An Analogue to SNR for Timing Synchronisation Motivated by Jitter and Insertion/Deletion Errors

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***Abstract*— This paper takes a closer look at the relationship between insertion/deletion errors and one of its most common causes, timing jitter. We consider two insertion/deletion channel models based on different assumptions about the timing jitter. In the literature, only channels with equal insertion/deletion probabilities for each bit have been considered. However, this assumption is only true under certain conditions. A channel, where the probability of insertion/deletion errors increases with time, is another feasible scenario. We derive an approximation for the probability of insertion/deletion errors for both models. With these ideas, we also define a possible counterpart in the time domain to the signal-to-noise ratio (defined in the amplitude domain). As an analogue to Gaussian noise in the amplitude domain, we only consider random Gaussian distributed jitter.**

1. Introduction

Although the literature on the individual themes of insertion/deletion errors and timing jitter is vast, the authors are unaware (to the best of their knowledge) of any paper in literature dealing with these two related themes concurrently. This is despite the fact that it has been known in literature for some time that the primary cause for insertion/deletion errors is timing jitter. Information theoretic approaches to timing jitter do exist, for example [1], [2], but they make no explicit mention of insertion/deletion errors.

In literature, channels where the probability of an insertion/deletion error is equal for every bit transmitted have been considered in many papers, for example [3]–[6] (many others could be cited). Yet this assumption is not universally true. If we consider the nature of timing jitter, other channel models are also possible. Timing jitter is the deviation in time between ideal positions (ideal sampling time) and the actual sampling times at the receiver. These deviations tend to accumulate through time, so that one may expect that the probability of error increases with time.

Timing jitter (or simply jitter) is classified into two broad categories: Deterministic Jitter (DJ) and Random Jitter (RJ). The primary difference is that the probability density function for DJ is bounded, while for RJ, it is unbounded. For a description of various types of jitter, see [7]. Random jitter has a Gaussian density function. Since amplitude noise is usually assumed to have a Gaussian distribution, we will only consider random jitter. This also simplifies the analysis, as will be seen later. Random jitter is the result of noise such as thermal and flicker noise [8]. Furthermore, by the central limit theorem, as the number of uncorrelated noise sources approaches infinity, its distribution approaches a Gaussian function. For an example of a circuit that can measure random jitter, see [9].

As the bit rate speed of communication systems increases, jitter is becoming an ever greater problem. However, at some point, the cost of eliminating or minimising jitter may become prohibitively expensive. At this point, insertion/deletion errors may become a dominant factor. Error control coding may then be a viable solution. In this paper, we consider the relationship between the bit rate, jitter variance and the probability of an insertion/deletion error. This may form a framework within which to determine the gain due to coding for insertion/deletion correction.

The paper is organised as follows. Section 2 defines the jitter model used in this paper. Section 3 considers the equal and unequal insertion/deletion probability channels and the conditions under which each occurs. Section 4 interprets the results from Section 3 and compares it to similar concepts in the amplitude domain, particularly the signal-to-noise ratio.

2. Timing Jitter Model

Assume that a binary bit of information is transmitted in a period T. The continuous time signal $i(t)$ is sampled at the half way point between transitions, which in time is represented by $\{T, 2T, 3T, \ldots, nT\}$.

At each sampling point nT, where n is a natural number, random jitter is represented by a random variable X. X has a Gaussian probability density function

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_X} e^{\frac{-x^2}{2\sigma_X^2}}, \tag{1}$$

where σ_X is the standard deviation. X has a mean of zero, which corresponds to the ideal sampling times $\{T, 2T, 3T, \ldots, nT\}$. Fig. 1 shows how random jitter is modeled. The continuous time information signal, $i(t)$, is represented by a periodic square wave (representing alternating zeros and ones). At each ideal sampling point, above the waveform, is the probability density function $f_X(x)$. The value x_1 is a realisation of the random variable X at $t = T$, x_2 is the realisation of X at $t = 2T$ and so on. Then, the total deviation in time, y_n, from the ideal sampling point at $t = nT$ is the total accumulative jitter up to and including $t = nT$. Therefore,

$$y_n = \sum_{i=1}^{n} x_i. \tag{2}$$

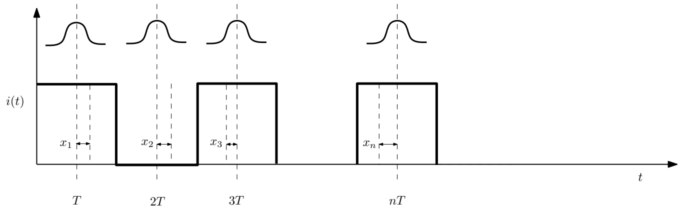

Fig. 1. Representation of jitter in the time domain.

Let y_n be a realisation of a random variable Y. Then,

$$Y = \underbrace{X + X + \ldots + X}_{n \text{ times}}. \tag{3}$$

A similar modeling of random jitter is present in [1], [10].

A deletion is said to occur when $y_n \geq 1/2T$ and an insertion when $y_n \leq -1/2T$.

3. Insertion/Deletion Channels

In this section, we are going to consider two insertion/deletion channels: the Equal Insertion/Deletion Probability Channel (EIDP) and the Unequal Insertion/Deletion Probability Channel (UIDP). EIDP channel is the standard insertion/deletion channel much used in literature, where the probability of an insertion/deletion error is equal for all transmitted bits.

3.1. Equal Insertion/Deletion Probability Channel

In order for the probability of an insertion/deletion error to be equal for all bits, it needs to be assumed that the jitter at time $t = nT$ is not dependent on the jitter at the previous sampling points, but only on the jitter at $t = nT$. In other words, the jitter is not accumulative. Such a situation is imaginable, for example, when there is some anti-jitter circuit such as a phase-lock loop (PLL). In an idealised scenario, the anti-jitter circuit would remove the effect of x_1 at $t = 2T$, so that only x_2 would comprise the total jitter at $t = 2T$. Similarly, at $t = 3T$, the effect of x_2 would be removed, and so on. Therefore, the total jitter at time $t = nT$ would be $y_n = x_n$. Since x_n, for all n, is the realisation of the same random variable X, the probability of an insertion/deletion error is not a function of n.

The next step is to determine the probability of an insertion/deletion error, $P_{\text{id}}(T, \sigma_X)$, as a function of T and σ_X. Let $P_{\text{d}}(T, \sigma_X)$ be the probability of a deletion error. Then

$$\begin{aligned} P_{\text{d}}(T, \sigma_X) &= P(x \geq 1/2T) \\ &= \int_{1/2T}^{\infty} f_X(x)\, dx. \end{aligned} \tag{4}$$

Unfortunately, no closed-form solution exists for the above integral. However, a good approximation does exist. Define $Q(z)$ as

$$Q(z) = \frac{1}{\sqrt{2\pi}} \int_z^{\infty} e^{-\zeta^2/2}\, d\zeta. \tag{5}$$

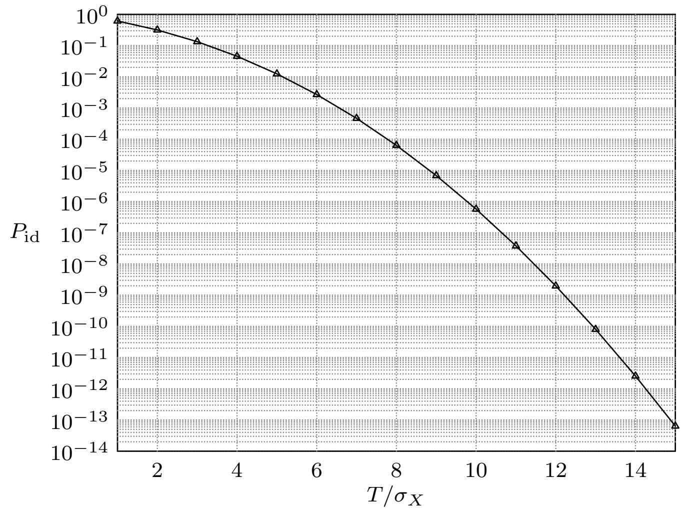

Fig. 2. The probability of an insertion/deletion as a function of the ratio T/σ_X.

Then $Q(z)$ can be approximated by [11]:

$$Q(z) \approx \frac{1}{(1-a)z + a\sqrt{z^2 + b}} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \tag{6}$$

where a and b can be optimised for a particular interval of z. For $z \geq 0$, $a = 0.339$ and $b = 5.51$. Let $\zeta = x/\sigma_X$. Then

$$\begin{aligned} P(x \geq 1/2T) &= \frac{1}{\sqrt{2\pi}} \int_{T/2\sigma_X}^{\infty} e^{-\zeta^2/2}\, d\zeta \\ &\approx \frac{1}{(1-a)\frac{T}{2\sigma_X} + a\sqrt{\frac{T^2}{4\sigma_X^2} + b}} \frac{1}{\sqrt{2\pi}} e^{\frac{-T^2}{8\sigma_X^2}}. \end{aligned}$$

Note that the probability of a deletion error is in fact only a function of the ratio T/σ_X. Since $f_X(x)$ is symmetric, the probability of an insertion error is equal to the probability of a deletion error. Therefore, $P_{\text{id}}(T/\sigma_X) = 2P_{\text{d}}(T/\sigma_X)$.

Fig. 2 shows the plot of P_{id} as a function of T/σ_X. It can be seen, for a modest $T/\sigma_X = 10$, $P_{\text{id}} \approx 10^{-6}$.

3.2. Unequal Insertion/Deletion Probability Channel

For the UIDP channel, we assume that there is no protection against jitter so that it accumulates through time, or alternatively, that the anti-jitter protection decreases the variance σ_X^2 of the jitter, which still accumulates. Then y_n is expressed by (2) and Y by (3).

Let $f_Y(y)$ be the probability density function (PDF) of the random variable Y. From statistics, for example [12], it is known that the PDF of a sum of two random variables is the convolution of the two PDFs. Therefore,

$$f_Y(y) = \underbrace{f_X(x) * f_X(x) * \ldots * f_X(x)}_{n \text{ times}}. \tag{7}$$

The convolution of two Gaussian distributions is itself a Gaussian distribution with a variance equal to the sum of the individual variances [13]. Therefore, $\sigma_Y^2 = n\sigma_X^2$. Mean values also add, but as the mean of $f_X(x)$ is zero, it is also zero for $f_Y(y)$. Let $P_{\text{id}}^{(1)}(n, T/\sigma_X)$ represent the probability of the first insertion/deletion occurring at $t = nT$ (nth bit). Let $P_{\text{d}}^{(1)}(n, T/\sigma_X)$ represent the probability of the first deletion occurring (assuming that no insertion has taken place previously) at $t = nT$. Then

$$\begin{aligned} P_{\text{d}}^{(1)}(n, T/\sigma_X) &= P(y_n \geq 1/2T) \\ &= \int_{1/2T}^{\infty} f_Y(y)\, dy. \end{aligned} \tag{8}$$

By setting $\zeta = y/\sigma_Y$,

$$P(y \geq 1/2T) = \frac{1}{\sqrt{2\pi}} \int_{T/2\sqrt{n}\sigma_X}^{\infty} e^{-\zeta^2/2}\, d\zeta. \tag{9}$$

Then, by (6),

$$P(y \geq 1/2T) \approx \frac{1}{(1-a)\frac{T}{2\sqrt{n}\sigma_X} + a\sqrt{\frac{T^2}{4n\sigma_X^2} + b}} \frac{1}{\sqrt{2\pi}} e^{\frac{-T^2}{8n\sigma_X^2}}.$$

Since $f_Y(y)$ is symmetrical, the probability of a first insertion is equal to the probability of a first deletion. Therefore, $P_{\text{id}}^{(1)}(n, T/\sigma_X) = 2P(y \geq 1/2T)$. It can be shown that as n tends to infinity, $P_{\text{id}}^{(1)}(n, T/\sigma_X)$ tends to one, for a finite T/σ_X. As $n \to \infty$, $z \to 0$. But $Q(0) = 1/2$, so that

$$\lim_{n\to\infty} P_{\text{id}}^{(1)}(n, T/\sigma_X) = 1. \tag{10}$$

Fig. 3 shows the plot of the probability of the first insertion/deletion as a function of n for $T/\sigma_X \in \{100, 200, 400, 800\}$. Table I shows the largest value of n for which $P_{\text{id}}^{(1)}$ is smaller than some probability for various values of T/σ_X.

Once the first insertion/deletion occurs, the dynamics change. Assume that the first insertion/deletion occurs at $t = n_1 T$. Let ϵ represent the amount of time exceeded beyond the transition point. If a deletion occurred, then $\epsilon = y_{n_1} - 1/2T$, while if an insertion occurred, then $\epsilon = -(y_{n_1} + 1/2T)$. This ϵ is a realisation of some random variable E. Then the probability of an insertion and the probability of a deletion are no longer symmetrical, in fact quite asymmetrical. If a deletion occurred first, then the probability of an insertion occurring at $t = (n_1 + n)T$ is

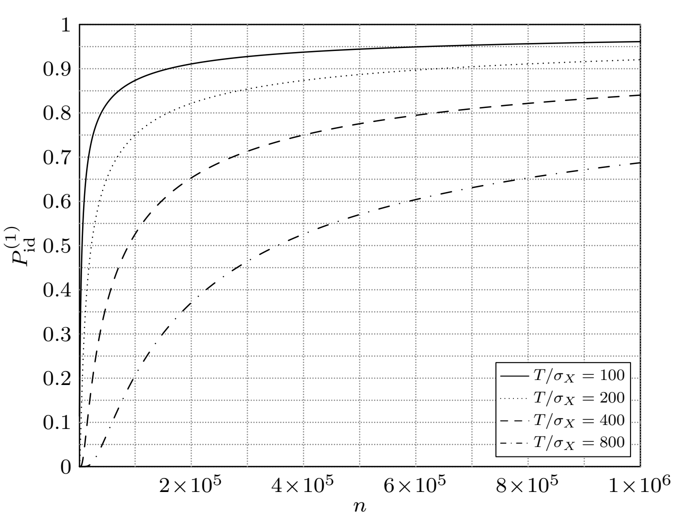

Fig. 3. The probability of the first insertion/deletion as a function of n for various values of T/σ_X.

TABLE I

VALUES OF n FOR WHICH $P_{\text{ID}}^{(1)}$ IS LESS THAN A GIVEN PROBABILITY FOR VARIOUS T/σ_X

	$P_{\text{id}}^{(1)}$		
T/σ_X	10^{-6}	10^{-5}	10^{-4}
100	104	128	165
200	417	512	660
400	1671	2049	2641
800	6685	8198	10567

$$P_{\text{i}}^{(2)}(n, T/\sigma_X) = \int_{-\epsilon}^{-\infty} f_Y(y)\, dy \tag{11}$$

and the probability of a deletion occurring that at $t = (n_1 + n)T$ is

$$P_{\text{d}}^{(2)}(n, T/\sigma_X) = \int_{T-\epsilon}^{\infty} f_Y(y)\, dy. \tag{12}$$

Usually, ϵ will be small, $\epsilon \approx 0$. Then, if a deletion occurred first, at $t = (n_1 + 1)T$, the probability of an insertion is approximately 0.5. Therefore, if a deletion occurs, it is highly likely that it will immediately be followed by an insertion, and vice versa. Then, after every insertion or deletion error, this pattern repeats. Therefore, after a deletion, $P_{\text{i}}^{(2)}(n, T/\sigma_X) = P_{\text{i}}^{(3)}(n, T/\sigma_X) = P_{\text{i}}^{(4)}(n, T/\sigma_X) = \ldots$ as defined in (11) and $P_{\text{d}}^{(2)}(n, T/\sigma_X) = P_{\text{d}}^{(3)}(n, T/\sigma_X) = P_{\text{d}}^{(4)}(n, T/\sigma_X) = \ldots$ as defined in (12). For the case if an insertion occurs first, the probabilities can be derived in the same way.

However, if one assumes that the random jitter has a non-zero mean, then the random drift will be inclined towards a drift in only one direction, eliminating the above scenario.

4. Discussion

The above section presented two insertion/deletion channel models based on different timing jitter models. The first, the EIDP channel, is equivalent to the channel often used in literature. The second, the UIDP channel, is more realistic and is based on the effect of accumulative jitter. The UIDP channel is more complex because the probability of a second, third, and so on, insertion or deletion depends on the occurrence of the previous insertion/deletion and its nature (an insertion or deletion error). We derived approximations of the probabilities of insertion/deletions errors. For both channel models, this probability is a function of the ratio T/σ_X. The UIDP channel is also a function of n.

This ratio, T/σ_X, functions as a parallel in the time domain to the signal-to-noise ratio (S/N) in the amplitude domain. In the amplitude domain, S/N defines the probability of a substitution error. It is often desired that S, the signal average power, be as small as possible, in order to conserve energy. Similarly, it is desired that T be as small as possible, as it defines the bit transmission rate. (Since one bit is transmitted in time T, the bit rate is $1/T$ bits per second). As $N \to 0$, the probability of a substitution error tends to zero. Similarly, as $\sigma_X \to 0$, the probability of an insertion/deletion error tends to zero (this is true for both insertion/deletion channels). However, in reality, a noise-free channel, in either the amplitude or time domain, does not exist. Since neither S nor T can tend to infinity, the probability of errors, either substitution or insertion/deletion, is finite. Therefore, T/σ_X possesses similar properties as S/N.

In the case of a channel with amplitude noise present, N is usually fixed (it cannot be decreased). In order for reliable communication to occur, there is a restriction on the probability of substitution error. Since the probability of a substitution error is a function of S/N, this also places a restriction on S. By using error-control coding, it is possible to decrease S/N for a given probability of substitution error in exchange for information redundancy. A decrease in S is associated with a decrease in cost.

Similar arguments can be presented to argue about the possible gain associated with coding for insertion/deletion correction. We will present this argument based on the EIDP channel, as it is the simpler channel. Note that the graph in Fig. 1 is reminiscent of probability of substitution error versus signal-to-noise ratio graphs. Fig. 1 corresponds to the probability of an insertion/deletion error with no coding. Assume that some insertion/deletion error correcting code of rate R is implemented. We assume that the correction of deletion error is accomplished by an insertion and vice versa. Therefore, erroneous decoding introduces insertion/deletion errors. One may assume that after some ratio T/σ_X, the probability of insertion/deletion error after decoding will improve on the uncoded insertion/deletion probability at a given ratio T/σ_X. Then the gain may be quantified as follows. Assume that P_{id} must be less or equal to some probability for reliable communication. Let this P_{id} be accomplished at $(T/\sigma_X)_u$ for the no coding case and $(T/\sigma_X)_c$ for the coding case. Then the gain, G, in T/σ_X is

$$G = \left(\frac{T}{\sigma_X}\right)_u - \left(\frac{T}{\sigma_X}\right)_c. \tag{13}$$

Fig. 4 presents this gain graphically. Unlike the signal-to-noise ratio gain, this gain can be realised in three ways: a decrease in T, an increase in σ_X or a mixture of the previous two. A decrease in T results in an increase in the bit transmission rate. An increase in σ_X allows for a greater jitter tolerance. This results in a cost gain, as it places less restriction on clock recovery, clock synchronisation and anti-jitter circuits, which are one of the most expensive components of a communication system.

As in the amplitude domain, this gain, G, comes at the cost of redundancy. Assume that we wish to transmit the same amount of information with coding as without coding in the same amount of time (the redundancy is not assumed to be information). In order to achieve this, T needs to be decreased

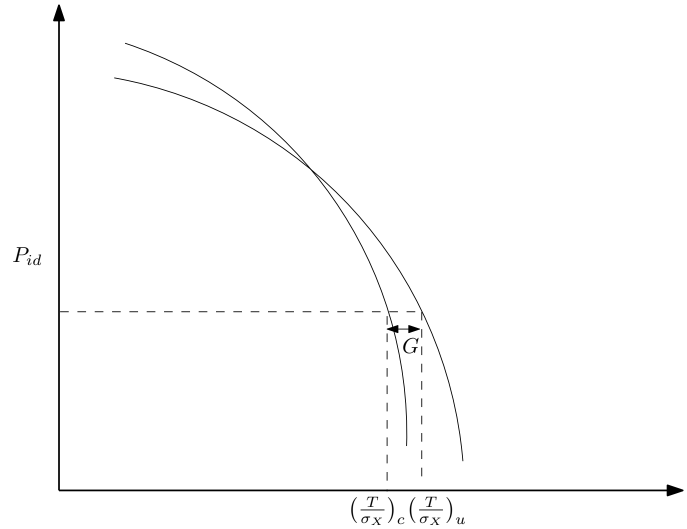

Fig. 4. A theoretical plot of P_{id} versus T/σ_X for the uncoded and coded cases.

by RT. Furthermore, assume that σ_X is constant, so that

$$G = \frac{T_u - T_c}{\sigma_X}.$$

Then, there will be an overall gain if

$$\sigma_X G = T_u - T_c > RT. \tag{14}$$

Unlike the case with substitution errors, insertion/deletion errors that are not corrected after decoding, lead to catastrophic error propagation, due to the loss of synchronisation between sender and receiver. To alleviate this, one can assume that there is some mechanism that can detect the type of error that occurred, so that after unsuccessful decoding, synchronisation is maintained.

These same general arguments can also be applied to the UIDP channel. This channel may be appropriate for asynchronous transmission. The start of asynchronous transmission is indicated by a starter bit, at which time we assume that there is perfect synchronisation. The information transmitted is then of a finite size, so that the clock does not have to be accurate for long periods.

The UIDP channel possesses its own peculiarities. Once the first insertion/deletion error occurs, the properties of the channel change. If the first error was a deletion, then the probability of an insertion for every bit thereafter is approximately 0.5, while the probability of a deletion is practically zero. A similar scenario is obtained when an insertion occurs. Therefore, after the occurrence of the first insertion/deletion, the probability of subsequent deletion/insertion errors becomes extremely high. However, the insertions and deletions will tend to alternate, which in fact result in bursts of substitution errors. If a consecutive insertion and deletion occur in the same run, there are no substitution errors, otherwise, there will be a burst of substitution errors. The probability that two consecutive insertion/deletion errors will be of the same type is extremely small. Therefore, if there is a loss of synchronisation at the end of the asynchronous transmission, it will most probably be a shift of one bit. Furthermore, since $P_{\text{id}}^{(1)}$ is a non-decreasing function of n, the burst of substitution errors is likely to occur towards the end of asynchronous transmission.

Then the gain can be formulated as follows. Assume that all asynchronous transmissions are of the same length of n bits. Let P_{id}^{*} be the probability of at least one insertion/deletion error occurring in an asynchronous transmission of length n. Then

$$P_{\text{id}}^{*} = \sum_{i=1}^{n} P_{\text{id}}^{(1)}(i, T/\sigma_X). \quad (15)$$

Let some value of P_{id}^{*} define an acceptable probability for reliable communication. Then, by coding for insertion/deletion errors, it is possible to attain this same P_{id}^{*} for a smaller T/σ_X.

In both insertion/deletion channel models, the ratio T/σ_X plays the fundamental role and defines the probability of insertion/deletion errors. It has all the fundamental characteristics of the signal-to-noise ratio in the amplitude domain. However, in practice, insertion/deletion errors have not been a problem in communication systems. This is, undoubtedly to be attributed to the fact that it possible to minimise the effects of timing jitter to within some tolerance, which is not the case with N in S/N. However, the ratio T/σ_X has an important interpretation. If T is halved, to maintain the same P_{id}, σ_X also needs to be halved. As T becomes ever smaller, so will the restrictions on σ_X. One can conceive the possibility of a time in the near future when the cost of minimising σ_X to an acceptable level may be prohibitively expensive. At that time, insertion/deletion errors will become a prominent factor.

5. Conclusion

New insight can be obtained by considering the fundamental cause of insertion/deletion errors, timing jitter. Based on timing jitter considerations, we investigate two insertion/deletion channel models. Both these models are based on some idealised conditions. The first channel, EIDP, is based on the assumption that all the jitter from previous sampling points is eliminated, and that the total jitter is equal to the jitter introduced at the current sampling point. The second channel, UIDP, is based on the assumption that there is no jitter correction, so that the total jitter is an accumulation of jitter from the all the previous sampling points. The true model probably lies somewhere in-between these two models. Nevertheless, the ratio T/σ_X emerges from both models as a natural counterpart in the time domain to the signal-to-noise ratio in the amplitude domain.

The interpretation of the gain in T/σ_X by coding for insertion/deletion errors follows the similar interpretation in the amplitude domain. However, this gain in the time domain can be realised with either T or σ_X or both. The reason why insertion/deletion errors have not been prevalent in actual communications systems, is that it is possible to combat (reduce) σ_X and make the probability of an insertion/deletion error negligibly small. Only when anti-jitter protection (used to minimise σ_X) cannot keep pace with T, will insertion/deletion errors (and hence bit synchronisation) become a problem.

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