Multi-objective optimization of the stack of a thermoacoustic engine using GAMS

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ABSTRACT

This work illustrates the use of a multi-objective optimization approach to model and optimize the performance of a simple thermoacoustic engine. System parameters and constraints that capture the underlying thermoacoustic dynamics have been used to define the model. Work output, viscous loss, conductive heat loss, convective heat loss and radiative heat loss have been used to measure the performance of the engine. The optimization task is formulated as a five-criterion mixed-integer non-linear programming problem. Since we optimize multiple objectives simultaneously, each objective component has been given a weighting factor to provide appropriate user-defined emphasis. A practical example is given to illustrate the approach. We have determined a design statement of a stack describing how the design would change if emphasis is given to one objective in particular. We also considered optimization of multiple objectives components simultaneously and identify global optimal solutions describing the stack geometry using the augmented \( r \)-constraint method. This approach has been implemented in GAMS (General Algebraic Modelling System).

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1. Introduction

This work demonstrates how multi-objective optimization techniques can be used to optimize the design and performance of small-scale thermoacoustic devices. Thermoacoustics relates to the physical phenomenon that a temperature difference can create and amplify a sound wave and vice versa \cite{1}. Hereo the sound wave is brought into interaction with a porous solid material with a much higher heat capacity compared to the gas through which the sound wave propagates. The solid material acts as a regenerator. When a temperature difference is applied across this stack and a sound wave passes through the stack from the cold to the hot side, a parcel of gas executes a thermoacoustic cycle. The gas will subsequently be compressed, displaced and heated, expanded, displaced again and cooled (Fig. 1). During this cycle the gas is being compressed at low temperature, while expansion takes place at high temperature. This means that work is performed on the gas.

The effect of this work is that the pressure amplitude of the sound wave is increased. In this way it is possible to create and amplify a sound wave by a temperature difference. The thermal energy is converted into acoustic energy. Within thermoacoustics, this is referred to as a thermoacoustic engine (TAE). In a thermoacoustic refrigerator (TAR), the thermodynamic cycle is run in the reverse way and heat is pumped from a low-temperature level to a high-temperature level by the acoustic power. The basic mechanics behind thermoacoustics are already well understood. A detailed explanation of the way thermoacoustic coolers work is given by Swift \cite{1} and Wheatley et al. \cite{2}. Recent research focuses on optimizing the modelling approach so that thermoacoustic coolers can compete with commercial refrigerators. After reviewing some fundamental physical properties and previous optimization effort underlying thermoacoustic devices, we will then proceed to discuss our approach to optimize their design.

1.1. Thermoacoustic engines

The most important part of the thermoacoustic system is the core, where the stack of plates is. Thermoacoustic effects actually occur within a very small layer next to the plate, the thermal boundary layer. It is defined as \cite{3}:

\[
\delta_b = \sqrt{\frac{2K}{\rho_m c_p \omega}}
\]

with \( K \) being the thermal conductivity, \( \rho_m \) the mean density, \( c_p \) the constant pressure specific heat of the working fluid. Heat transfer
by conduction is encouraged by a thick boundary layer during a period of \(1/\omega\), where \(\omega\) is the angular frequency of the vibrating fluid. However, another layer that occurs next to the plate, the viscous boundary layer, discourages the thermoacoustic effects. It is defined as [3]:

\[
\delta_v = \sqrt{\frac{2 \mu}{\rho_m \omega}}
\]

where \(\mu\) is the diffusivity of the gas. Losses due to viscous effects occur in this region. A thinner viscous boundary layer than the thermal boundary layer is desirable for effective thermoacoustic effects. Swift [1] started with the equation of heat transfer to come up with a theoretical critical mean temperature gradient, \(\nabla T_{\text{crit}}\), that describes the difference between a thermoacoustic heat engine and a refrigerator,

\[
\nabla T_{\text{crit}} = \frac{\alpha p_1}{\rho_m c_p \delta_v}
\]

This critical temperature gradient depends on the angular frequency \(\omega\), the first order pressure \(p_1\) and velocity \(u_1\) in the standing wave, as well as the mean gas density \(\rho_m\) and specific heat \(c_p\).

In a TAE, the imposed temperature gradient must be greater than this critical temperature gradient \(\nabla = (dT/dx)/(dT/dx_{\text{crit}}) > 1\), while in TARs the critical temperature gradient upper bounds its performance \(\nabla < 1\) [4]. Fig. 2 shows a very simple prototypical standing wave TAE.

The closed end of the resonance tube is the velocity node and the pressure antinode. The porous stack is located near the closed end and the interior gas experiences large pressure oscillations and relatively small displacement. Heat input is provided by a heating wire, causing a temperature gradient to be established across the stack (in the axial direction). A gas in the vicinity of the walls inside the regenerative unit experiences compression, expansion and displacement when it is subject to a sound wave. Over the course of the cycle, heat is added to the gas at high pressure, and heat is withdrawn from it at low pressure. This energy imbalance results in an increase of the pressure amplitude from one cycle to the next, until the acoustic dissipation of the sound energy equals the addition of heat to the system [4–8].

1.2. Optimization in thermoacoustics

Various parameters affecting the performance of thermoacoustic devices are well understood from previous studies. The optimization of a thermoacoustic system is considered by Minner et al. [9] through a parametric study. The design optimization programme developed with DELTAE shows that the efficiency of thermoacoustic coolers is sensitive to stack length, position, mean pressure and gas mixture (Prandtl number), and less sensitive to the stack spacing. A systematic design optimization of thermoacoustic coolers is proposed by Wetzel and Herman [10]. A model based on the boundary layer approximation and the short stack assumption has been developed to calculate the work flux and heat flux. Nineteen design variables are optimized to achieve the best COP. The influence of the plate spacing in the stack on the behaviour of the refrigerator was investigated systematically by Tijani et al. [3].

Optimal plate spacing was identified for thermoacoustic refrigeration. The effect of drive ratio, plate thickness, heat exchanger length and position on the performance of thermoacoustic refrigerator and the flow behaviour is analyzed by Besnoin [11]. The results indicate cooling load peak at a well-defined combination of heat exchanger thickness, length, and width of the gap between the heat exchanger and the stack plates. The optimization of an inerter segment used in a standing–wave type heat-driven thermoacoustic device is considered by Zoontjens et al. [12]. This study suggests that the vast array of variables which affect the system performance must all be considered as interdependent for robust device operation. This is by no means a complete list of the optimization of engine components, but it is a good overview of optimization targets.

A common trait of all these studies is the utilization of a linear approach while trying to optimize the device. Additionally, most studies (the exception being the Minner et al.’s [9] study) have been limited to parametric studies to estimate the effect of single design parameters on device performance and ignored thermal losses to the surroundings. These parametric studies are unable to capture the nonlinear interactions inherent in thermoacoustic models with multiple variables, and could only guarantee locally optimal solutions. Ueda et al. [13] consider varying the position, length, and flow channel radius of the regenerator of a travelling wave thermoacoustic refrigerator and optimize them simultaneously. The obtained results show an increase of the Carnot COP by 60%. Zink et al. [14] illustrate the optimization of thermoacoustic systems, while taking into account thermal losses to the surroundings that are typically disregarded. They have targeted a thermoacoustic engine as a starting point. A model has been built in order to develop an understanding of the importance of the trade-offs between the acoustic and thermal parameters. They use mathematical analysis and optimization, a design aid that is under-utilized in thermoacoustic community. The optimization considers four weighted objectives (these are the conductive heat flux from the stack’s outer surface, the conduction through the stack, acoustic work and viscous resistance) and is conducted with the Nelder–Mead Simplex method. A recent study by Trapp et al. [15] presented analytical solutions for cases of single objective optimization that identify globally optimal parameter levels. Optimization of multiple objective components (acoustic work, viscous resistance and heat fluxes) has been considered, efficient frontier of Pareto optimal solutions corresponding to selected weights have been generated and two profiles have been constructed to illustrate the conflicting nature of those objective component. Zink et al. [14] and Trapp et al. [15] studies show that geometrical parameters describing the stack are interdependent.

1.3. Motivations

Linear approximation used for the design modelling of thermoacoustic systems functions very well. A numerical model that couple one dimensional linear acoustics in the resonator with a low Mach number viscous and conducting flow in the stack/heat exchangers section was developed by Hireche et al. [16]. The results show that the model successfully captures the dynamics of the static process and allow exploring the engine efficiency. Interestingly, a study conducted at the Energy Research Centre of the Netherlands
[17] shows that the ratio between thermal losses and acoustic power changes with increasing acoustic power for thermoacoustic devices. It suggests that heat losses by convection, conduction and radiation need to be adequately covered in the modelling especially with regards to miniaturization of the devices where thermal losses are expected to increase [14].

McLaughlin [18] has thoroughly analyzed the heat transfer for a Helmholtz-like resonator, 1.91 cm in diameter and 3.28 cm in length. The loss to conduction has been estimated as 40% of the input power. The losses from convection inside and outside of the device have been estimated as 38%. Radiation accounts for 10% of the input power. This leaves only 12% of input power that can be used to produce acoustic work (Fig. 3). Although these losses are approximations not meant to be highly accurate determinations, they suggest that these losses are significant when compared to total heat input and should be considered as design criteria. Therefore, this work aims to highlight one methodology to incorporate thermal losses in the design process.

In spite of the introductory nature of Zink et al. [14] and Trapp et al. [15] studies with respect of their plans to expand it and include a driven thermoacoustic refrigerators, the presented works are important contributions to thermoacoustics as it merges the theoretical optimization approach with thermal investigation in thermoacoustics. However, through the objective function weights, a significant amount of personal preference is available to place desired emphasis. The scaling of the objective functions has strong influence in the obtained results. Therefore since several conflicting objectives have been identified, an effort to effectively implement the augmented ε-constraint method for producing the Pareto optimal solutions in a multi-objective optimization mathematical programming method is carried out in this approach. This has been implemented in the modelling language GAMS [19] (General Algebraic Modelling Language, www.gams.com). As a result, GAMS codes are written to define, to analyze, and solve optimization problems to generate sets of Pareto optimal solutions unlike previous studies.

1.4. Research goals

The purpose of the stack is to provide a medium where the walls are close enough so that each time a parcel of gas moves, the temperature differential is transferred to the wall of the stack. Their geometry and position are crucial for the performance of the device.

The primary objectives of the present research are as follows:

- optimizing its geometrical parameters—namely the stack length, the stack height, the stack position, the number of channels and the plate spacing.

Specific sub-objective is as follows:

- providing guidance to the decision maker on the choice of optimal geometry of thermoacoustic devices.

The remainder of this paper is organized in the following fashion: in Section 2, the modelling approach is presented. The fundamental components of our mathematical model characterizing the standing wave thermoacoustic heat engine are presented in Section 3. In Section 4, we discuss single objective optimization, using weighted sum method to find values of the variables that satisfy the constraints and are globally optimal with respect to the considered objective function. Section 5 considers multi-objective optimization using epsilon constraint method. Section 6 reports the contribution of this work.

2. Modelling approach

In this section, our modelling approach for the physical standing wave engine depicted in Fig. 2 is discussed; the development of our mathematical model and its corresponding optimization is included in Sections 3 and 4. The problem is reduced to a two dimensional domain, because of the symmetry present in the stack. Two constant temperature boundaries are considered namely one convective boundary and one adiabatic boundary, as shown in Fig. 4. For our model, only the stack geometry is considered; the model considers variation in operating condition and the independence of stack location and geometry. Five different parameters are considered to characterize the stack:

- \( L \): stack length.
- \( H \): stack height.
- \( \zeta \): stack placement (with \( \zeta = 0 \) corresponding to the closed end of the resonator tube).
- \( dc \): channel diameter, and
- \( N \): number of channels.

Those parameters have been allowed to vary simultaneously. Five different objectives as described by Trapp et al. [15] namely two acoustic objectives \( \text{acoustic work } W \) of the thermoacoustic engine and viscous resistance \( R_v \) through the stack [1,20] and three thermal objectives (convective heat flow \( Q_{\text{conv}} \), radiative heat flow \( Q_{\text{rad}} \), and conductive heat flow \( Q_{\text{cond}} \)) are considered to measure the quality of a given set of variable value that satisfies all of the constraints. Because work is the only objective to be maximized, we instead minimize its negative magnitude along with all of the other components. Ultimately, optimizing the resulting problem generates optimal objective function value \( G^* = \{ W^*, R_v^*, Q_{\text{conv}}^*, Q_{\text{rad}}^*, Q_{\text{cond}}^* \} \) and optimal solution \( x^* = \{ L^*, H^*, dc^*, \zeta^*, N^* \} \). Since the five objectives are conflicting in nature [15,21], a multi-objective optimization approach has been used. Since we optimize multiple objective components simultaneously, each objective component has been given a weighting factor \( w_i \) to provide appropriate user-defined emphasis.

According to Hwang and Masud [22], the methods for solving multi-objective mathematical programming problems can be classified into three categories, based on the phase in which the decision maker is involved in the decision making process expressing his/her preferences: the a priori methods, the interactive methods and the a posteriori or generation methods. The a
posteriori (or generation) methods give the whole picture (i.e. the Pareto set) to the decision maker, before his/her final choice, reinforcing thus, his/her confidence to the final decision. In general, the most widely used generation methods are the weighting method and the ε-constraint method. These methods can provide a representative subset of the Pareto set which in most cases is adequate. The basic step towards further penetration of the generation methods in our multi-objective mathematical problems is to provide appropriate codes in a GAMS environment and produce efficient solutions.

3. Illustration of the optimization procedure of the stack

3.1. Boundary conditions

The five variables L, H, dc, Za, N may only take values within the certain lower and upper bounds. The feasible domains for a thermoacoustic stack are defined as follow:

\begin{align}
L_{\text{min}} & \leq L \leq L_{\text{max}} \\
H_{\text{min}} & \leq H \leq H_{\text{max}} \\
dc_{\text{min}} & \leq dc \leq dc_{\text{max}} \\
Za_{\text{min}} & \leq Za \leq Za_{\text{max}} - L \\
N_{\text{min}} & \leq N \leq N_{\text{max}} \\
L, H, dc, Za, N & \in \mathbb{R} \quad \text{and} \quad N \in \mathbb{Z}^+ \\
\end{align}

(4)

with \(dc_{\text{min}} = 2\delta_k\) and \(dc_{\text{max}} = 4\delta_k\) [3].

Additionally, the total number of channels \(N\) of a given diameter \(d\) is limited by the cross-sectional radius of the resonance tube \(H\). Therefore the following constraint relation can be determined:

\[N(dc + t_w) \leq 2H\]

(6)

where \(t_w\) represents the wall thickness around a single channel and \(N_{\text{min}}\) and \(N_{\text{max}}\) predefined values respectively to \(H_{\text{min}}\) and \(H_{\text{max}}\).

The following boundary conditions must also be enforced:

1. Constant hot side temperature \((T_h)\).
2. Constant cold side temperature \((T_c)\).
3. Adiabatic boundary, modelling the central axis of the cylindrical stack:
   \[\frac{dT}{dr}\bigg|_{r=0} = 0;\]
4. Free convection and radiation to surroundings (at \(T_\infty\)) with temperature dependent heat transfer coefficient \(h\), emissivity \(\varepsilon\), and thermal conductivity \(k\):
   \[k \frac{dT}{dr}\bigg|_{r=H} = h(T_S - T_\infty) + \varepsilon k_0(T_S^4 - T_\infty^4)\]

(8)

3.2. Acoustic power

The acoustic power per channel has been derived by Swift [20]. The following equation can be derived for \(N\) channel:

\[W = \varepsilon \rho N \left(\frac{\pi H^2}{2(dc + t_w)}\right) \left[\frac{\delta_k}{\rho c^2(1 + \varepsilon)}(1 - \delta_v \mu)^2\right]\]

(9)

The relation between the stack perimeter \(P\) and the cross-sectional area \(A\) as determined by Swift [20] is given by:

\[P = \frac{2A}{dc + t_w}\]

(10)

The amplitudes of the dynamic pressure \(p\) and gas velocity \(u\) due to the standing wave in the tube are given by:

\[p = p_{\text{max}} \cos\left(\frac{2\pi Za}{\lambda}\right)\]

(11)

\[u = u_{\text{max}} \sin\left(\frac{2\pi Za}{\lambda}\right)\]

(12)

with

\[u_{\text{max}} = \frac{p_{\text{max}}}{\rho c}\]

(13)

The heat capacity ratio can be expressed by [9]:

\[\varepsilon = \left(\frac{\rho c_p \delta_k}{\rho c_s \tan h(l + 1)p_0/\delta_k}\right)\]

\[= \left(\frac{\rho c_p \delta_k}{\rho c_s \tan h(l + 1)p_0/\delta_k}\right)\]

(14)

This expression can be simplified to values of \(\varepsilon = p_0/\delta_k\) if \(y_0/\delta_k < 1\) and \(\varepsilon = 1\) if \(y_0/\delta_k > 1\) [14], where \(y_0\) half of the channel height is, \(l\) is half of the wall thickness and \(\delta_k\) is the solid’s thermal penetration depth.

3.3. Viscous resistance

Just as the total acoustic power of the stack was dependent on the total number of channels, the viscous resistance also depends on his value. The following equation can be derived [20]:

\[R_v = \frac{\mu}{\delta_k \rho N} \frac{1}{\delta_v (dc + t_w) \pi H^2 N}\]

(15)
3.4. Convective heat flux

The mechanism of convection for the thermoacoustic devices in this study is free convection with air at room temperature. The rate of heat transfer [23], \( \dot{Q}_{\text{conv}} \) to surround air due to convection is

\[
\dot{Q}_{\text{conv}} = hA(T_S - T_\infty) = \rho \gamma \nu L \beta (T_S - T_\infty)
\]

The heat transfer coefficient \( h \) and the heat flux to the surroundings were estimated using a linear temperature profile. In this model, the actual temperature distribution throughout the stack is taken into account by utilizing MATLAB finite element toolbox [24], which captures the temperature dependence of the heat transfer coefficient. Only the temperature distribution at the shell surface and the temperature gradient at the cold side are of interest. Trapp et al. [15] have derived the final surface temperature distribution as a function of axial direction \( Z_a \). It is given by:

\[
T_S = T_H \frac{\ln(T_C/T_H)2Z_a/L}{2} \ln(T_S/T_\infty)
\]  \hspace{1cm} (17)

The convective heat transfer coefficient and the radiative heat flux to the surroundings are assumed to be dependent on this temperature. The total convective heat transfer across the cylindrical shell in its integral form can be described by:

\[
\dot{Q}_{\text{conv}} = H \int_0^{2\pi} \int_0^1 h(T(z))(T(z) - T_\infty) dz d\rho
\]

\hspace{1cm} (18)

For the case of a horizontal tube subject to free convection [25], the heat transfer coefficient \( h \) is derived from the Nusselt number, which is a non-dimensional heat transfer coefficient as follows:

\[
h(T_S) = \frac{k_S}{2H} Nu
\]

\hspace{1cm} (19)

\[
Nu = 0.36 + \frac{0.518R_{d}^{1/4}}{1 + (0.559/Pr)^{9/16}} \left( \frac{H}{L} \right)^{9/8}
\]

\hspace{1cm} (20)

This expression depends on the Prandtl number, which can be expressed by:

\[
Pr = \frac{\nu}{\alpha}
\]

\hspace{1cm} (21)

\[
Ra = \frac{g\beta(T_S - T_\infty)8H^3}{\nu \alpha}
\]

\hspace{1cm} (22)

where \( Pr \) is the Prandtl number, \( T_S \) is the surface temperature, \( T_\infty \) is the (constant) temperature of the surroundings, \( \nu \) is the viscosity of the surrounding gas, and \( \alpha \) is the thermal diffusivity of the surrounding gas (air). The temperature distribution stated in Eq. (17) is then used to determine the convective heat transfer to the surroundings. After integrating, the following heat flow expression is derived:

\[
\dot{Q}_{\text{conv}} = 2\pi HLh \left( \frac{T_C - T_H}{\ln(T_C/T_H)} - T_\infty \right)
\]

\hspace{1cm} (23)

The following constraint can be derived from Eqs. (20) and (22):

\[
Z_a \geq L \log \left( \frac{T_\text{inf}}{T_C} \right)
\]

\hspace{1cm} (24)

3.5. Radiative heat flux

For an object having a surface area, \( A \), a temperature, \( T \), surrounded by air at temperature \( T_\infty \), the object will radiate heat at a rate, \( \dot{Q}_{\text{rad}} \) [26],

\[
\dot{Q}_{\text{rad}} = k_B \varepsilon A(T^4 - T_\infty^4)
\]

\hspace{1cm} (25)

The radiation heat flux becomes increasingly important as \( T_H \) increases, as shown in the following equation:

\[
\dot{Q}_{\text{rad}} = \frac{Hk_B}{2} \int_0^{2\pi} \int_0^1 \varepsilon(T(z)) \frac{T_C - T_H}{\ln(T_C/T_H)} - T_\infty dz d\rho
\]

\hspace{1cm} (26)

where \( k_B \) is the Stefan Boltzmann constant, and \( \varepsilon \) is the surface emissivity, which depends on the emitted wavelength, and in turn is a function of temperature. After integrating, the following heat flow expression is derived:

\[
\dot{Q}_{\text{rad}} = 2\pi HLk_B \varepsilon \left( \frac{T_C^4 - T_H^4}{4 \ln(T_C/T_H)} - T_\infty^4 \right)
\]

\hspace{1cm} (27)

3.6. Conductive heat flux

The temperature distribution is used to determine the temperature gradient on the top surface \( Z_a \), \( r = H \). According to Fourier’s law [23], the heat flow, \( \dot{Q}_{\text{cond}} \) in the \( z \) direction, through a material is expressed as:

\[
\frac{\Delta Q}{\Delta T} = -k_{\text{cond}} \frac{\Delta T}{\Delta x}
\]

\hspace{1cm} (28)

\[
\dot{Q}_{\text{cond}} = \int_0^{2\pi} \int_0^1 \left( k_{zz} \frac{dT}{dr} \right) dr d\rho
\]

\hspace{1cm} (29)

where the value of the axial thermal conductivity \( k_{zz} \) is determined by the following equation [14]:

\[
k_{zz} = \frac{k_{zz}}{t_w + dc}
\]

\hspace{1cm} (30)

Therefore

\[
\frac{dT}{dr} \bigg|_{z=-L} = \frac{\ln(T_C/T_H)/T_C}{T_C/T_H}
\]

\hspace{1cm} (31)

And after integration

\[
\dot{Q}_{\text{cond}} = \frac{k_{zz}}{L} \pi H^2 T_C \ln \left( \frac{T_H}{T_C} \right)
\]

\hspace{1cm} (32)

4. Single objective optimization

All the expressions involved in our mathematical model (MPF) have been presented in the previous section. Together with the following expressions, they represent a mixed-integer non-linear mixed integer programme:

\[
(MPF) \min_{L,H,Za,dc,N} \xi = w_1(-W) + w_2R_V + w_3Q_{\text{conv}} + w_4Q_{\text{rad}} + w_5Q_{\text{cond}}
\]

\hspace{1cm} (33)

This mathematical model characterizes the essential elements of a standing wave thermoacoustic engine. In the following discussion we analyze restricted cases of our objectives, and identify general tendencies of the structural variables to influence individual objective components. To illustrate our approach, we consider the thermoacoustic couple (TAC) as described in [27]. It consists of a parallel-plate stack placed in helium-filled resonator. All relevant parameters are given in Tables 1 and 2.
4.1. Emphasizing acoustic wave

All proposed models MINLP are solved by GAMS 23.8.1 [19], using LINDOGLOBAL solver on a personal computer Pentium IV 1.6 GHz with 0.99 GB RAM. The following constraints (upper and lower bounds) have been enforced on variables in order for the solver to carry out the search of the optimal solutions in those ranges:

\[
\begin{align*}
L_{\text{lo}} &= 0.005; & L_{\text{up}} &= 0.05; \\
Z_{\text{a,lo}} &= 0.005; & Z_{\text{a,up}} &= 0.050; \\
N_{\text{lo}} &= 20; & N_{\text{up}} &= 50; \quad (34) \\
H_{\text{lo}} &= 0.005; & H_{\text{up}} &= 0.050; \\
dc_{\text{lo}} &> 2.5\delta_{\text{c}}; & dc_{\text{up}} &< 4.5\delta_{\text{c}} \quad [10].
\end{align*}
\]

Setting the objective function weights to \(w_2 = w_3 = w_4 = w_5 = 0\) and \(w_2 = 1\), the problem reduces to Eqs. (1)–(3), (9) and (11)–(13), constraints (6) and (24) and variable restrictions (34). Objective function (33) becomes:

\[
\min_{L,H,dc,Za,N} \xi_W = (-W) \quad (35)
\]

In our approach, the geometry range is small in order to illustrate the behaviours of the objective functions and optimal solution of a small-scale thermoacoustic engine. In Table 3, the optimal solutions that maximize \(\xi_W\) are represented with letter subscripted with asterisk.

Physically, this optimal solution can be interpreted as:

- making the stack as long as possible \((L^* = L_{\text{max}})\),
- making the stack wide,
- moving the stack as near as possible to the closed end \((Za^* = Za_{\text{min}})\) maximizing the available pressure amplitude for the thermodynamic cycle and thus work output \(W\) and,

\[\begin{array}{cccccc}
L^* & H & Za^* & dc^* & N^* & W \\
0.050 & 0.034 & 0.005 & 0.001 & 50 & 4.5536E+9 & 18.171
\end{array}\]

4.2. Emphasizing viscous resistance

We emphasize \(R_V\) by setting objective function weights \(w_1 = w_3 = w_4 = w_5 = 0\) and \(w_2 = 1\). The problem then simplifies to Eqs. (2) and (15), constraints (6) and (24), and variable restrictions (34). Objective function (33) becomes:

\[
\min_{L,H,dc,Za,N} \xi_{R_V} = R_V \quad (36)
\]

In Table 4, the optimal solutions that minimize \(\xi_{R_V}\) are represented with letter subscripted with asterisk.

Physically, this optimal solution can be interpreted as:

- making the stack as small as possible \((L^* = L_{\text{min}})\), therefore reducing the individual (viscous) resistance of each channel to its minimum,
- moving the stack as near as possible to the closed end \((Za^* = Za_{\text{min}})\),
- decreasing the number of channels \((N^* = N_{\text{min}})\).

\[\begin{array}{ccccccc}
L^* & H^* & Za^* & dc^* & N^* & R_V & \text{CPU time (s)} \\
0.005 & 0.008 & 0.005 & 5.8140E-4 & 20 & 3.467 & 1.062
\end{array}\]

4.3. Emphasizing convective heat flux

We can emphasize \(Q_{\text{conv}}\) by setting objective function weights \(w_1 = w_2 = w_4 = w_5 = 0\) and \(w_3 = 1\). The problem then reduces to Eqs. (17), (19) and (20)–(22), constraints (6) and (24) and variable restrictions (34). Objective function (33) becomes:

\[
\min_{L,H,dc,Za,N} \xi_{Q_{\text{conv}}} = Q_{\text{conv}} \quad (37)
\]

In Table 5, the optimal solutions that minimize \(\xi_{Q_{\text{conv}}}\) are represented with letter subscripted with asterisk.

Physically, this optimal solution can be interpreted as:

- making the stack as large as possible \((L^* = L_{\text{max}})\),
- moving the stack as near as possible to the closed end \((Za^* = Za_{\text{min}})\),
- decreasing the number of channels \((N^* = N_{\text{min}})\).

\[\begin{array}{cccccccc}
L^* & H^* & Za^* & dc^* & N^* & Q_{\text{conv}} & \text{CPU time (s)} \\
0.005 & 0.008 & 0.005 & 5.93522E-4 & 20 & 0.1623 & 0.718
\end{array}\]

4.4. Emphasizing radiative heat flux

We can emphasize \(Q_{\text{rad}}\) by setting objective function weights \(w_1 = w_2 = w_3 = w_5 = 0\) and \(w_4 = 1\), so that only Eq. (27), constraints...
while the other objectives are added as constraints to the feasible solution space of $\Omega$ as follows:

$$SO_\varepsilon(X) = \min_{x \in \Omega} f_1(x)$$

subject to  
$$f_2(x) \leq \varepsilon_2, f_3(x) \leq \varepsilon_3, \ldots, f_p(x) \leq \varepsilon_p, \quad x \in \Omega$$

By solving iteratively problem $SO_\varepsilon(X)$ for different values of $\varepsilon_p$, different Pareto solutions can be obtained. The range of at least $p - 1$ objectives functions is necessary in order to determine grid points for $\varepsilon_1, \ldots, \varepsilon_p$ values and apply the $\varepsilon$-constraint method. The most common approach is to calculate these ranges from the payoff table (Fig. 5). Each objective function is optimized individually. The mathematical details of computing payoff table for a multi-objective mathematical programming (MMP) problem can be found in Cohon [28]. The payoff table for a MMP problem with $(p)$ competing objective functions is calculated as follow:

- The individual optima of the objective functions ($f_j$) are calculated. The optimum value of the objective functions ($f_j$) and the vector of decision variables which optimizes the objective function ($f_j$) are indicated respectively by $f_j^*(\vec{x}_j)$ and $\vec{x}_j$.

- Represent the payoff table including $f_1^*(\vec{x}_1), f_2^*(\vec{x}_2), \ldots, f_p^*(\vec{x}_p)$ as follows:

$$\Phi = \begin{pmatrix} f_1^*(\vec{x}_1) & \cdots & f_p^*(\vec{x}_1) \\ \vdots & \ddots & \vdots \\ f_1^*(\vec{x}_p) & \cdots & f_p^*(\vec{x}_p) \end{pmatrix}$$

- Determine the range of each objective function in the payoff table based on utopia and pseudo-nadir points. The Utopia point ($f^U$) refers to a specific point where all objectives are simultaneously at their best possible values. It is generally outside the feasible region. However, the Nadir point ($f^SN$) is a point where all objective functions are simultaneously at their worst values. It is generally in the objective space

$$f^U \leq f(X) \leq f^SN$$

- Divide the range of $p - 1$ objectives functions $f_2, \ldots, f_p$ to $q_2, \ldots, q_p$ into equal intervals using $(q_2 - 1), \ldots, (q_p - 1)$ intermediate equidistant points, respectively.

- Convert the MMP problem into $\prod_{i=2}^{p} (q_i + 1)$ single objective optimization sub-problems as follows:

$$\min_{x_j} f_1(x_j)$$

subject to  
$$f_2(x) \leq \varepsilon_{2,n_2}, \ldots, f_p(x) \leq \varepsilon_{p,n_p}$$

where

$$\varepsilon_{2,n_2} = f_{2}^{SN} - \frac{(f_{2}^{SN} - f_{2}^{U})}{q_2} \times n_2, \quad n_2 = 0, 1, \ldots, q_2$$

$$\varepsilon_{2,n_2} = f_{2}^{SN} - \frac{(f_{2}^{SN} - f_{2}^{U})}{q_2} \times n_2, \quad n_2 = 0, 1, \ldots, q_2$$

- Each sub-problem is a candidate solution or Pareto optimal solution of the MMP problem. At the same time, some of these optimization sub-problems may have infeasible solution space due to the added constraints for $f_2, \ldots, f_p$; such sub-problems are discarded.

- Selection of the most preferred solution out of the obtained Pareto optimal solutions by the decision maker.
5.2. Augmented $\varepsilon$-constraint method

In the ordinary $\varepsilon$-constraint method, the efficiency of Pareto solutions is not guaranteed. Inefficient solutions can be generated. The obtained solution is considered inefficient if there is another Pareto solution that can improve at least one objective function without deteriorating the other objectives functions. In order to overcome this drawback, we consider the following:

a. The objective functions constraints in Eq. (43) are transformed into equality constraints by means of the slack variable technique [31,32]. Therefore, the augmented $\varepsilon$-constraint method can be formulated as follows:

$$
\min \left( f_1(\mathbf{x}) - \delta \left( \frac{s_2}{r_2} + \frac{s_3}{r_3} + \cdots + \frac{s_p}{r_p} \right) \right)
$$

subject to

$$
f_2(\mathbf{x}) + s_2 = \varepsilon_2, \ldots, f_p(\mathbf{x}) + s_p = \varepsilon_p, \quad s_2, \ldots, s_p \in \mathbb{R}^+
$$

where $s_2, \ldots, s_p$ represent the slack variables for the constraints in Eq. (43) of the multi-objective mathematical programming (MMP) problem and $\delta$ is a small number usually between $10^{-3}$ and $10^{-6}$ [33]. This formulation (Eq. (44)), preventing the generation of an inefficient solution, is known as ‘augmented $\varepsilon$-constraint method’ due to the augmentation of the objective function ($f_1$) by the second term. Its proof can be found in [33].

Adapted from [29].

Fig. 5. Flowchart of the lexicographic optimization for calculation of payoff table.
b. The concept of relative importance of objective in generating the Pareto solutions is introduced to be consistent with the decision maker policy. Although each objective has its own relative importance in the MMP problem, the previous formulations consider all slack variables with equivalent importance. In MMP problems, the concept of optimality stipulates that we search for the most preferred solution among the generated Pareto set. To remedy the inconsistency in the decision making process, the formulation of the augmented $\varepsilon$-constraint method is modified by the use of a lexicographic optimisation of these series of objective functions. Practically, the first objective function of higher priority is optimized, obtaining $\min f_1 = x_1^*$. Then the second objective function is optimized in order to keep the solution of the first optimization. Assume that we obtain $\min f_2 = x_2^*$. Subsequently, the constraints $f_1 = x_1^*$ and $f_2 = x_2^*$ are added to optimize the third objective function in order to keep the previous optimal solutions and so on, until all objective functions are dealt with. The flowchart of the lexicographic optimization of a series of objective functions is illustrated in Fig. 6.

By the combination of the lexicographic optimization and augmented $\varepsilon$-constraint method, the range of the objective functions in the payoff table is optimized and results in the generation of only efficient solutions within the identified ranges. This is illustrated by the flowchart in Fig. 6.
The proposed augmented ε-constraint method is expected to provide a representative subset of the Pareto set when in most cases is adequate. The basic step towards further penetration of the generation methods in our multi-objective mathematical problems is to provide appropriate codes in a GAMS environment and produce efficient solutions.

5.3. Model implementation

Lastly, we simultaneously consider all five objective components by regarding acoustic work $W$, viscous resistance $R_v$, convective heat flux $Q_{conv}$, radiative heat flux $Q_{rad}$ and conductive heat flux $Q_{cond}$ as five distinct objective components. All the expressions involved in the multi-objective mathematical programming (MMP) have been presented in the previous section. The optimization task is formulated as a five-criteria mixed-integer non-linear programming problem (MPF) that simultaneously minimize the negative magnitude of the acoustic work $W$ (since it is the only objective to be maximized), the viscous resistance $R_v$, the convective heat flux $Q_{conv}$, the radiative heat flux $Q_{rad}$ and the conductive heat flux $Q_{cond}$.

\[
\begin{align*}
\text{(MPF)} \quad \min_{L,H,Za,dc,N} & \quad \xi = [-W(L,H,Za,dc,N), R_v(L,H,Za,dc,N), \\
& \quad Q_{conv}(L,H,Za,dc,N), Q_{rad}(L,H,Za,dc,N), Q_{cond}(L,H,Za,dc,N)] \\
\text{subject to} & \quad (6), (24) \text{ and } (47) \\
\end{align*}
\]

In this formulation, $(L, H, Za, dc, N)$ denotes the geometric parameters.

There is no single optimal solution that simultaneously optimizes all the five objectives functions. In these cases, the decision makers are looking for the “most preferred” solution. To find the most preferred solution of this multi-objective model, the augmented ε-constraint method (AUGMECON) as proposed by Mavrotas [33] is applied. The AUGMECON method has been coded in GAMS. The code is available in the GAMS library (http://www.gams.com/modlib/libhtml/epsclm.htm) with an example. While the part of the code that has to do with the example (the specific objective functions and constraints), as well as the parameters of AUGMECON have been modified in this case, the part of the code that performs the calculation of payoff table with lexicographic optimization and the production of the Pareto optimal solutions is fully parameterized in order to be ready to use.

Practically, the ε-constraint method is applied as follows: from the payoff table the range of each one of the $p - 1$ objective functions that are going to be used as constraints is obtained. Then the range of the $i$th objective function to $q_i$ equal intervals using $(q_i - 1)$ intermediate equidistant grid points is divided. Thus in total $(q_i - 1)$ grid points that are used to vary parametrically the right hand side $(q_i)$ of the $i$th objective function is obtained. The total number of runs becomes $(q_2 + 1) \times (q_3 + 1) \times \ldots \times (q_p + 1)$. The ε-constraint method has several important advantages over traditional weighted method. These advantages are listed in Mavrotas [33]. In the conventional ε-constraint method, there is no guarantee that the obtained solutions from the individual optimization of the objective functions are Pareto optima or efficient solutions. In order to overcome this deficiency, the lexicographic optimization for each objective functions to construct the payoff table for the multi-objective mathematical programming (MMP) is proposed in order to yield only Pareto optimal solutions (it avoids the generation of weakly efficient solutions) [29]. The mathematical details of computing payoff table for MMP problem can be found in [29]. The augmented ε-constraint method for solving model (Eq. (40)) can be formulated as follows:

\[
\begin{align*}
\text{max} \quad W(L,H,Za,dc,N) + & \quad \ dir_1 r_1 \times \left( \frac{S_2}{r_2} + \frac{S_3}{r_3} + \frac{S_4}{r_4} + \frac{S_5}{r_5} \right) \\
\text{subject to} & \quad R_v(L,H,Za,dc,N) - dir_2 S_2 = \varepsilon_2 \\
& \quad Q_{conv}(L,H,Za,dc,N) - dir_3 S_3 = \varepsilon_3 \\
& \quad Q_{rad}(L,H,Za,dc,N) - dir_4 S_4 = \varepsilon_4 \\
& \quad Q_{cond}(L,H,Za,dc,N) - dir_5 S_5 = \varepsilon_5 \\
& \quad S_i \in \mathbb{N}^+ \\
\end{align*}
\]

where $dir_i$ is the direction of the $i$th objective function, which is equal to $-1$ when the $i$th function should be minimized, and equal to $+1$, when it should be maximized. Efficient solutions of the problem are obtained by parametrical iterative variations in the $\varepsilon_i$. $S_i$ are the introduced surplus variables for the constraints of the MMP problem. $r_j S_j/r_j$ is used in the second term of the objective function, in order to avoid any scaling problem. The formulation of Eq. (41) is known as the augmented ε-constraint method due to the augmentation of the objective function $W$ by the second term. The following constraints (upper and lower bounds) have been enforced on variables in other for the solver to carry out the search of the optimal solutions in those ranges:

\[
\begin{align*}
L.lo &= 0.005; & L.up &= 0.05; \\
Za.lo &= 0.005; & H.lo &= 0.005; \\
dc.lo &> 2.\delta_k; & dc.up &< 4.\delta_k \\
\end{align*}
\]

We use lexicographic optimization for the payoff table; the application of model (Eq. (46)) will provide only the Pareto optimal solutions, avoiding the weakly Pareto optimal solutions. Efficient solutions of the proposed model have been found using AUGMECON method and the LINDOGLOBAL solver. To save computational time, the early exit from the loops as proposed by Mavrotas [23] has been applied. The range of each five objective functions is divided in four intervals (5 grid points). The integer variable $N$ has been given values from 20 to 50. This process generates optimal solutions corresponding to each integer variable. The maximum CPU time taken to complete the results is 1029.700 s. The following section report only sets of Pareto solutions obtained:

Fig. 7 represents the Pareto optimal solutions graphically; it shows that there is not one single optimal solution that optimizes the geometry of the stack and highlights the fact that the geometrical parameters are interdependent, supporting the use of a multi-objective approach for optimization of thermoacoustic engines. To maximize acoustic work $W$ and minimize viscous resistance and thermal losses simultaneously, there is a specific stack length $(L)$ to which correspond a specific stack height $(H)$, a specific stack spacing $(dc)$ and a specific number of channels $(N)$. This study highlights the fact that the geometrical parameters are interdependent, which support the use of a multi-objective approach for optimization. It should be noted that in all cases, locating the stack closer to the closed end produced the desired effect. All Pareto optimal solutions can be identified to reinforce the decision maker’s final decision and preferred choice.

These optimal solutions are then used to construct Figs. 8–10 representing respectively acoustic work, viscous resistance, conductive, and thermal losses plotted as a function of $N, L, dc$ and $H$.

It can be seen that there is a similar trend for acoustic work when considering the number of channels $N$ and the stack height $H$ with the maximum values expected respectively for $N = 43$ (Fig. 8a) and $H = 0.028$ (Fig. 8d). The results obtained for the stack length and the
stack spacing suggest to increase the stack length to \( L_{\text{max}} \) (Fig. 8b) or decrease the stack spacing to \( d_{c_{\text{min}}} \) (Fig. 8c) in order to maximize the acoustic work.

The viscous resistance results show that minimal values are expected by maximizing number of channels \((N = 42)\) (Fig. 9a) and increasing the stack height \((H = 0.028)\) (Fig. 9d). Minimal values of viscous resistance function of the stack length and the stack spacing are observed by decreasing the stack length \((L = 0.005)\) (Fig. 9b) and increasing the stack spacing to \( d_{c_{\text{max}}} \) \((d_{c} = 4\delta_k)\) (Fig. 9c).

The analyses of results obtained for the conductive, convective and radiative heat fluxes show a similar trend. An increase in number of channels, stack spacing and stack height result in a similar increase of these thermal losses with minimal values observed when geometrical parameters are minimized (Fig. 10a, c and d). However, the influence of the stack length show a different trend with minimal value observed for \( L = 0.047 \) (Fig. 10b).

5.4. Results comparison

An alternative way to simultaneously maximize acoustic work and minimize losses (viscous resistance as well as heat flows) is to consider the thermal efficiency \((\eta)\) which can be defined as the ratio of the work output over the sum of the work output and losses as follows:

\[
\eta = \frac{W}{W + R_V + Q_{\text{conv}} + Q_{\text{rad}} + Q_{\text{cond}}} \tag{48}
\]

The viscous resistance in Eq. (15) has the units \([\text{kg/m}^4 \text{s}]\). In order to express this in terms similar to the other variables used in Eq. (48), we multiply Eq. (15) (subsequently \( R_V \) in Table 10) by the volumetric velocity \([\text{m}^3/\text{s}]\) and the oscillating frequency \([1/\text{s}]\), yielding \([\text{W/m}]\) as final unit for the viscous resistance per channel used in Eq. (48). This ratio can be used to compare the results obtained.

![Fig. 7. Optimal structural variables.](image)

![Fig. 8. Acoustic power plotted as a function of N, L, dc and H.](image)
by the proposed augmented ε-constraint method and identify the preferred solution. These solutions are presented in Table 11. For the sake of conciseness, optimal solutions with identical number of channels obtained with ordinary ε-constraint method (Table 11) are compared to solutions obtained with the proposed method (Table 10) to demonstrate the efficiency of the solutions obtained with the proposed method (Fig. 11).

Based on the magnitude of work output, viscous resistance and heat fluxes, these results suggest that the solution reported in Table 10 corresponding to 42 and 43 channels are the best
Table 10
Non-dominated solutions found by AUGMENCON.

<table>
<thead>
<tr>
<th>N</th>
<th>( L' )</th>
<th>( dc' )</th>
<th>( H' )</th>
<th>( Za' )</th>
<th>( W' )</th>
<th>( R_c )</th>
<th>( Q_{\text{conv}}^* )</th>
<th>( Q_{\text{conv}}^* )</th>
<th>( Q_{\text{rad}}^* )</th>
<th>CPU time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>21 Channels</td>
<td>0.047</td>
<td>0.00098521</td>
<td>0.012</td>
<td>0.005</td>
<td>13752.67</td>
<td>671.112</td>
<td>1.803</td>
<td>9.301</td>
<td>1.312</td>
<td>1029.700</td>
</tr>
<tr>
<td>25 Channels</td>
<td>0.043</td>
<td>0.00100000</td>
<td>0.016</td>
<td>0.005</td>
<td>23061.14</td>
<td>292.007</td>
<td>3.010</td>
<td>10.230</td>
<td>1.552</td>
<td>1024.988</td>
</tr>
<tr>
<td>26 Channels</td>
<td>0.043</td>
<td>0.00100000</td>
<td>0.017</td>
<td>0.005</td>
<td>25924.46</td>
<td>243.410</td>
<td>3.367</td>
<td>10.464</td>
<td>1.614</td>
<td>1017.766</td>
</tr>
<tr>
<td>35 Channels</td>
<td>0.044</td>
<td>0.00100000</td>
<td>0.022</td>
<td>0.005</td>
<td>63219.01</td>
<td>110.627</td>
<td>5.783</td>
<td>13.081</td>
<td>2.171</td>
<td>47.409</td>
</tr>
<tr>
<td>36 Channels</td>
<td>0.045</td>
<td>0.00100000</td>
<td>0.022</td>
<td>0.005</td>
<td>68794.08</td>
<td>113.043</td>
<td>5.790</td>
<td>13.461</td>
<td>2.233</td>
<td>94.506</td>
</tr>
<tr>
<td>37 Channels</td>
<td>0.047</td>
<td>0.00097718</td>
<td>0.022</td>
<td>0.005</td>
<td>74687.64</td>
<td>125.216</td>
<td>5.557</td>
<td>13.927</td>
<td>2.295</td>
<td>118.498</td>
</tr>
<tr>
<td>42 Channels</td>
<td>0.040</td>
<td>0.00100000</td>
<td>0.028</td>
<td>0.005</td>
<td>108740.00</td>
<td>47.243</td>
<td>9.780</td>
<td>14.533</td>
<td>2.594</td>
<td>446.288</td>
</tr>
<tr>
<td>43 Channels</td>
<td>0.049</td>
<td>0.00092050</td>
<td>0.024</td>
<td>0.005</td>
<td>117420.00</td>
<td>97.351</td>
<td>6.763</td>
<td>15.798</td>
<td>2.672</td>
<td>386.976</td>
</tr>
</tbody>
</table>

Table 11
Solutions found by ordinary \( e \)-constraint method.

<table>
<thead>
<tr>
<th>N</th>
<th>( L' )</th>
<th>( dc' )</th>
<th>( H' )</th>
<th>( Za' )</th>
<th>( W' )</th>
<th>( R_c )</th>
<th>( Q_{\text{conv}}^* )</th>
<th>( Q_{\text{conv}}^* )</th>
<th>( Q_{\text{rad}}^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>21 Channels</td>
<td>0.005</td>
<td>9.73E-04</td>
<td>0.012</td>
<td>0.005</td>
<td>1446.994</td>
<td>73.615</td>
<td>1.975</td>
<td>0.407</td>
<td>0.138</td>
</tr>
<tr>
<td>25 Channels</td>
<td>0.005</td>
<td>6.33E-04</td>
<td>0.010</td>
<td>0.005</td>
<td>1730.644</td>
<td>123.442</td>
<td>16.401</td>
<td>0.218</td>
<td>0.117</td>
</tr>
<tr>
<td>26 Channels</td>
<td>0.005</td>
<td>6.00E-04</td>
<td>0.010</td>
<td>0.005</td>
<td>1865.801</td>
<td>123.67</td>
<td>17.031</td>
<td>0.223</td>
<td>0.116</td>
</tr>
<tr>
<td>35 Channels</td>
<td>0.005</td>
<td>5.81E-04</td>
<td>0.014</td>
<td>0.005</td>
<td>4445.136</td>
<td>54.421</td>
<td>30.103</td>
<td>0.045</td>
<td>0.153</td>
</tr>
<tr>
<td>36 Channels</td>
<td>0.005</td>
<td>5.81E-04</td>
<td>0.014</td>
<td>0.005</td>
<td>4837.137</td>
<td>50.011</td>
<td>31.848</td>
<td>0.045</td>
<td>0.157</td>
</tr>
<tr>
<td>37 Channels</td>
<td>0.005</td>
<td>5.81E-04</td>
<td>0.014</td>
<td>0.005</td>
<td>5251.533</td>
<td>46.065</td>
<td>33.642</td>
<td>0.045</td>
<td>0.161</td>
</tr>
<tr>
<td>42 Channels</td>
<td>0.005</td>
<td>7.52E-04</td>
<td>0.020</td>
<td>0.005</td>
<td>9.38E+03</td>
<td>17.300</td>
<td>53.546</td>
<td>0.564</td>
<td>0.224</td>
</tr>
<tr>
<td>43 Channels</td>
<td>0.005</td>
<td>6.70E-04</td>
<td>0.019</td>
<td>0.005</td>
<td>9.192.205</td>
<td>21.164</td>
<td>50.976</td>
<td>0.539</td>
<td>0.209</td>
</tr>
</tbody>
</table>

Fig. 11. Pareto solutions comparisons.

solutions for this application. As seen from Fig. 11, the augmented \( e \)-constraint produces more efficient candidate solutions and so the decision maker can select a better final solution among the generated Pareto sets.

6. Conclusion

Optimization as a design aid is required for thermoacoustic engine to be competitive on the current market. Previous studies have relied heavily upon parametric studies. This work targets the geometry of the thermoacoustic stack and uses multi-objective optimization approach to find the optimal set of geometrical parameters that optimizes the device. Five different parameters (stack length, stack height, stack placement, stack spacing and number of channels) describing the geometry of the device have been studied. Five different objectives have been identified; a weight has been given to each of them to allow the designer to place desired emphasis. A mixed-integer non-linear programming problem for thermoacoustic stack has been implemented in GAMS. We have determined design statements for the each single objective emphasis case. For the case of multiple objectives considered simultaneously, we have applied an improved version of a multi-objective solution method, i.e., the epsilon constraint method called augmented epsilon constraint method (AUGMENCON). This process generates optimal solutions which are then use to illustrate the conflicting nature of objective functions. The results found shows the interdependence between the geometrical parameters of the stack which support the use of our multi-objective approach to optimize the geometry of thermoacoustic engine. The magnitude of acoustic work, viscous resistance and thermal losses are computed for all optimal solutions and guidance is provided to minimize thermal losses unlike previous studies. This is useful for electronic cooling application where the magnitude of these thermal losses is expected to increase. The proposed method not only gives the efficient solutions corresponding to the optimal thermoacoustic engine stack geometry but also results in solutions that are more preferred than those obtained using the ordinary \( e \)-constraint technique.

References