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Binary permutation sequences as subsets of Levenshtein codes, spectral null codes, run-length limited codes and constant weight codes

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Abstract We investigate binary sequences which can be obtained by concatenating the columns of (0,1)-matrices derived from permutation sequences. We then prove that these binary sequences are subsets of a surprisingly diverse ensemble of codes, namely the Levenshtein codes, capable of correcting insertion/deletion errors; spectral null codes, with spectral nulls at certain frequencies; as well as being subsets of run-length limited codes, Nyquist null codes and constant weight codes.

Keywords Permutation codes · Insertion/deletion correcting codes · Constant weight codes · Spectral null codes · Run-length limited codes

AMS Classifications 20B30 · 20B35 · 68P30 · 94A05 · 94A24 · 94B50 · 94B60

1 Introduction

A permutation sequence of non repetitive symbols of length M is an ordered arrangement of a specified number of symbols selected from a set, such as $\{1, 2, \ldots, M\}$. The total number of different permutations of M symbols from the total sequence length which is M is $M!$.

S_M denotes the symmetric group, which consists of all the possible $M!$ permutations of length M. As example, $S_3 = \{123, 132, 213, 231, 312, 321\}$.

Recent work in power-line communications [5,14] has renewed interest in permutation codes, which led to several papers regarding permutation mappings [3,13], as well as papers for constructing new permutation arrays [4]. In this paper we investigate some interesting properties of binary sequences which are derived from permutation codes.

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Blake, Cohen and Deza [2] made use of a permutation to binary transformation to decode permutation array codes. (The same transformation will be used in this paper.) They showed that the permutation array forms an (n, k, d) q-ary code, which is transformed to an $(nq, k, 2d)$ binary code which is decodable using majority logic.

Related work was done by Ferreira et al. [6], where it was proved that some higher order spectral null codes are subcodes of the balanced Levenshtein codes. In this paper we will show that the binary sequences obtained from permutation sequences are subsets of the Levenshtein codes, the spectral null codes and other codes.

We consider permutation sequences written in the passive form, such as $12\ldots M$, where each of the symbols are written as a binary sequence of length M, with the symbol value indicating where a one is to appear and zeros everywhere else (similar to pulse position modulation). As example for $M = 3$ we have

$$\begin{array}{l} 1 \rightarrow 100 \\ 2 \rightarrow 010. \\ 3 \rightarrow 001 \end{array} \tag{1}$$

The permutation sequences for $M = 3$ are thus changed to

$$\begin{Bmatrix} 123, 132 \\ 213, 231 \\ 312, 321 \end{Bmatrix} \rightarrow \begin{Bmatrix} 100010001, 100001010 \\ 010100001, 010001100 \\ 001100010, 001010100 \end{Bmatrix}. \tag{2}$$

Therefore, each of the $M!$ permutation sequences can be converted to binary sequences of length M^2.

An alternative representation is that of (0,1)-matrices, where only a single 1 is allowed in every column and every row. As example, the permutation sequence 2431 will be

$$\begin{bmatrix} 0&0&0&1 \\ 1&0&0&0 \\ 0&0&1&0 \\ 0&1&0&0 \end{bmatrix}$$

and the binary sequence is then constructed by concatenating the columns to form 0100000100101000. We will use $\omega = 1$ to denote that only a single 1 is allowed in each row and each column.

After converting all the permutation sequences of length M to binary sequences, we define the binary permutation code, $\mathscr{P}_1(M^2)$, as the code containing all these binary sequences of length M^2. The cardinality of $\mathscr{P}_1(M^2)$ is $|\mathscr{P}_1(M^2)| = M!$.

Similarly, (0,1)-matrices with ω 1s in every column and every row can be regarded as an extension of permutations. For $\omega = 2$, we can have (12)(13)(24)(34), which is

$$\begin{bmatrix} 1&1&0&0 \\ 1&0&1&0 \\ 0&1&0&1 \\ 0&0&1&1 \end{bmatrix},$$

and by permuting the columns we can obtain other possible matrices.

In general, we will use $\mathscr{P}_\omega(M^2)$ to denote the code containing all the possible binary sequences that is obtained from (0,1)-matrices with ω ones in each row and each column.

2 Levenshtein codes

Levenshtein [9] showed that the 2^n binary sequences of length n can be partitioned into codebooks, each capable of correcting a single insertion or deletion error. If $\mathbf{x} = x_1 x_2 \ldots x_n$, then the binary sequences can be partitioned by using

$$\sum_{i=1}^{n} i x_i \equiv a \pmod{m},$$

for a fixed a, where $0 \leq a \leq m-1$. The sequences are thus partitioned into m distinct codebooks where each codebook is denoted by the integer a. Let $\mathscr{L}(n, a, m)$ denote the Levenshtein codebook in partition a. For Levenshtein's first class of codes, we require that $m \geq n+1$ and for Levenshtein's second class of codes, we require that $m \geq 2n$. For the highest cardinality, $m = n+1$ and $m = 2n$ are used respectively. To simplify the calculations to follow, we let $\sigma = \sum_{i=1}^{n} i x_i$.

2.1 Levenshtein's first class of codes

2.1.1 Sequences with $\omega = 1$

As example, for the $\mathscr{P}_1(3^2)$ sequence 100010001 we get

$$\sigma = 1 + 5 + 9 = 15,$$

which can be verified for the other sequences in (2) as well. With $n = M^2 = 9$ and $m = n+1 = 10$, we have $\sigma \equiv a \pmod{10}$ which results in all the sequences being a subset of the first class Levenshtein code words in the $a = 5$ partition.

We will now prove for the general case that $\mathscr{P}_1(M^2) \subset \mathscr{L}(n, a, m)$, with $n = M^2$, $m = n+1$ and some a, $0 \leq a \leq n$.

Proposition 1 *The $\mathscr{P}_1(M^2)$ code is a subset of the Levenshtein code, $\mathscr{L}(M^2, a, M^2+1)$, as follows*

1. *If M is even, then $\mathscr{P}_1(M^2) \subset \mathscr{L}(M^2, 0, M^2+1)$.*
2. *If M is odd, then $\mathscr{P}_1(M^2) \subset \mathscr{L}(M^2, \frac{M^2+1}{2}, M^2+1)$.*

Proof As we saw in (1), each sequence of length M^2 consists of M subsequences of length M. Since it is a permutation, these subsequences will always be present, just in different positions. Therefore, each subsequence is shifted multiples of M relative to each other in the large sequence. If we use any arbitrary permutation sequence, $p_1 p_2 \ldots p_M$, then we have

$$\begin{aligned}\sigma &= p_1 + (p_2 + M) + (p_3 + 2M) + (p_4 + 3M) + \cdots + (p_M + (M-1)M) \\ &= (p_1 + p_2 + p_3 + \cdots + p_M) + (M + 2M + 3M + \cdots + (M-1)M),\end{aligned}$$

also showing that the actual position of the symbols in the permutation plays no role in the sum.

We have for

$$\begin{aligned}&M = 2 \rightarrow \sigma = (1+2) + 2 \\ &M = 3 \rightarrow \sigma = (1+2+3) + 3(1+2) \\ &M = 4 \rightarrow \sigma = (1+2+3+4) + 4(1+2+3) \\ &\quad\vdots\end{aligned}$$

Using identities for sums, we have for M in general

$$\sigma = \frac{M}{2}(M^2 + 1).$$

For Levenshtein codes, $\sigma \equiv a \pmod{n+1}$, then with $n = M^2$ we require that

$$\frac{M}{2}(M^2 + 1) \equiv a \quad (\text{mod } M^2 + 1).$$

1. If M is even, then $M/2$ will be some integer, say r, and σ will be divisible by $M^2 + 1$ such that

$$r(M^2 + 1) \equiv a \quad (\text{mod } M^2 + 1) \Rightarrow a = 0.$$

   Thus, for M even, $\mathscr{P}_1(M^2) \subset \mathscr{L}(M^2, 0, M^2 + 1)$.
2. If M is odd, then $M/2$ will be some value, say $r + 1/2$ where r is some integer. Then,

$$r(M^2 + 1) + \frac{M^2 + 1}{2} \equiv a \ (\text{mod } M^2 + 1) \Rightarrow a = \frac{M^2 + 1}{2}.$$

   Thus, for M odd, $\mathscr{P}_1(M^2) \subset \mathscr{L}(M^2, \frac{M^2+1}{2}, M^2 + 1)$. □

2.1.2 Sequences with any ω

In a similar manner, we can show that (0,1)-matrices with other ω-values are also subsets of the Levenshtein code.

Proposition 2 *The $\mathscr{P}_\omega(M^2)$ code is a subset of the Levenshtein code, $\mathscr{L}(M^2, a, M^2 + 1)$, as follows*

1. *If M is even and for any ω, then $\mathscr{P}_\omega(M^2) \subset \mathscr{L}(M^2, 0, M^2 + 1)$,*
2. *If M is odd and ω is even, then $\mathscr{P}_\omega(M^2) \subset \mathscr{L}(M^2, 0, M^2 + 1)$,*
3. *If M is odd and ω is odd, then $\mathscr{P}_\omega(M^2) \subset \mathscr{L}(M^2, \frac{M^2+1}{2}, M^2 + 1)$.*

Proof Any $\omega = 2$ (0,1)-matrix can be constructed from two $\omega = 1$ (0,1)-matrices by XORing them, as in

$$\begin{bmatrix} 1\,0\,0\,0 \\ 0\,1\,0\,0 \\ 0\,0\,1\,0 \\ 0\,0\,0\,1 \end{bmatrix} \oplus \begin{bmatrix} 0\,1\,0\,0 \\ 0\,0\,1\,0 \\ 0\,0\,0\,1 \\ 1\,0\,0\,0 \end{bmatrix} = \begin{bmatrix} 1\,1\,0\,0 \\ 0\,1\,1\,0 \\ 0\,0\,1\,1 \\ 1\,0\,0\,1 \end{bmatrix} \tag{3}$$

or equivalently 1000010000100001 ⊕ 0001100001000010 = 100111000110001 for the binary sequences. For each $\omega = 1$ sequence we know that $\sigma = M(M^2 + 1)/2$ and thus for any $\omega = 2$ sequence we will have $\sigma = M(M^2 + 1)$. For ω in general, $\sigma = \omega M(M^2 + 1)/2$, therefore we require $\omega M(M^2 + 1)/2 \equiv a \ (\text{mod } M^2 + 1)$.

1. As before, with M even, $M/2$ will be some integer r, then

$$\omega r(M^2 + 1) \equiv a \ (\text{mod } M^2 + 1) \Rightarrow a = 0.$$

   Thus, for M even and any ω, $\mathscr{P}_\omega(M^2) \subset \mathscr{L}(M^2, 0, M^2 + 1)$.

2. Similarly, with M odd, $M/2$ will be some value $r + 1/2$ where r is some integer, then

$$\sigma = \omega(r + \frac{1}{2})(M^2 + 1)$$
$$= (\omega r + \frac{\omega}{2})(M^2 + 1).$$

If ω is even, then $\omega/2$ is some integer q, resulting in

$$(\omega r + q)(M^2 + 1) \equiv a \pmod{M^2 + 1} \Rightarrow a = 0.$$

Thus, for M odd and ω even, $\mathscr{P}_\omega(M^2) \subset \mathscr{L}(M^2, 0, M^2 + 1)$.

3. If ω is odd, $\omega/2$ will be some value $q + 1/2$ where q is some integer, then

$$\sigma = \omega r(M^2 + 1) + (q + \frac{1}{2})(M^2 + 1),$$

with

$$(\omega r + q)(M^2 + 1) + \frac{1}{2}(M^2 + 1) \equiv a \pmod{M^2 + 1} \Rightarrow a = \frac{M^2 + 1}{2}.$$

Thus, for M odd and ω odd, $\mathscr{P}_\omega(M^2) \subset \mathscr{L}(M^2, \frac{M^2+1}{2}, M^2 + 1)$. □

It is interesting to note that for M even and ω any integer, $1 \leq \omega \leq M - 1$, all the binary sequences are a subset of the first class Levenshtein code in partition $a = 0$. Thus, for M even, we have

$$\mathscr{P}_1(M^2) \cup \mathscr{P}_2(M^2) \cup \cdots \cup \mathscr{P}_{M-1}(M^2) \subset \mathscr{L}(M^2, 0, M^2 + 1).$$

For M odd, the binary sequences are split between partitions $a = 0$ and $a = \frac{M^2+1}{2}$, depending on ω being even or odd. For M odd we have

$$\mathscr{P}_2(M^2) \cup \mathscr{P}_4(M^2) \cup \cdots \cup \mathscr{P}_{M-1}(M^2) \subset \mathscr{L}(M^2, 0, M^2 + 1),$$

$$\mathscr{P}_1(M^2) \cup \mathscr{P}_3(M^2) \cup \cdots \cup \mathscr{P}_{M-2}(M^2) \subset \mathscr{L}(M^2, \frac{M^2 + 1}{2}, M^2 + 1).$$

2.2 Levenshtein's second class of codes

We will now prove for the general case that $\mathscr{P}_\omega(M^2) \subset \mathscr{L}(M^2, a, 2M^2)$, for some a, $0 \leq a \leq 2n$. Since $\omega = 1$ is a special case, we will go directly to the case for any ω.

Proposition 3 *Let* $M \equiv r \pmod 4$ *and* $\omega \equiv q \pmod 4$*, then the* $\mathscr{P}_\omega(M^2)$ *code is a subset of the Levenshtein code,* $\mathscr{L}(M^2, a, 2M^2)$*, as follows*

1. *if* $(r, q) = (0,0), (0,1), (0,2), (0,3), (1,0), (2,0), (2,2)$ *or* $(3,0)$*, then* $\mathscr{P}_\omega(M^2) \subset \mathscr{L}(M^2, \frac{\omega M}{2}, 2M^2)$*,*
2. *if* $(r, q) = (1,2), (2,1), (2,3)$ *or* $(3,2)$*, then* $\mathscr{P}_\omega(M^2) \subset \mathscr{L}(M^2, \frac{2M^2+\omega M}{2}, 2M^2)$*,*
3. *if* $(r, q) = (1,1)$ *or* $(3,3)$*, then* $\mathscr{P}_\omega(M^2) \subset \mathscr{L}(M^2, \frac{M^2+\omega M}{2}, 2M^2)$*,*
4. *if* $(r, q) = (1,3)$ *or* $(3,1)$*, then* $\mathscr{P}_\omega(M^2) \subset \mathscr{L}(M^2, \frac{3M^2+\omega M}{2}, 2M^2)$*.*

Proof For Levenshtein codes, $\sigma \equiv a \pmod{2n}$, then with $n = M^2$ we require that

$$\frac{\omega M}{2}(M^2 + 1) \equiv a \pmod{2M^2}.$$

We will only prove selected cases as the other cases follow in a similar manner.

1. If $(r, q) = (0, 0)$ then $M = 4r'$ where r' is some integer, then
$$\omega r'(2M^2) + \frac{\omega M}{2} \equiv a \pmod{2M^2} \Rightarrow a = \frac{\omega M}{2}.$$
Thus, $\mathscr{P}_\omega(M^2) \subset \mathscr{L}(M^2, \frac{\omega M}{2}, 2M^2)$.
2. If $(r, q) = (1, 2)$ then $M = 4r' + 1$ and $\omega = 4q' + 2$, where r' and q' are some integers, then
$$\begin{aligned}\sigma &= \frac{1}{2}(4q' + 2)(4r' + 1)(M^2) + \frac{\omega M}{2} \\ &= (8r'q' + 2q' + 4r' + 1)(M^2) + \frac{\omega M}{2} \\ &= (4r'q' + q' + 2r')(2M^2) + \frac{\omega M}{2} + M^2,\end{aligned}$$
resulting in
$$(4r'q' + q' + 2r')(2M^2) + \frac{\omega M + 2M^2}{2} \equiv a \pmod{2M^2} \Rightarrow a = \frac{\omega M + 2M^2}{2}.$$
Thus, $\mathscr{P}_\omega(M^2) \subset \mathscr{L}(M^2, \frac{2M^2+\omega M}{2}, 2M^2)$.
3. If $(r, q) = (1, 1)$ then $M = 4r' + 1$ and $\omega = 4q' + 1$, where r' and q' are some integers, then
$$\begin{aligned}\sigma &= \frac{1}{2}(4q' + 1)(4r' + 1)(M^2) + \frac{\omega M}{2} \\ &= (8r'q' + 2r' + 2q' + \frac{1}{2})(M^2) + \frac{\omega M}{2} \\ &= (4r'q' + q' + r')(2M^2) + \frac{\omega M}{2} + \frac{M^2}{2},\end{aligned}$$
resulting in
$$(4r'q' + q' + r')(2M^2) + \frac{\omega M + M^2}{2} \equiv a \pmod{2M^2} \Rightarrow a = \frac{\omega M + M^2}{2}.$$
Thus, $\mathscr{P}_\omega(M^2) \subset \mathscr{L}(M^2, \frac{M^2+\omega M}{2}, 2M^2)$.
4. If $(r, q) = (1, 3)$ then $M = 4r' + 1$ and $\omega = 4q' + 3$, where r' and q' are some integers, then
$$\begin{aligned}\sigma &= \frac{1}{2}(4q' + 3)(4r' + 1)(M^2) + \frac{\omega M}{2} \\ &= \left(8r'q' + 2q' + 6r' + \frac{3}{2}\right)(M^2) + \frac{\omega M}{2} \\ &= (4r'q' + q' + 3r')(2M^2) + \frac{\omega M}{2} + \frac{3M^2}{2},\end{aligned}$$
resulting in
$$(4r'q' + q' + 3r')(2M^2) + \frac{\omega M + 3M^2}{2} \equiv a \pmod{2M^2} \Rightarrow a = \frac{\omega M + 3M^2}{2}.$$
Thus, $\mathscr{P}_\omega(M^2) \subset \mathscr{L}(M^2, \frac{3M^2+\omega M}{2}, 2M^2)$. □

3 Spectral null codes

The technique of designing codes to have a spectrum with nulls occurring at certain frequencies started with Gorog [7], when he considered the vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $x_i \in \{-1, +1\}$, $1 \leq i \leq n$, to be an element of a set S. The Fourier transform,

$$X(w) = \sum_{i=1}^{n} x_i e^{-jiw}, \quad -\pi \leq w \leq \pi, \tag{4}$$

is applied to all codewords in S. Having nulls at certain frequencies are then the same as having the power spectral density function equal to zero at those frequencies. This means that $H(w) = 0$, where

$$H(w) = \frac{1}{n|S|} \sum_{i=0}^{n-1} \left| X^{(i)}(w) \right|^2, \tag{5}$$

where $X^{(i)}(w)$ is the Fourier transform of the i-th element in S. It can be seen from (5), that the power spectral density depends on the frequency value, so a codebook can be designed to generate nulls at certain frequencies.

Usually for simplification the codeword length, n, is represented as an integer multiple of N, then

$$n = Nz,$$

where $f = r/N$ represents the spectral nulls at rational sub multiples r/N [8]. We have to satisfy

$$A_1 = A_2 = \cdots = A_N, \tag{6}$$

where

$$A_i = \sum_{\lambda=0}^{z-1} x_{i+\lambda N}, \quad i = 1, 2, \ldots, N. \tag{7}$$

In the case where N is not prime, suppose that $N = cd$, where c and d are integer factors of N. The equation which leads to nulls is

$$A_u = A_{u+vc},$$

where $u = 1, 2, \ldots, c$, $v = 1, 2, \ldots, d-1$ and A_u is the same as in (7).

If all the codewords in a codebook satisfy these equations, the codebook will exhibit nulls at the required frequencies.

As example, for $\mathscr{P}_1(3^2)$ with 123 written as 100010001 and mapping $\{0, 1\} \to \{-1, +1\}$ we find

$$\begin{aligned} A_1 &= 1 + (-1) + (-1) = -1 \\ A_2 &= (-1) + 1 + (-1) = -1 \\ A_3 &= (-1) + (-1) + 1 = -1, \end{aligned}$$

and checking all sequences in $\mathscr{P}_1(3^2)$, we get

$$A_1 = A_2 = A_3.$$

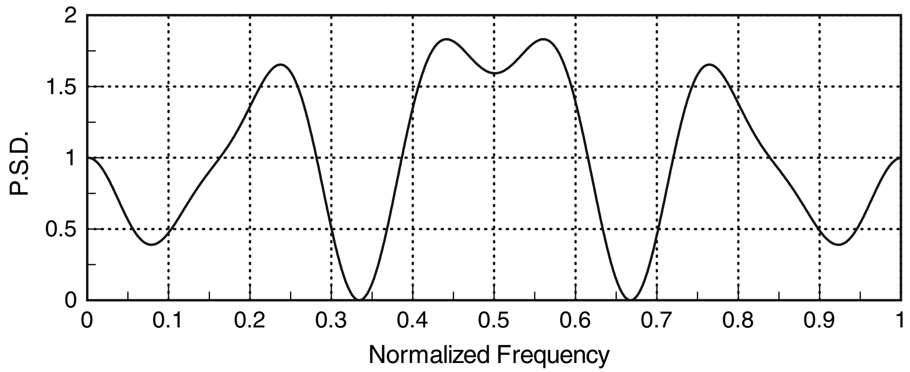

Fig. 1 Power spectral density of $\mathscr{P}_1(3^2)$

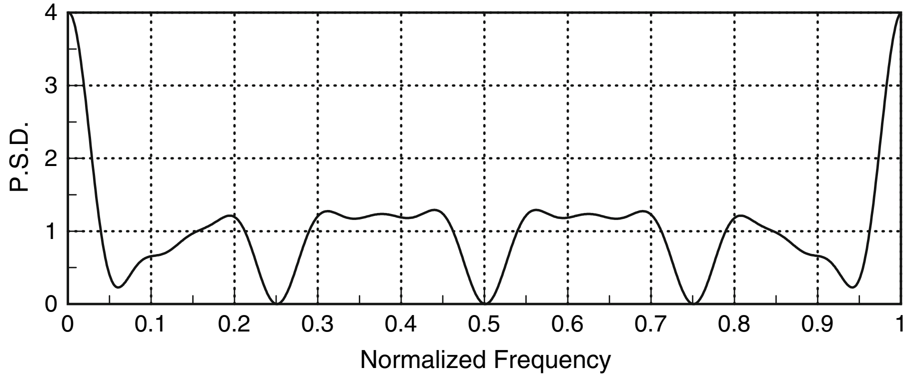

Fig. 2 Power spectral density of $\mathscr{P}_1(4^2)$

In the case of M not a prime number, we have as example $M = 4,\ c = 2$ and $d = 2$, where for 1234 written as 1000010000100001 and mapping $\{0, 1\} \to \{-1, +1\}$ we find

$$\begin{aligned} A_1 &= 1 + (-1) + (-1) + (-1) = -2 \\ A_2 &= (-1) + 1 + (-1) + (-1) = -2 \\ A_3 &= (-1) + (-1) + 1 + (-1) = -2 \\ A_4 &= (-1) + (-1) + (-1) + 1 = -2, \end{aligned}$$

and for all sequences in $\mathscr{P}_1(4^2)$, we have

$$\left.\begin{aligned} A_1 &= A_3 \\ A_2 &= A_4 \end{aligned}\right\} \Rightarrow A_1 = A_2 = A_3 = A_4.$$

It is clear that for $\mathscr{P}_1(3^2)$ and $\mathscr{P}_1(4^2)$, we have spectral null codes with nulls at frequency multiples of $1/3$ and $1/4$ respectively, as depicted in Figs. 1 and 2.

Proposition 4 *The $\mathscr{P}_\omega(M^2)$ code is a subset of the spectral null codes with nulls at the frequency $f = r/M$. When M is even, a null is present at the Nyquist frequency. In addition, the codes are also dc-free when M is even and $\omega = M/2$.*

Proof From $n = Nz$ we have $n = M^2,\ N = M$ and $z = M$. Using an arbitrary matrix

$$\begin{bmatrix} b_{11} & b_{21} & \cdots & b_{M1} \\ b_{12} & b_{22} & \cdots & b_{M2} \\ \vdots & \vdots & \ddots & \vdots \\ b_{1M} & b_{2M} & \cdots & b_{MM} \end{bmatrix},$$

where the numbers of ones in each column and each row sums to ω.

When we concatenate the columns we obtain a $\mathscr{P}_\omega(M^2)$ sequence, x, as in

$$\begin{aligned} x &= x_1 x_2 \ldots x_{M^2} \\ &= b_{11} \ldots b_{1M} b_{21} \ldots b_{2M} \ldots b_{M1} \ldots b_{MM}, \end{aligned}$$

where

$$x_{j+(i-1)M} = b_{ij}. \tag{8}$$

We know that for each matrix the sum of each row and column are the same, thus the sum R_j for the j-th row is

$$R_j = \sum_{i=1}^{M} b_{ij}, \quad 1 \le j \le M.$$

Using (8), this sum becomes

$$R_j = \sum_{i=1}^{M} x_{j+(i-1)M}$$

and this is the same as the sum in (7). Therefore,

$$A_1 = A_2 = \cdots = A_M,$$

irrespective of M being prime or not, proving that $\mathscr{P}_\omega(M^2)$ is a spectral null code.

We use the mapping $\{0, 1\} \to \{-1, +1\}$ and calculate the running alternate sum (RAS). When M is even, let $\alpha \in \{2, 4, \ldots, M^2\}$ and $\beta \in \{1, 3, \ldots, M^2 - 1\}$. We have $i \in \{\alpha, \beta\},\ \alpha = 2r$ and $\beta = 2r - 1,\ 1 \le r \le M^2/2$.

$$\begin{aligned} \text{RAS} &= \sum_{i=1}^{M^2} (-1)^i x_i \\ &= \sum_{r=1}^{M^2/2} (-1)^{2r} x_{2r} + \sum_{r=1}^{M^2/2} (-1)^{2r-1} x_{2r-1} \\ &= \sum_{r=1}^{M^2/2} x_{2r} - \sum_{r=1}^{M^2/2} x_{2r-1} \\ &= \frac{M}{2}(2\omega - M) - \frac{M}{2}(2\omega - M) \\ &= 0. \end{aligned}$$

Therefore a null will be present at the Nyquist frequency when M is even.

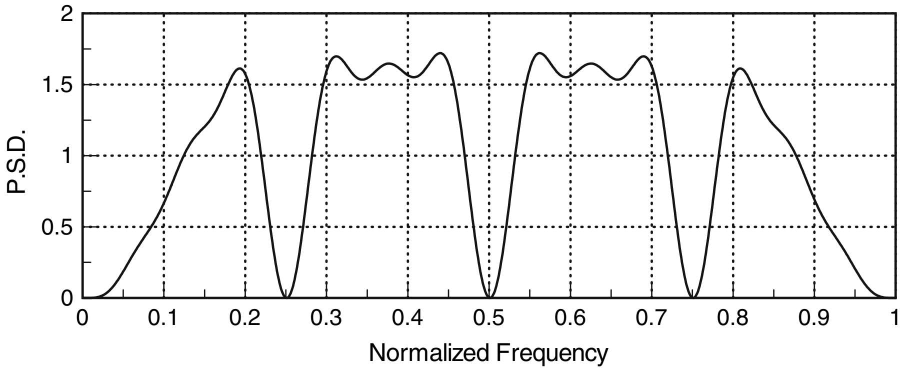

Fig. 3 Power spectral density of $\mathscr{P}_2(4^2)$

We now calculate the running digital sum (RDS) as

$$\begin{aligned} \mathrm{RDS} &= \sum_{i=1}^{M^2} x_i \\ &= R_1 + R_2 + \ldots + R_M \\ &= M R_1 \\ &= M(\omega - (M - \omega)) \\ &= M(2\omega - M). \end{aligned}$$

Clearly when M is even and $\omega = M/2$ we have RDS $= 0$, proving that $\mathscr{P}_{M/2}(M^2)$ are dc-free codes. □

As example, for $\mathscr{P}_2(4^2)$ with the sequence (12)(23)(34)(14), (which is the matrix in (3)) and mapping $\{0, 1\} \rightarrow \{-1, +1\}$, we have

$$\begin{aligned} A_1 &= 1 + 1 + (-1) + (-1) = 0 \\ A_2 &= (-1) + 1 + 1 + (-1) = 0 \\ A_3 &= (-1) + (-1) + 1 + 1 = 0 \\ A_4 &= 1 + (-1) + (-1) + 1 = 0, \end{aligned}$$

and checking all sequences in $\mathscr{P}_2(4^2)$, we get

$$A_1 = A_2 = A_3 = A_4.$$

Figure 3 shows that the $\mathscr{P}_2(4^2)$ code is dc-free, in addition to having nulls at multiples of $f = 1/4$.

It is clear that for $\mathscr{P}_1(4^2)$, $\mathscr{P}_2(4^2)$ and $\mathscr{P}_1(6^2)$, we have a spectral null at the Nyquist frequency, as depicted in Figs. 2, 3 and 4.

4 Constant weight codes

The $\mathscr{P}_\omega(M^2)$ codes are a class of constant weight codes. The cardinalities of these codes are usually denoted by $|\mathscr{A}(M^2, d_{\min}, \omega M)|$, where the length of the binary block code is equal to M^2 and the minimum Hamming distance $d_{\min}$ is equal to 4 and the weight is ωM.

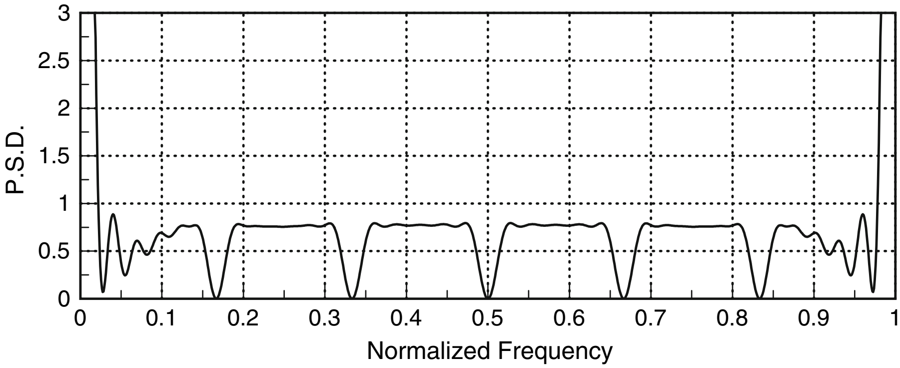

Fig. 4 Power spectral density of $\mathscr{P}_1(6^2)$

Proposition 5 *All codewords in* $\mathscr{P}_\omega(M^2)$ *has a weight of* ωM *and* $d_{\min} = 4$.

Proof The proof is straightforward, we have a codeword with length M^2 and in each M bits we have ω bits. So the weight of the codeword is ωM.

From [2] we know that any (n, k, d) M-ary permutation code is changed to an $(nq, k, 2d)$ binary code using the previous transformation. All symmetric permutation groups have minimum Hamming distances of 2, thus the transformed binary codes will have a minimum Hamming distance of 4.

Also, as we saw in (3), any ω sequence can be created from $\omega = 1$ sequences, and therefore in general $\mathscr{P}_\omega(M^2)$ will have $d_{\min} = 4$. $\square$

5 Run-length limited codes

Codes that have a restriction on the number of consecutive 1s or 0s in a sequence are generally called run-length limited codes [8]. They are described with two parameters denoted as d and k, which respectively represent the minimum and the maximum number of 0s between two 1s in a sequence.

Proposition 6 $\mathscr{P}_\omega(M^2)$ *is a subset of the fixed-length* $(d, k) = (0, 2M - 2\omega)$*-code.*

Proof For $\omega = 1$, we know from the constraints of permutations and (1) that a 1 will appear somewhere in every M bits. The maximum number of 0s will thus occur when the 1s are on the extremes of $2M$ bits, as in 10000001 for $M = 4$. The minimum number of 0s will obviously occur when the 1s are next to each other, as in 00011000. The maximum number of 0s that can occur at the start and end of the sequences is $M - 1$, thus concatenating any two such sequences will result in $2(M - 1)$ 0s next to each other. Therefore, for $\omega = 1$ we have $d = 0$ and $k = 2M - 2$.

For any ω, we know that ω 1s will appear somewhere in every M bits. Again, the maximum number of 0s occur when these 1s are on the extremes of the $2M$ bits,

$$\overbrace{\underbrace{1\ldots1}_{\omega}0\ldots00\ldots0\underbrace{1\ldots1}_{\omega}}^{2M}.$$

The minimum number of 0s occur when 1s are next to each other. The maximum number of 0s that can occur at the start and end of the sequences is $M - \omega$, thus concatenation will

Table 1 $|\mathscr{P}_\omega(M)|$ compared to cardinalities of other codes

Code	M			
	3	4	5	6
$\|\mathscr{P}_1(M^2)\|$	6	24	120	720
$\|\mathscr{P}_2(M^2)\|$	6	90	2040	67950
$\|\mathscr{P}_3(M^2)\|$	1	24	2040	297200
$\|\mathscr{P}_4(M^2)\|$	0	1	120	67950
$\|\mathscr{P}_5(M^2)\|$	0	0	1	720
$\|\mathscr{A}(M^2, M, 4)\|$	12	140	2334	–
$\|\mathscr{A}(M^2, 2M, 4)\|$	12	1170	140605	–
$\|\mathscr{A}(M^2, 3M, 4)\|$	1	140	–	–
$\|\mathscr{L}(M^2, 0, M^2+1)\|$	52	3856	1290556	1857283156
$\|\mathscr{N}(M^2, M)\|$	56	1810	17318417	–
$\|\mathscr{R}(M^2, 0, 2M-2)\|$	376	55906	31049952	66212034562

result in a maximum of $2(M-\omega)$ 0s next to each other. Therefore, for $\omega,\ 1 \le \omega \le M-1$ we have $d = 0$ and $k = 2M - 2\omega$. □

6 Cardinalities

Table 1 compares the cardinalities for the various codes with that of selected $\mathscr{P}_\omega(M^2)$ codes.

The values for $|\mathscr{P}_\omega(M^2)|,\ 2 \le \omega \le 5$ can be found as the number of $M \times M$ arrays containing ω ones in each row and each column. $|\mathscr{P}_2(M^2)|,\ |\mathscr{P}_3(M^2)|,\ |\mathscr{P}_4(M^2)|$ and $|\mathscr{P}_5(M^2)|$ can be found on the "On-line Encyclopedia of Integer Sequences" [12] as sequences A001499, A001501, A058528 and A075754 respectively.

$|\mathscr{A}(n, w, d)|$ is the cardinality of the constant weight binary codes of length n, weight w and minimum Hamming distance of d. Tables with the largest cardinalities found for this class of codes can be found in [1] and [10]. Note that for some of the entries in Table 1 no value could be found in the references cited. The cardinalities of the $d = 4$ constant weight codes are included, as the $\mathscr{P}_\omega(M)$ codes also have a minimum Hamming distance of 4.

The values for $|\mathscr{L}(n, 0, n+1)|$ can be found using the following equation [11]

$$|\mathscr{L}(n+1, 0, n+2)| = \frac{1}{2n} \sum_{\text{odd } d|n} \phi(d) 2^{n/d}, \quad n \ge 1,$$

where the sum is over all odd divisors d of n and ϕ is the Euler totient function. Only the $a = 0$ partition is considered, as this partition provides the maximum cardinality.

Finally, $|\mathscr{N}(n, N)|$ is used to denote the spectral null codes for length n with nulls at submultiples of $1/N$ and $|\mathscr{R}(n, d, k)|$ is used to denote the fixed length run-length limited codes of length n, with the d and k constraints as defined earlier.

We see that the cardinality of the $\mathscr{P}_1(M^2)$ codes are lower when compared with the three other codes with similar properties. However, considering that the $\mathscr{P}_1(M^2)$ is a subset of all the other four codes, we must consider the intersection of the other four codes, as depicted

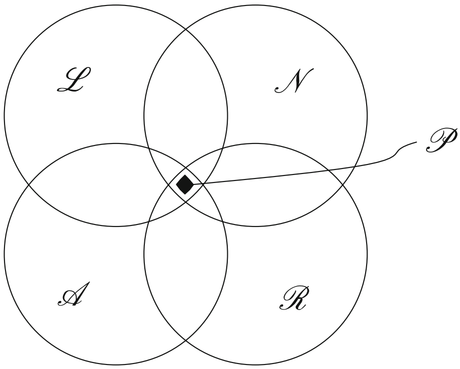

Fig. 5 $\mathscr{P}$ as subsets of other codes

in Fig. 5. For $M = 3$ and $M = 4$ we find that

$$\mathscr{L}(9, 5, 10) \cap \mathscr{A}(9, 3, 4) \cap \mathscr{N}(9, 3) \cap \mathscr{R}(9, 0, 4) = \mathscr{P}_1(3^2)$$

and

$$\mathscr{L}(16, 0, 17) \cap \mathscr{A}(16, 4, 4) \cap \mathscr{N}(16, 4) \cap \mathscr{R}(16, 0, 6) = \mathscr{P}_1(4^2).$$

Unfortunately this does not hold true for all values of M, as for $M = 5$ and $M = 6$ we find that

$$|\mathscr{L}(25, 13, 26) \cap \mathscr{A}(25, 5, 4) \cap \mathscr{N}(25, 5) \cap \mathscr{R}(25, 0, 8)| = 142$$

and

$$|\mathscr{L}(36, 0, 37) \cap \mathscr{A}(36, 6, 4) \cap \mathscr{N}(36, 6) \cap \mathscr{R}(36, 0, 10)| = 1144.$$

For $\omega = 2$, we find that the cardinalities for the intersection of the four codes for $M = 3$, $M = 4$ and $M = 5$, respectively are 6, 158 and 7648.

7 Conclusion

We believe an interesting new insight into coding theory has been presented by showing how binary sequences, derived from (0,1)-matrices, are subsets of various other well-known codes. Although we did not include $\mathscr{P}_0(M^2)$ and $\mathscr{P}_M(M^2)$ (all zeros and all ones sequences, respectively), both are also subsets of the codes discussed, except for $\mathscr{P}_0(M^2)$ which does not fall in the (d, k)-codes.

The $\mathscr{P}_{M/2}(M^2)$ code would be of most interest, since it possesses all the properties discussed, namely single insertion/deletion error correction, single additive error correction, constant weight, run-length limited, dc-free, null at the Nyquist frequency, as well as nulls at submultiples of the frequency $1/M$. Additionally, the rate for this code is good: for $M = 4$ the rate is 0.375, for $M = 6$ the rate is 0.5 and for $M = 8$ the rate is 0.563.

References

1. Agrell E., Vardy A., Zeger K.: Upper bounds for constant-weight codes. IEEE. Trans. Inf. Theory **46**(7), 2373–2395 (2000).

2. Blake I.F., Cohen G., Deza M.: Coding with permutations. Inf. Contr. **43**(1), 1–19 (1979).
3. Chang J.C., Chen R.J., Kløve T., Tsai S.C.: Distance-preserving mappings from binary vectors to permutations. IEEE. Trans. Inf. Theory **49**(4), 1054–1059 (2003).
4. Ding C., Fu F.-W., Kløve T., Wei V.F.-W.: Construction of permutation arrays. IEEE. Trans. Inf. Theory **48**(4), 977–980 (2002).
5. Ferreira H.C., Vinck A.J.H.: Interference cancellation with permutation trellis codes. In: Proc 2000 IEEE Vehicular Tech Conf, Boston, MA, USA, pp. 2401–2407 (2000).
6. Ferreira H.C., Clarke W.A., Helberg A.S.J., Abdel-Ghaffar K.A.S., Vinck A.J.H.: Insertion/deletion correction with spectral nulls. IEEE. Trans. Inf. Theory **43**(2), 722–732 (1997).
7. Gorog E.: Alphabets with desirable frequency spectrum properties. IBM J. Res. Dev. **12**, 234–241 (1968).
8. Immink K.A.S.: Codes for Mass Data Storage Systems. Shannon Foundation Publishers, The Netherlands (1999).
9. Levenshtein V.I.: Binary codes capable of correcting deletions, insertions and reversals. Sov. Phys.-Doklady **10**(8), 707–710 (1966).
10. Rains E., Sloane N.J.A.: Table of constant weight binary codes. http://www.research.att.com/~njas/codes/Andw/ (2005).
11. Sloane N.J.A.: On single-deletion-correcting codes. In: Arasu KT, Seress A (eds.) Codes and Designs, pp. 273–291. Ray-Chaudhuri Festschrift, Walter de Gruyter, Berlin (2002).
12. Sloane N.J.A.: The on-line encyclopedia of integer sequences. http://www.research.att.com/~njas/sequences/ (2005).
13. Swart T.G., de Beer I., Ferreira H.C.: On the distance optimality of permutation mappings. In: Proc. 2005 IEEE Intl. Symp. Inf. Theory, Adelaide, Australia, pp. 1068–1072 (2005).
14. Vinck A.J.H.: Coded modulation for powerline communications. AEÜ Int. J. Electron. Commun. **54**(1), 45–49 (2000).