x, y) = x \otimes y \quad (x \in Ap, y \in pA).$$

Next we consider the mapping $\psi : Ap \oplus pA \to J_p$ given by

$$\psi(x, y) = xpy \quad (x \in Ap, y \in pA).$$

It is routine to show that ψ is a bilinear mapping. Thus, by [4, Theorem 42.6] there exists a unique linear mapping $\phi : Ap \otimes pA \to J_p$ such that $\psi = \phi \circ \tau$.

It will therefore suffice to show that ϕ is bijective: Let $x \in J_p$ be arbitrary. Then $x = \sum_{j=1}^{n} u_j p v_j$ for some $u_1, \ldots, u_n, v_1, \ldots, v_n \in A$. By the linearity of ϕ and the fact that $p = p^2$, it readily follows that

$$\phi\left(\sum_{j=1}^{n} u_j p \otimes p v_j\right) = x.$$

This proves that ϕ is surjective. To see that ϕ is injective, suppose that

$$\phi\left(\sum_{j=1}^{n} x_j \otimes y_j\right) = 0, \tag{1.2.1}$$

where $x_1, \ldots, x_n \in Ap$ and $y_1, \ldots, y_n \in pA$. By [4, Lemma 42.3] we may assume without loss of generality that $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ are linearly independent subsets of Ap and pA, respectively. From (1.2.1) it follows that

$$x_1 p y_1 + \cdots x_n p y_n = 0. \tag{1.2.2}$$

Using (1.2.2), the minimality of p, the fact that $x_i p = x_i$ for each $i \in \{1, \ldots, n\}$, and the linear independence of $\{x_1, \ldots, x_n\}$, we may conclude that for any $j \in \{1, \ldots, n\}$, we have

$$p y_j w p = 0 \text{ for all } w \in A. \tag{1.2.3}$$

Fix any $j \in \{1, \ldots, n\}$ and let $y \in A$ be arbitrary. Since $y_j = p y_j$, it follows from (1.2.3) and Jacobson's Lemma that $\sigma(y y_j) = \{0\}$. Hence, by the semisimplicity of A it follows that $y_j = 0$ for each $j \in \{1, \ldots, n\}$. From this it now follows that $x_j \otimes y_j = 0$ for all $j \in \{1, \ldots, n\}$. Thus, $\sum_{j=1}^{n} x_j \otimes y_j = 0$, and so, ϕ is injective. This completes the proof. □

By definition $Ap \otimes pA$ is a vector space. However, Theorem 1.2.3 readily gives the following result:

Corollary 1.2.4. *Let $\phi : Ap \otimes pA \to J_p$ be the linear isomorphism obtained in Theorem 1.2.3. Define multiplication in $Ap \otimes pA$ by letting*

$$uv = \phi^{-1}\left(\phi(u)\phi(v)\right) \quad (u, v \in Ap \otimes pA). \tag{1.2.4}$$

Then $Ap \otimes pA$ equipped with the above multiplication scheme is an algebra.

Proof. There is nothing ambiguous about the right-side of (1.2.4), so the multiplication is well-defined. Moreover, the algebra properties are inherited from J_p which gives the result. □

It is interesting to note that the multiplication scheme above for elementary tensors in $Ap \otimes pA$ is actually given by

$$(x_1 \otimes y_1)(x_2 \otimes y_2) = \mathrm{Tr}\,(y_1 x_2)(x_1 \otimes y_2) \quad (x_1, x_2 \in Ap, y_1, y_2 \in pA)\,.$$

Moreover, since ϕ obtained in Theorem 1.2.3 is a linear isomorphism and

$$\phi^{-1}(\phi(u))\,\phi^{-1}(\phi(v)) = uv = \phi^{-1}(\phi(u)\phi(v))$$

for all $u, v \in Ap \otimes pA$, we readily obtain the following result:

Corollary 1.2.5. *Let p be a projection of A with* $\mathrm{rank}\,(p) \leq 1$. *Then* $Ap \otimes pA \cong J_p$.

Chapter 2

Trace Characterizations and Socle Identifications in Banach Algebras

2.1 Trace Characterizations and Socle Identifications

If $x, y \in A$, then we define the *commutator* of x, y by

$$[x, y] := xy - yx.$$

The next lemma can be obtained as a direct consequence of a deep result by K. Shoda (see [17]). However, we show that this result is not necessary for the development of the theory in this section.

Lemma 2.1.1. *Let* $A = M_n(\mathbb{C})$*. Then* $\operatorname{span}\{[a, b] : a, b \in A\} = \operatorname{Ker} \operatorname{Tr}$.

Proof. By Corollary 1.1.4 and the linearity of the trace it follows that

$$\operatorname{span}\{[a, b] : a, b \in A\} \subseteq \operatorname{Ker} \operatorname{Tr}.$$

We prove the reverse containment. Notice that the traceless matrices are precisely those matrices which have arbitrary entries off the main diagonal and whose entries on the main diagonal sum to zero. Thus, if the (i, i)-entry is denoted by $\lambda_{i,i}$ for each $i \in \{1, \dots, n\}$, then for the aforementioned traceless matrices we have $\lambda_{n,n} = -(\lambda_{1,1} + \cdots + \lambda_{n-1,n-1})$. We show that we can find a spanning set for the traceless matrices using only commutators. Let $e_{i,j}$ denote the matrix which has zeros at each entry except at the (i, j)-entry where it has a 1. Consider any $e_{i,j}$ where $i \neq j$ (i.e. off the main diagonal). Then

$$[e_{i,i}, e_{i,j}] = e_{i,i}e_{i,j} - e_{i,j}e_{i,i} = e_{i,j}.$$

So we can generate all 0 diagonal matrices with commutators. To deal with the diagonal consider $[e_{1,2}, e_{2,1}] = e_{1,1} - e_{2,2}$; $[e_{2,3}, e_{3,2}] = e_{2,2} - e_{3,3}$; $[e_{3,4}, e_{4,3}] = e_{3,3} - e_{4,4}$; $\cdots$; $[e_{n-1,n}, e_{n,n-1}] = e_{n-1,n-1} - e_{n,n}$. Thus, to get

$$\lambda_{1,1}, \lambda_{2,2}, \ldots, \lambda_{n-1,n-1}, -(\lambda_{1,1} + \cdots + \lambda_{n-1,n-1})$$

as entries on the main diagonal just build the linear combination

$$\begin{aligned}&\lambda_{1,1}[e_{1,2}, e_{2,1}] + (\lambda_{1,1} + \lambda_{2,2})[e_{2,3}, e_{3,2}] + (\lambda_{1,1} + \lambda_{2,2} + \lambda_{3,3})[e_{3,4}, e_{4,3}] \\ &+ \cdots + (\lambda_{1,1} + \cdots + \lambda_{n-1,n-1})[e_{n-1,n}, e_{n,n-1}].\end{aligned}$$

This completes the proof. □

Lemma 2.1.2. *Let $x \in pAp = B$, where $p = p_1 + \cdots + p_n$ with $p_1, \ldots, p_n$ orthogonal rank one projections of A. Then $\mathrm{Tr}_A(x) = \mathrm{Tr}_B(x)$.*

Proof. We have $x = pyp = \sum_{i=1}^{n}\sum_{j=1}^{n} p_i y p_j$. So, by the properties of the trace it will suffice to show that $\mathrm{Tr}_A(p_i y p_i) = \mathrm{Tr}_B(p_i y p_i)$ for each $i \in \{1, \ldots, n\}$. But this follows immediately from (1.1.3) and (1.1.4). □

Theorem 2.1.3. *For any linear functional f on $\mathrm{Soc}(A)$ we have that*

(a) *$f = \alpha \mathrm{Tr}$ for some $\alpha \in \mathbb{C}$,*

(b) *$f(ab) = f(ba)$ for all $a, b \in \mathrm{Soc}(A)$, and*

(c) *There exists a real number $c > 0$ such that $|f(a)| \leq c \cdot \rho(a)$ for all $a \in \mathrm{Soc}(A)$*

are all equivalent if and only if $\mathrm{Soc}(A) \cong M_n(\mathbb{C})$.

Proof. For the reverse implication, by the remark on p. 9, and Lemma 2.1.2, it will suffice to prove that (a), (b) and (c) are all equivalent when $A = M_n(\mathbb{C})$. Since $\mathrm{Soc}(A)$ is finite-dimensional, Theorem 1.1.5 readily gives that (a) $\Rightarrow$ (c). Suppose now that f satisfies (c). Then, in particular,

$$|f(a)| \leq c \cdot \rho(a) \leq c \cdot \mathrm{rank}(a) \cdot \rho(a)$$

for all $a \in \mathrm{Soc}(A)$. Hence, f satisfies condition (b) by Theorem 1.1.3. This shows that (c) $\Rightarrow$ (b). Finally, if f satisfies (b), then by Lemma 2.1.1 we have Ker Tr $\subseteq$ Ker f. So f is constant on the rank one projections of A, say $f(p) = \alpha$ for each rank one projection p. Let $a \in \mathrm{Soc}(A) - \{0\}$ be arbitrary. By the density of $E(a)$ and the Diagonalization Theorem, it follows that $a = \lambda_1 up_1 + \cdots + \lambda_m up_m$, where $\lambda_1, \ldots, \lambda_m \in \mathbb{C} - \{0\}$, $u \in G(A)$ and $p_1, \ldots, p_m$ are orthogonal rank one projections. Thus, since Corollary 1.1.4 and the minimality of the p_i gives $\lambda_i p_i up_i = \mathrm{Tr}\,(\lambda_i up_i)\, p_i$ for each $i \in \{1, \ldots, m\}$, we obtain

$$\begin{aligned} f(a) &= f(\lambda_1 up_1) + \cdots + f(\lambda_m up_m) \\ &= f(\lambda_1 p_1 up_1) + \cdots + f(\lambda_m p_m up_m) \\ &= \mathrm{Tr}\,(\lambda_1 up_1)\, f(p_1) + \cdots + \mathrm{Tr}\,(\lambda_m up_m)\, f(p_m) \\ &= \mathrm{Tr}\,(\lambda_1 up_1 + \cdots + \lambda_m up_m) \cdot \alpha \;=\; \alpha \mathrm{Tr}\,(a). \end{aligned}$$

Thus, (b) $\Rightarrow$ (a), which establishes the desired equivalence. For the forward implication we note that $|\mathrm{Tr}\,(a)| \leq c \cdot \rho(a)$ for all $a \in \mathrm{Soc}(A)$, so $\mathrm{Soc}(A)$ is finite-dimensional by Theorem 1.1.5. Thus, by the remark on p. 9, it follows that $\mathrm{Soc}(A)$ is isomorphic as an algebra to

$$B = M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})\,. \tag{2.1.1}$$

If $\mathrm{Soc}(A) = \{0\}$, then the result is trivially true. So assume that $n_i \geq 1$ for each $i \in \{1, \ldots, n\}$. We claim that $B = M_{n_1}(\mathbb{C})$, that is, that the direct sum above contains only one term: Suppose this is false, say B appears as in (2.1.1) with $k \geq 2$. Let $f : B \to \mathbb{C}$ be defined by $f((a_1, \ldots, a_k)) = \mathrm{tr}\,(a_1)$, where tr denotes the trace in $M_{n_1}(\mathbb{C})$. The linearity of f on B follows readily from that of tr on $M_{n_1}(\mathbb{C})$. Also, $f \neq 0$ since $f((\mathbf{1}, 0, \ldots, 0)) = n_1$. Moreover, if $a = (a_1, \ldots, a_k)$ and $b = (b_1, \ldots, b_k)$ are in B, then by Corollary 1.1.4 it follows that

$$f(ab) = \mathrm{tr}\,(a_1 b_1) = \mathrm{tr}\,(b_1 a_1) = f(ba).$$

However, $f((0, \mathbf{1}, 0, \ldots, 0)) = 0$, whereas $\mathrm{Tr}_B((0, \mathbf{1}, 0, \ldots, 0)) \neq 0$. This is a contradiction, for it shows that $f \neq \alpha \mathrm{Tr}_B$ for all $\alpha \in \mathbb{C}$. So $B = M_{n_1}(\mathbb{C})$ as advertised. This completes the proof. $\square$

Remarkably it is possible to show that the condition that f is constant on the rank one projections is enough to characterize the trace in general Banach algebras:

Theorem 2.1.4. *Suppose that* $\mathrm{Soc}(A) \neq \{0\}$. *For any linear functional f on* $\mathrm{Soc}(A)$ *we have that $f = \alpha \mathrm{Tr}$ for some $\alpha \in \mathbb{C}$ if and only if f is constant on the rank one projections of A.*

Proof. The forward implication follows from the definition of the trace. For the converse, suppose that f is constant on the rank one projections of A. Let p be any rank one projection of A and let $x \in A$ be arbitrary. Then $(p + px - pxp)^2 = p + px - pxp$. Moreover, $p + px - pxp \neq 0$, for otherwise $p = (p + px - pxp)\,p = 0$ which is absurd. Thus, since

$$p + px - pxp = p\,(p + px - pxp)\,,$$

it follows from the properties of the rank that $\operatorname{rank}(p + px - pxp) = 1$. Hence, by hypothesis we have

$$f(p) = f\,(p + px - pxp)\,,$$

and so $f(px) = f(pxp)$. This, together with the fact that f is constant on the rank one projections, is enough to prove that $f = \alpha\mathrm{Tr}$ for some $\alpha \in \mathbb{C}$. Indeed, simply use a similar argument as in the proof of Theorem 2.1.3. □

The next few results will be used to characterize those socles for which conditions (a) and (b) from Theorem 2.1.3 are equivalent. This will lead to some insights about the trace of finite rank operators.

Lemma 2.1.5. *Let $\mathcal{M}$ be the collection of ideals obtained in Lemma 1.2.1. If for any linear functional f on $\mathrm{Soc}(A)$ we have that $f(ab) = f(ba)$ for each $a, b \in \mathrm{Soc}(A)$ implies $f = \alpha\mathrm{Tr}$ for some $\alpha \in \mathbb{C}$, then it must be the case that $\mathcal{M}$ contains at most one nontrivial member.*

Proof. Suppose $\mathcal{M}$ has more than one nontrivial member. Let $J_{p_0} \in \mathcal{M}$ be arbitrary but not zero. Define f on $\mathrm{Soc}(A)$ first implicitly as follows: $f(J_{p_0}) = \{0\}$ and $f(a) = \mathrm{Tr}\,(a)$ if $a \notin J_{p_0}$ but $a \in J_p$ for some $J_p \in \mathcal{M}$. In particular, $f(0) = 0$. The next step is to show that f is well-defined on the union of the members of $\mathcal{M}$: It will suffice to show that distinct nonzero members of $\mathcal{M}$ intersect only at 0. If $0 \neq a \in J_p \cap J_q$, then we can write

$$a = x_1py_1 + \cdots + x_npy_n = u_1qv_1 + \cdots + u_kqv_k,$$

and so for each $x \in A$, $xa \in J_p \cap J_q$. But notice now that $(xa)^2 = 0$ since $J_pJ_q = J_qJ_p = \{0\}$. Thus, $\sigma(xa) = \{0\}$ so that $a \in \mathrm{Rad}(A) = \{0\}$. This shows that f is indeed well-defined on the union of the members of $\mathcal{M}$. Now, by using a similar argument as above, it follows that every nonzero element of the socle can be written uniquely as a finite sum of nonzero elements from the members of $\mathcal{M}$. So f extends linearly to all of $\mathrm{Soc}(A)$. Now take any $a, b \in \mathrm{Soc}(A)$. Without loss of generality we can write $a = w_{p_0} + w_{p_1} + \cdots + w_{p_n}$, where $w_{p_j} \in J_{p_j}$ and $b = v_{p_0} + v_{q_1} + \cdots + v_{q_n}$, where $v_{p_0} \in J_{p_0}$ and $v_{q_j} \in J_{q_j}$.

From the orthogonality of the members of $\mathcal{M}$ and Corollary 1.1.4 it follows that $f(ab) = f(ba)$. But it is clear that $f \neq \alpha\mathrm{Tr}$ for all $\alpha \in \mathbb{C}$ because $f(p_0) = 0$ and $f(q_0) = \mathrm{Tr}(q_0) = 1$ for some rank one projection $q_0 \neq p_0$ (which exists by hypothesis). This contradiction completes the proof. $\square$

Lemma 2.1.6. *If $pAp \ncong M_n(\mathbb{C})$ for some finite-rank projection p of A, then $\mathcal{M}$ contains at least two distinct members.*

Proof. By the Wedderburn-Artin Theorem,

$$pAp \cong M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$$

and, by assumption, we may assume at least two nonzero terms in the direct sum. This means that we can find rank one projections, say p and q, such that $pxq = 0$ and $qxp = 0$ for all $x \in A$. Suppose that $\mathcal{M}$ has only one nontrivial member, say J_r, where r is a rank one projection. If $(xry)\,p = 0$ for all $x, y \in A$, then certainly $p \notin J_r$ which is contradictive to the fact that $\mathrm{Soc}(A) = J_r$. So $x_1ry_1p \neq 0$ for some $x_1, y_1 \in A$. By minimality of p, we have $A(x_1ry_1p) = Ap$, and in turn the minimality of r implies that $Ary_1p = Ap$. So $ry_1p = zp$ for some $z \in A$. Again minimality of r gives $(ry_1p)\,A = rA$, and so $rA = zpA$. Thus, $r \in ApA$. So, since $\mathrm{Soc}(A) = J_r$, we have shown that

$$J_r = \left\{ \sum_{j=1}^{n} x_j p y_j : x_j, y_j \in A, n \geq 1 \text{ an integer} \right\} = J_p.$$

Similarly, it can be shown that $J_r = J_q$. But this is not possible since $J_p \cap J_q = \{0\}$. Thus, the lemma is proved. $\square$

Remark. Notice that the argument used in the proof of Lemma 2.1.6 can be simplified significantly by using Lemma 1.2.2. Indeed, Lemma 1.2.2 readily gives that J_r is a minimal two-sided ideal which implies that $J_p = J_r = J_q$, and produces a contradiction. However, the author has chosen to stick with the original proof which appears in the literature as [16, Lemma 3.7]. A similar remark can be made about the proof of the next theorem which appears as [16, Theorem 3.8]:

Theorem 2.1.7. *For any linear functional f on $\mathrm{Soc}(A)$ we have that*

(a) $f = \alpha\mathrm{Tr}$ *for some* $\alpha \in \mathbb{C}$, *and*

(b) $f(ab) = f(ba)$ *for all* $a, b \in \mathrm{Soc}(A)$

are equivalent if and only if $\mathrm{Soc}(A)$ *is a minimal two-sided ideal.*

Proof. If $\mathrm{Soc}(A) = \{0\}$, then the result trivially holds true. So assume that $\mathrm{Soc}(A) \neq \{0\}$. If (a) and (b) are equivalent, then by Lemma 2.1.5 $\mathrm{Soc}(A) = J_p$ for some rank one projection p. Let $I \subseteq \mathrm{Soc}(A)$ be a two-sided ideal and suppose that $a \in I - \{0\}$. By the density of $E(a)$ and the Diagonalization Theorem, there exist a $u \in G(A)$, orthogonal rank one projections $p_1, \ldots, p_n$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C} - \{0\}$ such that $ua = \lambda_1 p_1 + \cdots + \lambda_n p_n$. Consequently, if $x, y \in A$, then $\frac{1}{\lambda_1} x p_1 u a y = x p_1 y \in I$. This shows that $J_{p_1} \subseteq I$. But since $p_1 \in J_p$, it follows by the argument in the proof of Lemma 2.1.6 that $J_{p_1} = J_p$. So $I = \mathrm{Soc}(A)$. This proves the forward implication. Conversely, if $\mathrm{Soc}(A)$ is minimal, then $\mathrm{Soc}(A) = J_p$ for some rank one projection p. Let $a \in \mathrm{Soc}(A)$ be abitrary. Then $a = x_1 p y_1 + \cdots + x_n p y_n$ for some $x_1, \ldots, x_n, y_1, \ldots, y_n \in A$. So if f satisfies (b), then, by Corollary 1.1.4 and the fact that $p_i x p_i = \mathrm{Tr}\,(x p_i)\, p_i$ for all $x \in A$, it follows that

$$\begin{aligned} f(a) &= f\,(x_1 p y_1) + \cdots + f\,(x_n p y_n) \\ &= f\,(p y_1 x_1 p) + \cdots + f\,(p y_n x_n p) \\ &= \mathrm{Tr}\,(y_1 x_1 p)\, f(p) + \cdots + \mathrm{Tr}\,(y_n x_n p)\, f(p) \\ &= [\mathrm{Tr}\,(x_1 p y_1) + \cdots + \mathrm{Tr}\,(x_n p y_n)]\, f(p) \\ &= f(p) \mathrm{Tr}\,(a). \end{aligned}$$

This shows that (b) $\Rightarrow$ (a). The implication (a) $\Rightarrow$ (b) is of course a consequence of Corollary 1.1.4, so we have the result. $\square$

Theorem 2.1.8. *For each linear functional f on $\mathrm{Soc}(A)$ we have that*

(a) $f = \alpha \mathrm{Tr}$ *for some* $\alpha \in \mathbb{C}$, *and*

(b) $f(ab) = f(ba)$ *for all* $a, b \in \mathrm{Soc}(A)$

are equivalent if and only if $pAp \cong M_{n_p}(\mathbb{C})$ *for each finite-rank projection* p *of* A.

Proof. If (a) and (b) are equivalent, then by Lemma 2.1.5 it follows that $\mathcal{M}$ contains at most one nontrivial member. So Lemma 2.1.6 gives the forward implication. For the converse, we note that (a) $\Rightarrow$ (b) by Corollary 1.1.4. So suppose that f satisfies condition (b). Let $x \in \mathrm{Soc}(A) - \{0\}$ be arbitrary. By the density of $E(x)$ and the Diagonalization Theorem, there exist a $u \in G(A)$, orthogonal rank one projections $p_1, \ldots, p_n$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C} - \{0\}$ such that $x = \lambda_1 u p_1 + \cdots + \lambda_n u p_n$. Consequently, if $p = p_1 + \cdots + p_n$, then $xp = x$. Thus, by hypothesis we have that $f(x) = f(xp) = f(pxp)$. Moreover, since $B = pAp \cong M_{n_p}(\mathbb{C})$, it follows from Theorem 2.1.3 that $f|_B = \alpha_B \mathrm{Tr}_B$ for some $\alpha_B \in \mathbb{C}$. Hence, by Lemma 2.1.2 and Corollary 1.1.4, we obtain

$$\begin{aligned} f(x) = f(pxp) &= \alpha_B \mathrm{Tr}_B(pxp) = \alpha_B \mathrm{Tr}_A(pxp) \\ &= \alpha_B \mathrm{Tr}_A(xp) = \alpha_B \mathrm{Tr}_A(x). \end{aligned}$$

Since $x \in \mathrm{Soc}(A) - \{0\}$ was arbitrary, this implies in particular that $\mathrm{Ker}\mathrm{Tr}_A \subseteq \mathrm{Ker}\, f$. Consequently, f is constant on the rank one projections of A. So, by Theorem 2.1.4 we have that (b) $\Rightarrow$ (a). This completes the proof. □

Although the trace on the socles of the Banach algebras in Theorem 2.1.7 and Theorem 2.1.8 is characterized, up to scalar multiples, as precisely those linear functionals on the socle which are 0 on the commutators, it is possible to characterize the trace here by other properties. First, however, we show that these properties characterize the trace when $A = M_n(\mathbb{C})$:

Lemma 2.1.9. *Let $A = M_n(\mathbb{C})$. For any linear functional f on A the following are equivalent:*

(a) $f = \alpha \mathrm{Tr}$ *for some* $\alpha \in \mathbb{C}$.

(b) $f(a) = 0$ *for each nilpotent* $a \in A$.

(c) *For each* $a \in A$, $f(a) = 0$ *whenever* $a^2 = 0$.

Proof. It is obvious that (a) $\Rightarrow$ (b) $\Rightarrow$ (c). So it will suffice to show that (c) $\Rightarrow$ (a): For each $i, j \in \{1, \ldots, n\}$, as before we let $e_{i,j}$ denote the matrix which has zeros at each entry except at the (i, j)-entry where it has a 1. Consider any $e_{i,j}$ with $i \neq j$. In particular, $e_{i,j}^2 = 0$. Consequently, for any $b \in A$ we have $\left(e^{\lambda b} e_{i,j} e^{-\lambda b}\right)^2 = 0$ for all $\lambda \in \mathbb{C}$. So, by hypothesis $f\left(e^{\lambda b} e_{i,j} e^{-\lambda b}\right) = 0$ for all $\lambda \in \mathbb{C}$. Thus, $\lambda \mapsto f\left(e^{\lambda b} e_{i,j} e^{-\lambda b}\right)$ is a constant function from $\mathbb{C}$ into itself. Therefore, since f is automatically continuous on A, it follows from a similar argument as the one used in Theorem 1.1.3 that $f(be_{i,j} - e_{i,j}b) = 0$. Thus, $f(be_{i,j}) = f(e_{i,j}b)$ for all $b \in A$. Hence,

$$f(e_{j,j}) = f(e_{j,i}e_{i,j}) = f(e_{i,j}e_{j,i}) = f(e_{i,i}).$$

So, $f(e_{1,1}) = f(e_{j,j})$ for all $j \in \{1, \ldots, n\}$ and $f(e_{i,j}) = 0$ for $i \neq j$. Thus, if $a \in A$ is given by $a = \sum_{i=1}^n \sum_{j=1}^n \lambda_{i,j} e_{i,j}$, then

$$f(a) = f(e_{1,1})(\lambda_{1,1} + \cdots + \lambda_{n,n}) = f(e_{1,1})\,\mathrm{Tr}(a).$$

Hence, the lemma holds true. □

Theorem 2.1.10. *Suppose that* $\mathrm{Soc}(A)$ *is a minimal two-sided ideal of* A. *Then for every linear functional* f *on* $\mathrm{Soc}(A)$ *the following are equivalent:*

(a) $f = \alpha \mathrm{Tr}$ *for some* $\alpha \in \mathbb{C}$.

(b) $f(a) = 0$ *for each nilpotent* $a \in \mathrm{Soc}(A)$.

(c) *For each* $a \in \mathrm{Soc}(A)$, $f(a) = 0$ *whenever* $a^2 = 0$.

(d) *There exists a real number* $c > 0$ *such that* $|f(a)| \leq c \cdot \mathrm{rank}\,(a) \cdot \rho(a)$ *for all* $a \in \mathrm{Soc}(A)$.

Proof. It is clear that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) and that (a) $\Rightarrow$ (d). Moreover, by Theorem 2.1.7 and Theorem 1.1.3 it follows that (d) $\Rightarrow$ (a). It will therefore suffice to show that (c) $\Rightarrow$ (a): Let $x \in \mathrm{Soc}(A) - \{0\}$ be arbitrary. As in the proof of Theorem 2.1.8 we can find a finite-rank projection p such that $xp = x$. Since $(xp - pxp)^2 = 0$, the assumption on f gives $f(x) = f(xp) = f(pxp)$. Moreover, by Theorem 2.1.7 and Theorem 2.1.8 we have $B = pAp \cong M_{n_p}(\mathbb{C})$. So, by Lemma 2.1.9, and the argument used in the proof of Theorem 2.1.8, we have the desired implication. □

The result after the next lemma implies the classical fact that, for any Banach space X, $\mathrm{Soc}\, B(X)$ is a minimal two-sided ideal.

Lemma 2.1.11. *Let* P, Q *be rank one projections of* $B(X)$, *where* X *is some Banach space. Then there exist* $S, T \in B(X)$ *such that* $P - Q = ST - TS$ *and* S *and* T *are both of rank one.*

Proof. By hypothesis there exist $x, y \in X - \{0\}$ such that $Pu = f(u)x$ and $Qu = g(u)y$ for each $u \in X$, where $f, g \in X'$ (the dual space of X). Moreover, since $P^2 = P$ and $Q^2 = Q$, it follows that $f(x)x = x$ and $g(y)y = y$. Hence, since $x, y \in X - \{0\}$, it follows that $f(x) = g(y) = 1$. Let $S : X \to X$ and $T : X \to X$ be defined by $Su = g(u)x$ and $Tu = f(u)y$ for each $u \in X$. Then $S, T \in B(X)$ and S and T are both of rank one. Furthermore,

$$(ST)(u) = S(f(u)y) = f(u)g(y)x = f(u)x = Pu$$

and

$$(TS)(u) = T(g(u)x) = g(u)f(x)y = g(u)y = Qu$$

for each $u \in X$. Thus, $P - Q = ST - TS$ as desired. □

Theorem 2.1.12. *Let $A = B(X)$ for some Banach space X. Then for every linear functional f on $\mathrm{Soc}(A)$ the following are equivalent:*

(a) $f = \alpha \mathrm{Tr}$ *for some* $\alpha \in \mathbb{C}$.

(b) $f(ab) = f(ba)$ *for all* $a, b \in \mathrm{Soc}(A)$.

(c) *There exists a real number* $c > 0$ *such that* $|f(a)| \leq c \cdot \mathrm{rank}\,(a) \cdot \rho(a)$ *for all* $a \in \mathrm{Soc}(A)$.

(d) $f(a) = 0$ *for each nilpotent* $a \in \mathrm{Soc}(A)$.

(e) *For each* $a \in \mathrm{Soc}(A)$, $f(a) = 0$ *whenever* $a^2 = 0$.

Proof. By Theorem 2.1.7 and Theorem 2.1.10 it will suffice to show that (a) and (b) are equivalent. By Corollary 1.1.4 we know that (a) $\Rightarrow$ (b). Conversely, by Lemma 2.1.11 it follows that f is constant on the rank one projections. So by Theorem 2.1.4 it follows that (b) $\Rightarrow$ (a), establishing the result. $\square$

2.2 The Wedderburn-Artin Theorem

Let $\mathcal{P}$ be a class of projections generating the J_p in Lemma 1.2.1. By the direct sum $\oplus_{p \in \mathcal{P}} Ap \otimes pA$ we denote the subset of the Cartesian product $\times_{p \in \mathcal{P}} Ap \otimes pA$ consisting of all cross sections which are zero except at a finite number of elements of $\mathcal{P}$, equipped with pointwise scalar multiplication, addition and multiplication. From Lemma 1.2.1 and Corollary 1.2.5 we obtain the following theorem concerning the structure of the socle:

Theorem 2.2.1. $\mathrm{Soc}(A) \cong \oplus_{p \in \mathcal{P}} Ap \otimes pA$.

The result above can be viewed as a generalization of the celebrated *Wedderburn-Artin Theorem*. Of course, any claim of generality should inherently contain the classical result in some or other form. We now proceed to show that Theorem 2.2.1 is indeed a generalized version of the Wedderburn-Artin Theorem. Noteworthy is that this approach not only yields further interesting consequences, but it also avoids the use of representation theory altogether, that is, we manage to prove the Wedderburn-Artin Theorem without the use of continuous irreducible representations of A. Firstly, however, a little preparation is needed:

Lemma 2.2.2. *Let p be a rank one projection of A and let S be a linearly independent subset of Ap such that $p \in S$. Then*

$$S' = \{p\} \cup \{(\mathbf{1} - p)\, xp : x \in S - \{p\}\}$$

is a linearly independent subset of Ap and $\operatorname{span} S = \operatorname{span} S'$. *A similar result is true for pA.*

Proof. Let $x_2, \ldots, x_n \in S - \{p\}$ be distinct. For the first part it will suffice to show that the set $\{p, (\mathbf{1} - p)\, x_2 p, \ldots, (\mathbf{1} - p)\, x_n p\}$ is linearly independent. To this end, suppose that

$$\lambda_1 p + \lambda_2 (\mathbf{1} - p)\, x_2 p + \cdots + \lambda_n (\mathbf{1} - p)\, x_n p = 0 \tag{2.2.1}$$

for some $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$. Since p has rank one, it is minimal. Hence, for each $i \in \{2, \ldots, n\}$ we have that $px_i p = \alpha_i p$, where $\alpha_i \in \mathbb{C}$. Moreover, since $x_i \in Ap$, it follows that $x_i p = x_i$ for each $i \in \{2, \ldots, n\}$. Hence, (2.2.1) becomes

$$(\lambda_1 - \lambda_2 \alpha_2 - \cdots - \lambda_n \alpha_n)\, p + \lambda_2 x_2 + \cdots + \lambda_n x_n = 0.$$

Thus, since $\{p, x_2, \ldots, x_n\}$ is a linearly independent set, it readily follows that $\lambda_i = 0$ for each $i \in \{1, \ldots, n\}$. The second part follows from the minimality of p and the observation that

$$\lambda_1 p+\lambda_2 x_2+\cdots+\lambda_n x_n = \left(\lambda_1 + \sum_{j=2}^{n} \lambda_j \alpha_j\right) p+\lambda_2 \left(\mathbf{1}-p\right) x_2 p+\cdots+\lambda_n \left(\mathbf{1}-p\right) x_n p$$

and

$$\lambda_1 p+\lambda_2 \left(\mathbf{1}-p\right) x_2 p+\cdots+\lambda_n \left(\mathbf{1}-p\right) x_n p = \left(\lambda_1 - \sum_{j=2}^{n} \lambda_j \alpha_j\right) p+\lambda_2 x_2+\cdots+\lambda_n x_n$$

for any $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, where $px_jp = \alpha_j p$ for each $j \in \{2, \ldots, n\}$. This gives the result. □

From [4, pp. 155–157] we recall that associated to every rank one element, a, there exists a characteristic functional $\tau_a \in A'$ (the dual space of A) such that

$$axa = \tau_a\left(x\right) a \text{ for all } x \in A.$$

Observe that $\alpha\tau_a = \tau_{\alpha a}$ for all $\alpha \in \mathbb{C} - \{0\}$. Moreover, by the density of $E(a)$ and the Diagonalization Theorem, there exist a minimal projection p in A and a $u \in G(A)$ such that $a = pu$. Thus, by Jacobson's Lemma, and the definitions of Tr and τ_a, it readily follows that

$$\mathrm{Tr}\left(ax\right) = \tau_a(x) \text{ for all } x \in A.$$

Theorem 2.2.3. *Let $\{b, a_1, \ldots, a_n\}$ be a linearly independent set of rank one elements in A. Then there exists a $y \in A$ such that $\sigma(by) \neq \{0\}$ and $\sigma\left(a_i y\right) = \{0\}$ for each $i \in \{1, \ldots, n\}$.*

Proof. For the sake of a contradiction, suppose that

$$\sigma\left(a_i x\right) = \{0\} \text{ for all } i \in \{1, \ldots, n\} \Rightarrow \sigma(bx) = \{0\}. \tag{2.2.2}$$

Now, (2.2.2) is clearly equivalent to

$$x \in \bigcap_{i=1}^{n} \mathrm{Ker}\, \tau_{a_i} \Rightarrow x \in \mathrm{Ker}\, \tau_b,$$

which means that τ_b vanishes on $\bigcap_{i=1}^{n} \mathrm{Ker}\, \tau_{a_i}$ i.e. $\mathrm{Ker}\, \tau_b \supseteq \bigcap_{i=1}^{n} \mathrm{Ker}\, \tau_{a_i}$. Consequently, from linear algebra (see [10, p. 10]) it follows that

$$\tau_b = \alpha_1 \tau_{a_1} + \cdots + \alpha_n \tau_{a_n},$$

where $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$. Next, we let $x \in A$ be arbitrary, and consider

$$\begin{aligned}\operatorname{Tr}(bx) = \tau_b(x) &= (\alpha_1\tau_{a_1} + \cdots + \alpha_n\tau_{a_n})(x) \\ &= \alpha_1\tau_{a_1}(x) + \cdots + \alpha_n\tau_{a_n}(x) \\ &= \tau_{\alpha_1 a_1}(x) + \cdots + \tau_{\alpha_n a_n}(x) \\ &= \operatorname{Tr}(\alpha_1 a_1 x) + \cdots + \operatorname{Tr}(\alpha_n a_n x) \\ &= \operatorname{Tr}(\alpha_1 a_1 x + \cdots + \alpha_n a_n x) \\ &= \operatorname{Tr}((\alpha_1 a_1 + \cdots + \alpha_n a_n)x).\end{aligned}$$

From this it now follows that

$$\operatorname{Tr}\left(\left(b - \sum_{i=1}^{n} \alpha_i a_i\right) x\right) = 0.$$

However, since $x \in A$ was arbitrary, it follows from the properties of the trace (see p. 6) that $b = \sum_{i=1}^{n} \alpha_i a_i$. But this is absurd since $\{b, a_1, \ldots, a_n\}$ is linearly independent. We conclude that there is at least one $y \in A$ for which $\sigma(by) \neq \{0\}$ and $\sigma(a_i y) = \{0\}$ for each $i \in \{1, \ldots, n\}$. □

Lemma 2.2.4. *Let p be a rank one projection of A and let*

$$\{p, (\mathbf{1} - p)x_2 p, \ldots, (\mathbf{1} - p)x_n p\}$$

be a linearly independent subset of Ap. For each $j \in \{2, \ldots, n\}$, there exists a $y_j \in A$ such that $1 \in \sigma((\mathbf{1} - p)x_j p y_j)$ and $\sigma((\mathbf{1} - p)x_i p y_j) = \{0\}$ for $i \neq j$. A similar result holds true if we start with a linearly independent subset of pA.

Proof. This is an immediate consequence of Theorem 2.2.3 and the Spectral Mapping Theorem. □

Lemma 2.2.5. *Let $\{y_2, \ldots, y_n\}$ be the set obtained in the conclusion of Lemma 2.2.4. Then $\{p, py_2(\mathbf{1} - p), \ldots, py_n(\mathbf{1} - p)\}$ is a linearly independent subset of pA. A similar result holds true if we start with a linearly independent subset of pA in Lemma 2.2.4.*

Proof. By Lemma 2.2.2 it suffices to show that the set $\{p, py_2, \ldots, py_n\}$ is linearly independent in pA: Firstly, note that $py_j \neq 0$ for each $j \in \{2, \ldots, n\}$, for otherwise we obtain a contradiction with the fact that $1 \in \sigma((\mathbf{1} - p)x_j p y_j)$ for each $j \in \{2, \ldots, n\}$. Secondly, by Jacobson's Lemma and the minimality of p it readily follows that $py_j(\mathbf{1} - p)x_j p = p$ for each $j \in \{2, \ldots, n\}$, and, moreover, that $py_i(\mathbf{1} - p)x_j p = 0$ for $i \neq j$. Now, suppose that

$$\lambda_1 p + \lambda_2 p y_2 + \cdots + \lambda_n p y_n = 0 \tag{2.2.3}$$

for some $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$. Let $j \in \{2, \ldots, n\}$ be arbitrary but fixed. Multiplying both sides of (2.2.3) on the right by $(\mathbf{1} - p)\, x_j p$ yields $\lambda_j p = 0$. Hence, $\lambda_j = 0$ for each $j \in \{2, \ldots, n\}$. Thus, (2.2.3) becomes $\lambda_1 p = 0$ from which we get $\lambda_1 = 0$. Therefore, $\{p, py_2, \ldots, py_n\}$ is a linearly independent set as desired. $\square$

Lemma 2.2.6. *Let p be a rank one projection of A. Then* $\dim Ap = \dim pA$.

Proof. If $\dim pA > \dim Ap$ or $\dim Ap > \dim pA$, then in either case we may use Lemma 2.2.2 and Lemma 2.2.5 to construct a linearly independent set of sufficiently large cardinality in Ap or pA, respectively, to produce a contradiction. Hence, it must be the case that $\dim pA \leq \dim Ap$ and $\dim Ap \leq \dim pA$, establishing the result. $\square$

Theorem 2.2.7. *Let p be a rank one projection of A and suppose that either* $\dim Ap$ *or* $\dim pA$ *is finite and at least 2. Then there exist a basis $\{p, u_2, \ldots, u_n\}$ of Ap and a basis $\{p, v_2, \ldots, v_n\}$ of pA such that the following properties hold:*

(i) $pu_i = v_i p = 0$ *and* $u_i^2 = v_i^2 = 0$ *for each* $i \in \{2, \ldots, n\}$.

(ii) $u_i p = u_i$ *and* $pv_i = v_i$ *for each* $i \in \{2, \ldots, n\}$.

(iii) $v_i u_i = p$ *for each* $i \in \{2, \ldots, n\}$ *and* $v_i u_j = 0$ *for* $i \neq j$.

Proof. From Lemma 2.2.6 it follows that $\dim Ap = \dim pA$. If $\dim Ap = 1$, then $\{p\}$ is a basis for Ap and for pA. However, then the result is trivially true. So assume that $\dim Ap \geq 2$ and let $\{p, x_2, \ldots, x_n\}$ be any basis for Ap. From Lemma 2.2.2 it follows that $\{p, (\mathbf{1} - p)\, x_2 p, \ldots, (\mathbf{1} - p)\, x_n p\}$ is a basis for Ap. Let $\{p, py_2\,(\mathbf{1} - p), \ldots, py_n\,(\mathbf{1} - p)\}$ be the basis for pA constructed in Lemma 2.2.4 and Lemma 2.2.5 using the aforementioned basis for Ap. Recall that Jacobson's Lemma and the minimality of p gives that $py_i\,(\mathbf{1} - p)\, x_i p = p$ for each $i \in \{2, \ldots, n\}$ and that $py_i\,(\mathbf{1} - p)\, x_j p = 0$ for $i \neq j$. Consequently, if we take $u_i = (\mathbf{1} - p)\, x_i p$ and $v_i = py_i\,(\mathbf{1} - p)$ for each $i \in \{1, \ldots, n\}$, then a moment's investigation shows that $\{p, u_2, \ldots, u_n\}$ and $\{p, v_2, \ldots, v_n\}$ satisfy properties (i) to (iii). $\square$

Lemma 2.2.8. *Let p be a rank one projection of A and suppose that either* $\dim Ap$ *or* $\dim pA$ *is finite. Then* $J_p \cong M_n(\mathbb{C})$.

Proof. From Lemma 2.2.6 it follows that $\dim Ap = \dim pA$. If $\dim Ap = 1$, then $J_p = \mathbb{C}p \cong \mathbb{C}$ and we are done. So assume that $\dim Ap \geq 2$. By Theorem 2.2.7 we can find a basis $\{p, u_2, \ldots, u_n\}$ of Ap and a basis $\{p, v_2, \ldots, v_n\}$ of

pA satisfying properties (i) to (iii) as listed there. From Corollary 1.2.5 it follows that $J_p \cong Ap \otimes pA$. Moreover, by [4, Lemma 42.5] it follows that

$$\{a \otimes b : a \in \{p, u_2, \ldots, u_n\}, b \in \{p, v_2, \ldots, v_n\}\}$$

is a basis for $Ap \otimes pA$. Hence,

$$\{p\} \cup \{u_i p v_j, u_i p, p v_j : i, j \in \{2, \ldots, n\}\}$$

is a basis for J_p. Let $e_{i,j}$ be the $n \times n$ matrix with 1 in its (i,j)-entry and 0 everywhere else. Then, in particular, $\{e_{i,j} : i, j \in \{1, \ldots, n\}\}$ is a basis for $M_n(\mathbb{C})$. We define a linear mapping $\phi : J_p \to M_n(\mathbb{C})$ in terms of basis elements as follows: $\phi(p) = e_{1,1}$, $\phi(u_i p) = e_{i,1}$, $\phi(p v_j) = e_{1,j}$ and $\phi(u_i p v_j) = e_{i,j}$ for each $i, j \in \{2, \ldots, n\}$. It is not hard to show that ϕ is bijective. Moreover, by properties (i) to (iii) in Theorem 2.2.7 it readily follows that ϕ is multiplicative. This gives the result. □

Theorem 2.2.9. *Let* Soc(A) *be finite-dimensional. Then*

$$\mathrm{Soc}(A) \cong M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C}).$$

Proof. By Theorem 1.1.5 there exists a $c > 0$ such that

$$|\mathrm{Tr}\,(a)| \leq c \cdot \rho(a) \tag{2.2.4}$$

for all $a \in \mathrm{Soc}(A)$. Suppose that the collection of two-sided ideals $\{J_p : p \in \mathcal{P}\}$, which exists by Lemma 1.2.1, contains infinitely many elements. However, then $\mathcal{P}$ contains a subset of n distinct pairwise orthogonal rank one projections, say $\{q_1, \ldots, q_n\}$, where $n > c$. By the orthogonality of the q_i, it follows that $q = q_1 + \cdots + q_n$ is also a projection. But then

$$|\mathrm{Tr}\,(q)| = n > c = c \cdot 1 = c \cdot \rho(q),$$

contradicting (2.2.4). We may therefore conclude that $\mathcal{P}$ is a finite set, say $\mathcal{P} = \{p_1, \ldots, p_k\}$. Since the two-sided ideals are pairwise orthogonal, it follows that

$$\mathrm{Soc}(A) \cong J_{p_1} \oplus \cdots \oplus J_{p_k}.$$

But by Lemma 2.2.8 it follows that $J_{p_i} \cong M_{n_i}(\mathbb{C})$ for each $i \in \{1, \ldots, k\}$. This yields the result. □

Recall that $\mathrm{Soc}(A) = A$ if and only if A is finite-dimensional. Hence, Theorem 2.2.9 readily gives the following:

Corollary 2.2.10. (Wedderburn-Artin.)
Let A be finite-dimensional. Then

$$A \cong M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C}).$$

To conclude this section, we shall prove two more lemmas which will lead to an illuminative theorem. In particular, this theorem states that any finite collection of socle elements is contained in a subalgebra B of $\mathrm{Soc}(A)$ which has the Wedderburn-Artin structure; that is,

$$B \cong M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C}).$$

Lemma 2.2.11. *Let $S = \{p, (\mathbf{1}-p)\, a_2 p, \ldots, (\mathbf{1}-p)\, a_n p\}$ and*

$$T = \{p, pb_2(\mathbf{1}-p), \ldots, pb_m(\mathbf{1}-p)\}$$

be linearly independent subsets of Ap and pA, respectively. Then there exist two linearly independent subsets $S' = \{p, u_2, \ldots, u_k\}$ and $T' = \{p, v_2, \ldots, v_k\}$ of Ap and pA, respectively, such that span $S \subseteq$ span S', span $T \subseteq$ span T' *and properties* (i) *to* (iii) *in Theorem* 2.2.7 *holds for S' and T'.*

Proof. Apply Lemma 2.2.4 and Lemma 2.2.5 to S in order to obtain a corresponding linearly independent set $\{p, pc_2(\mathbf{1}-p), \ldots, pc_n(\mathbf{1}-p)\}$ in pA such that the sets together satisfy properties (i) to (iii) in Theorem 2.2.7. We shall use these sets to construct S' and T': To this end, set $S' := S$ and $T' := \{p, pc_2(\mathbf{1}-p), \ldots, pc_n(\mathbf{1}-p)\}$. Consider the element $pb_2(\mathbf{1}-p)$. If $pb_2(\mathbf{1}-p) \in$ span T', then we leave S' and T' unchanged. On the other hand, if $pb_2(\mathbf{1}-p) \notin$ span T', then we proceed with the following scheme: For each $i \in \{2, \ldots, n\}$, let $\alpha_i = -\mathrm{Tr}\,(pb_2(\mathbf{1}-p)\, a_i p)$. Let $c_{n+1} = b_2 + \sum_{i=2}^{n} \alpha_i c_i$ and notice that $pc_{n+1}(\mathbf{1}-p) \notin$ span T'. Thus, in particular, the set $\{p, pc_2(\mathbf{1}-p), \ldots, pc_{n+1}(\mathbf{1}-p)\}$ is linearly independent. Moreover, for any $j \in \{2, \ldots, n\}$ we have that

$$\begin{aligned} pc_{n+1}(\mathbf{1}-p)\, a_j p &= p\left(b_2 + \sum_{i=2}^{n} \alpha_i c_i\right)(\mathbf{1}-p)\, a_j p \\ &= pb_2(\mathbf{1}-p)\, a_j p + \alpha_j p \\ &= \mathrm{Tr}\,(pb_2(\mathbf{1}-p)\, a_j p)\, p + \alpha_j p \;=\; 0. \end{aligned}$$

Next we apply Theorem 2.2.3 and the Spectral Mapping Theorem to obtain an element $a_{n+1} \in A$ such that $pc_i(\mathbf{1}-p)\, a_{n+1} p = 0$ for each $i \in \{2, \ldots, n\}$ and $pb_2(\mathbf{1}-p)\, a_{n+1} p = p$. We claim that the set

$$\{p, (\mathbf{1}-p)\, a_2 p, \ldots, (\mathbf{1}-p)\, a_{n+1} p\}$$

is linearly independent: Assume that

$$\lambda_1 p + \lambda_2 (\mathbf{1} - p) a_2 p + \cdots + \lambda_{n+1} (\mathbf{1} - p) a_{n+1} p = 0 \tag{2.2.5}$$

for some $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$. Let $i \in \{2, \ldots, n+1\}$ be arbitrary but fixed. Multiplying both sides of (2.2.5) on the left by $pc_i (\mathbf{1} - p)$ yields $\lambda_i p = 0$. Consequently, we may conclude that $\lambda_i = 0$ for each $i \in \{2, \ldots, n+1\}$. Thus, (2.2.5) becomes $\lambda_1 p = 0$ which gives $\lambda_1 = 0$ and proves our claim. We now set $S' := \{p, (\mathbf{1} - p) a_2 p, \ldots, (\mathbf{1} - p) a_{n+1} p\}$ and

$$T' = \{p, pc_2 (\mathbf{1} - p), \ldots, pc_{n+1} (\mathbf{1} - p)\}.$$

A quick inspection shows that S' and T' satisfy properties (i) to (iii) from Theorem 2.2.7. Next we consider the element $pb_3 (\mathbf{1} - p)$ and repeat the iteration scheme above. After a total of $m - 1$ iterations, we obtain the desired sets S' and T'. □

For any $z_1, \ldots, z_r \in A$, we denote by $C[z_1, \ldots, z_r]$ the algebra generated by $z_1, \ldots, z_r$, that is,

$$C[z_1, \ldots, z_r] = \{q(z_1, \ldots, z_r) : q \text{ is a polynomial without constant term}\}.$$

Lemma 2.2.12. *Let p be a rank one projection and let $z_1, \ldots, z_r \in J_p$. Then there exists a subalgebra B of J_p such that $z_1, \ldots, z_r \in B \cong M_k(\mathbb{C})$. Moreover, $C[z_1, \ldots, z_r] \subseteq B$ and $z_j A z_j \subseteq B$ for each $j \in \{1, \ldots, r\}$.*

Proof. If $\dim Ap$ or $\dim pA$ is finite, then the result follows from Lemma 2.2.8. So we may assume that $\dim Ap = \dim pA = \infty$. Let V and W be bases for Ap and pA, respectively, both containing p. By Lemma 2.2.2 we may assume that any element $y \in V - \{p\}$ is of the form $y = (\mathbf{1} - p) xp$ for some $x \in A$. A similar observation holds for W. Now, for any $j \in \{1, \ldots, r\}$ it follows that $z_j = \sum_{i=1}^{N_j} \alpha_{j,i} x_{j,i} p y_{j,i}$, where $\alpha_{j,1}, \ldots, \alpha_{j,N_j} \in \mathbb{C}$, $x_{j,1}, \ldots, x_{j,N_j} \in V$ and $y_{j,1}, \ldots, y_{j,N_j} \in W$. To simplify our notation we may assume that

$$\begin{aligned} S &:= \{p\} \cup \{x_{j,1}, \ldots, x_{j,N_j} : j \in \{1, \ldots, r\}\} \\ &= \{p, (\mathbf{1} - p) a_2 p, \ldots, (\mathbf{1} - p) a_n p\}, \end{aligned}$$

and that

$$\begin{aligned} T &:= \{p\} \cup \{y_{j,1}, \ldots, y_{j,N_j} : j \in \{1, \ldots, r\}\} \\ &= \{p, pb_2 (\mathbf{1} - p), \ldots, pb_m (\mathbf{1} - p)\}. \end{aligned}$$

By Lemma 2.2.11 there are two linearly independent subsets S' and T' of Ap and pA, respectively, such that span $S \subseteq$ span S', span $T \subseteq$ span T'

and properties (i) to (iii) in Theorem 2.2.7 holds for S' and T'. Say $S' = \{p, u_2, \ldots, u_k\}$ and $T' = \{p, v_2, \ldots, v_k\}$. If we take

$$B = \text{span}\ (\{p\} \cup \{u_i p v_j, u_i p, p v_j : i, j \in \{2, \ldots, k\}\}),$$

then B is the desired subalgebra of J_p. Finally, we note that the containments $C[z_1, \ldots, z_r] \subseteq B$ and $z_j A z_j \subseteq B$ for each $j \in \{1, \ldots, r\}$ follows from properties (i) to (iii) for S' and T' and the minimality of p. □

Theorem 2.2.13. *Let $z_1, \ldots, z_r \in \operatorname{Soc}(A)$. Then there exists a subalgebra B of $\operatorname{Soc}(A)$ such that*

$$z_1, \ldots, z_r \in B \cong M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C}).$$

Moreover, $C[z_1, \ldots, z_r] \subseteq B$ and $z_j A z_j \subseteq B$ for each $j \in \{1, \ldots, r\}$.

Proof. This is an immediate consequence of Lemma 1.2.1 and Lemma 2.2.12. □

As in [8], we denote by $M_{r,n}$, where $r \leq n \leq 2r$, the algebra of $n \times n$ matrices $S = [\alpha_{ij}]$ satisfying $\alpha_{ij} = 0$ whenever $i > r$ or $j \leq n - r$. In [8, Lemma 2.7], Brešar and Šemrl managed to prove that an operator $T \in B(X)$ has rank r if and only if $TB(X)T \cong M_{r,n}$ for some n. Moreover, recall that $\operatorname{Soc} B(X)$ is a minimal two-sided ideal. Thus, although the structure of $TB(X)T$ may be very complicated, by Lemma 2.2.12 it is possible to find a full matrix algebra in which $TB(X)T$ can be embedded.

2.3 Commutators in the Socle

For the convenience of the reader we recall here that Theorem 2.1.7 and Theorem 2.1.8 collectively gives:

Theorem 2.3.1. *The following are equivalent:*

(a) *For any linear functional f on* $\mathrm{Soc}(A)$ *we have that*

$$f(ab) = f(ba) \text{ for all } a, b \in \mathrm{Soc}(A) \Leftrightarrow f = \alpha \mathrm{Tr} \text{ for some } \alpha \in \mathbb{C}.$$

(b) $\mathrm{Soc}(A)$ *is a minimal two-sided ideal.*

(c) $pAp \cong M_{n_p}(\mathbb{C})$ *for each finite-rank projection p of A.*

Shoda's Theorem (see [17]) says that if $A = M_k(\mathbb{C})$, then the traceless matrices in A are precisely those matrices that can be expressed as commutators. As a particular example, we obtain the classical fact that this result can be generalized to $B(X)$. However, Shoda's Theorem fails in general. We can in fact give a precise characterization of those Banach algebras in which Shoda's Theorem holds:

Theorem 2.3.2. (Generalized Shoda's Theorem.)
Every traceless element $a \in$ $\mathrm{Soc}(A)$ *can be expressed as the commutator of two elements belonging to* $\mathrm{Soc}(A)$ *if and only if* $\mathrm{Soc}(A)$ *is a minimal two-sided ideal. Moreover, the rank of each of the two elements in the commutator does not exceed* $\mathrm{rank}\,(a)$*. In particular, the Generalized Shoda's Theorem is valid for $A = B(X)$.*

Proof. Suppose that $\mathrm{Soc}(A)$ is a minimal two-sided ideal of A, and suppose that $a \in \mathrm{Soc}(A)$ has $\mathrm{Tr}\,(a) = 0$. If $a = 0$, then a is a commutator. So assume that $a \neq 0$. By the density of $E(a)$ and the Diagonalization Theorem there exist a $u \in G(A)$, orthogonal rank one projections $p_1, \ldots, p_n$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C} - \{0\}$ such that $a = \lambda_1 u p_1 + \cdots + \lambda_n u p_n$. So, by the orthogonality of the p_i, a can be expressed as $a = ap$, where $p = p_1 + \ldots + p_n$ is a finite-rank projection. Setting $r_a = ap - pap$, write

$$a = pap + (ap - pap) = pap + r_a$$

and notice that $pr_a = 0$ and $r_a p = r_a$. Observe also that $\mathrm{Tr}\,(pap) = \mathrm{Tr}\,(ap) = \mathrm{Tr}\,(a) = 0$. Now, by Theorem 2.3.1 it follows that $pAp \cong M_k(\mathbb{C})$. Thus, by Lemma 2.1.2 and the classical Shoda's Theorem for matrices, we have

$pap = [pxp, pyp]$ for some $x, y \in A$. Finally pick $|\lambda|$ sufficiently large so that $\lambda p + pyp \in G(pAp)$. Then

$$\begin{aligned} & \left[pxp + r_a (\lambda p + pyp)^{-1}, \lambda p + pyp\right] \\ = \; & (pxp)(pyp) + \lambda pxp + r_a p - (pyp)(pxp) - \lambda pxp \\ = \; & [pxp, pyp] + r_a p = pap + r_a p = a. \end{aligned}$$

Since

$$\text{rank}\,(p) = \text{rank}\,(p_1) + \cdots + \text{rank}\,(p_n) = n = \text{rank}\,(a),$$

it follows from the properties of the rank that the rank of each of the two elements in the commutator does not exceed rank (a). This proves the reverse implication. For the forward implication, let f be any linear functional on $\text{Soc}(A)$ such that $f(ab) = f(ba)$ for all $a, b \in \text{Soc}(A)$. Now, for any rank one projections p and q of A we have $\text{Tr}\,(p - q) = \text{Tr}\,(p) - \text{Tr}\,(q) = 0$. Hence, $p - q$ is a commutator. Therefore, by the linearity of f we may conclude that $f(p) = f(q)$. In fact, since p and q were arbitrary, it follows that f is constant on the rank one projections of A. Thus, by Theorem 2.1.4 it follows that $f = \alpha\text{Tr}$ for some $\alpha \in \mathbb{C}$. By Theorem 2.3.1 this shows that $\text{Soc}(A)$ is indeed a minimal two-sided ideal. This proves the forward implication. The last observation is true since Soc $B(X)$ is a minimal two-sided ideal. □

Of course it is possible to obtain the reverse implication of Theorem 2.3.2 by means of Lemma 1.2.1, Lemma 1.2.2 and Lemma 2.2.12. However, this approach does not yield the upper bound for the rank of each of the two elements in the commutator.

By [3, Theorem 2.12] it follows that

$$\text{rank}\,(a) \leq \dim aAa \leq [\text{rank}\,(a)]^2 \quad \text{for all } a \in A.$$

We also have the following:

Theorem 2.3.3. *Suppose that* $a \in \text{Soc}(A)$ *and that* $\text{Tr}\,(a) = 0$. *If* $\dim aAa = [\text{rank}\,(a)]^2$, *then* $a = [x, y]$ *for some* $x, y \in \text{Soc}(A)$.

Proof. If $a = 0$, then the result is obviously true. So assume that $a \neq 0$. By the density of $E(a)$ and the Diagonalization Theorem there exist a $u \in G(A)$, orthogonal rank one projections $p_1, \ldots, p_n$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C} - \{0\}$ such that $a = \lambda_1 u p_1 + \cdots + \lambda_n u p_n$, where rank $(a) = n \geq 1$. Let $p = p_1 + \cdots + p_n$ and observe that $ap = a$ and that

$$a \left(\frac{1}{\lambda_1} p_1 + \cdots + \frac{1}{\lambda_n} p_n \right) = up.$$

Let $w \in A$ be arbitrary and consider awa. Then

$$\begin{aligned} awa = apwap &= a\left(\frac{1}{\lambda_1}p_1 + \cdots + \frac{1}{\lambda_n}p_n\right)(\lambda_1 p_1 + \cdots + \lambda_n p_n)\, wap \\ &= up\,(\lambda_1 p_1 + \cdots + \lambda_n p_n)\, wap. \end{aligned}$$

This shows that $aAa \subseteq upAp$. Conversely, if we let $v \in A$ be arbitrary, then

$$\begin{aligned} upvp &= u\,(\lambda_1 p_1 + \cdots + \lambda_n p_n)\left(\frac{1}{\lambda_1}p_1 + \cdots + \frac{1}{\lambda_n}p_n\right) vp \\ &= a\left(\frac{1}{\lambda_1}p_1 + \cdots + \frac{1}{\lambda_n}p_n\right) v \left(\frac{1}{\lambda_1}p_1 + \cdots + \frac{1}{\lambda_n}p_n\right) u^{-1}u\,(\lambda_1 p_1 + \cdots + \lambda_n p_n) \\ &= a\left(\frac{1}{\lambda_1}p_1 + \cdots + \frac{1}{\lambda_n}p_n\right) v \left(\frac{1}{\lambda_1}p_1 + \cdots + \frac{1}{\lambda_n}p_n\right) u^{-1}a. \end{aligned}$$

This shows that $upAp \subseteq aAa$. Hence, equality holds. In particular, this implies that

$$\dim\,(pAp) = \dim\,(upAp) = \dim\,(aAa) = [\mathrm{rank}\,(a)]^2\,.$$

Moreover, we have

$$\mathrm{rank}\,(p) = \mathrm{rank}\,\,(p_1 + \cdots + p_n) = \sum_{i=1}^{n} \mathrm{rank}\,\,(p_i) = n = \mathrm{rank}\,(a).$$

Now, recall that pAp is a finite-dimensional closed semisimple subalgebra of A with identity p. Also, $\mathrm{rank}_{pAp}(pxp) = \mathrm{rank}_A(pxp)$ for all $x \in A$. By the Wedderburn-Artin Theorem, it follows that

$$pAp \cong M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})\,,$$

where we may assume without loss of generality that $k \geq 1$ and $n_i \geq 1$ for each $i \in \{1, \ldots, k\}$. We claim that $k = 1$: Suppose not. We have $n = \mathrm{rank}\,(p) = n_1 + \cdots + n_k$ and $n^2 = \dim pAp = n_1^2 + \cdots + n_k^2$. However, then

$$\begin{aligned} n^2 = (n_1 + \cdots + n_k)^2 &= n_1^2 + \cdots + n_k^2 + \sum_{i<j} 2n_i n_j \\ &> n_1^2 + \cdots + n_k^2 = \dim pAp = n^2, \end{aligned}$$

which is clearly a contradiction. Hence, $pAp \cong M_{n_1}(\mathbb{C})$ for some integer $n_1 \geq 1$. Using the argument in the proof of Theorem 2.3.2, it readily follows that $a = pap + r_a$ is a commutator as desired. $\square$

Example 2.3.4. Let $A = M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$ and let

$$a = \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right).$$

Then $\operatorname{rank}(a) = 4$ and by the classical Shoda's Theorem for matrices it follows that $a = [x, y]$ for some $x, y \in A = \operatorname{Soc}(A)$. However, $aAa = A$, so

$$\dim aAa = 8 \neq 16 = [\operatorname{rank}(a)]^2.$$

This shows that the converse of Theorem 2.3.3 does not hold in general.

The aim of the next few results is to prove the highly nontrivial fact that

$$\mathcal{C} = \{[x, y] : x, y \in \operatorname{Soc}(A)\}$$

is a vector subspace of $\operatorname{Soc}(A)$.

Lemma 2.3.5. *Let p be a rank one projection of A. If $q \in J_p - \{0\}$ is a projection, then $qAq \cong M_k(\mathbb{C})$ for some integer $k \geq 1$.*

Proof. By the Wedderburn-Artin Theorem,

$$qAq \cong M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C}). \tag{2.3.1}$$

Assume that the direct sum in (2.3.1) contains at least two nonzero terms. This means that we can find distinct rank one projections r and s such that $rxs = 0$ and $sxr = 0$ for all $x \in A$. Since $q \in J_p$, it follows that $r \in J_p$. Consequently, $\{0\} \neq J_r \subseteq J_p$. Thus, by Lemma 1.2.2 it follows that $J_r = J_p$. Similarly, $J_s = J_p$. However, then $r = \sum_{i=1}^{m} x_i s y_i$ for some $x_1, \ldots, x_m, y_1, \ldots, y_m \in A$ and integer $m \geq 1$. But this implies that

$$r = r^2 = \sum_{i=1}^{m} r x_i s y_i = 0$$

which is absurd. From this contradiction, the result now follows. □

Lemma 2.3.6. *Let p be a rank one projection. If $a \in J_p$ and $\operatorname{Tr}(a) = 0$, then $a = [x, y]$ for some $x, y \in J_p$. Hence, if $b, c \in \mathcal{C} \cap J_p$, then $b + c \in \mathcal{C}$. In particular, $b + c = [u, v]$ for some $u, v \in J_p$.*

Proof. If $a = 0$, then the result trivially holds true. So assume that $a \neq 0$. By the density of $E(a)$ and the Diagonalization Theorem there exist a $u \in G(A)$, mutually orthogonal rank one projections $q_1, \ldots, q_n \in Aa$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C} - \{0\}$ such that $a = \lambda_1 u q_1 + \cdots + \lambda_n u q_n$, where $\operatorname{rank}(a) = n \geq 1$. Let

$q = q_1 + \cdots + q_n$. Then $aq = a$, and so, by the properties of the trace and Lemma 2.1.2,

$$0 = \mathrm{Tr}_A(a) = \mathrm{Tr}_A(qaq) = \mathrm{Tr}_{qAq}(qaq).$$

Moreover, by Lemma 2.3.5 it follows that $qAq \cong M_k(\mathbb{C})$. So, by the classical Shoda's Theorem for matrices, we have $qaq = [qxq, qyq]$ for some $x, y \in A$. Consequently, by the argument used in the proof of Theorem 2.3.2 with $|\lambda|$ sufficiently large we get

$$a = \left[qxq + r_a\left(\lambda q + qyq\right)^{-1}, \lambda q + qyq\right],$$

where $r_a = aq - qaq$. Of course, since $q \in Aa \subseteq J_p$, it follows that $qxq + r_a\left(\lambda q + qyq\right)^{-1}$ and $\lambda q + qyq$ both belong to J_p. This proves the first part of the lemma. The last part now follows from the fact that $b, c \in \mathcal{C} \cap J_p$ implies that $b + c \in J_p$ and $\mathrm{Tr}\,(b + c) = 0$. □

Theorem 2.3.7. $\mathcal{C} = \{[x, y] : x, y \in \mathrm{Soc}(A)\}$ *is a vector subspace of* $\mathrm{Soc}(A)$.

Proof. If $a \in \mathcal{C}$, then it readily follows that $\lambda a \in \mathcal{C}$ for all $\lambda \in \mathbb{C}$. So it remains to show that if $a, b \in \mathcal{C}$, then $a + b \in \mathcal{C}$: Let $a, b \in \mathcal{C}$ be arbitrary. Let $\mathcal{P}$ be the class of minimal projections from Lemma 1.2.1 generating the J_p. By Lemma 1.2.1 it follows that

$$a = [w_1, x_1] + \cdots + [w_n, x_n],$$

where $w_i, x_i \in J_{q_i}$ and $q_i \in \mathcal{P}$ for each $i \in \{1, \ldots, n\}$, and where $q_i \neq q_j$ for $i \neq j$. Similarly,

$$b = [y_1, z_1] + \cdots + [y_k, z_k],$$

where $y_i, z_i \in J_{p_i}$ and $p_i \in \mathcal{P}$ for each $i \in \{1, \ldots, k\}$, and where $p_i \neq p_j$ for $i \neq j$. Consequently,

$$a + b = [w_1, x_1] + \cdots + [w_n, x_n] + [y_1, z_1] + \cdots + [y_k, z_k].$$

Now, if $q_i = p_j$ for some $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, k\}$, then by Lemma 2.3.6 it follows that

$$[w_i, x_i] + [y_j, z_j] = [x, y]$$

for some $x, y \in J_{p_i} = J_{q_j}$. So, in order to simplify our notation we may assume, without loss of generality, that $q_i \neq p_j$ for all $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, k\}$. Thus, using the fact that the two-sided ideals in $\{J_p : p \in \mathcal{P}\}$ are pairwise orthogonal, we get

$$a + b = [w_1 + \cdots + w_n + y_1 + \cdots + y_k, x_1 + \cdots + x_n + z_1 + \cdots + z_k].$$

The elements in this commutator are of course in $\mathrm{Soc}(A)$, and so, $a + b \in \mathcal{C}$. This completes the proof. □

Corollary 2.3.8. *Let $a \in \mathcal{C}$ and let p be any finite-rank projection such that $ap = a$. Then $pap \in \mathcal{C}$.*

Proof. Simply observe that $pap = pap - ap + a$ and that $pap - ap = [pap - ap, p]$. Now apply Theorem 2.3.7. □

Corollary 2.3.9. *Let $\mathcal{P}$ be the class of minimal projections from Lemma* 1.2.1 *generating the J_p. Then $a \in \mathcal{C}$ if and only if $a = a_1 + \cdots + a_n$, where $a_i \in J_{p_i}$, $p_i \in \mathcal{P}$ and $\mathrm{Tr}_A(a_i) = 0$ for each $i \in \{1, \dots, n\}$.*

Proof. The forward implication follows from Lemma 1.2.1 since

$$a = [w_1, x_1] + \cdots + [w_n, x_n],$$

where $w_i, x_i \in J_{p_i}$, $p_i \in \mathcal{P}$ for each $i \in \{1, \dots, n\}$ and $p_i \neq p_j$ for $i \neq j$. The reverse implication follows from Lemma 2.3.6 and Theorem 2.3.7. □

Lemma 2.3.10. *Let $a \in \mathrm{Soc}(A)$ and suppose that $\sigma(a) = \{0\}$. Then $a \in \mathcal{C}$.*

Proof. If $a = 0$, then the result is obviously true. So assume that $a \neq 0$. By the density of $E(a)$ and the Diagonalization Theorem there exist a $u \in G(A)$, orthogonal rank one projections $p_1, \dots, p_n$ and $\lambda_1, \dots, \lambda_n \in \mathbb{C} - \{0\}$ such that $a = \lambda_1 u p_1 + \cdots + \lambda_n u p_n$, where $\mathrm{rank}\,(a) = n \geq 1$. Let $p = p_1 + \cdots + p_n$ and observe that $ap = a$. Hence, by Jacobson's Lemma it follows that $\{0\} = \sigma_A(pap) = \sigma_{pAp}(pap)$. Now, by the Wedderburn-Artin Theorem it follows that pAp is isomorphic as an algebra to

$$M = M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C}).$$

Let $\phi : pAp \to M$ be the algebra isomorphism, and let $M_i = M_{n_i}(\mathbb{C})$ for each $i \in \{1, \dots, k\}$. We claim that $pap \in \mathcal{C}$: To prove our claim it will suffice to show that $\phi(pap) = (a_1, \dots, a_k)$ is a commutator in M. Since

$$\{0\} = \sigma_{pAp}(pap) = \sigma_M(\phi(pap)) = \bigcup_{i=1}^{k} \sigma_{M_i}(a_i),$$

it readily follows that a_i is a nilpotent and hence traceless element of M_i for each $i \in \{1, \dots, k\}$. So by Shoda's Theorem for matrices, it follows that a_i is a commutator of M_i for each $i \in \{1, \dots, k\}$. By the pointwise definition of multiplication in the direct sum, this proves our claim. It is now possible to proceed as in the proof of Theorem 2.3.2 and conclude that $a = pap + r_a \in \mathcal{C}$. So the lemma is proved. □

Theorem 2.3.11. *Let $a \in \mathrm{Soc}(A)$ with $\mathrm{Tr}\,(a) = 0$, and let p be the Riesz projection associated with $\sigma'(a)$ and a. If $\dim pAp = [\mathrm{rank}\,(p)]^2$, then $a \in \mathcal{C}$.*

Proof. By the Holomorphic Functional Calculus it follows that $p \in Aa$, $ap = pa = pap$ and that $\sigma\left(\left(\mathbf{1} - p\right)a\right) = \{0\}$. Thus, in particular, $\mathrm{Tr}\left(\left(\mathbf{1} - p\right)a\right) = 0$. Hence, by the linearity of the trace it follows that $\mathrm{Tr}\left(pap\right) = 0$. Now, since $\dim pAp = \left[\mathrm{rank}\left(p\right)\right]^2$, it follows from the argument used in the proof of Theorem 2.3.3 that $pAp \cong M_n\left(\mathbb{C}\right)$. Hence, by Lemma 2.1.2 and the classical Shoda's Theorem for matrices that $pap = [pxp, pyp]$ for some $x, y \in A$. Moreover, since $\left(\mathbf{1} - p\right)a \in \mathrm{Soc}(A)$ and $\sigma\left(\left(\mathbf{1} - p\right)a\right) = \{0\}$, it follows from Lemma 2.3.10 that $\left(\mathbf{1} - p\right)a \in \mathcal{C}$. Therefore, since $\mathcal{C}$ is a vector space by Theorem 2.3.7 and $a = \left(\mathbf{1} - p\right)a + pap$, it follows that $a \in \mathcal{C}$ as desired. $\square$

2.4 Characterizations of Central Socles

If $\mathrm{Soc}(A) = \{0\}$, then obviously $\mathrm{Soc}(A) \subseteq Z(A)$. For this reason we shall assume throughout this section that $\mathrm{Soc}(A) \neq \{0\}$.

Lemma 2.4.1. $\mathrm{Soc}(A) \subseteq Z(A)$ *if and only if*

$$xp = x \Leftrightarrow px = x \tag{2.4.1}$$

for all $x \in \mathrm{Soc}(A)$ *and rank one projections* p *of* A.

Proof. The forward implication is obvious. Assume next that $\mathrm{Soc}(A) \not\subseteq Z(A)$. Then there exists a rank one projection p such that $p \notin Z(A)$. Hence, $xp \neq px$ for some $x \in A$. However, if (2.4.1) holds, then

$$xp^2 = xp \Rightarrow pxp = xp \text{ and } p^2x = px \Rightarrow pxp = px.$$

But then $xp = px$ which is absurd. So (2.4.1) does not hold. This completes the proof. $\square$

Lemma 2.4.2. $\mathrm{Soc}(A) \subseteq Z(A)$ *if and only if for all rank one projections* p *and* q *of* A, $\dim pAq = 1 \Rightarrow p = \alpha q$ *for some* $\alpha \in \mathbb{C}$.

Proof. Suppose that $\mathrm{Soc}(A) \subseteq Z(A)$. Let p and q be any rank one projections of A such that $\dim pAq = 1$. Then, in particular, $pq = qp \neq 0$. Hence, by the minimality of p and q it follows that $pq = qpq = \lambda q$ and $pq = pqp = \beta p$ for some $\lambda, \beta \in \mathbb{C}$. Thus, $p = \frac{\lambda}{\beta} q$. For the converse, suppose that $\dim pAq = 1$ implies that $p = \alpha q$ for some $\alpha \in \mathbb{C}$ for all rank one projections p and q of A but that $\mathrm{Soc}(A) \not\subseteq Z(A)$. By Lemma 2.4.1 there exist an $x \in \mathrm{Soc}(A)$ and a rank one projection p such that either $xp = x$ and $px \neq x$, or, $px = x$ and $xp \neq x$. Say it is the former. In particular, $x \neq 0$. Moreover, since $xp = x$, it follows that $\mathrm{rank}\,(x) = 1$. Thus, by the density of $E(x)$ and the Diagonalization Theorem, there exist a $u \in G(A)$ and a rank one projection q such that $x = qu$. Hence, $x = xp = qup$, and so, $0 \neq x \in qAp$. It can be shown that $\dim qAp \leq 1$ (see [13, Lemma 4.2]), so it must be the case that $\dim qAp = 1$. Hence, by hypothesis, $q = \lambda p$ for some $\lambda \in \mathbb{C} - \{0\}$. However, then

$$px = p\,(qu) = p\,(\lambda pu) = \lambda pu = qu = x$$

which is a contradiction. Thus, it must be the case that $\mathrm{Soc}(A) \subseteq Z(A)$. This gives the result. $\square$

Lemma 2.4.3. $\mathrm{Soc}(A) \subseteq Z(A)$ *if and only if for all rank one projections* p *and* q *of* A, $pq = 0 \Rightarrow \dim pAq = 0$ *and* $pq \neq 0 \Rightarrow p = \alpha q$ *for some* $\alpha \in \mathbb{C}$.

Proof. The forward implication is obvious. For the reverse implication, let p and q be any rank one projections of A such that dim$pAq = 1$. By hypothesis, $pq \neq 0$ and hence $p = \alpha q$ for some $\alpha \in \mathbb{C}$. Thus, by Lemma 2.4.2 we have the result. □

Theorem 2.4.4. $\mathrm{Soc}(A) \subseteq Z(A)$ *if and only if for all* $x \in \mathrm{Soc}(A)$*,* $x^2 = 0 \Rightarrow x = 0$.

Proof. Suppose first that $\mathrm{Soc}(A) \subseteq Z(A)$. Let $x \in \mathrm{Soc}(A)$ such that $x^2 = 0$. Since x is von Neumann regular, it follows that $x = xyx$ for some $y \in A$. Thus, $x = x^2y = 0$. Conversely, suppose that $x = 0$ whenever $x^2 = 0$ for all $x \in \mathrm{Soc}(A)$. Let p and q be any rank one projections such that $pq = 0$. Then, in particular, $(qp)^2 = qpqp = 0$. Thus, $qp = 0$. Consequently, for any $y \in A$, we have $(pyq)^2 = pyqpyq = 0$, and so, $pyq = 0$. In other words, $\dim pAq = 0$. This shows that $pq = 0 \Rightarrow \dim pAq = 0$. Next, assume that $pq \neq 0$ for some rank one projections p and q of A. Now, $(pqp - pq)^2 = (qpq - pq)^2 = 0$, and so, by hypothesis, $pqp = pq$ and $qpq = pq$. Hence, by the minimality of p and q and the fact that $pq \neq 0$, we get $p = \alpha q$ for some $\alpha \in \mathbb{C} - \{0\}$. This shows that $pq \neq 0 \Rightarrow p = \alpha q$ for some $\alpha \in \mathbb{C}$. By Lemma 2.4.3 we conclude that $\mathrm{Soc}(A) \subseteq Z(A)$, establishing the result. □

Lemma 2.4.5. $\dim aAa = \mathrm{rank}\,(a)$ *for all* $a \in \mathrm{Soc}(A)$ *if and only if for all rank one projections* p *and* q *of* A*,* $pq = 0 \Rightarrow \dim pAq = 0$.

Proof. Suppose first that $\dim aAa = \mathrm{rank}\,(a)$ for all $a \in \mathrm{Soc}(A)$. Let p and q be any rank one projections such that $pq = 0$. We claim that $\dim pAq = 0$: Suppose this is false. Then there exists an $x_0 \in pAq$ such that $px_0q \neq 0$. Since $pq = 0$, it follows that $p \neq \alpha q$ for all $\alpha \in \mathbb{C}$. Moreover, by the subadditivity of the rank we have $\mathrm{rank}\,(p+q) \leq 2$. So, by hypothesis, $\{p, q\}$ is a spanning set of $(p+q)A(p+q)$ since $(p+q)p(p+q)$, $(p+q)q(p+q)$ and $(p+q)^2$ in $(p+q)A(p+q)$ implies that $p, q \in (p+q)A(p+q)$. Consequently, we can find $\lambda, \beta \in \mathbb{C}$ such that

$$\lambda p + \beta q = (p+q)x_0(p+q) = px_0p + px_0q + qx_0p + qx_0q.$$

However, then

$$px_0q = p^2x_0q^2 = \lambda pq + \beta pq - px_0pq - pqx_0pq - pqx_0q = 0,$$

contradicting our choice of x_0. This proves our claim. So $pq = 0$ implies $\dim pAq = 0$. For the converse, let $a \in \mathrm{Soc}(A) - \{0\}$ be arbitrary. By the density of $E(a)$ and the Diagonalization Theorem there exist a $u \in G(A)$, mutually orthogonal rank one projections $p_1, \ldots, p_n$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C} - \{0\}$

such that $ua = \lambda_1 p_1 + \cdots + \lambda_n p_n$, where $\operatorname{rank}(a) = n \geq 1$. Observe that $uaAua = pAp$, where $p = p_1 + \cdots + p_n$, by the orthogonality of the p_i. Moreover, by hypothesis, $\dim p_i A p_j = 0$ for $i \neq j$. Hence,

$$\dim pAp \leq \dim \left(\sum_{i=1}^{n} \sum_{j=1}^{n} p_i A p_j \right) \leq \sum_{i=1}^{n} \sum_{j=1}^{n} \dim p_i A p_j = n,$$

where the final equality follows from the minimality of the p_i. Furthermore, by orthogonality it follows that $\{p_1, \ldots, p_n\}$ is a linearly independent set contained in pAp. Consequently,

$$\dim aAa = \dim uaAua = \dim pAp = n = \operatorname{rank}(a),$$

as desired. This completes the proof. □

Theorem 2.4.6. $\operatorname{Soc}(A) \subseteq Z(A)$ *if and only if* $\dim aAa = \operatorname{rank}(a)$ *for all* $a \in \operatorname{Soc}(A)$.

Proof. The forward implication readily follows from Lemma 2.4.5. Conversely, suppose that $\dim aAa = \operatorname{rank}(a)$ for all $a \in \operatorname{Soc}(A)$. Let $x \in \operatorname{Soc}(A)$ with $x^2 = 0$ be arbitrary. We claim that $x = 0$: Suppose this is false. By the density of $E(x)$ and the Diagonalization Theorem there exist $u, v \in G(A)$, mutually orthogonal rank one projections $p_1, \ldots, p_n$, mutually orthogonal rank one projections $q_1, \ldots, q_n$, and $\alpha_1, \ldots, \alpha_n, \lambda_1, \ldots, \lambda_n \in \mathbb{C} - \{0\}$ such that $ux = \alpha_1 p_1 + \cdots + \alpha_n p_n$ and $xv = \lambda_1 q_1 + \cdots + \lambda_n q_n$, where $\operatorname{rank}(x) = n \geq 1$. It is useful to recall here that the p_i all belong to $Aux \cap uxA$, and similarly, that the q_i all belong to $Axv \cap xvA$. Hence, for each $i \in \{1, \ldots, n\}$ there exist $x_i, y_i \in A$ such that $p_i = x_i ux$ and $q_i = xvy_i$. Hence, $p_j q_i = 0$ for all $i, j \in \{1, \ldots, n\}$. Thus, for any $i, j \in \{1, \ldots, n\}$, we have $p_j \neq \alpha q_i$ for all $\alpha \in \mathbb{C}$. We now show that $q_i p_j = 0$ for all $i, j \in \{1, \ldots, n\}$: Let $i, j \in \{1, \ldots, n\}$ be arbitrary but fixed. Since $(p_j + q_i)^2$, $(p_j + q_i)\, p_j\, (p_j + q_i)$ and $(p_j + q_i)\, q_i\, (p_j + q_i)$ are all in $(p_j + q_i)\, A\, (p_j + q_i)$, it follows that $p_j, q_i, q_i p_j \in (p_j + q_i)\, A\, (p_j + q_i)$. Moreover, by the subadditivity of the rank we get $\operatorname{rank}(p_j + q_i) \leq 2$. Hence, by hypothesis, we have $q_i p_j = \beta p_j + \lambda q_i$ for some $\beta, \lambda \in \mathbb{C}$. Consequently, since $q_i p_j q_i = \beta p_j q_i + \lambda q_i$, $p_j q_i p_j = \beta p_j + \lambda p_j q_i$ and $p_j q_i = 0$, we get $\lambda q_i = 0$ and $\beta p_j = 0$. Hence, $q_i p_j = 0$. So, by Lemma 2.4.5 we may infer that $\dim q_i A p_j = 0$ for all $i, j \in \{1, \ldots, n\}$. However, this means that for any $y \in A$ we have

$$xyx = xvv^{-1}yu^{-1}ux \in \sum_{i=1}^{n} \sum_{j=1}^{n} q_i A p_j = \{0\}\,.$$

But x is von Neumann regular, so $x = 0$. This contradiction now proves our claim. Hence, $\operatorname{Soc}(A) \subseteq Z(A)$ by Theorem 2.4.4. This establishes the result. □

Theorem 2.4.7. $\mathcal{C} = \{0\}$ *if and only if* $\operatorname{Soc}(A) \subseteq Z(A)$.

Proof. The reverse implication is obvious. For the forward implication, let p be a minimal projection of A. By hypothesis, for all $x, y \in A$ it follows that $[xp, py] = 0$. Hence, $xpy \in pAp$ for all $x, y \in A$. Thus, J_p is one-dimensional and equals pAp. Consequently, every element of $\operatorname{Soc}(A)$ is of the form $\lambda_1 p_1 + \cdots + \lambda_n p_n$ for some minimal projections $p_1, \ldots, p_n$ and complex numbers $\lambda_1, \ldots, \lambda_n$. Next pick $x \in A$ and let p be any projection in $\operatorname{Soc}(A)$. Since $[p, px] = [xp, p] = 0$, we infer that $xp = pxp = px$. Thus, $\operatorname{Soc}(A) \subseteq Z(A)$. □

To conclude we show that the inequality

$$\operatorname{rank}(p) \leq \dim pAp \leq [\operatorname{rank}(p)]^2$$

for all $p = p^2 \in \operatorname{Soc}(A)$ can be "solved" at the extremities:

Theorem 2.4.8.

(1) $\dim pAp = \operatorname{rank}(p)$ *for all finite-rank projections p of A if and only if* $\operatorname{Soc}(A) \subseteq Z(A)$.

(2) $\dim pAp = [\operatorname{rank}(p)]^2$ *for all finite-rank projections p of A if and only if the Generalized Shoda's Theorem holds for A.*

Proof. We firstly consider the characterization which appears as (1) above. By Theorem 2.4.6 it suffices to show that for each $a \in \operatorname{Soc}(A)$, there exists a projection $p \in \operatorname{Soc}(A)$ such that $\dim aAa = \dim pAp$ and $\operatorname{rank}(a) = \operatorname{rank}(p)$. But this is exactly what was shown in the first part of the proof of Theorem 2.3.3. Next we consider the characterization which appears as (2) above. By Theorem 2.3.1 and Theorem 2.3.3, it will suffice to show that $\dim pAp = [\operatorname{rank}(p)]^2$ for all finite-rank projections p of A if and only if $pAp \cong M_{n_p}(\mathbb{C})$ for all finite-rank projections p of A: The reverse implication is obvious since p is the identity of pAp, and the forward implication follows from the last part of the argument in the proof of Theorem 2.3.3. This completes the proof. □

Theorem 2.4.8 suggests that the dimension of certain subalgebras of the socle is to some extent a measure of commutativity.

2.5 Uniqueness under Spectral Variation in the Socle

In [7] M. Brešar and Š. Špenko consider two interesting problems which resulted from certain questions centered around Kaplansky's problem on spectrum preserving maps [11]:

Problem 1. Suppose that $a, b \in A$ satisfy $\sigma(ax) = \sigma(bx)$ for all $x \in A$. Does this imply $a = b$?

Problem 2. Suppose that $a, b \in A$ satisfy

$$\rho(ax) \leq \rho(bx) \text{ for all } x \in A. \tag{2.5.1}$$

What is the relation between a and b?

The first problem has been settled by G. Braatvedt and R. Brits in [5]:

Theorem 2.5.1. [5, Theorem 2.1, Theorem 2.6] *Let $a, b \in A$. Then the following are equivalent:*

(i) $a = b$.

(ii) $\sigma(ax) = \sigma(bx)$ *for all* $x \in A$ *such that* $\rho(x - \mathbf{1}) < 1$.

(iii) $\sigma(a + x) = \sigma(b + x)$ *for all* x *in some open neighbourhood of* $-b$.

Problem 2, as to be expected, is slightly more intricate. Evidence such as [7, Example 3.3] suggests that the answer to this question may depend on the algebra or on the elements under consideration. Indeed, in the special situation where $b = \mathbf{1}$ it was found in [6] that a must then belong to $Z(A)$. Moreover, in [7] Brešar and Špenko investigated the special case where A is a prime C^*-algebra. The conclusion in this case is that the elements a and b satisfying (2.5.1) are necessarily linearly dependent. We recall that A is a *prime algebra* if all nonzero two-sided ideals I and J of A satisfy $IJ \neq \{0\}$. In particular, we will see that the linear dependence obtained in the prime C^*-algebra case extends to the case where A is assumed to be prime with a nonzero socle.

The following theorem will be crucial in the proofs of some results. We list it here with its popular name for convenient referencing later on:

Scarcity Theorem [1, Theorem 3.4.25]: Let f be an analytic function from a domain D of $\mathbb{C}$ into A. Then either the set of $\lambda \in D$ such that $\sigma\left(f(\lambda)\right)$ is finite is a Borel set having zero capacity, or there exist an integer $n \geq 1$ and a closed discrete subset E of D such that $\#\left(\sigma\left(f(\lambda)\right)\right) = n$ for $\lambda \in D - E$ and $\#\left(\sigma\left(f(\lambda)\right)\right) < n$ for $\lambda \in E$. In the latter case the n points of $\sigma\left(f(\lambda)\right)$ are locally holomorphic functions on $D - E$.

Let $a \in A$. A standard argument using Baire's Category Theorem and the Scarcity Theorem can be used to show that if $\sigma(ax)$ is finite for all x in some nonempty open set N of A, then a has finite rank. This observation will be useful later on.

Another useful result appears in [1]:

Lemma 2.5.2. [1, Exercise 3.9] *Given n elements $x_1, \ldots, x_n \in A$, suppose that $x_i x_j = 0$ for $i \neq j$. Then*

$$\sigma'\left(x_1 + \cdots + x_n\right) = \sigma'\left(x_1\right) \cup \cdots \cup \sigma'\left(x_n\right).$$

Let $a \in A$. J. Zemánek has shown that $\rho(a + x) = 0$ for all quasinilpotent x in A if and only if $a \in \mathrm{Rad}(A)$ [1, Theorem 5.3.1]. In order to get some feeling for the subject matter, we start by utilizing the aforementioned result to show that condition (iii) in Theorem 2.5.1 can be substancially relaxed:

Theorem 2.5.3. *Let $a, b \in A$. Then the following are equivalent:*

(i) $a = b$.

(ii) $\rho\left(a + x\right) \leq \rho\left(b + x\right)$ *for all x in some open neighbourhood of $-b$.*

Proof. Certainly, (i) $\Rightarrow$ (ii). We therefore proceed to show that (ii) $\Rightarrow$ (i). We claim that $\rho\left(a - b + q\right) = 0$ for all quasinilpotent elements q in A: Let q be any quasinilpotent element in A. Consider the analytic function $f : \mathbb{C} \to A$ defined by $f(\lambda) = a - b + \lambda q$. By hypothesis and the Spectral Mapping Theorem, there exists a real number $k > 0$ such that $\rho\left(a - b + \lambda q\right) \leq \rho\left(\lambda q\right) = 0$ whenever $|\lambda| < k$. Hence, $\sigma\left(f\left(\lambda\right)\right) = \{0\}$ whenever $|\lambda| < k$. By the Scarcity Theorem we may therefore conclude that $\sigma\left(f\left(\lambda\right)\right) = \{\alpha\left(\lambda\right)\}$ for all $\lambda \in \mathbb{C}$, where α is a mapping from $\mathbb{C}$ into $\mathbb{C}$. By [1, Corollary 3.4.18], α is an entire function. However, $\alpha\left(\lambda\right) = 0$ whenever $|\lambda| < k$, and so, from basic Complex Analysis it must be the case that $\alpha\left(\lambda\right) = 0$ for all $\lambda \in \mathbb{C}$. This proves our claim. Consequently, $a - b \in \mathrm{Rad}(A)$ by [1, Theorem 5.3.1]. Thus, by semisimplicity we have the result. $\square$

Theorem 2.5.4. *Let $a, b \in \mathrm{Soc}(A)$. Then $a = b$ if and only if any one of the following holds true:*

(i) $\sigma(ax) = \sigma(bx)$ *for all rank one elements* $x \in A$.

(ii) $\sigma(a + x) = \sigma(b + x)$ *for all rank one elements* $x \in A$.

Proof. Obviously, if $a = b$ then conditions (i) and (ii) both hold. So let $a, b \in \mathrm{Soc}(A)$ and assume that condition (i) holds. Then $\mathrm{Tr}\,(ax) = \mathrm{Tr}\,(bx)$ for all rank one elements $x \in A$. Let $y \in \mathrm{Soc}(A)$ be arbitrary. Clearly, $\mathrm{Tr}\,(ay) = \mathrm{Tr}\,(by)$ if $y = 0$. So assume that $y \neq 0$. By the Diagonalization Theorem and the density of $E(y)$ there exist rank one projections $p_1, \ldots, p_n$, $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ and a $u \in G(A)$ such that $y = \alpha_1 u p_1 + \cdots + \alpha_n u p_n$. Thus, by the linearity of the trace we readily obtain $\mathrm{Tr}\,(ay) = \mathrm{Tr}\,(by)$ for all $y \in \mathrm{Soc}(A)$. Consequently, $\mathrm{Tr}\,((a - b)\,y) = 0$ for all $y \in \mathrm{Soc}(A)$. Thus, since $a - b \in \mathrm{Soc}(A)$, it follows from the properties of the trace (see p. 6) that $a - b = 0$. Next take $a, b \in \mathrm{Soc}(A)$ and assume that condition (ii) holds. Fix any $\lambda \notin \sigma(a) \cup \sigma(b)$ and $0 \neq \alpha \in \mathbb{C}$. If $x \in A$ has rank one, then we have

$$\lambda\mathbf{1} - \left(a + \alpha^{-1}x\right) \in G(A) \Leftrightarrow \lambda\mathbf{1} - \left(b + \alpha^{-1}x\right) \in G(A).$$

Consequently,

$$(\lambda\mathbf{1} - a)\left(\mathbf{1} + (\lambda\mathbf{1} - a)^{-1}\alpha^{-1}x\right) \in G(A)$$

if and only if

$$(\lambda\mathbf{1} - b)\left(\mathbf{1} + (\lambda\mathbf{1} - b)^{-1}\alpha^{-1}x\right) \in G(A).$$

Since the first term on the left of each expression is invertible, it follows that

$$\alpha \in \sigma\left((\lambda\mathbf{1} - a)^{-1}x\right) \Leftrightarrow \alpha \in \sigma\left((\lambda\mathbf{1} - b)^{-1}x\right).$$

Hence, $\sigma'\left((\lambda\mathbf{1} - a)^{-1}x\right) = \sigma'\left((\lambda\mathbf{1} - b)^{-1}x\right)$ for all rank one elements $x \in A$. Thus, $\mathrm{Tr}\,\left((\lambda\mathbf{1} - a)^{-1}x\right) = \mathrm{Tr}\,\left((\lambda\mathbf{1} - b)^{-1}x\right)$ for all rank one elements $x \in A$. Moreover, since

$$(\lambda\mathbf{1} - a)^{-1} - (\lambda\mathbf{1} - b)^{-1} = (\lambda\mathbf{1} - a)^{-1}(a - b)(\lambda\mathbf{1} - b)^{-1} \in \mathrm{Soc}(A),$$

it follows as before from the properties of the trace (see p. 6) that

$$(\lambda\mathbf{1} - a)^{-1} - (\lambda\mathbf{1} - b)^{-1} = 0.$$

Hence, $a = b$, which establishes the result. ∎

Let J be a two-sided ideal of A. Denote by $l(J)$ the *left-annihilator* of J, that is,

$$l(J) := \{x \in A : xJ = \{0\}\}.$$

Similarly, we define the *right-annihilator* of J by

$$r(J) := \{x \in A : Jx = \{0\}\}.$$

Theorem 2.5.5. *Suppose that* $\mathrm{Soc}(A) \neq \{0\}$. *Then* $l(\mathrm{Soc}(A)) = \{0\}$ *if and only if the following are equivalent for any* $a, b \in A$*:*

(i) $a = b$.

(ii) $\sigma(ax) = \sigma(bx)$ *for all rank one elements* $x \in A$.

(iii) $\sigma(a + x) = \sigma(b + x)$ *for all rank one elements* $x \in A$.

Proof. Suppose first that $l(\mathrm{Soc}(A)) = \{0\}$ and let $a, b \in A$. Certainly, (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii). Using the argument in the proof of Theorem 2.5.4 we see that (ii) implies $\mathrm{Tr}\,((a-b)\,y) = 0$ for all $y \in \mathrm{Soc}(A)$. Hence, from the properties of the trace (see p. 6) it follows that $(a-b)\,\mathrm{Soc}(A) = \{0\}$. Hence, $a - b \in l(\mathrm{Soc}(A)) = \{0\}$, so (ii) $\Rightarrow$ (i). Similarly, the argument in the proof of Theorem 2.5.4 can also be used to show that (iii) implies $\mathrm{Tr}\left(\left((\lambda\mathbf{1} - a)^{-1} - (\lambda\mathbf{1} - b)^{-1}\right) y\right) = 0$ for all $y \in \mathrm{Soc}(A)$, where $\lambda \notin \sigma(a) \cup \sigma(b)$ is fixed. Hence, the properties of the trace (see p. 6) yields

$$(\lambda\mathbf{1} - a)^{-1} - (\lambda\mathbf{1} - b)^{-1} \in l(\mathrm{Soc}(A)) = \{0\}.$$

Thus, (iii) $\Rightarrow$ (i). This proves the forward implication. For the converse, we argue contrapositively. Suppose that $l(\mathrm{Soc}(A)) \neq \{0\}$. Let $0 \neq a \in l(\mathrm{Soc}(A))$ be fixed. Moreover, pick $y \in \mathrm{Soc}(A)$. Since $a \neq 0$, $a + y \neq y$. However, since $a\mathrm{Soc}(A) = \{0\}$, it follows that $\sigma((a+y)\,x) = \sigma(yx)$ for all rank one elements $x \in A$. Hence, (ii) $\not\Rightarrow$ (i). This completes the proof. $\square$

In [5, Theorem 2.5] it was shown that properties (i) to (iii) are equivalent for any two bounded linear operators on a Banach space X. Consequently, Theorem 2.5.5 implies the well-known fact that $l(\mathrm{Soc}\,B(X)) = \{0\}$.

Lemma 2.5.6. *Suppose that* $\mathrm{Soc}(A)$ *is a minimal two-sided ideal. Let* $a, b \in A$ *and suppose that* $b = pt$*, where* $p = p^2 \in \mathrm{Soc}(A)$ *and* $t \in G(A)$*. If*

$$\rho(ax) \leq \rho(bx) \text{ for all } x \in A,$$

then $a = \lambda b$ *for some* $\lambda \in \mathbb{C}$ *with* $|\lambda| \leq 1$.

Proof. If $p = 0$, then by semisimplicity $a = 0$ and we are done. So assume that $p \neq 0$. By hypothesis, $\rho(a'x) \leq \rho(px)$ for all $x \in A$, where $a' = at^{-1}$. It will suffice to show that $a' = \lambda p$ for some $\lambda \in \mathbb{C}$, since of course the assumption in conjunction with the Spectral Mapping Theorem automatically yields $|\lambda| \leq 1$. Replacing x by $(\mathbf{1} - p)\,x$, we get $\rho(a'(\mathbf{1} - p)\,x) = 0$ for all $x \in A$. Thus, $a'(\mathbf{1} - p) \in \mathrm{Rad}(A) = \{0\}$, and so, $a' = a'p$. Moreover, if we replace x by $x\,(\mathbf{1} - p)$, then by Jacobson's Lemma we have $\rho((\mathbf{1} - p)\,a'x) = 0$ for all $x \in A$. Hence, as before, the semisimplicity of A yields $a' = pa'$. Consequently, $a' = pa'p$. Now, pAp is a closed semisimple subalgebra of A with identity p. Moreover, $\sigma'_{pAp}(pxp) = \sigma'_A(pxp)$ for all $x \in A$. Hence, by hypothesis, we have

$$\rho_{pAp}\left((pa'p)\,(pxp)\right) \leq \rho_{pAp}\,(pxp) \text{ for all } x \in A.$$

Hence, by the result in [6] it follows that $a' \in Z(pAp)$. However, since $\mathrm{Soc}(A)$ is a minimal two-sided ideal, by Theorem 2.3.1 we may infer that $pAp \cong M_n(\mathbb{C})$. Consequently, $Z(pAp) = \mathbb{C}p$. So, $a' = \lambda p$ for some $\lambda \in \mathbb{C}$. The result now follows. □

Theorem 2.5.7. $\mathrm{Soc}(A)$ *is a minimal two-sided ideal if and only if the following are equivalent for any $a \in A$ and $b \in \mathrm{Soc}(A)$:*

(i) $\rho(ax) \leq \rho(bx)$ *for all* $x \in A$

(ii) $a = \lambda b$ *for some* $\lambda \in \mathbb{C}$ *with* $|\lambda| \leq 1$.

Proof. Suppose first that $\mathrm{Soc}(A)$ is a minimal two-sided ideal and let $a \in A$ and $b \in \mathrm{Soc}(A)$. Obviously, (ii) $\Rightarrow$ (i), so assume that condition (i) holds. If $b = 0$, then by hypothesis and the semisimplicity of A we have $a = 0$. So assume $b \neq 0$. By the Diagonalization Theorem and the density of $E(b)$ we can find mutually orthogonal rank one projections $p_1, \dots, p_n$, $\alpha_1, \dots, \alpha_n \in \mathbb{C} - \{0\}$ and a $u \in G(A)$ such that $b = \alpha_1 p_1 u + \cdots + \alpha_n p_n u$. Observe firstly that if we set $p := p_1 + \cdots + p_n$, then $p^2 = p$ and $pb = b$. Consequently, by hypothesis and Jacobson's Lemma it follows that $\rho((\mathbf{1} - p)\,ax) = 0$ for all $x \in A$. Hence, $(\mathbf{1} - p)\,a \in \mathrm{Rad}(A) = \{0\}$, and so, $a = pa$. By orthogonality it follows that $\left(\alpha_1^{-1}p_1 + \cdots + \alpha_n^{-1}p_n\right) b = pu$. Thus, by hypothesis and Jacobson's Lemma it follows that

$$\rho\left(\left(\alpha_1^{-1}p_1 + \cdots + \alpha_n^{-1}p_n\right) ax\right) \leq \rho\,(pux) \text{ for all } x \in A.$$

Thus, by Lemma 2.5.6 we may infer that

$$\left(\alpha_1^{-1}p_1 + \cdots + \alpha_n^{-1}p_n\right) a = \lambda pu \text{ for some } \lambda \in \mathbb{C}.$$

Hence,

$$\begin{aligned} a \;=\; pa &= (\alpha_1 p_1 + \cdots + \alpha_n p_n)\left(\alpha_1^{-1} p_1 + \cdots + \alpha_n^{-1} p_n\right) a \\ &= (\alpha_1 p_1 + \cdots + \alpha_n p_n)(\lambda p u) \\ &= \lambda (\alpha_1 p_1 u + \cdots + \alpha_n p_n u) \;=\; \lambda b. \end{aligned}$$

This proves the forward implication. For the reverse implication we argue contrapositively. Suppose that $\mathrm{Soc}(A)$ is not a minimal two-sided ideal. Then by Lemma 1.2.1 and Lemma 1.2.2 we may infer the existence of two rank one projections, say p and q, such that $J_pJ_q = J_qJ_p = \{0\}$. In particular, $p \neq \lambda(p+q)$ for all $\lambda \in \mathbb{C}$. Let $x \in A$ be arbitrary. Then

$$(px)(qx) = (qx)(px) = 0.$$

Hence, by Lemma 2.5.2 it follows that $\sigma'((p+q)x) = \sigma'(px) \cup \sigma'(qx)$. So, $\rho(px) \leq \rho((p+q)x)$. Since $x \in A$ was arbitrary, this shows that (i) $\not\Rightarrow$ (ii), which completes the proof. □

Lemma 2.5.8. *Suppose that for any $a, b \in A$ we have that the following are equivalent:*

(i) *$\rho(ax) \leq \rho(bx)$ for all $x \in A$*

(ii) *$a = \lambda b$ for some $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$.*

Then A is a prime algebra.

Proof. We shall argue contrapositively. If A is not prime, then we can find two nonzero two-sided ideals I and J such that $IJ = \{0\}$. Let $0 \neq a \in I$. If $a \in J$, then $aAa = \{0\}$. But then, by semisimplicity, it follows that $a = 0$ which is absurd. Hence, $a \notin J$. Pick $0 \neq b \in J$. In particular then, $a \neq \lambda b$ for all $\lambda \in \mathbb{C}$. We firstly claim that $I \subseteq r(J)$. Let $x \in l(J)$ and let $y \in J$ be arbitrary. By Jacobson's Lemma and the fact that J is a two-sided ideal, it follows that $\rho(yxw) = 0$ for all $w \in A$. Hence, $yx \in \mathrm{Rad}(A) = \{0\}$. Since $y \in J$ was arbitrary, it follows that $I \subseteq r(J)$ as claimed. Since $a \neq \lambda b$ for all $\lambda \in \mathbb{C}$ and $b \neq 0$, we may infer that $a \neq \lambda(b+a)$ for all $\lambda \in \mathbb{C}$. Let $x \in A$ be arbitrary. Then $ax \in l(J) \cap r(J)$. Consequently, $(ax)(bx) = (bx)(ax) = 0$. Thus, by Lemma 2.5.2 it follows that $\sigma'((a+b)x) = \sigma'(ax) \cup \sigma'(bx)$. Hence, $\rho(ax) \leq \rho((a+b)x)$. Since $x \in A$ was arbitrary, this gives the result. □

Theorem 2.5.9. *Let A be a C^*-algebra. Then A is prime if and only if for any $a, b \in A$ we have that the following are equivalent:*

(i) *$\rho(ax) \leq \rho(bx)$ for all $x \in A$*

(ii) $a = \lambda b$ *for some* $\lambda \in \mathbb{C}$ *with* $|\lambda| \leq 1$.

Proof. This is immediate from [7, Theorem 3.7] and Lemma 2.5.8. □

Lemma 2.5.10. *Suppose that* $\mathrm{Soc}(A)$ *is a minimal two-sided ideal and that* $l\,(\mathrm{Soc}(A)) = \{0\}$. *Then for any* $a, b \in A$ *we have that the following are equivalent:*

(i) $\rho(ax) \leq \rho(bx)$ *for all* $x \in A$

(ii) $a = \lambda b$ *for some* $\lambda \in \mathbb{C}$ *with* $|\lambda| \leq 1$.

Proof. Let $a, b \in A$. If $a = 0$, then we are done. So assume $a \neq 0$. It suffices to show that (i) $\Rightarrow$ (ii). Let $y \in \mathrm{Soc}(A)$ be arbitrary but fixed. By hypothesis, $\rho(ayx) \leq \rho(byx)$ for all $x \in A$. Hence, by Theorem 2.5.7 there exists a $\lambda_y \in \mathbb{C}$ such that $ay = \lambda_y by$. Let $f_b : \mathrm{Soc}(A) \to \mathbb{C}$ and $f_a : \mathrm{Soc}(A) \to \mathbb{C}$ be defined as follows: $f_b(y) = \mathrm{Tr}\,(by)$ and $f_a(y) = \mathrm{Tr}\,(ay)$ for $y \in \mathrm{Soc}(A)$. Then f_b and f_a are nonzero linear functionals on the linear space $\mathrm{Soc}(A)$. Moreover, by our first observation it follows that $\mathrm{Ker}\ f_b \subseteq \mathrm{Ker}\ f_a$. Hence, from linear algebra (see [10, p. 10]), it follows that $f_a = \lambda f_b$ for some $\lambda \in \mathbb{C}$. Thus, by the linearity of the trace, $\mathrm{Tr}\,((a - \lambda b)\,y) = 0$ for all $y \in \mathrm{Soc}(A)$. Hence, from the properties of the trace (see p. 6) it follows that $a - \lambda b \in l\,(\mathrm{Soc}(A)) = \{0\}$ which gives the result. □

Theorem 2.5.11. *Suppose that* $\mathrm{Soc}(A) \neq \{0\}$. *Then* A *is prime if and only if for any* $a, b \in A$ *we have that the following are equivalent:*

(i) $\rho(ax) \leq \rho(bx)$ *for all* $x \in A$

(ii) $a = \lambda b$ *for some* $\lambda \in \mathbb{C}$ *with* $|\lambda| \leq 1$.

Proof. The reverse implication follows immediately from Lemma 2.5.8. So assume that A is prime. Since $\mathrm{Soc}(A) \neq \{0\}$, we may infer that $l\,(\mathrm{Soc}(A)) = \{0\}$. Moreover, since A is prime, it readily follows from Lemma 1.2.1 that $\mathrm{Soc}(A)$ is a minimal two-sided ideal. The forward implication therefore follows from Lemma 2.5.10. □

Corollary 2.5.12. *Suppose that* $\mathrm{Soc}(A) \neq \{0\}$. *Then* A *is prime if and only if* $\mathrm{Soc}(A)$ *is a minimal two-sided ideal and* $l\,(\mathrm{Soc}(A)) = \{0\}$.

Proof. This is a direct consequence of Lemma 2.5.10 and Theorem 2.5.11. □

Corollary 2.5.13. *Suppose that* A *is a prime algebra with* $\mathrm{Soc}(A) \neq \{0\}$. *Then* $Z(A)$ *is trivial, i.e.* $Z(A) = \mathbb{C}\mathbf{1}$.

Proof. If A is prime and $\operatorname{Soc}(A) \neq \{0\}$, then by Theorem 2.5.11 we may infer that the following are equivalent for any $a, b \in A$:

(i) $\rho(ax) \leq \rho(bx)$ for all $x \in A$

(ii) $a = \lambda b$ for some $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$.

Let $u \in Z(A)$ be arbitrary. In particular, we have $\rho(ux) \leq \rho(u)\rho(x)$ for all $x \in A$. If $\rho(u) = 0$, then $\rho(ux) = 0$ for all $x \in A$, and so, by semisimplicity $u = 0 \in \mathbb{C}\mathbf{1}$. Hence, we may assume that $\rho(u) \neq 0$. Set $\alpha := \rho(u)^{-1}$. Then $\rho(\alpha ux) \leq \rho(x)$ for all $x \in A$. Thus, $\alpha u = \lambda \mathbf{1}$ for some $\lambda \in \mathbb{C}$. Hence, $u \in \mathbb{C}\mathbf{1}$. Since the other set inclusion is obvious, we conclude that $Z(A) = \mathbb{C}\mathbf{1}$. $\square$

It is known that $\operatorname{Soc} B(X)$ is a minimal two-sided ideal, and, as mentioned earlier, that $l(\operatorname{Soc} B(X)) = \{0\}$. Hence, we obtain the well-known result that $Z(B(X))$ is trivial. Moreover, Theorem 2.5.9 and Corollary 2.5.13 also implies the well-established fact that all prime C^*-algebras have trivial centers.

Let $0 \neq a \in A$ and $0 \neq b \in \operatorname{Soc}(A)$. It turns out that the condition

$$\sigma'(ax) \subseteq \sigma'(bx) \text{ for all } x \in A \Rightarrow a = b$$

can also be used to characterize socles which are minimal two-sided ideals. Firstly, however, we will prove some related results.

The next result was obtained by G. Braatvedt and R. Brits in [5]. We state it together with a short new proof based on the spectral trace:

Theorem 2.5.14. [5, Corollary 2.3] *Let N be an arbitrary nonempty open subset of A and let $a, b \in A$. If $\sigma(ax)$ and $\sigma(bx)$ are finite and equal for all $x \in N$, then $a = b$.*

Proof. By the remark following the statement of the Scarcity Theorem on p. 44, we may infer that a and b both have finite rank. Furthermore, since $E(a)$ and $E(b)$ are both open dense subsets of A, it readily follows that $E(a) \cap E(b)$ is a dense subset of A. Consequently, we can find an $x_0 \in N$ such that ax_0 and bx_0 are both maximal finite-rank elements. Let $y \in A$ be arbitrary but fixed, and define analytic functions from $\mathbb{C}$ into $\operatorname{Soc}(A)$ as follows:

$$f(\lambda) = a\left[(1-\lambda)\, x_0 + \lambda y\right] \text{ and } g(\lambda) = b\left[(1-\lambda)\, x_0 + \lambda y\right] \ (\lambda \in \mathbb{C}).$$

Since $(E(a) \cap E(b)) \cap N$ is a nonempty open set and x_0 belongs to this set, there exists a real number $\epsilon > 0$ such that for all $\lambda \in B(0, \epsilon)$ we have that

$f(\lambda)$ and $g(\lambda)$ are maximal finite-rank elements and $\sigma(f(\lambda)) = \sigma(g(\lambda))$. By the Diagonalization Theorem the functions

$$\lambda \mapsto \operatorname{Tr}(f(\lambda)) \text{ and } \lambda \mapsto \operatorname{Tr}(g(\lambda))$$

agree on $B(0,\epsilon)$. Thus, since these functions are entire by the properties of the trace (see p. 6), it must be the case that they agree on all of $\mathbb{C}$. With the particular value $\lambda = 1$ we get $\operatorname{Tr}(ay) = \operatorname{Tr}(by)$. Since $y \in A$ was arbitrary we conclude that $a = b$ by the properties of the trace (see p. 6). $\square$

From Theorem 2.5.14 it is clear that if $\sigma(ax)$ and $\sigma(bx)$ are finite and equal for all x in some nonempty open set N, then $\sigma(ax) = \sigma(bx)$ for all $x \in A$. In fact, we have the following result:

Lemma 2.5.15. *Let N be an arbitrary nonempty open subset of A and let $a, b \in A$. If $\sigma(bx)$ is finite and $\sigma'(ax) \subseteq \sigma'(bx)$ for all $x \in N$, then $\sigma'(ax) \subseteq \sigma'(bx)$ for all $x \in A$.*

Proof. The hypotheses allows us to infer that both a and b are finite-rank elements. Recall that $E(a)$ and $E(b)$ are open and dense in A. Hence, $E(a) \cap E(b)$ is open and dense in A. Fix any $x_0 \in (E(a) \cap E(b)) \cap N$ and let $x \in A$ be arbitrary. Define the following analytic functions from $\mathbb{C}$ into $\operatorname{Soc}(A)$:

$$f(\lambda) = a[(1-\lambda)x_0 + \lambda x] \text{ and } g(\lambda) = b[(1-\lambda)x_0 + \lambda x] \quad (\lambda \in \mathbb{C}).$$

Let $\operatorname{rank}(a) = k$ and $\operatorname{rank}(b) = n$ and note that $k \leq n$ (since $(E(a) \cap E(b)) \cap N \neq \emptyset$). By the Scarcity Theorem there exist two closed and discrete subsets of $\mathbb{C}$, say F_a and F_b, such that $\#\sigma'(f(\lambda)) = k$ for all $\lambda \in \mathbb{C} - F_a$ and $\#\sigma'(g(\lambda)) = n$ for all $\lambda \in \mathbb{C} - F_b$. Moreover, by the Scarcity Theorem, our choice of x_0, and the definitions of f and g, there exists a real number $\epsilon > 0$ such that for all $\lambda \in B(0,\epsilon)$,

$$\sigma'(f(\lambda)) = \{\alpha_1(\lambda), \ldots, \alpha_k(\lambda)\}, \; \sigma'(g(\lambda)) = \{\gamma_1(\lambda), \ldots, \gamma_n(\lambda)\},$$

$\sigma'(f(\lambda)) \subseteq \sigma'(g(\lambda))$, and the α_i's and γ_i's are all holomorphic on $B(0,\epsilon)$. Let $i \in \{1, \ldots, k\}$ be arbitrary but fixed. We claim that $\alpha_i = \gamma_j$ for some $j \in \{1, \ldots, n\}$: Fix any β_0 in $B(0,\epsilon)$ and let (λ_m) be any sequence in $B(0,\epsilon) - \{\beta_0\}$ which converges to β_0. Since $\sigma'(f(\lambda)) \subseteq \sigma'(g(\lambda))$ for each $\lambda \in B(0,\epsilon)$, it follows that $\alpha_i(\lambda_m) = \gamma_{j_m}(\lambda_m)$ for some $j_m \in \{1, \ldots, n\}$. However, by the Pigeon Hole Principle we may infer the existence of a subsequence, denoted by (λ_m) for convenience, and a $j \in \{1, \ldots, n\}$ such that $\alpha_i(\lambda_m) = \gamma_j(\lambda_m)$. However, then the set $\{\lambda \in B(0,\epsilon) : \alpha_i(\lambda) - \gamma_j(\lambda) = 0\}$

contains a limit point. So, from elementary Complex Analysis we conclude that $\alpha_i = \gamma_j$. This proves our claim. Without loss of generality we may therefore assume that $\sigma'(f(\lambda)) = \{\gamma_1(\lambda), \ldots, \gamma_k(\lambda)\}$ for each $\lambda \in B(0,\epsilon)$. Pick any $\lambda_0 \in \partial B(0,\epsilon) \cap [\mathbb{C} - (F_a \cup F_b)]$ (which exists since F_a and F_b are discrete), and let $z \in \mathbb{C} - (F_a \cup F_b)$ be arbitrary. We claim that $\sigma'(f(z)) \subseteq \sigma'(g(z))$: Since F_a and F_b are discrete, we can find a path Γ in $\mathbb{C} - (F_a \cup F_b)$ which connects λ_0 and z. Now, for each $\lambda \in \Gamma$, there exists a nonempty open disk $B_\lambda := B(\lambda, r_\lambda)$ such that for $\beta \in B_\lambda$,

$$\sigma'(f(\beta)) = \left\{\alpha_1^{(\lambda)}(\beta), \ldots, \alpha_k^{(\lambda)}(\beta)\right\} \text{ and } \sigma'(g(\beta)) = \left\{\gamma_1^{(\lambda)}(\beta), \ldots, \gamma_n^{(\lambda)}(\beta)\right\},$$

where the $\alpha_i^{(\lambda)}$'s and $\gamma_i^{(\lambda)}$'s are all holomorphic on B_λ. By compactness we can find $\lambda_1, \ldots, \lambda_m \in \Gamma$ such that $B_{\lambda_i} \cap B_{\lambda_{i+1}} \neq \emptyset$ for $i \in \{0, \ldots, m-1\}$ and $B_{\lambda_m} \cap B_z \neq \emptyset$. Now, observe that

$$\sigma'(f(\beta)) = \{\gamma_1(\beta), \ldots, \gamma_k(\beta)\} \text{ and } \sigma'(g(\beta)) = \{\gamma_1(\beta), \ldots, \gamma_n(\beta)\}$$

for each $\beta \in B(0,\epsilon) \cap B_{\lambda_0}$. Since $B(0,\epsilon) \cap B_{\lambda_0}$ is a nonempty open set, it follows in a similar way as before that

$$\sigma'(f(\beta)) = \left\{\gamma_1^{(\lambda_0)}(\beta), \ldots, \gamma_k^{(\lambda_0)}(\beta)\right\}$$

for each $\beta \in B_{\lambda_0}$. Hence, $\sigma'(f(\beta)) \subseteq \sigma'(g(\beta))$ for each $\beta \in B_{\lambda_0}$. Repeating this argument with the chain of intersecting open disks we may conclude that $\sigma'(f(\beta)) \subseteq \sigma'(g(\beta))$ for each $\beta \in B_z$. This proves our claim. Since

$$z \in \mathbb{C} - (F_a \cup F_b)$$

was arbitrary, $\sigma'(f(z)) \subseteq \sigma'(g(z))$ for all $z \in \mathbb{C} - (F_a \cup F_b)$. Thus, by a straightforward argument, using the upper semicontinuity of the spectrum and Newburgh's Theorem [1, Theorem 3.4.4], we may conclude that the spectral containment extends to all of $\mathbb{C}$. Hence,

$$\sigma'(ax) = \sigma'(f(1)) \subseteq \sigma'(g(1)) = \sigma'(bx).$$

Since $x \in A$ was arbitrary, this establishes the result. $\square$

The Jacobson radical formula is really only a particular case of a more general type of spectral calculus: Suppose $\sigma(bx)$ is finite for all $x \in A$. If for each $x \in A$ we have that $\sigma'(ax)$ is a portion of $\sigma'(bx)$, then "a is a portion of b" in the following sense:

Theorem 2.5.16. *Let N be an arbitrary nonempty open subset of A and let $a, b \in A$. If $\sigma(bx)$ is finite for each $x \in N$, and $\sigma'(ax) \subseteq \sigma'(bx)$ for each $x \in N$ then a commutes with b and, either $a = 0$, or there exist rank one elements $a_1, \ldots, a_n$, and $k \leq n$ such that*

$$a = a_1 + \cdots + a_k \text{ and } b = a_1 + \cdots + a_n.$$

Moreover, a is orthogonal to $b - a$.

Proof. As before, by the hypotheses above, it follows that both a and b have finite rank. Moreover, by Lemma 2.5.15 it follows that the spectral containment assumption "for all $x \in N$" may be replaced by "for all $x \in A$". Now, if $\sigma(ax) = \{0\}$ for all $x \in A$, then by the semisimplicity of A we may infer that $a = 0$. We may therefore assume that $a \neq 0$ and conclude that $\operatorname{rank}(b) = n \geq 1$. Recall that $E(a) \cap E(b)$ is an open dense subset of A since $E(a)$ and $E(b)$ are both open and dense. Further, since $\sigma'(ax) \subseteq \sigma'(bx)$ for each $x \in A$, it follows in particular that $\operatorname{rank}(a) \leq \operatorname{rank}(b)$. Since $G(A)$ is open and $E(a) \cap E(b)$ is dense, we can fix an $x \in (E(a) \cap E(b)) \cap G(A)$. By the Diagonalization Theorem and our hypothesis on the spectrums of ax and bx, we can find n mutually orthogonal rank one projections $p_1, \ldots, p_n$, k mutually orthogonal rank one projections $q_1, \ldots, q_k$ (with $k \leq n$), and nonzero complex numbers $\alpha_1, \ldots, \alpha_n$ such that

$$bx = \alpha_1 p_1 + \cdots + \alpha_n p_n \text{ and } ax = \alpha_1 q_1 + \cdots + \alpha_k q_k. \tag{2.5.2}$$

Set $b' = bx$ and $a' = ax$. Let p be any rank one projection such that $a'p \neq 0$. Then $a'p$ has rank one. Moreover, by the containment above and the fact that $E(a'p)$ is dense, it follows that $\sigma'(a'py) = \sigma'(b'py)$ for all y in a dense subset of A. Thus, $\operatorname{Tr}(a'py) = \operatorname{Tr}(b'py)$ for all y in a dense subset of A. However, by [16, Lemma 2.3] the trace is continuous on the set of rank one elements. Hence, $\operatorname{Tr}(a'py) = \operatorname{Tr}(b'py)$ for all $y \in A$, and so, $a'p = b'p$ by the properties of the trace (see p. 6). A similar statement is valid for multiplication on the left. We shall use this to show that $q_j = p_j$ for each $j \in \{1, \ldots, k\}$. For $j \in \{1, \ldots, k\}$ we have (see p. 7)

$$q_j = \frac{1}{2\pi i} \int_{\Gamma_j} (\lambda \mathbf{1} - a')^{-1} \, d\lambda \tag{2.5.3}$$

and

$$p_j = \frac{1}{2\pi i} \int_{\Gamma_j} (\lambda \mathbf{1} - b')^{-1} \, d\lambda, \tag{2.5.4}$$

where Γ_j is a small circle surrounding α_j and separating it from 0 and the remaining spectrum of b'. From (2.5.2) it follows that $q_j a' = a' q_j \neq 0$ so, by the preceding paragraph, we have $q_j b' = q_j a'$ and $b' q_j = a' q_j$ and hence that

$$q_j p_j = \frac{1}{2\pi i}\int_{\Gamma_j} q_j\left(\lambda\mathbf{1} - b'\right)^{-1} d\lambda = \frac{1}{2\pi i}\int_{\Gamma_j} q_j\left(\lambda\mathbf{1} - a'\right)^{-1} d\lambda = q_j^2 = q_j, \tag{2.5.5}$$

and similarly $p_j q_j = q_j$. Now, if $p_j a' = 0$ then

$$p_j q_j = \frac{1}{2\pi i}\int_{\Gamma_j} p_j\left(\lambda\mathbf{1} - a'\right)^{-1} d\lambda = \frac{1}{2\pi i}\int_{\Gamma_j} \frac{p_j}{\lambda}\, d\lambda = 0$$

which contradicts the first calculation that $p_j q_j = q_j \neq 0$. Thus, $p_j a' \neq 0$ from which we have $p_j a' = p_j b'$. From a similar argument we have $a' p_j = b' p_j$. As in (2.5.5), but now using (2.5.3), we have $p_j q_j = p_j = q_j p_j$. Hence, $ax = \alpha_1 p_1 + \cdots + \alpha_k p_k$. Since x is invertible we can solve for a and b in (2.5.2) and our result follows with $a_j = \alpha_j p_j x^{-1}$. Now, since $E(a) \cap E(b)$ is dense and open in A we can find a sequence $(x_n) \subseteq E(a) \cap E(b)$ such that $x_n \to \mathbf{1}$ as $n \to \infty$. But for each x_n the first part of the proof shows that

$$ax_n\left(bx_n - ax_n\right) = \left(bx_n - ax_n\right)ax_n = 0.$$

So in the limit we obtain $a(b-a) = (b-a)a = 0$ and hence also $ab = ba$. □

It is immediate from the above result that if we add to the assumptions the requirement that $\operatorname{rank}(a) = \operatorname{rank}(b)$, then $a = b$. With the hypothesis of Theorem 2.5.16 a inherits analytic properties from b. Firstly, however, we recall a lemma from [5] which will be useful to prove Corollary 2.5.18 and Theorem 2.5.19:

Lemma 2.5.17. [5, Theorem 2.1] *If $b \in G(A)$ and $\#\sigma(ax) \leq \#\sigma(bx)$ for all x in some neighbourhood N of b^{-1}, then $ab^{-1} = \alpha\mathbf{1}$ for some $\alpha \in \mathbb{C}$.*

Corollary 2.5.18. *Suppose a and b satisfy the hypothesis of Theorem 2.5.16 and that $a \neq 0$. If $f(\lambda)$ is holomorphic on a domain D containing $\sigma(b)$ and $f(b) = 0$, then also $f(a) = 0$ and $f(b-a) = 0$. In particular, if b is a projection then so is a.*

Proof. If b is invertible, then by Lemma 2.5.15 and Lemma 2.5.17 we have $a = \alpha b$ for some $\alpha \in \mathbb{C}$. So $\operatorname{rank}(a) = \operatorname{rank}(b)$, and the comment following Theorem 2.5.16 readily yields $a = b$. We may therefore assume that $b \notin G(A)$ and that $\operatorname{rank}(a) < \operatorname{rank}(b) = n \neq 0$. Moreover, we may also assume that f is not identically 0. By hypothesis and the Spectral Mapping Theorem, $f(\sigma(b)) = \sigma(f(b)) = \{0\}$. Hence, f has zeroes at the spectral points of b. By [9, Corollary 4.3.9] there exists a polynomial $h(\lambda)$ without a constant term and a holomorphic function $g(\lambda)$ on D such that $f(\lambda) = h(\lambda)\, g(\lambda)$ and $g(\alpha) \neq 0$ for all $\alpha \in \sigma(b)$. In particular then, $g(b)$ is invertible by the Spectral Mapping Theorem. Hence, since $0 = f(b) = h(b)g(b)$ (by the Holomorphic Functional Calculus), we have $h(b) = 0$. Since a and $b - a$ are orthogonal, it follows that $h(a) = -h(b-a)$. For the sake of a contradiction suppose that $h(a)$ and $h(b-a)$ are not zero. Then since $a = a_1 + \cdots + a_k$ and $b - a = a_{k+1} + \cdots + a_n$ by Theorem 2.5.16, it follows that there is a largest integer $k+1 \leq m \leq n$ and $x_1, \ldots, x_m \in A$ such that

$$0 \neq a_m x_m = a_1 x_1 + \cdots + a_{m-1} x_{m-1}.$$

Since a_m has rank one the minimal right ideal $a_m A = a_m x_m A$ which shows that $a_m \in a_1 A + \cdots + a_{m-1} A$. However, by the subadditivity of the rank we then obtain that $\operatorname{rank}(b) < n$ which is absurd. Thus, $h(a) = h(b-a) = 0$ and the result follows from the Holomorphic Functional Calculus. The last part of the statement is obvious. □

The next result is similar in spirit to Theorem 2.5.16:

Theorem 2.5.19. *Let p be a projection of A, let $q \in A$, and suppose there exist a neighbourhood N_p of p and a neighbourhood $N_{\mathbf{1}-p}$ of $\mathbf{1} - p$ such that*

$$\#\sigma(qx) = \#\sigma(px) < \infty \text{ for all } x \in N_p \cup N_{\mathbf{1}-p}.$$

Then q is a scalar multiple of p or q is a scalar multiple of the identity.

Proof. If $p = \mathbf{1}$, then by Lemma 2.5.17 q is a scalar multiple of the identity. If $p = 0$ and $q \notin G(A)$, then $q \in \operatorname{Rad}(A) = \{0\}$. If $p = 0$ and $q \in G(A)$, then for all x in a neighbourhood N_0 of 0 we have that $\#\sigma(qx) = 1$ which by the Scarcity Theorem implies that every element of A has one point spectrum. Thus, since A is semisimple, $A \cong \mathbb{C}$ and hence q is a scalar multiple of the identity. So assume that p is neither 0 nor $\mathbf{1}$, and that q is not a scalar multiple of the identity. The hypothesis implies that, for all x in some neighbourhood of $\mathbf{1}$, say N_1, we have $\#\sigma_A(qpxp) = \#\sigma_A(pxp)$. Moreover, since $yqpxp \in G(A)$ or $pxp \in G(A)$ implies $p = \mathbf{1}$ which contradicts our hypothesis on p, it follows that 0 belongs to both $\sigma_A(yqpxp)$ and $\sigma_A(pxp)$ for all $x, y \in A$. Hence, by Jacobson's Lemma we may infer that $\#\sigma_A((pqp)(pxp)) = \#\sigma_A(pxp)$ for all $x \in N_1$. So it follows that

$$\#\sigma'_{pAp}((pqp)(pxp)) = \#\sigma'_{pAp}(p(pxp)) \text{ when } x \in N_1.$$

Applying the Open Mapping Theorem to the continuous linear operator $x \mapsto pxp$ from A onto pAp we have that

$$\#\sigma'_{pAp}((pqp)(pxp)) = \#\sigma'_{pAp}(p(pxp))$$

for all pxp in some neighbourhood of p in pAp. Hence, by the density of $E_{pAp}(pqp)$ and $E_{pAp}(p)$ in pAp we may conclude that

$$\mathrm{rank}_{pAp}(pqp) = \mathrm{rank}_{pAp}(p).$$

Whence, since pAp is finite-dimensional, it follows that $pqp \in G(pAp)$. Thus,

$$\#\sigma_{pAp}\left((pqp)(pxp)\right) = \#\sigma_{pAp}\left(p(pxp)\right)$$

for all pxp in some neighbourhood of p in pAp. Hence, by Lemma 2.5.17 it follows that $pqp = \alpha p$ for some $\alpha \in \mathbb{C}$. On the other hand, using the hypothesis with the neighbourhood $N_{\mathbf{1}-p}$ and the fact that q is not a scalar multiple of the identity, it follows, for all x in some neighbourhood of $\mathbf{1}$, that

$$\sigma\left(q\left(\mathbf{1}-p\right)x\right) = \sigma\left(\left(\mathbf{1}-p\right)qx\right) = \{0\}.$$

Hence, by the Scarcity Theorem and the semisimplicity of A we get $q = pq = qp$. Therefore, $q = \alpha p$, which completes the proof. □

Theorem 2.5.20. Soc(A) *is a minimal two-sided ideal if and only if the following are equivalent for any* $0 \neq a \in A$ *and* $0 \neq b \in \mathrm{Soc}(A)$*:*

(i) $\sigma'(ax) \subseteq \sigma'(bx)$ *for all* x *in some nonempty open set* N.

(ii) $\sigma'(ax) \subseteq \sigma'(bx)$ *for all* $x \in A$.

(iii) $a = b$.

Proof. Suppose first that Soc(A) is a minimal two-sided ideal and let $0 \neq a \in A$ and $b \in \mathrm{Soc}(A)$. Obviously (iii) ⇒ (i). Moreover, by Lemma 2.5.15, (i) ⇒ (ii). So assume that condition (ii) holds. Since (ii) implies that $\rho(ax) \leq \rho(bx)$ for all $x \in A$, it readily follows from Theorem 2.5.7 and hypothesis that $a = \lambda b$ for some $\lambda \in \mathbb{C} - \{0\}$. Hence, $\mathrm{rank}\,(a) = \mathrm{rank}\,(b)$, and so, by Theorem 2.5.16 and the remark following it, $a = b$. This proves the forward implication. For the other direction, we argue contrapositively. Suppose that Soc(A) is not a minimal two-sided ideal. Then by Lemma 1.2.1 and Lemma 1.2.2 we may infer the existence of two rank one projections p and q such that $J_pJ_q = J_qJ_p = \{0\}$. However, as in the proof of Theorem 2.5.7 this implies that $p \neq p + q$ and $\sigma'(px) \subseteq \sigma'\left(\left(p+q\right)x\right)$ for all $x \in A$. Hence, (ii) $\not\Rightarrow$ (iii), which establishes the result. □

Moreover, we obtain a similar characterization of prime Banach algebras as was done in Theorem 2.5.11:

Theorem 2.5.21. *Suppose that* $\mathrm{Soc}(A) \neq \{0\}$*. Then* A *is prime if and only if for any* $a, b \in A - \{0\}$ *we have that the following are equivalent:*

(i) $\sigma'(ax) \subseteq \sigma'(bx)$ *for all* $x \in A$.

(ii) $a = b$.

Proof. If A is not prime then we may proceed as in the proof of Lemma 2.5.8 and expose two elements a and b such that $a \neq a+b$ and $\sigma'(ax) \subseteq \sigma'((a+b)x)$ for all $x \in A$. This proves the reverse implication. Conversely, if A is prime, then since $\mathrm{Soc}(A) \neq \{0\}$ it follows that $\mathrm{Soc}(A)$ is a minimal two-sided ideal and that $l(\mathrm{Soc}(A)) = \{0\}$. Let $a, b \in A - \{0\}$ be arbitrary. Obviously (ii) $\Rightarrow$ (i). So assume that condition (i) holds and let $y \in \mathrm{Soc}(A)$ be arbitrary but fixed. Then by Theorem 2.5.20 we may infer that $ay = by$. Since $y \in \mathrm{Soc}(A)$ was arbitrary, we conclude that $\mathrm{Tr}\,((a-b)y) = 0$ for all $y \in \mathrm{Soc}(A)$. Hence, from the properties of the trace (see p. 6) it follows that $a - b \in l(\mathrm{Soc}(A)) = \{0\}$. Therefore, (i) $\Rightarrow$ (ii), so the theorem is true. $\square$

To conclude we will show that if $\mathrm{Soc}(A)$ is a minimal two-sided ideal, then conditions (i) and (ii) in Theorem 2.5.21 are equivalent whenever b belongs to some *inessential ideal*; that is, a two-sided ideal in which the spectrum of all elements contain at most 0 as an accumulation point. Before that, however, we will need a little preparation:

Lemma 2.5.22. *Let* $s \in A$ *and for each* $x \in A$ *suppose that* $\sigma(sx)$ *contains at most* 0 *as an accumulation point for all* $x \in A$*. Then the Riesz projections of* s *corresponding to nonzero spectral values have finite rank.*

Proof. Let $\sigma'(s) = \{\lambda_1, \lambda_2, \dots\}$ and set, for $i \in \mathbb{N}$, $p := p(\lambda_i, s)$. Recall that pAp is a semisimple Banach algebra with identity p. There exists an open neighborhood V of $\mathbf{1}$ in A such that pxp is invertible in pAp for each $x \in V$. Now suppose $x \in V$ and $\#\sigma_A(px) = \infty$. Then, by Jacobson's Lemma, $\#\sigma_A(pxp) = \infty = \#\sigma'_A(pxp)$, and, since $p \in sA$, it follows from our hypothesis on s that $\sigma'_A(pxp)$ is a sequence converging to 0. But this means $\sigma_{pAp}(pxp)$ contains a sequence converging to zero, from which it follows (since the spectrum is closed) that pxp cannot be invertible in pAp giving a contradiction. So $\#\sigma_A(px) < \infty$ for all $x \in V$ and a standard application of the Scarcity Theorem then says $\#\sigma_A(px) < \infty$ for all $x \in A$. Thus $\mathrm{rank}\,(p) < \infty$. $\square$

Theorem 2.5.23. *Suppose that* $\mathrm{Soc}(A)$ *is a minimal two-sided ideal. Let* $0 \neq a \in A$ *and let* $0 \neq b \in A$ *such that* $\sigma(bx)$ *has at most* 0 *as an accumulation point for all* $x \in A$*. Then the following are equivalent:*

(i) $\sigma'(ax) \subseteq \sigma'(bx)$ *for all* $x \in A$.

(ii) $a = b$.

Proof. Let $0 \neq a \in A$ and $b \in A$. Surely, (ii) $\Rightarrow$ (i). So assume that condition (i) holds. We claim that $\sigma(ax) = \sigma(bx)$ for all $x \in A$: Let $x \in A$ be arbitrary. It will suffice to show that $\sigma'(ax) = \sigma'(bx)$ and $0 \in \sigma(ax) \Leftrightarrow 0 \in \sigma(bx)$. If $\sigma(bx) = \{0\}$, then $\sigma'(ax) = \sigma'(bx) = \emptyset$. So assume that $\sigma(bx) \neq \{0\}$ and let $\lambda \in \sigma'(bx)$. Since $\sigma'(bx)$ is either finite or a sequence converging to zero, we may consider the Riesz projection of bx associated with λ, say $p := p(\lambda, bx)$. Now, by Lemma 2.5.22 it follows that $p \in \mathrm{Soc}(A)$. Consequently, by hypothesis and Theorem 2.5.20, we may infer that $axp = bxp$. Moreover, since $bxp = pbx$, by condition (i), Jacobson's Lemma and Theorem 2.5.20 it follows that $pax = pbx = axp$. Hence,

$$(ax(\mathbf{1} - p))(axp) = (axp)(ax(\mathbf{1} - p)) = 0.$$

Thus, since $ax = ax(\mathbf{1} - p) + axp$, it follows from Lemma 2.5.2 that

$$\sigma'(ax) = \sigma'(ax(\mathbf{1} - p)) \cup \sigma'(axp) = \sigma'(ax(\mathbf{1} - p)) \cup \sigma'(bxp).$$

But by the Holomorphic Functional Calculus it follows that $\sigma'(bxp) = \{\lambda\}$. Hence, $\lambda \in \sigma'(ax)$. This shows that $\sigma'(ax) = \sigma'(bx)$. Suppose now that $0 \notin \sigma(bx)$. Then, by hypothesis on b it must be the case that $\sigma(bx)$ is finite, say $\sigma(bx) = \{\alpha_1, \ldots, \alpha_r\}$. For each $i \in \{1, \ldots, r\}$, let p_i denote the Riesz projection of bx associated with α_i. By condition (i) and Theorem 2.5.20 it follows that $axp_i = bxp_i$ for all $i \in \{1, \ldots, r\}$. But by the Holomorphic Functional Calculus $p_1 + \cdots + p_r = \mathbf{1}$. Hence,

$$\begin{aligned} ax &= ax(p_1 + \cdots + p_r) = axp_1 + \cdots + axp_r \\ &= bxp_1 + \cdots + bxp_r = bx(p_1 + \cdots + p_r) = bx. \end{aligned}$$

So, $0 \notin \sigma(ax)$. Similarly, $0 \notin \sigma(ax)$ yields $bx = ax$ and consequently $0 \notin \sigma(bx)$. Hence, $0 \in \sigma(ax) \Leftrightarrow 0 \in \sigma(bx)$. This proves our claim. By Theorem 2.5.1 we may therefore conclude that $a = b$, which completes the proof. $\square$

Chapter 3

Shoda-Completion via Projective Tensor Product Extensions

3.1 Motivation

Throughout this chapter we shall assume that A has a nontrivial socle, i.e. $\mathrm{Soc}(A) \neq \{0\}$. Moreover, we shall denote by $G_1(A)$ the *principal component* of $G(A)$, that is, the connected component of $G(A)$ containing $\mathbf{1}$. By [1, Theorem 3.3.7] it follows that

$$G_1(A) = \{e^{x_1} \cdots e^{x_n} : x_1, \ldots, x_n \in A, n \geq 1 \text{ an integer}\}.$$

We also note here that $G_1(A)$ is open in A. Central to our discussion in this chapter will be the following:

Suppose that $\mathrm{Soc}(A) \neq \{0\}$. Then A is called *Shoda-complete* if every traceless element of $\mathrm{Soc}(A)$ can be expressed as a commutator of two elements belonging to $\mathrm{Soc}(A)$.

Chapter 2 contains a number of characteristic properties of Shoda-complete Banach algebras, some of which will be useful in the current chapter. We now proceed to motivate the notion of Shoda-completeness:

The canonical example of a Shoda-complete Banach algebra is $B(X)$, but many other Banach algebras do not have this property. For instance, consider

$$A := M_n(\mathbb{C}) \oplus M_k(\mathbb{C}), \quad (n, k \in \mathbb{N}),$$

where A is a Banach algebra under pointwise operations. It is not hard to find traceless elements of A which cannot be expressed as the commutator of two elements of A; so A is not Shoda-complete. But the algebra $B := M_{n+k}(\mathbb{C})$, which is Shoda-complete, contains an algebraically isomorphic copy of A. In fact, with a little effort, one may show that B is the smallest semisimple extension of A such that every traceless element of A becomes a commutator. By the classical Wedderburn-Artin Theorem we therefore know how to extend a finite dimensional algebra to a Shoda-complete algebra. The aim of this chapter is to show that this can be done for Banach algebras with infinite dimensional socles as well. However, it should be emphasized that it will be desirable that the Shoda-completion preserves the socle in the following sense: The image of socle elements in A must be socle elements in the Shoda-completion of A. This immediately rules out the option of simply embedding A in $B(A)$ (see the remark after Theorem 3.2.5). To address the problem of finding such a Shoda-completion, we start in the next section on minimal ideals, and obtain some new results which will be used in the forthcoming construction; almost all of these statements are of independent interest as well. From here we travel along the familiar road of extensions: In Section 3.3 we algebraically extend A to an algebra which is Shoda-complete. Then, in Section 3.4, we introduce a submultiplicative norm on the algebraic extension which extends the norm on A. The subsequent norm completion is unfortunately not necessarily semisimple. However, because of the location of the radical, the quotient algebra turns out to be a semisimple Banach algebra extension of A which is simultaneously Shoda-complete. Finally, in the last section of this chapter, we show how the particular extension is independent of the choice of rank one projection representatives generating $\mathrm{Soc}(A)$.

3.2 Minimal Ideals

In particular, for some fixed set of projections $\mathcal{P}$ generating $\mathrm{Soc}(A)$, Theorem 2.2.3 can be used to show that for each rank one element $a \in \mathrm{Soc}(A)$, there exists a unique $p \in \mathcal{P}$ such that $a \in J_p$. In fact, more can be said if a is a rank one projection:

Lemma 3.2.1. *If q is a rank one projection belonging to J_p then $q \in ApA$.*

Proof. It is well-known and easy to prove that if p and q are rank one projections then $\dim(qAp) \leq 1$ and $\dim(pAq) \leq 1$ (see for instance [13, Theorem 4.2]). But $q \in J_p$ forces $\dim(qAp) = 1 = \dim(pAq)$. It therefore follows that

$$q = \sum_{i=1}^{n} x_i p y_i \Rightarrow q = \sum_{i=1}^{n} (q x_i p)(p y_i q) \Rightarrow q = qxpyq \in ApA,$$

for some x and y in A. $\square$

In [18] J. Zemánek showed that the components of the set of projections of A are arcwise connected, and, moreover, that the component containing the projection $p \in A$ has the form $G_1(A)pG_1(A)$. These observations will prove useful in establishing the next two lemmas:

Lemma 3.2.2. *If q is a rank one projection belonging to J_p then*

$$q \in G_1(A)pG_1(A).$$

Proof. If $p = \mathbf{1}$, then $A = \mathbb{C}$. But then $q = p$ and we are done. We may therefore assume that $p \notin G(A)$. By Lemma 3.2.1 we have that $q = xpy$ for some $x, y \in A$. By the density of $E(xp)$ and the Diagonalization Theorem we can find a $z \in G_1(A)$ such that zxp is a rank one projection. Now consider the analytic function $f : \mathbb{C} \to Ap$ defined by

$$f(\lambda) = (1 - \lambda)p + \lambda zxp.$$

We observe that $\#\sigma(f(\lambda)) \leq 2$ for all $\lambda \in \mathbb{C}$. Moreover, $\#\sigma(f(\lambda)) = 2$ is actually attained for some $\lambda \in \mathbb{C}$. The Scarcity Theorem now says that the set

$$D = \{\lambda \in \mathbb{C} : \#\sigma(f(\lambda)) = 1\}$$

is closed and discrete in $\mathbb{C}$. If we notice that $D = \{\lambda \in \mathbb{C} : \mathrm{Tr}\,(f(\lambda)) = 0\}$, and we define $g : \mathbb{C} - D \to Ap$ by

$$g(\lambda) = \frac{f(\lambda)}{\mathrm{Tr}\,(f(\lambda))}$$

then it follows that we can find a path of projections connecting the two projections p and zxp. Thus, by Zemánek's result [18, Theorem 3.3] we infer that $zxp = vpv^{-1}$ for some $v \in G_1(A)$. In exactly the same manner we can find $\tilde{z} \in G_1(A)$ such that $py\tilde{z} = wpw^{-1}$ for some $w \in G_1(A)$. Together we see that $zq\tilde{z} \in G_1(A)pG_1(A)$. Since $G_1(A)$ is a group we can solve for q to get the result. □

Theorem 3.2.3. *A rank one projection q belongs to J_p if and only if q belongs to the similarity orbit of p, that is, $q \in E_p := \{upu^{-1} : u \in G_1(A)\}$.*

Proof. As before we may exclude the case where $p = \mathbf{1}$ and assume that $p \notin G(A)$. By Lemma 3.2.2 we can write

$$q = e^{x_1} \cdots e^{x_n} p e^{y_1} \cdots e^{y_k},$$

where $x_1, \ldots, x_n, y_1, \ldots, y_k \in A$. We now define the analytic function f from $\mathbb{C}$ into $\mathrm{Soc}(A)$ by

$$f(\lambda) = e^{\lambda x_1} \cdots e^{\lambda x_n} p e^{\lambda y_1} \cdots e^{\lambda y_k},$$

and we notice that for each $\lambda \in \mathbb{C}$, $f(\lambda)$ is a rank one element. As in the preceding lemma we observe $\#\sigma(f(\lambda)) \leq 2$ for all $\lambda \in \mathbb{C}$, and that $\#\sigma(f(\lambda)) = 2$ is actually attained for some $\lambda \in \mathbb{C}$. The Scarcity Theorem together with the argument in Lemma 3.2.2 guarantees the existence a path of projections connecting the two projections p and q. Hence, by Zemánek's result [18, Theorem 3.3] we obtain $q = upu^{-1}$ for some $u \in G_1(A)$. □

Theorem 3.2.4. *All minimal left ideals contained in J_p are isomorphic as Banach algebras. Similarly all minimal right ideals contained in J_p are isomorphic as Banach algebras.*

Proof. The minimal left ideal Ap is contained in J_p. Let I be any minimal left ideal in J_p. Then $I = Aq$ for some rank one projection q. Obviously $q \in J_p$ and so Lemma 3.2.3 says that $q = vpv^{-1}$ for some $v \in G_1(A)$. Since v is invertible we have

$$Aq = Avpv^{-1} = Apv^{-1} = vApv^{-1}.$$

Define $T_q : Ap \to Aq$ by $T(xp) = vxpv^{-1}$. Then T_q is obviously linear, T_q is surjective since $Aq = vApv^{-1}$, and T_q is injective since v is invertible. To show that T_q is multiplicative:

$$T_q((xp)(yp)) = vxpypv^{-1} = vxpv^{-1}vypv^{-1} = T_q(xp)T_q(yp).$$

This gives the result, and a similar argument works for minimal right ideals. □

Theorem 3.2.5. *Let $A = B(X)$, where X is a Banach space. If $P \in A$ is any rank one projection, then*

$$AP \cong X \text{ and } PA \cong X' \text{ (as Banach spaces).}$$

Proof. Since P has rank one, there exist $0 \neq f_P \in X'$ and $0 \neq u_P \in X$ such that $P(x) = f_P(x)u_P$ for each $x \in X$. From this it follows that for each $S \in A$ we have that $(SP)(x) = f_P(x)S(u_P)$ for $x \in X$. We are therefore naturally led to the map $\Omega : \mathcal{A}P \to X$ defined by $\Omega(SP) = (SP)(u_P)$. It is clear that Ω is linear and injective. To see that Ω is surjective, let $v \in X$ be arbitrary and simply pick $T \in A$, where $T(x) = f_P(x)v$ for each $x \in X$. Then $T(u_P) = v$, and the surjectivity follows. Furthermore, observe that

$$\|\Omega(SP)\| = \|(SP)(u_P)\| \leq \|u_P\| \|SP\|.$$

So it follows, from the Open Mapping Theorem, that $AP \cong X$ as Banach spaces. For the isomorphism $PA \cong X'$, we consider the map $\Gamma : PA \to X'$ defined by $\Gamma(PS) = f_P \circ PS$. It is trivial that Γ is linear and injective. To show that Γ is surjective, note that $f_P(u_P) = 1$ and let $S_f \in A$ be defined by $S_f(x) = f(x)u_P$ for each $x \in X$, where $f \in X'$. It then follows that $\Gamma(PS_f) = f$. To prove continuity, simply note that

$$\|\Gamma(PS)\| \leq \|f_P\| \|PS\|.$$

So the isomorphism $P\mathcal{A} \cong X'$ follows from the Open Mapping Theorem. □

Remark. Let X be any infinite Banach space, and consider $A := B(X) \oplus B(X)$, where the operations are all pointwise. In particular, we note that A is a semisimple Banach algebra with identity element. Fix any rank one projection P in $B(X)$. Then $(P, 0)$ and $(0, P)$ are both rank one projections of A. Moreover, $J_{(P,0)}$ and $J_{(0,P)}$ are mutually orthogonal minimal two-sided ideals. Consequently, $\mathrm{Soc}(A)$ is not Shoda-complete. Isometrically and algebraically embed A in $B(A)$ using right multiplication. Then $A(P, 0) \cong (X, 0)$ as Banach spaces by Theorem 3.2.5. Hence, $A(P, 0)$ is infinite-dimensional; so the image of $(P, 0)$ is not a socle element in $B(A)$. This shows that $B(A)$ is not a suitable candidate for a Shoda-completion of A.

Chapter 2 contains various algebraic and spectral characterizations of Shoda-completeness. Theorem 3.2.3 can be used to give a topological characterization:

Theorem 3.2.6. *A is Shoda-complete if and only if any one of the following holds:*

(i) *The collection of rank one projections is connected.*

(ii) *For each* $n \in \mathbb{N}$, $\mathcal{R}_n := \{a \in \mathrm{Soc}(A) : \mathrm{rank}\,(a) = n\}$ *is connected.*

(iii) $\mathcal{R}_1$ *is connected.*

Proof. (i) If A is Shoda complete then by Theorem 2.3.2 and Lemma 1.2.2 there exists a rank one projection p such that $\mathrm{Soc}(A) = J_p$. So any rank one projection q belongs to J_p, which, by Theorem 3.2.3, means that $q = upu^{-1}$ for some $u \in G_1(A)$. The result then follows from [18, Theorem 3.3]. Conversely, if the rank one projections is a connected space then, by [18, Theorem 3.3], there exists some rank one projection p such that any rank one projection q takes the form $q = upu^{-1}$ for some $u \in G_1(A)$. Thus, since $E(a)$ is dense for all $a \in \mathrm{Soc}(A)$, it follows from the Diagonalization Theorem that $\mathrm{Soc}(A) = J_p$. So $\mathrm{Soc}(A)$ is minimal by Lemma 1.2.2 and hence A is Shoda-complete by Theorem 2.3.2.
(ii) Let A be Shoda-complete and suppose $a, b \in \mathcal{R}_n$. Since $E(a) \cap E(b)$ is dense in A, we can pick a $u \in G_1(A)$ such that au and bu are maximal finite rank elements. By the Diagonalization Theorem we can write

$$bu = \beta_1 p_1 + \cdots + \beta_n p_n \text{ and } au = \alpha_1 q_1 + \cdots + \alpha_n q_n,$$

where the β_i are distinct nonzero complex numbers and the p_i are pairwise orthogonal rank one projections. A similar statement is valid for the α_i and the q_i. Fix $i \in \{1, \ldots, n\}$. Since A is Shoda-complete, $\mathrm{Soc}(A) = J_p$ for some rank one projection p. Thus, by Theorem 3.2.3 we can find a set $\{x_1, \ldots, x_k\} \subseteq A$ and a set $\{y_1, \ldots, y_l\} \subseteq A$ such that

$$p_i = e^{x_1} \cdots e^{x_k} p e^{-x_1} \cdots e^{-x_k} \text{ and } q_i = e^{y_1} \cdots e^{y_l} p e^{-y_1} \cdots e^{-y_l}.$$

Without loss of generality we may assume $k \geq l$. Define $f_i : \mathbb{C} \to A$ by

$$f_i(\lambda) = [(1-\lambda)\beta_i + \lambda\alpha_i] \left[\prod_{j=1}^{k} e^{(1-\lambda)x_j + \lambda y_j}\right] p \left[\prod_{j=1}^{k} e^{(\lambda-1)x_j - \lambda y_j}\right],$$

where $y_j = 0$ if $j > l$. Then define $g : \mathbb{C} \to A$ by

$$g(\lambda) = \sum_{i=1}^{n} f_i(\lambda).$$

The function g is analytic from $\mathbb{C}$ into $\mathrm{Soc}(A)$ and satisfies $g(0) = bu$, $g(1) = au$. Since the rank is subadditive we have that $\mathrm{rank}\,(g(\lambda)) \leq n$ for all $\lambda \in \mathbb{C}$ with $\mathrm{rank}\,(bu) = \mathrm{rank}\,(au) = n$. It follows from the Scarcity Theorem for

rank that $\{\lambda \in \mathbb{C} : \operatorname{rank}(g(\lambda)) < n\}$ is closed and discrete in $\mathbb{C}$. So there exists an arc $\mathcal{C}$ in A, each of whose members has rank n, connecting bu and au. Since $u \in G_1(A)$ a similar argument implies that b and bu (as well as a and au) can be connected by an arc consisting of rank n elements. This suffices to prove that $\mathcal{R}_n$ is connected. For the converse suppose that A is not Shoda-complete. Then we can find at least two distinct minimal two-sided ideals generated by rank one projections, say J_p and J_q, which are orthogonal and properly contained in $\operatorname{Soc}(A)$. We shall use this to show that $\mathcal{R}_1$ is not connected. It suffices to show that $J_p \cap \mathcal{R}_1$ is both open and closed in $\mathcal{R}_1$. To see that $J_p \cap \mathcal{R}_1$ is closed, let a_n be a sequence in $J_p \cap \mathcal{R}_1$ which converges to $a \in \mathcal{R}_1$. For the sake of a contradiction suppose that $a \notin J_p \cap \mathcal{R}_1$. Then $\lim a_n = a \in J_q$, where $J_p J_q = \{0\}$. Hence, for each $x \in A$ we have that $\lim a_n x a x = (ax)^2$. But $a_n x a x = 0$, for each n, means that $(ax)^2 = 0$ from which the Spectral Mapping Theorem together with the semisimplicity of A then imply that $a = 0 \notin \mathcal{R}_1$. Thus, $J_p \cap \mathcal{R}_1$ is closed in $\mathcal{R}_1$. Similarly, $(\operatorname{Soc}(A) - J_p) \cap \mathcal{R}_1$ is closed, and so, $J_p \cap \mathcal{R}_1$ is open in $\mathcal{R}_1$. Thus, if A is not Shoda-complete, then $\mathcal{R}_1$ is not connected.
(iii) In view of the preceding argument this is now trivial. $\square$

3.3 Algebraic Extension

Throughout this section $\mathcal{A}$ will denote an algebra over a scalar field K. In the ensuing discussion we will firstly show how to obtain a multiplication scheme for $\mathcal{A}\otimes\mathcal{A}$ by utilizing any linear functional f on $\mathcal{A}$. This will turn the vector space $\mathcal{A}\otimes\mathcal{A}$ into an algebra. We will then consider the specific case where $\mathcal{A} = \mathrm{Soc}(A)$ and $f = \mathrm{Tr}$, and use this to construct a Shoda Completion for A:

Let $\tau : \mathcal{A}\oplus\mathcal{A} \to \mathcal{A}\otimes\mathcal{A}$ be the tensor map, that is, let

$$\tau(c,d) = c\otimes d \quad (c,d\in\mathcal{A}),$$

and let f be a linear functional on $\mathcal{A}$. For $a,b\in\mathcal{A}$, we define $\psi_{a,b} : \mathcal{A}\oplus\mathcal{A}\to\mathcal{A}\otimes\mathcal{A}$ by

$$\psi_{a,b}(c,d) = f(bc)a\otimes d \quad (c,d\in\mathcal{A}).$$

By the linearity of f and the properties of the tensor product, it readily follows that $\psi_{a,b}$ is bilinear. Thus, by [4, Theorem 42.6] there exists a unique linear mapping $\phi_{a,b}$ such that $\psi_{a,b} = \phi_{a,b}\circ\tau$. Next we define $\Lambda : \mathcal{A}\oplus\mathcal{A}\to L(\mathcal{A}\otimes\mathcal{A})$ by

$$\Lambda(a,b) = \phi_{a,b} \quad (a,b\in\mathcal{A}),$$

where $L(\mathcal{A}\otimes\mathcal{A})$ denotes the algebra of linear operators on $\mathcal{A}\otimes\mathcal{A}$ into itself. We claim that Λ is bilinear: Fix $b\in\mathcal{A}$ and let $\lambda,\beta\in K$ and $a_1,a_2\in\mathcal{A}$. Then $\Lambda(\lambda a_1+\beta a_2, b) = \phi_{\lambda a_1+\beta a_2,b}$. By the linearity of the $\phi_{c,d}$'s and the properties of the tensor product, for any $\sum_{i=1}^n c_i\otimes d_i\in\mathcal{A}\otimes\mathcal{A}$, we have

$$\begin{aligned}
\phi_{\lambda a_1+\beta a_2,b}\left(\sum_{i=1}^n c_i\otimes d_i\right) &= \sum_{i=1}^n \phi_{\lambda a_1+\beta a_2,b}(c_i\otimes d_i)\\
&= \sum_{i=1}^n f(bc_i)(\lambda a_1+\beta a_2)\otimes d_i\\
&= \sum_{i=1}^n \left[\lambda f(bc_i)(a_1\otimes d_i)+\beta f(bc_i)(a_2\otimes d_i)\right]\\
&= \sum_{i=1}^n \lambda\phi_{a_1,b}(c_i\otimes d_i)+\sum_{i=1}^n \beta\phi_{a_2,b}(c_i\otimes d_i)\\
&= \lambda\phi_{a_1,b}\left(\sum_{i=1}^n c_i\otimes d_i\right)+\beta\phi_{a_2,b}\left(\sum_{i=1}^n c_i\otimes d_i\right).
\end{aligned}$$

This shows that $\phi_{\lambda a_1+\beta a_2,b} = \lambda\phi_{a_1,b}+\beta\phi_{a_2,b}$. Hence,

$$\Lambda(\lambda a_1+\beta a_2,b) = \lambda\Lambda(a_1,b)+\beta\Lambda(a_2,b).$$

Consequently, $a \mapsto \Lambda(a,b)$ is linear. Next we fix $a \in \mathcal{A}$ and we take $\lambda, \beta \in K$ and $b_1, b_2 \in \mathcal{A}$. Then $\Lambda(a, \lambda b_1 + \beta b_2) = \phi_{a,\lambda b_1+\beta b_2}$. Moreover, by the linearity of f, the linearity of the $\phi_{c,d}$'s, and the properties of the tensor product, for any $\sum_{i=1}^{n} c_i \otimes d_i \in \mathcal{A} \otimes \mathcal{A}$, we obtain

$$\begin{aligned}
\phi_{a,\lambda b_1+\beta b_2}\left(\sum_{i=1}^{n} c_i \otimes d_i\right) &= \sum_{i=1}^{n} \phi_{a,\lambda b_1+\beta b_2}(c_i \otimes d_i) \\
&= \sum_{i=1}^{n} f\left((\lambda b_1 + \beta b_2) c_i\right) a \otimes d_i \\
&= \sum_{i=1}^{n} \left[\lambda f(b_1 c_i) + \beta f(b_2 c_i)\right] a \otimes d_i \\
&= \sum_{i=1}^{n} \lambda f(b_1 c_i) a \otimes d_i + \sum_{i=1}^{n} \beta f(b_2 c_i) a \otimes d_i \\
&= \sum_{i=1}^{n} \lambda \phi_{a,b_1}(c_i \otimes d_i) + \sum_{i=1}^{n} \beta \phi_{a,b_2}(c_i \otimes d_i) \\
&= \lambda \phi_{a,b_1}\left(\sum_{i=1}^{n} c_i \otimes d_i\right) + \beta \phi_{a,b_2}\left(\sum_{i=1}^{n} c_i \otimes d_i\right).
\end{aligned}$$

This shows that $\phi_{a,\lambda b_1+\beta b_2} = \lambda \phi_{a,b_1} + \beta \phi_{a,b_2}$, and so,

$$\Lambda(a, \lambda b_1 + \beta b_2) = \lambda \Lambda(a, b_1) + \beta \Lambda(a, b_2).$$

Thus, $b \mapsto \Lambda(a,b)$ is also linear, so Λ is bilinear as advertised. Hence, by [4, Theorem 42.6] we may infer the existence of a unique linear mapping γ such that $\Lambda = \gamma \circ \tau$.

The next step is to use γ to define a possible multiplication scheme for $\mathcal{A} \otimes \mathcal{A}$. We propose the following definition:

$$uv = \gamma(u)v \quad (u, v \in \mathcal{A} \otimes \mathcal{A}).$$

The operation above is well-defined: Indeed, if $u = u'$ and $v = v'$, then since γ is a function $\gamma(u) = \gamma(u')$. Moreover, since $\gamma(u)$ is a linear operator, $\gamma(u)v = \gamma(u)v'$. Thus, we readily obtain

$$uv = \gamma(u)v = \gamma(u)v' = \gamma(u')v' = u'v'.$$

Next we observe that for elementary tensors we have

$$\begin{aligned}
(a \otimes b)(c \otimes d) &= \gamma(a \otimes b)(c \otimes d) \\
&= \phi_{a,b}(c \otimes d) \\
&= f(bc) a \otimes d.
\end{aligned}$$

We now proceed to show that the definition above satisfies the axioms of multiplication: Let $u, v, w \in \mathcal{A} \otimes \mathcal{A}$ and let $\alpha \in K$. By the linearity of γ and $\gamma(u)$ it readily follows that $\alpha(uv) = (\alpha u)\, v = x\,(\alpha v)$. So the multiplication scheme satisfies associativity with scalars. Furthermore, we have

$$u\,(v+w) = \gamma(u)\,(v+w) = \gamma(u)v + \gamma(u)w = uv + uw$$

and

$$(v+w)\,u = \gamma\,(v+w)\,(u) = [\gamma(v)+\gamma(w)]\,(u) = \gamma(v)u + \gamma(w)u = vu + wu,$$

so the multiplication scheme is distributive over addition. Now, for elementary tensors we have

$$\begin{aligned} [(a_1 \otimes b_1)\,(a_2 \otimes b_2)]\,(a_3 \otimes b_3) &= [f\,(b_1 a_2)\, a_1 \otimes b_2]\,(a_3 \otimes b_3) \\ &= f\,(b_1 a_2)\, f\,(b_2 a_3)\, a_1 \otimes b_3, \end{aligned}$$

and

$$\begin{aligned} (a_1 \otimes b_1)\,[(a_2 \otimes b_2)\,(a_3 \otimes b_3)] &= (a_1 \otimes b_1)\,[f\,(b_2 a_3)\, a_2 \otimes b_3] \\ &= f\,(b_2 a_3)\, f\,(b_1 a_2)\, a_1 \otimes b_3, \end{aligned}$$

and so,

$$[(a_1 \otimes b_1)\,(a_2 \otimes b_2)]\,(a_3 \otimes b_3) = (a_1 \otimes b_1)\,[(a_2 \otimes b_2)\,(a_3 \otimes b_3)]\,.$$

This shows that associativity holds for elementary tensors. Thus, since left and right distributivity also holds, it readily follows that $(uv)w = u(vw)$; that is, the multiplication scheme is associative. We have therefore shown the following:

Proposition 3.3.1. *Let $\mathcal{A}$ be an algebra over the field K and let f be a linear functional on $\mathcal{A}$. Then $\mathcal{A} \otimes \mathcal{A}$ is an algebra where multiplication of elementary tensors is defined by*

$$(a \otimes b)\,(c \otimes d) = f(bc) a \otimes d \quad (a, b, c, d \in \mathcal{A})\,.$$

With $\mathcal{A} = \mathrm{Soc}(A)$ and $f = \mathrm{Tr}$, Proposition 3.3.1 now readily gives that $\mathrm{Soc}(A) \otimes \mathrm{Soc}(A)$ is an algebra where multiplication of elementary tensors is given by

$$(a \otimes b)\,(c \otimes d) = \mathrm{Tr}\,(bc) a \otimes d \quad (a, b, c, d \in \mathrm{Soc}(A))\,.$$

Let $\mathcal{P}$ be a fixed set of projections generating $\mathrm{Soc}(A)$. For $p, q \in \mathcal{P}$, define $J_{p,q} := Ap \otimes qA$. Moreover, let

$$A_J := \left\{ \sum_{i=1}^{n} x_i : x_i \in J_{p,q} \text{ for some } p, q \in \mathcal{P} \text{ with } p \neq q, n \geq 1 \text{ an integer} \right\},$$

$$A'_J := \left\{ \sum_{i=1}^{n} x_i : x_i \in J_{p,q} \text{ for some } p, q \in \mathcal{P}, n \geq 1 \text{ an integer} \right\},$$

and

$$A''_J := \left\{ \sum_{i=1}^{n} x_i : x_i \in J_{p,p} \text{ for some } p \in \mathcal{P}, n \geq 1 \text{ an integer} \right\}.$$

It is not hard to see that A_J is a vector subspace but not a subalgebra of $\mathrm{Soc}(A) \otimes \mathrm{Soc}(A)$, since it is not closed under multiplication. However, A'_J is indeed a subalgebra of $\mathrm{Soc}(A) \otimes \mathrm{Soc}(A)$. Also note that $A'_J = A''_J + A_J$, and that $A''_J \cong \oplus_{p \in \mathcal{P}} Ap \otimes pA$. The following lemma will be useful later on:

Lemma 3.3.2. *Let $u \in A'_J$. Then there exist a unique element $u_S \in A''_J$ and a unique element $u_j \in A_J$ such that $u = u_S + u_j$.*

Proof. Suppose that

$$u = \sum_{i=1}^{n} x_i p_i \otimes p_i y_i + \sum_{i=1}^{m} z_i r_i \otimes q_i w_i \tag{3.3.1}$$

and

$$u = \sum_{i=1}^{n'} x'_i p'_i \otimes p'_i y'_i + \sum_{i=1}^{m'} z'_i r'_i \otimes q'_i w'_i, \tag{3.3.2}$$

where $p_i, q_i, r_i, p'_i, q'_i, r'_i \in \mathcal{P}$, $r_i \neq q_i$ and $r'_i \neq q'_i$ for all i. To prove uniqueness it will suffice to show that

$$\sum_{i=1}^{n} x_i p_i \otimes p_i y_i = \sum_{i=1}^{n'} x'_i p'_i \otimes p'_i y'_i. \tag{3.3.3}$$

For the sake of a contradiction, suppose (3.3.3) is false. Let $u_S = \sum_{i=1}^{n} x_i p_i \otimes p_i y_i$, $u_j = \sum_{i=1}^{m} z_i r_i \otimes q_i w_i$, $u'_S = \sum_{i=1}^{n'} x'_i p'_i \otimes p'_i y'_i$ and $u'_j = \sum_{i=1}^{m'} z'_i r'_i \otimes q'_i w'_i$. From (3.3.1) and (3.3.2) it follows that

$$0 = (u_S - u'_S) + \left(u_j - u'_j\right). \tag{3.3.4}$$

Without loss of generality we may assume that

$$u_S - u'_S = \left(\sum_{i=1}^{l_1} x_{1,i}s_1 \otimes s_1 y_{1,i}\right) + \cdots + \left(\sum_{i=1}^{l_k} x_{k,i}s_k \otimes s_k y_{k,i}\right), \qquad (3.3.5)$$

where $s_1, \ldots, s_k \in \mathcal{P}$ and $s_i \neq s_j$ for $i \neq j$. Moreover, by [4, Lemma 42.3] we may assume that $\{x_{i,j}s_i\}$ and $\{s_i y_{i,j}\}$ are linearly independent sets for all $i \in \{1, \ldots, k\}$. Since (3.3.3) is false, one of the terms in (3.3.5) must be nonzero, say

$$\sum_{i=1}^{l_1} x_{1,i}s_1 \otimes s_1 y_{1,i} \neq 0. \qquad (3.3.6)$$

Similarly, one of the terms in (3.3.6) must be nonzero, say $x_{1,1}s_1 \otimes s_1 y_{1,1} \neq 0$. In particular, this means that $x_{1,1}s_1 \neq 0$, $s_1 y_{1,1} \neq 0$ and that s_1 is a rank one projection of A. By Theorem 2.2.3, the Spectral Mapping Theorem and Jacobson's Lemma, there exist $a_1, a_2 \in A$ such that $s_1 a_1 x_{1,1} s_1 = s_1$, $s_1 y_{1,1} a_2 s_1 = s_1$, $s_1 a_1 x_{1,j} s_1 = 0$ and $s_1 y_{1,j} a_2 s_1 = 0$ for all $j \in \{2, \ldots, l_1\}$. Let $s = a_2 s_1 \otimes s_1 a_1$. In particular, $s \neq 0$. Moreover, from (3.3.4) it now follows that

$$0 = s\left(u_S - u'_S\right)s + s\left(u_j - u'_j\right)s. \qquad (3.3.7)$$

Thus, by the orthogonality of the J_p, the definition of multiplication in $\mathrm{Soc}(A) \otimes \mathrm{Soc}(A)$ and our choice of s, (3.3.7) reduces to $0 = a_2 s_1 \otimes s_1 a_1$. But this is absurd since $s \neq 0$. Hence, (3.3.3) holds which completes the proof. $\square$

Let $u, v, w \in A_J$ and let $\alpha \in \mathbb{C}$. In particular, since any product of elements from A_J belongs to A'_J, Lemma 3.3.2 implies the following:

$$\alpha\,[uv]_S = [(\alpha u)\,v]_S = [u\,(\alpha v)]_S\,,$$

$$[u\,(v+w)]_S = [uv]_S + [uw]_S\,, \quad [(v+w)\,u]_S = [vu]_S + [wu]_S\,,$$

and

$$[u(vw)]_S = [(uv)w]_S\,.$$

A similar observation is true if J is used as a subscript instead of S above.

Let $B := A \oplus A_J$. If we define addition and scalar multiplication in B pointwise, then surely B is a vector space over $\mathbb{C}$. However, for B to be an algebra, it is crucial that multiplication in B is not pointwise. In fact, general multiplication in B should be definable in a natural way provided we also know how to multiply elements of A with elements of A_J. This goes back to multiplication of $a \in A$ with elementary tensors $xp \otimes qy \in A_J$; we readily think of

$$a\,(xp \otimes qy) = axp \otimes qy$$

and

$$(xp \otimes qy)\,a = xp \otimes qya.$$

We can make the following observation in this regard:

Lemma 3.3.3. *Let $\phi : A''_J \to \mathrm{Soc}(A)$ be the algebra isomorphism obtained from Theorem 2.2.1 and the remark preceding Lemma 3.3.2, and let $u \in A''_J$. Then*

$$u\,(xp \otimes qy) = \phi(u)\,(xp \otimes qy)$$

and

$$(xp \otimes qy)\,u = (xp \otimes qy)\,\phi(u)$$

for all $x, y \in A$ and $p, q \in \mathcal{P}$.

Proof. By the distributivity of both multiplication schemes, we may assume, without loss of generality, that $u = z_1 r \otimes r z_2$ for some $r \in \mathcal{P}$. If $r = 0$, then equality trivially holds true. So assume that $r \neq 0$. Recall that $\phi\,(z_1 r \otimes r z_2) = z_1 r z_2$. Now, if $r \neq p$, then by the orthogonality of J_p and J_r it follows that $r z_2 x p = 0$ and hence $\mathrm{Tr}\,(r z_2 x p) = 0$. Thus,

$$u\,(xp \otimes qy) = \phi(u)\,(xp \otimes qy) = 0.$$

Similarly,

$$(xp \otimes qy)\,u = (xp \otimes qy)\,\phi(u) = 0$$

whenever $r \neq q$. We may therefore assume that $r = p$ and $r = q$. By the minimality of r and the definition of the trace we have $r z_2 x r = \mathrm{Tr}\,(r z_2 x r)\,r$ and $r y z_1 r = \mathrm{Tr}\,(r y z_1 r)\,r$. Consequently,

$$u\,(xr \otimes qy) = (z_1 r \otimes r z_2)\,(xr \otimes qy) = \mathrm{Tr}\,(r z_2 x r)\,z_1 r \otimes qy$$

and

$$\begin{aligned} \phi(u)\,(xr \otimes qy) &= z_1 r z_2\,(xr \otimes qy) = z_1 r z_2 x r \otimes qy \\ &= \mathrm{Tr}\,(r z_2 x r)\,z_1 r \otimes qy, \end{aligned}$$

and so, $u\,(xr \otimes qy) = \phi(u)\,(xr \otimes qy)$. Similarly, $(xp \otimes ry)\,u = (xp \otimes ry)\,\phi(u)$, so the lemma is proved. □

For $a \in A$ and $u = \sum_{i=1}^{n} x_i p_i \otimes q_i y_i \in A_J$ we now define

$$au = \sum_{i=1}^{n} a x_i p_i \otimes q_i y_i$$

and

$$ua = \sum_{i=1}^{n} x_i p_i \otimes q_i y_i a.$$

Moreover, if $u = \sum_{i=1}^{n} x_i p_i \otimes q_i y_i \in A_J$ and $u = u' = \sum_{i=1}^{n'} x_i' p_i' \otimes q_i' y_i' \in A_J$, then

$$0 = a\left(u - u'\right) = \sum_{i=1}^{n} a x_i p_i \otimes q_i y_i - \sum_{i=1}^{n'} a x_i' p_i' \otimes q_i' y_i' = au - au',$$

and, similarly, $0 = ua - u'a$. This shows that this operation is well-defined. This leads us to the following operation for multiplication in B:

$$(x,u)(y,v) = \Big(xy + \phi\left([uv]_S\right), uy + xv + [uv]_j\Big), \tag{3.3.8}$$

where $x, y \in A$, $u, v \in A_J$ and $\phi : A_J'' \to \operatorname{Soc}(A)$ is the isomorphism from Lemma 3.3.3 above. By Lemma 3.3.2, and the observation above, the operation in (3.3.8) is well-defined. Moreover, by Lemma 3.3.2 and the remark following it, associativity, distributivity and associativity with scalars are all satisfied by the operation in (3.3.8). So B with the multiplication described in (3.3.8) is an algebra over $\mathbb{C}$. If $\mathbf{1} \in A$ is the identity element of A, then $(\mathbf{1}, 0)$ is the identity element of B. An important subalgebra of B is described in the next lemma:

Proposition 3.3.4. $\operatorname{Soc}(A) \oplus A_J$ *is a subalgebra of* B*. Moreover,* $A_J' \cong \operatorname{Soc}(A) \oplus A_J$*.*

Proof. It is routine to check that $\operatorname{Soc}(A) \oplus A_J$ is closed under addition, scalar multiplication and multiplication. Define $\psi : A_J' \to \operatorname{Soc}(A) \oplus A_J$ by

$$\psi(u) = \left(\phi\left(u_S\right), u_j\right) \quad \left(u \in A_J'\right),$$

where $\phi : A_J'' \to \operatorname{Soc}(A)$ is the algebra isomorphism from Lemma 3.3.3 and $u = u_S + u_j$ as defined in Lemma 3.3.2. We claim that ψ is an algebra isomorphism: Let $u, v \in A_J'$ and let $\alpha \in \mathbb{C}$. Then,

$$\begin{aligned} \psi\left(\alpha u\right) &= \Big(\phi\left([\alpha u]_S\right), [\alpha u]_j\Big) = \left(\alpha\phi\left(u_S\right), \alpha u_j\right) \\ &= \alpha\left(\phi\left(u_S\right), u_j\right) = \alpha\psi(u), \end{aligned}$$

$$\begin{aligned}\psi(u+v) &= \left(\phi([u+v]_S), [u+v]_j\right) = (\phi(u_S+v_S), u_j+v_j) \\ &= (\phi(u_S), u_j) + (\phi(v_S), v_j) = \psi(u)+\psi(v),\end{aligned}$$

and

$$\begin{aligned}\psi(u)\psi(v) &= (\phi(u_S), u_j)(\phi(v_S), v_j) \\ &= \left(\phi(u_S)\phi(v_S) + \phi\left([u_jv_j]_S\right), \phi(u_S)v_j + u_j\phi(v_S) + [u_jv_j]_j\right) \\ &= \left(\phi\left(u_Sv_S + [u_jv_j]_S\right), u_Sv_j + u_jv_S + [u_jv_j]_j\right) \\ &= (\phi([uv]_S), [uv]_j) = \psi(uv),\end{aligned}$$

where we have used Lemma 3.3.3 and the fact that

$$uv = (u_S+u_j)(v_S+v_j) = u_Sv_S + u_jv_S + u_Sv_j + u_jv_j,$$

and so, since u_jv_S and u_Sv_j must be in A_J, it follows that $[uv]_S = u_Sv_S + [u_jv_j]_S$ and $[uv]_j = u_jv_S + u_Sv_j + [u_jv_j]_j$. This shows that ψ is an algebra homomorphism. To complete the proof we must therefore show that ψ is bijective: Let (x,u) be any element in $\mathrm{Soc}(A) \oplus A_J$. Then $\psi(\phi^{-1}(x)+u) = (x,u)$ by Lemma 3.3.2. This shows that ψ is surjective. To see that ψ is injective, we assume that $\psi(v) = 0$ and prove that $v = 0$. Now, $\psi(v) = 0$ implies that $(\phi(v_S), v_j) = 0$. Hence, $\phi(v_S) = 0$ and $v_j = 0$. But ϕ is an algebra isomorphism, so $v_S = 0$. Hence, $v = v_S + v_j = 0$. This completes the proof. $\square$

3.4 Norm Extension

With the basic algebraic structure of the Shoda-completion defined for some fixed projection representative class $\mathcal{P}$ generating $\mathrm{Soc}(A)$, we are now in a position to extend the algebra norm on A to an algebra norm on $B = A \oplus A_J$. Firstly, however, we need the following lemma:

Lemma 3.4.1. *Let $0 \neq u \in A_J$. Then u can be uniquely expressed as $u = \sum_{i=1}^{n} u_{p_i,q_i}$, where $0 \neq u_{p_i,q_i} \in J_{p_i,q_i}$, $p_i, q_i \in \mathcal{P}$ and $p_i \neq q_i$ for each $i \in \{1, \ldots, n\}$, and where $(p_i, q_i) \neq (p_j, q_j)$ for $i \neq j$ (as ordered sets).*

Proof. Suppose that $u = \sum_{i=1}^{n} u_{p_i,q_i}$ as described above and that $u = \sum_{i=1}^{m} v_{r_i,s_i}$, where $0 \neq v_{r_i,s_i} \in J_{r_i,s_i}$, $r_i, s_i \in \mathcal{P}$ and $r_i \neq s_i$ for each $i \in \{1, \ldots, m\}$, and where $(r_i, s_i) \neq (r_j, s_j)$ for $i \neq j$. Then,

$$0 = \sum_{i=1}^{n} u_{p_i,q_i} - \sum_{i=1}^{m} v_{r_i,s_i}. \tag{3.4.1}$$

For the sake of a contradiction, suppose that there exists an $i \in \{1, \ldots, n\}$ such that $(p_i, q_i) \neq (r_j, s_j)$ for all $j \in \{1, \ldots, m\}$. Recall that $u_{p_i,q_i} \neq 0$ can be written as

$$u_{p_i,q_i} = \sum_{j=1}^{k} x_j p_i \otimes q_i y_j,$$

where $\{x_j p_i\}$ and $\{q_i y_j\}$ are linearly independent sets. By Theorem 2.2.3, the Spectral Mapping Theorem, Jacobson's Lemma and the definition of multiplication in $\mathrm{Soc}(A) \otimes \mathrm{Soc}(A)$, there exist $x, y \in A$ such that

$$(p_i \otimes p_i x)\, u_{p_i,q_i}\, (y q_i \otimes q_i) = p_i \otimes q_i \neq 0.$$

However, from (3.4.1) it then follows that

$$\begin{aligned} 0 &= (p_i \otimes p_i x) \left(\sum_{i=1}^{n} u_{p_i,q_i} - \sum_{i=1}^{m} v_{r_i,s_i} \right) (y q_i \otimes q_i) \\ &= p_i \otimes q_i, \end{aligned}$$

producing a contradiction. Hence, for every $i \in \{1, \ldots, n\}$ there exists a $j \in \{1, \ldots, m\}$ such that $(p_i, q_i) = (r_j, s_j)$. Similarly, it can be shown that for every $j \in \{1, \ldots, m\}$ there exists an $i \in \{1, \ldots, n\}$ such that $(r_j, s_j) = (p_i, q_i)$. Hence, (3.4.1) can be written as

$$0 = \sum_{i=1}^{n} \left(u_{p_i,q_i} - v_{p_i,q_i} \right).$$

So the lemma is true if $u_{p_i,q_i} = v_{p_i,q_i}$ for each $i \in \{1, \ldots, n\}$. But if $u_{p_i,q_i} \neq v_{p_i,q_i}$ for some $i \in \{1, \ldots, n\}$, then we can obtain a contradiction using the same argument as before. We therefore have the result. $\square$

Lemma 3.4.1 now allows us to write

$$A_J = \bigoplus_{\substack{p,q \in \mathcal{P} \\ p \neq q}} Ap \otimes qA.$$

This point of view, together with Lemma 3.3.2, is crucial for the norm to be well-defined.

For $p \neq q$ denote by $Ap \otimes_\pi qA$ the algebraic tensor product endowed with the projective tensor norm (see for instance [14, p. 15]), and by $Ap \,\hat{\otimes}_\pi\, qA$ its norm completion. The norm on each $Ap \,\hat{\otimes}_\pi\, qA$ will be denoted by $\| \cdot \|_{\pi,p,q}$. We denote then the algebraic direct sum of the normed algebras $Ap \otimes_\pi qA$ under the l_1 norm by

$$A_{J,\pi} = \bigoplus_{\substack{p,q \in \mathcal{P} \\ p \neq q}} Ap \otimes_\pi qA,$$

and then by $\bar{A}_{J,\pi}$ the norm completion of $A_{J,\pi}$. It is worthwhile to notice that

$$\bigoplus_{\substack{p,q \in \mathcal{P} \\ p \neq q}} Ap \,\hat{\otimes}_\pi\, qA \subseteq \bar{A}_{J,\pi},$$

but that the containment may be strict. With the norm on A denoted by $\| \cdot \|_A$ and the l_1 norm on $\bar{A}_{J,\pi}$ by $\| \cdot \|_1$, we have:

Lemma 3.4.2. *$A \oplus A_{J,\pi}$ is a normed algebra under*

$$\|(x, u)\| = \|x\|_A + \|u\|_1,$$

where $x \in A$ and $u \in A_{J,\pi}$.

Proof. It has been established that $A \oplus A_{J,\pi}$ is a complex unital algebra. Since vector addition is pointwise, $\| \cdot \|$ is a vector norm. But vector multiplication is not pointwise so we need to verify that $\| \cdot \|$ is submultiplicative: Let $a, b \in A$ and $u, v \in A_{J,\pi}$. We show that

$$\|(a, u)(b, v)\| \leq \|(a, u)\|\|(b, v)\|.$$

We can write

$$u = \sum_{i=1}^{n} u_{p_i,q_i} \text{ and } v = \sum_{i=1}^{m} v_{\tilde{p}_i,\tilde{q}_i},$$

where, for each i,

$$u_{p_i,q_i} \in Ap_i \otimes q_iA \text{ and } v_{\tilde{p}_i,\tilde{q}_i} \in A\tilde{p}_i \otimes \tilde{q}_iA.$$

As was shown earlier, the product $(a,u)(b,v)$ takes the form

$$(a,u)(b,v) = (ab + \Omega(uv), ub + av + \Gamma(uv)).$$

We have

$$\begin{aligned} \|(a,u)(b,v)\| &= \|ab + \Omega(uv)\|_A + \|ub + av + \Gamma(uv)\|_1 \\ &\le \|ab\|_A + \|\Omega(uv)\|_A + \|ub\|_1 + \|av\|_1 + \|\Gamma(uv)\|_1 \\ &\le \|a\|_A\|b\|_A + \|\Omega(uv)\|_A + \|ub\|_1 + \|av\|_1 + \|\Gamma(uv)\|_1. \end{aligned}$$

On the other hand,

$$\|(a,u)\|\|(b,v)\| = \|a\|_A\|b\|_A + \|u\|_1\|b\|_A + \|a\|_A\|v\|_1 + \|u\|_1\|v\|_1,$$

from which we now observe that $\|\cdot\|$ will be submultiplicative provided we can establish:

(i) $\|ub\|_1 \le \|u\|_1\|b\|_A$.

(ii) $\|av\|_1 \le \|v\|_1\|a\|_A$.

(iii) $\|\Omega(uv)\|_A + \|\Gamma(uv)\|_1 \le \|u\|_1\|v\|_1$.

We proceed as follows:
(i) We can write

$$u = \sum_{i=1}^{n} u_{p_i,q_i} = \sum_{i=1}^{n}\sum_{j=1}^{k(i)} x_{i,j}p_i \otimes q_iy_{i,j},$$

where $k(i) \in \mathbb{N}$ for $i \in \{1,\ldots,n\}$ and $x_{i,j}, y_{i,j} \in A$. Multiplication dictates that

$$ub = \sum_{i=1}^{n}\sum_{j=1}^{k(i)} x_{i,j}p_i \otimes q_iy_{i,j}b.$$

Notice that, for each i, $\sum_{j=1}^{k(i)} x_{i,j}p_i \otimes q_iy_{i,j}b$ is a representation of the tensor $u_{p_i,q_i}b \in Ap_i \otimes q_iA$. So it follows that

$$\|u_{p_i,q_i}b\|_{\pi,p_i,q_i} \le \sum_{j=1}^{k(i)} \|x_{i,j}p_i\|_A\|q_iy_{i,j}b\|_A \le \|b\|_A \sum_{j=1}^{k(i)} \|x_{i,j}p_i\|_A\|q_iy_{i,j}\|_A.$$

But the preceding inequality is valid for any representation

$$\sum_{j=1}^{k(i)} x_{i,j} p_i \otimes q_i y_{i,j} \text{ of } u_{p_i,q_i} \in Ap_i \otimes q_i A.$$

So we infer, by the definition of the projective tensor product, that

$$\|u_{p_i,q_i} b\|_{\pi,p_i,q_i} \leq \|b\|_A \|u_{p_i,q_i}\|_{\pi,p_i,q_i}.$$

This being true for each i, we obtain

$$\|ub\|_1 \leq \|u\|_1 \|b\|_A.$$

(ii) Similar to (i).
(iii) With $u = \sum_{i=1}^n u_{p_i,q_i}$ and $v = \sum_{i=1}^m v_{\tilde{p}_i,\tilde{q}_i}$ we observe that

$$(0,u)(0,v) = (\Omega(uv), \Gamma(uv)),$$

where $\Omega(uv)$ is a sum of elements belonging to A and $\Gamma(uv)$ is a sum of elements belonging to $A_{J,\pi}$. Furthermore:

(a) $u_{p_i,q_i} v_{\tilde{p}_j,\tilde{q}_j} = 0$ whenever $q_i \neq \tilde{p}_j$.

(b) $\Omega\left(u_{p_i,q_i} v_{\tilde{p}_j,\tilde{q}_j}\right) \in A$ whenever $q_i = \tilde{p}_j$ and $p_i = \tilde{q}_j$.

(c) $u_{p_i,q_i} v_{\tilde{p}_j,\tilde{q}_j} \in A_{J,\pi}$ whenever $q_i = \tilde{p}_j$ and $p_i \neq \tilde{q}_j$.

Consider

$$\|u\|_1 \|v\|_1 = \sum_{i,j} \|u_{p_i,q_i}\|_{\pi,p_i,q_i} \|v_{\tilde{p}_j,\tilde{q}_j}\|_{\pi,\tilde{p}_j,\tilde{q}_j}.$$

Now we take any term in the sum on the right for which $q_i = \tilde{p}_j$. If the situation in (b) prevails then $p_i = \tilde{q}_j$, and with $u_{p_i,q_i} = \sum_{k=1}^n x_k p_i \otimes q_i y_k$ and $v_{\tilde{p}_j,\tilde{q}_j} = \sum_{l=1}^m \tilde{y}_l q_i \otimes p_i \tilde{x}_l$, it then follows that

$$\begin{aligned}
\|u_{p_i,q_i} v_{\tilde{p}_j,\tilde{q}_j}\| &:= \|\Omega\left(u_{p_i,q_i} v_{\tilde{p}_j,\tilde{q}_j}\right)\|_A \\
&= \left\| \sum_{k,l} \mathrm{Tr}\,(q_i y_k \tilde{y}_l q_i) x_k p_i \tilde{x}_l \right\|_A \\
&\leq \sum_{k,l} |\mathrm{Tr}\,(q_i y_k \tilde{y}_l q_i)| \, \|x_k p_i\|_A \|p_i \tilde{x}_l\|_A \\
&= \sum_{k,l} |\rho(q_i y_k \tilde{y}_l q_i)| \, \|x_k p_i\|_A \|p_i \tilde{x}_l\|_A \\
&\leq \sum_{k,l} \|q_i y_k\|_A \|\tilde{y}_l q_i\|_A \|x_k p_i\|_A \|p_i \tilde{x}_l\|_A \\
&= \left(\sum_{k=1}^n \|x_k p_i\|_A \|q_i y_k\|_A \right) \left(\sum_{l=1}^m \|\tilde{y}_l q_i\|_A \|p_i \tilde{x}_l\|_A \right).
\end{aligned}$$

By definition of the projective tensor product we conclude that

$$\|u_{p_i,q_i}v_{\tilde{p}_j,\tilde{q}_j}\| \leq \|u_{p_i,q_i}\|_{\pi,p_i,q_i}\|\|v_{\tilde{p}_j,\tilde{q}_j}\|_{\pi,\tilde{p}_j,\tilde{q}_j}.$$

If the situation in (c) occurs, then $p_i \neq \tilde{q}_j$, and we have

$$\begin{aligned}
\|u_{p_i,q_i}v_{\tilde{p}_j,\tilde{q}_j}\| &:= \|u_{p_i,q_i}v_{\tilde{p}_j,\tilde{q}_j}\|_1 \\
&= \left\|\sum_{k,l} \mathrm{Tr}\,(q_i y_k \tilde{y}_l q_i) x_k p_i \otimes \tilde{q}_j \tilde{x}_l\right\|_{\pi,p_i,\tilde{q}_j} \\
&\leq \sum_{k,l} \|q_i y_k\|_A \|\tilde{y}_l q_i\|_A \|x_k p_i\|_A \|\tilde{q}_j \tilde{x}_l\|_A \\
&= \left(\sum_{k=1}^{n} \|x_k p_i\|_A \|q_i y_k\|_A\right)\left(\sum_{l=1}^{m} \|\tilde{y}_l \tilde{p}_j\|_A \|\tilde{q}_j \tilde{x}_l\|_A\right).
\end{aligned}$$

Again the definition of the projective tensor norms on $Ap_i \otimes_\pi q_i A$ and $A\tilde{p}_j \otimes_\pi \tilde{q}_j A$ implies that

$$\|u_{p_i,q_i}v_{\tilde{p}_j,\tilde{q}_j}\| \leq \|u_{p_i,q_i}\|_{\pi,p_i,q_i}\|\|v_{\tilde{p}_j,\tilde{q}_j}\|_{\pi,\tilde{p}_j,\tilde{q}_j}.$$

With the partial multiplicative inequalities established and the triangle inequality we now have:

$$\begin{aligned}
\|\Omega(uv)\|_A + \|\Gamma(uv)\|_1 &\leq \sum_{i,j} \|u_{p_i,q_i}v_{\tilde{p}_j,q_j}\| \\
&\leq \sum_{i,j} \|u_{p_i,q_i}\|_{\pi,p_i,q_i}\|v_{\tilde{p}_j,\tilde{q}_j}\|_{\pi,\tilde{p}_j,\tilde{q}_j} \\
&= \|u\|_1\|v\|_1,
\end{aligned}$$

which proves (iii). □

It follows from Lemma 3.4.2 that the norm completion of $A \oplus A_{J,\pi}$ is a Banach algebra. If we denote the completion by $\tilde{A}_S$ then of course

$$\tilde{A}_S = A \oplus \bar{A}_{J,\pi}.$$

It is unlikely that the unital Banach algebra $\tilde{A}_S$ is semisimple. So we factor out the radical to obtain the semisimple

$$A_S := \tilde{A}_S/\mathrm{Rad}(\tilde{A}_S).$$

We next want to show that A is isometrically embeddable into A_S. This will be easy once we have:

Lemma 3.4.3.

$$\mathrm{Rad}(\tilde{A}_S) \subseteq \{(0,u) : u \in \bar{A}_{J,\pi}\}.$$

Proof. Suppose $(a,u) \in \mathrm{Rad}(\tilde{A}_S)$. Then we can write $u = \lim_n u_n$, where, for each n, $u_n \in A_{J,\pi}$. Let $x \in A$ be arbitrary and observe that

$$(a,u)(x,0) = (ax, ux) \in \mathrm{Rad}(\tilde{A}_S).$$

So if p is a rank one projection belonging to the class $\mathcal{P}$, and $z \in A$ is arbitrary, then, since $ux = \lim_n u_n x$, we deduce that

$$(pz,0)(ax,ux)(p,0) = (pzaxp, 0) \in \mathrm{Rad}(\tilde{A}_S).$$

It follows from the semisimplicity of A that

$$pzaxp = 0 \text{ for all } x, z \in A.$$

Thus, by Jacobson's Lemma, $\sigma_A(zaxp) = \{0\}$ for all $x, z \in A$. Hence, since A is semisimple, we have that $axp = 0$ for all $x \in A$. Therefore, since $p \in \mathcal{P}$ was arbitrary, we have that $axux = 0$ for each $x \in A$, and hence that

$$(ax,0)(ax,ux) = ((ax)^2, 0) \in \mathrm{Rad}(\tilde{A}_S)$$

for each $x \in A$. But this means that $\sigma_A(ax) = \sigma_{\tilde{A}_S}((ax,0)) = \{0\}$ holds for each $x \in A$ from which the semisimplicity of A yields $a = 0$. □

Theorem 3.4.4. *A is isometrically embeddable into A_S.*

Proof. Let $T : A \to A_S$ be the canonical map given by

$$a \mapsto (a,0) + \mathrm{Rad}(\tilde{A}_S).$$

It is obvious that T is a homomorphism into A_S. It remains to prove that T is an isometry. Using Lemma 3.4.3 we have

$$\begin{aligned}
\|Ta\| &= \|(a,0) + \mathrm{Rad}(\tilde{A}_S)\| \\
&= \inf\{\|(a,0) + (0,u)\| : (0,u) \in \mathrm{Rad}(\tilde{A}_S)\} \\
&= \inf\{\|a\|_A + \|u\|_1 : (0,u) \in \mathrm{Rad}(\tilde{A}_S)\} \\
&= \|a\|_A + \inf\{\|u\|_1 : (0,u) \in \mathrm{Rad}(\tilde{A}_S)\} \\
&= \|a\|_A,
\end{aligned}$$

which gives the result. □

The Banach algebra A_S is thus a semisimple extension of A. We want to show that A_S satisfies Shoda's Theorem. Let p be a rank one projection of $\mathrm{Soc}(A)$, and consider the element $(p,0)+\mathrm{Rad}(\tilde{A}_S)\in A_S$. Let $(x,u)+\mathrm{Rad}(\tilde{A}_S)$ be arbitrary in A_S. Then,

$$\begin{aligned}
& \left[(p,0)+\mathrm{Rad}(\tilde{A}_S)][(x,u)+\mathrm{Rad}(\tilde{A}_S)\right]\left[(p,0)+\mathrm{Rad}(\tilde{A}_S)\right] \\
= \;& (p,0)(x,u)(p,0)+\mathrm{Rad}(\tilde{A}_S) \\
= \;& (pxp,0)+\mathrm{Rad}(\tilde{A}_S) \\
= \;& (\lambda p,0)+\mathrm{Rad}(\tilde{A}_S) \\
= \;& \lambda\left[(p,0)+\mathrm{Rad}(\tilde{A}_S)\right],
\end{aligned}$$

which proves that $(p,0)+\mathrm{Rad}(\tilde{A}_S)\in\mathrm{Soc}\,A_S$.
For any pair $p,q\in\mathcal{P}$ where $p\neq q$, we now consider the element $(0,p\otimes q)+\mathrm{Rad}(\tilde{A}_S)$. Since

$$\left[(p,0)+\mathrm{Rad}(\tilde{A}_S)\right]\left[(0,p\otimes q)+\mathrm{Rad}(\tilde{A}_S)\right]=(0,p\otimes q)+\mathrm{Rad}(\tilde{A}_S),$$

it follows that $(0,p\otimes q)+\mathrm{Rad}(\tilde{A}_S)$ is a rank one element of A_S. Moreover, since

$$\begin{aligned}
& \left[(x,0)+\mathrm{Rad}(\tilde{A}_S)\right]\left[(0,p\otimes q)+\mathrm{Rad}(\tilde{A}_S)\right]\left[(y,0)+\mathrm{Rad}(\tilde{A}_S)\right] \\
= \;& (0,xp\otimes qy)+\mathrm{Rad}(\tilde{A}_S),
\end{aligned}$$

for all $x,y\in A$, it follows that $(0,xp\otimes qy)+\mathrm{Rad}(\tilde{A}_S)$ has rank less than or equal to one for all $x,y\in A$. Together we have proved the following:

Proposition 3.4.5. $\mathrm{Soc}(A)\oplus A_{J,\pi}+\mathrm{Rad}(\tilde{A}_S)$ *is a vector subspace, but not necessarily an ideal of* $\mathrm{Soc}\,A_S$.

So, to simplify, we shall from now on write (a,u) for elements belonging to A_S, with the understanding that we actually mean $(a,u)+\mathrm{Rad}(\tilde{A}_S)$.

Theorem 3.4.6. *The semisimple Banach algebra A_S is Shoda-complete.*

Proof. Let p be any rank one projection of A such that $p\in\mathcal{P}$. Then $(p,0)$ is a rank one projection in A_S. To prove the result it suffices to show that any rank one projection of A_S belongs to the two-sided ideal generated by $(p,0)$ i.e. to $J_{(p,0)}$. First let p' be any other rank one projection of A. If $p'\in E_p$, then there is a $w\in G_1(A)$ such that $p'=wpw^{-1}$, and so

$$(p',0)=(w,0)(p,0)(w^{-1},0)\in J_{(p,0)}.$$

If $p' \notin E_p$, then $p' = vqv^{-1}$ where $v \in G_1(A)$ and $q \in \mathcal{P}$ with $q \neq p$. Observe now that

$$(q, 0) = (0, q \otimes p)(p, 0)(0, p \otimes q) \in J_{(p,0)},$$

whence

$$(p', 0) = (v, 0)(q, 0)(v^{-1}, 0) \in J_{(p,0)}.$$

Now let (a, u) be an arbitrary rank one projection of A_S. Since u is the limit of a sequence of elements in $A_{J,\pi}$, it follows from Proposition 3.4.5 that $(0, u) \in \mathrm{cl}(\mathrm{Soc}(A_S))$. Hence, it follows that $(a, 0) \in \mathrm{cl}(\mathrm{Soc}(A_S))$. So the two-sided ideal generated by a, namely J_a, is inessential in A. It follows from Theorem 2.5.22 that for any member, say b, of J_a the Riesz projections corresponding to the isolated nonzero spectral values of b are finite rank elements of A. Now take $x \in A$ such that $\sigma(ax) \neq \{0\}$. Then there is a finite rank projection, p', corresponding to $0 \neq \lambda \in \sigma(ax)$ such that $\sigma(axp') = \{0, \lambda\}$ (or possibly $\{\lambda\}$ in the finite dimensional case). But we can write $p' = \sum_{i=1}^{k} p_i$ where each p_i is a rank one projection of A. If $p_i axp_i = 0$ for each i, then $\mathrm{Tr}\,(axp') = 0$ which is absurd, since $\mathrm{Tr}\,(axp')$ is a positive integer multiple of $\lambda \neq 0$. So we may assume without loss of generality that, say $p_1 axp_1 = \alpha p_1$, where $\alpha \neq 0$ and moreover then that $\alpha = 1$. Thus, if we take (a, u) and then perform the operations (i) $(a, u)(x, 0) = (ax, ux)$, (ii) $(p_1, 0)(ax, ux)(p_1, 0) = (p_1, 0)$, we see that $(p_1, 0) \in J_{(a,u)}$. However, it was also shown in the first part of the proof that $(p_1, 0) \in J_{(p,0)}$. So $J_{(a,u)} = J_{(p,0)}$, which shows that $\mathrm{Soc}\, A_S$ is a minimal two-sided ideal. □

3.5 Independence Property

What we have shown thus far is that any semisimple unital Banach algebra A has a Shoda-completion, which we called A_S. This completion was obtained by adjoining to A a "hybrid" collection of tensor products using representatives from each minimal two-sided ideal $\{J_p : p \in \mathcal{P}\}$; where $\mathcal{P}$ is a fixed class of rank one projections each of which belongs to precisely one connected component of E (the set of projections in A). By definition of J_p it follows that $E_p \subset J_p$ for each $p \in \mathcal{P}$. The Shoda-completion then has $\mathrm{Soc}(A_S) = J_{(p,0)}$ as the only minimal two-sided ideal containing finite rank elements. At a first glance it may seem that the Shoda-completion is independent of any particular choice of representatives taken from the collection of orbits $E_p \subseteq E$; this is certainly true for the algebraic extension of A which was obtained in Section 3.3 of this chapter (the algebraic extension is automatically semisimple as well). If we look at the construction of the norm in Section 3.4, then it is evident that a uniqueness problem may arise when we take the norm completion of the algebra obtained in Lemma 3.4.2. What we want to show finally is that this problem never manifests. Let $\mathcal{P}$ be a fixed collection of rank one projection representatives generating $\mathrm{Soc}(A)$ as in Lemma 1.2.1, and then construct $B = A \oplus A_{J,\pi}$ the normed algebra with identity $(\mathbf{1}, 0)$ in Theorem 3.4.2. We obviously need a relative notation to go further: For any representative class $\mathcal{P}'$ we denote the Shoda-completion relative to $\mathcal{P}'$ by $A_S(\mathcal{P}')$. Substructures of $A_S(\mathcal{P}')$ will be indicated correspondingly by using a superscript $\mathcal{P}'$ if needed; i.e. $B^{\mathcal{P}'}$, $A_{J,\pi}^{\mathcal{P}'}$ etc. We shall first show that if $\mathcal{P}'$ is any collection of rank one projections generating $\mathrm{Soc}(A)$ then $B^{\mathcal{P}}$ and $B^{\mathcal{P}'}$ are topologically and algebraically equivalent i.e. that the construction is essentially independent of the choice of representative class. Let $X = \prod_{p \in \mathcal{P}} E_p$. For each $p \in \mathcal{P}$ denote by π_p the projection map onto E_p. That is, if $f \in X$ then $\pi_p(f) = f(p) \in E_p$. Notice that for each function f its corresponding graph values take the form (p, wpw^{-1}), where w varies over $G_1(A)$ and p over $\mathcal{P}$. Every choice of representatives corresponds to some function belonging to X with $\mathcal{P}$ itself corresponding to the identity function $e(p) = p$ for each $p \in \mathcal{P}$. We adopt this notation, replacing a class of representatives by its corresponding function in X, in the remainder of our work. The next Definition 3.5.1 will be convenient, but we first need to recall the generalized Wedderburn-Artin Theorem (Theorem 2.2.1) which says $\oplus_{p' \in \mathcal{P}'} Ap' \otimes p'A \cong \mathrm{Soc}(A)$. If $f \in X$ then we denote the Wedderburn-Artin isomorphism onto $\mathrm{Soc}(A)$, corresponding to f, by ϕ_f.

Definition 3.5.1. We say that a subset U of X is *homogeneous* if the identity function e belongs to U and if for any $f, g \in U$, there exists a linear operator

$$\Phi_{f,g} : B^f \to B^g$$

such that:

(1) $\Phi_{f,g}(x, 0) = (x, 0)$ for $x \in A$.

(2) For $p, q \in \mathcal{P}$, there exist bounded linear isomorphisms

$$\psi_{f,g,p} : Af(p) \to Ag(p) \text{ and } \gamma_{f,g,q} : f(q)A \to g(q)A$$

such that:

(i) If $p \neq q$, then restriction of $\Phi_{f,g}$ to the subspace $Af(p) \otimes f(q)A$ of B^f satisfies the decomposition $\Phi_{f,g} = \psi_{f,g,p} \otimes \gamma_{f,g,q}$.

(ii) If $p = q$, then $\phi_f = \phi_g \circ (\psi_{f,g,p} \otimes \gamma_{f,g,q})$.

(iii) The trace is invariant under the operators $\psi_{f,g,p}$ and $\gamma_{f,g,q}$ in the following sense: $\mathrm{Tr}\ (f(q)yxf(p)) = \mathrm{Tr}\ (\gamma_{f,g,q}\,(f(q)y)\,\psi_{f,g,p}\,(xf(p)))$ for all $x, y \in A$.

(iv) For all $x, y \in A$, we have

$$y\psi_{f,g,p}\,(xf(p)) = \psi_{f,g,p}\,(yxf(p)) \text{ and } \gamma_{f,g,q}\,(f(q)x)\,y = \gamma_{f,g,q}\,(f(q)xy)\,.$$

(v) There is a real number $K > 0$ such that for all $p \in \mathcal{P}$, we have

$$\max\{\|\psi_{f,g,p}\|, \|\gamma_{f,g,p}\|\} \leq K.$$

Proposition 3.5.2. *The operator $\Phi_{f,g} : B^f \to B^g$ in Definition* 3.5.1 *is a continuous algebra isomorphism.*

Proof. By property (1), (i) and (v), and [14, Proposition 2.3], it readily follows that

$$\|\Phi_{f,g}\,(x, u)\| \leq \max\left\{1, K^2\right\}\|(x, u)\|$$

for all $(x, u) \in B^f$. Hence, $\Phi_{f,g}$ is continuous. Moreover, by property (1) and (i), and [14, p. 7], we may infer that $\Phi_{f,g}$ is bijective. It therefore remains to show that $\Phi_{f,g}$ is multiplicative: Let $x, y, x_1, x_2, y_1, y_2 \in A$ be arbitrary. Moreover, take any $p_1, p_2, q_1, q_2 \in \mathcal{P}$ with $p_i \neq q_i$ for each $i \in \{1, 2\}$. Set $u := x_1 f(p_1) \otimes f(q_1)\, y_1$ and $v := x_2 f(p_2) \otimes f(q_2)\, y_2$. Now, since $\Phi_{f,g}$ is linear, in order to prove that $\Phi_{f,g}$ is multiplicative it will suffice to show that

$$\Phi_{f,g}\,((x, u)\,(y, v)) = \Phi_{f,g}\,((x, u))\,\Phi_{f,g}\,((y, v))\,.$$

Observe that

$$\begin{aligned}
& \Phi_{f,g}\left((x,u)(y,v)\right) \\
= \ & \Phi_{f,g}\left([(x,0)+(0,u)]\,[(y,0)+(0,v)]\right) \\
= \ & \Phi_{f,g}\left((xy,0)\right)+\Phi_{f,g}\left((0,uy)\right)+\Phi_{f,g}\left((0,xv)\right)+\Phi_{f,g}\left((0,u)(0,v)\right) \\
= \ & (xy,0)+\Phi_{f,g}\left((0,u)\right)(y,0)+(x,0)\,\Phi_{f,g}\left((0,v)\right)+\Phi_{f,g}\left((0,u)(0,v)\right),
\end{aligned}$$

where the last equality sign follows from property (1) and (iv) in Definition 3.5.1. Moreover, note that

$$\begin{aligned}
& \Phi_{f,g}\left((x,u)\right)\Phi_{f,g}\left((y,v)\right) \\
= \ & \Phi_{f,g}\left((x,0)+(0,u)\right)\Phi_{f,g}\left((y,0)+(0,v)\right) \\
= \ & \left[(x,0)+\Phi_{f,g}\left((0,u)\right)\right]\left[(y,0)+\Phi_{f,g}\left((0,v)\right)\right] \\
= \ & (xy,0)+\Phi_{f,g}\left((0,u)\right)(y,0)+(x,0)\,\Phi_{f,g}\left((0,v)\right)+\Phi_{f,g}\left((0,u)\right)\Phi_{f,g}\left((0,v)\right),
\end{aligned}$$

where we have used property (1) in Definition 3.5.1. Consequently, we need only verify that

$$\Phi_{f,g}\left((0,u)(0,v)\right)=\Phi_{f,g}\left((0,u)\right)\Phi_{f,g}\left((0,v)\right).$$

If $q_1 \neq p_2$, then

$$\Phi_{f,g}\left((0,u)(0,v)\right)=\Phi_{f,g}(0)=0 \text{ and } \Phi_{f,g}\left((0,u)\right)\Phi_{f,g}\left((0,v)\right)=0,$$

which yields the result. So assume that $q_1 = p_2$. If $p_1 = q_2$, then equality follows from properties (1), (i), (ii) and (iii) in Definition 3.5.1. On the other hand, if $p_1 \neq q_2$, then equality follows from properties (i) and (iii) in Definition 3.5.1. This establishes that $\Phi_{f,g}$ is indeed multiplicative, which gives the result. $\square$

Theorem 3.5.3. *The normed algebras B^f and B^g generated by any two functions $f, g \in X$ are algebraically and topologically equivalent.*

Proof. Let $\mathcal{U}$ be the collection of all homogeneous subsets of X. Then $\mathcal{U} \neq \emptyset$ since $\{e\} \in \mathcal{U}$. Partially order $\mathcal{U}$ by set containment. If $\mathcal{C}$ is a chain in $\mathcal{U}$, then as usual $\cup\mathcal{C}$ furnishes an upperbound for $\mathcal{C}$; from which we infer the existence of a maximal element, say M. We claim that $M = X$: In order to prove our claim, it suffices to establish the following:

(i) If $\tilde{p}$ and q are rank one projections such that $\tilde{p} \in \mathcal{P}$ and $q \in E_{\tilde{p}}$, then there exists an $\tilde{f} \in M$ such that $\tilde{f}(\tilde{p}) = q$.

(ii) $M = \prod_{p \in \mathcal{P}} M_p$, where $M_p \subseteq E_p$ for each $p \in \mathcal{P}$.

We proceed as follows:
(i) Fix any $f \in M$ and define $\tilde{f} \in X$ by:

$$\tilde{f}(p) = \begin{cases} f(p) & \text{if } p \neq \tilde{p} \\ q & \text{if } p = \tilde{p}. \end{cases}$$

We shall prove that $\tilde{f} \in M$. By the maximality of M it suffices to show that $M \cup \left\{ \tilde{f} \right\}$ is homogeneous. To this end, let $g \in M$ be arbitrary but fixed. Since $f, g \in M$, there exists a linear operator $\Phi_{f,g} : B^f \to B^g$ satisfying properties (1) and (2) in Definition 3.5.1. By Theorem 3.2.3, $\tilde{f}\,(\tilde{p}) = q = tg\,(\tilde{p})\,t^{-1}$ for some $t \in G_1(A)$. If we define $\psi_{\tilde{f},g,\tilde{p}} : A\tilde{f}\,(\tilde{p}) \to Ag\,(\tilde{p})$ by

$$\psi_{\tilde{f},g,\tilde{p}}(y) = yt^{-1} \text{ for each } y \in A\tilde{f}\,(\tilde{p})\,,$$

it readily follows that $\psi_{\tilde{f},g,\tilde{p}}$ is a continuous linear isomorphism with $\left\|\psi_{\tilde{f},g,\tilde{p}}\right\| \leq \left\|t^{-1}\right\|_A$. Similarly, if we define $\gamma_{\tilde{f},g,\tilde{p}} : \tilde{f}\,(\tilde{p})\,A \to g\,(\tilde{p})\,A$ by

$$\gamma_{\tilde{f},g,\tilde{p}}(y) = ty \text{ for each } y \in \tilde{f}\,(\tilde{p})\,A,$$

then $\gamma_{\tilde{f},g,\tilde{p}}$ is a continuous linear isomorphism with $\left\|\gamma_{\tilde{f},g,\tilde{p}}\right\| \leq \|t\|_A$. In particular, we observe that the restriction of $\phi_{\tilde{f}}$ to $A\tilde{f}\,(\tilde{p}) \otimes \tilde{f}\,(\tilde{p})\,A$ satisfies

$$\phi_{\tilde{f}} = \phi_g \circ \left(\psi_{\tilde{f},g,\tilde{p}} \otimes \gamma_{\tilde{f},g,\tilde{p}}\right),$$

and that

$$y\psi_{\tilde{f},g,\tilde{p}}\,(xf\,(\tilde{p})) = \psi_{\tilde{f},g,\tilde{p}}\,(yxf\,(\tilde{p})) \text{ and } \gamma_{\tilde{f},g,\tilde{p}}\,(f\,(\tilde{p})\,x)\,y = \gamma_{\tilde{f},g,\tilde{p}}\,(f\,(\tilde{p})\,xy)$$

for all $x, y \in A$. Moreover, by Corollary 1.1.4 we have that

$$\text{Tr }(f\,(\tilde{p})\,yxf\,(\tilde{p})) = \text{Tr }\left(\gamma_{\tilde{f},g,\tilde{p}}\,(f\,(\tilde{p})\,y)\,\psi_{\tilde{f},g,\tilde{p}}\,(xf\,(\tilde{p}))\right)$$

for all $x, y \in A$. By Lemma 3.4.1, we can now use $\Phi_{f,g}$, $\psi_{\tilde{f},g,\tilde{p}}$ and $\gamma_{\tilde{f},g,\tilde{p}}$ to define a linear operator from $B^{\tilde{f}}$ into B^g satisfying all the required properties in Definition 3.5.1. Indeed, simply let $\Phi_{\tilde{f},g} : B^{\tilde{f}} \to B^g$ be the linear map defined as follows:

(a) $\Phi_{\tilde{f},g}(x, 0) = (x, 0)$ for $x \in A$.

(b) For each $p, r \in \mathcal{P}$ with $p \neq r$, the restriction of $\Phi_{\tilde{f},g}$ to the subspace $A\tilde{f}(p) \otimes \tilde{f}(r)A$ of $B^{\tilde{f}}$ is given by

$$\Phi_{\tilde{f},g} = \Delta_{\tilde{f},g,p} \otimes \nabla_{\tilde{f},g,r},$$

where

$$\Delta_{\tilde{f},g,p} = \begin{cases} \psi_{f,g,p} & \text{if } p \neq \tilde{p} \\ \psi_{\tilde{f},g,p} & \text{if } p = \tilde{p} \end{cases}$$

and

$$\nabla_{\tilde{f},g,r} = \begin{cases} \gamma_{f,g,r} & \text{if } r \neq \tilde{p} \\ \gamma_{\tilde{f},g,r} & \text{if } r = \tilde{p}. \end{cases}$$

Thus, since $g \in M$ was arbitrary, it follows that $M \cup \left\{\tilde{f}\right\}$ is homogeneous which proves (i).
(ii) Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be any nonempty disjoint subsets of $\mathcal{P}$ such that $\mathcal{Q}_1 \cup \mathcal{Q}_2 = \mathcal{P}$. Fix any f and g in M. Define $\tilde{g} \in X$ as follows:

$$\tilde{g}(p) = \begin{cases} f(p) & \text{if } p \in \mathcal{Q}_1 \\ g(p) & \text{if } p \in \mathcal{Q}_2. \end{cases}$$

In order to prove (ii) it suffices to show that $\tilde{g} \in M$. We shall once again utilize the maximality of M and simply show that the set $M \cup \{\tilde{g}\}$ is homogeneous. To this end, let $h \in M$ be arbitrary. Since $f, g, h \in M$, there exist linear operators $\Phi_{f,h} : B^f \to B^h$ and $\Phi_{g,h} : B^g \to B^h$ satisfying the properties in Definition 3.5.1. Now, if we once again use Lemma 3.4.1 and define the linear operator $\Phi_{\tilde{g},h} : B^{\tilde{g}} \to B^h$ by:

(a) $\Phi_{\tilde{g},h}(x, 0) = (x, 0)$ for $x \in A$.

(b) For each $p, r \in \mathcal{P}$ with $p \neq r$, the restriction of $\Phi_{\tilde{g},h}$ to the subspace $A\tilde{g}(p) \otimes \tilde{g}(r)A$ of $B^{\tilde{g}}$ is given by

$$\Phi_{\tilde{g},h} = \Delta_{\tilde{g},h,p} \otimes \nabla_{\tilde{g},h,r},$$

where

$$\Delta_{\tilde{g},h,p} = \begin{cases} \psi_{f,h,p} & \text{if } p \in \mathcal{Q}_1 \\ \psi_{g,h,p} & \text{if } p \in \mathcal{Q}_2 \end{cases}$$

and

$$\nabla_{\tilde{f},g,r} = \begin{cases} \gamma_{f,h,r} & \text{if } p \in \mathcal{Q}_1 \\ \gamma_{g,h,r} & \text{if } p \in \mathcal{Q}_2, \end{cases}$$

then $\Phi_{\tilde{g},h}$ satisfies all the conditions in Definition 3.5.1. Thus, since $h \in M$ was arbitrary, it follows that $M \cup \{\tilde{g}\}$ is homogeneous which proves (ii) and hence establishes our claim. By Proposition 3.5.2, this gives the result. □

Corollary 3.5.4. *If $A_S(\mathcal{P})$ and $A_S(\mathcal{P}')$ are Shoda-completions with respect to P and P', respectively, then $A_S(\mathcal{P})$ and $A_S(\mathcal{P}')$ are algebraically and topologically equivalent.*

Proof. If $\mathcal{P}'$ corresponds to the function, say $f \in X$, then by Theorem 3.5.3 B^f and B^e are algebraically and topologically equivalent. Therefore, since they have the same Cauchy sequences, B^f and B^e have norm completions which are algebraically and topologically equivalent. Factorizing the radicals, if necessary, we obtain the conclusion. $\square$

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Index