ON RIESZ OPERATORS

by

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Introduction

Our objective in this thesis is to investigate two fundamental questions concerning Riesz operators defined on a Banach space. Recall that Riesz operators are generalizations of compact operators in the sense that Riesz operators have the same spectral properties as compact operators. However, they do not possess the same algebraic properties as compact operators. Our first question that we investigate is: When is a Riesz operator a finite rank operator? This question is motivated from the fact that if a compact operator defined on a Banach space has closed range, then it is a finite rank operator. Also, Ghahramani proved that a compact homomorphism defined on a C^*-algebra is a finite rank operator, see [17]. Martin Mathieu, in his paper [23], generalized the result of Ghahramani by proving that a weakly compact homomorphism defined on a C^*-algebra is a finite rank operator.

The next question that we investigate is the following: Let S and T be operators defined on a Banach lattice and suppose $0 \leq S \leq T$. If T is a Riesz operator, does it follow that S is a Riesz operator? We refer to this question as the domination problem for Riesz operators. Amongst other things, Troitsky investigated the domination problem for Riesz operators in [40]. The domination problem originates from the work of Dodds and Fremlin [14] where they investigated the question whether or not S is compact if T is compact, i.e., they investigated the domination problem for compact operators. In this regard, see also the work of Aliprantis and Burkinshaw [2]. The domination problem for weakly compact operators was investigated by Wickstead [41] and Aliprantis and Burkinshaw [3]. Since the 1980's the domination problem was researched extensively for many classes of operators. It was communicated to us by Professor Tony Wickstead that the domination problem for Riesz operators on Banach lattices was first explicitly posed by T.T. West to a Ph.D. student of his, Ed Bach, who later gave a talk on the problem at the 1985 conference in Tübingen to mark the 60th birthday of H.H. Schaeffer.

This thesis is organized as follows: Chapter 1 contains preparatory material needed for our study. Results in this chapter are generally well known and will only be mentioned and not proved. In Chapter 2 we focus on the question when is a Riesz operator finite rank and in Chapter 3 we focus on the domination problem for Riesz operators. Chapters 2 and 3 contain new results.

In the thesis, Theorems, Definitions and other results are numbered successively in each section. When we refer to Theorem x.y.z, we mean Theorem z in Section y of Chapter x. The symbol "■" indicates the end of a proof.

Chapter 1

Preliminaries

As previously mentioned, in this chapter we are going to focus on the necessary tools for our study which are: a brief overview of Banach algebras, spectral theory and related concepts. Included here are some reminders and remarks that will be useful in persuading our discussions in Chapter 2 and 3.

1.1 Banach Algebras

An *algebra* A is a vector space over a field $\mathbb{K}$ such that for all elements $x, y, z \in A$ and $\alpha \in \mathbb{K}$ a unique product xy satisfies the following properties:

(i) $(xy)z = x(yz)$;

(ii) $x(y+z) = xy + xz$ and $(x+y)z = xz + yz$;

(iii) $\alpha(xy) = (\alpha x)y = x(\alpha y)$.

A is said to be *commutative* (or *abelian*) if the multiplication is commutative (i.e., $xy = yx$, for all $x, y \in A$). A is called a *unital algebra* if A contains a unit $\mathbf{1}$ such that $\mathbf{1}x = x\mathbf{1} = x$, for all $x \in A$. An element x of a unital algebra A is said to be *invertible* if there exists a unique element $y \in A$ such that $xy = yx = \mathbf{1}$. In this case, we write $y = x^{-1}$. We denote the set of invertible elements in A by A^{-1}. Note that A^{-1} forms a group under multiplication and it contains the unit $\mathbf{1}$.

A *normed algebra* A is a normed space which is an algebra equipped with a norm $\|\cdot\|$ that satisfies the following norm inequality:

$$\|xy\| \leq \|x\| \cdot \|y\|, \text{ for all } x, y \in A$$

and if A is unital, then we can assume $\|\mathbf{1}\| = 1$. A *Banach algebra* A is a complete normed algebra. If A is a non-unital Banach algebra, it is always possible to imbed it isometrically in the Banach algebra $A \oplus \mathbb{C}$ with unit. The operations and the norm in $A \oplus \mathbb{C}$ are defined by

$$\begin{aligned}(x, \alpha)(y, \beta) &= (xy + \beta x + \alpha y, \alpha\beta), \\ (x, \alpha) + (y, \beta) &= (x + y, \alpha + \beta), \\ \lambda(x, \alpha) &= (\lambda x, \lambda\alpha) \text{ and} \\ \|(x, \alpha)\| &= \|x\| + |\alpha|\end{aligned}$$

for all $\alpha, \beta \in \mathbb{C}$ and $x, y \in A$. Thus $A \oplus \mathbb{C}$ is a unital algebra with unit $(0, 1)$. We call B a *subalgebra* of an algebra A if B is a subspace of A such that $xy \in B$, for each $x, y \in B$. A *two-sided ideal* in an algebra A is a subspace $I \subseteq A$ so that for all $x \in A$ and $y \in I$, we always have $xy \in I$ and $yx \in I$. Similarly we define the notion of a $left$ ideal and a $right$ ideal by $AI \subseteq I$ and $IA \subseteq I$, respectively.

In this thesis, all Banach algebras will be assumed to have a unit $\mathbf{1}$ and to be over the field $\mathbb{C}$ of complex numbers. Let I be a closed two-sided ideal of a Banach algebra A. Then A/I with the quotient norm

$$\|x + I\| = \inf_{u \in I} \|x + u\|,$$

where $x + I$ denotes the coset containing x, is a Banach algebra. Among Banach algebras there are very interesting ones called C^*-algebras and we shall briefly discuss them. An *involution* in a complex Banach algebra A is a map $x \mapsto x^*$ of A into itself such that for all $x, y \in A$ and $\lambda \in \mathbb{C}$ we have:

(i) $(x+y)^* = x^* + y^*$;

(ii) $(xy)^* = y^*x^*$;

(iii) $(\lambda x)^* = \overline{\lambda}x^*$, where $\overline{\lambda}$ is the complex conjugate of λ;

(iv) $(x^*)^* = x$.

An algebra over $\mathbb{C}$ endowed with an involution is called an *involutive algebra*. The element x^* is often called the *adjoint* of x. Then any $x \in A$ such that $x = x^*$ is called *self-adjoint*, any $x \in A$ such that $xx^* = x^*x$ is called *normal* and any $x \in A$ such that $xx^* = x^*x = \mathbf{1}$ is called *unitary*. If A is a Banach algebra with involution such that $\|x^*x\| = \|x\|^2$, for each $x \in A$, then A is called a C^*-*algebra*.

Example 1.1.1 Let X be a compact space and let $C(X)$ be the Banach space of all complex-valued continuous functions on X with the usual norm

$$\|f\| = \sup_{x \in X} |f(x)|.$$

Multiplication in $C(X)$ is defined pointwise (i.e., $(f \cdot g)(x) = f(x) \cdot g(x)$) and the involution is defined by complex conjugation (i.e., $f^*(x) = \overline{f(x)}$). One can easily show that $C(X)$ is a unital commutative C^*-algebra.

Another example is the algebra $\mathcal{L}(H)$ of all bounded linear operators on a Hilbert space H. For each $T \in \mathcal{L}(H)$ let T^* be the usual adjoint. Since $\|T^*T\| = \|T\|^2$, for all $T \in \mathcal{L}(H)$, $\mathcal{L}(H)$ is a C^*-algebra.

Let A be a C^*-algebra. An operator $T \in \mathcal{L}(A)$ is said to be a *homomorphism* if $Tab = TaTb$, for all $a, b \in A$ and this notion will play a crucial role in section 2.2.

If a C^*-algebra A has a unit $\mathbf{1}$ and $T : A \to A$ is an onto homomorphism then $T\mathbf{1} = \mathbf{1}$. If a C^*-algebra A is non-unital and there is a need for an identity then we will adjoin an identity to A. Recall that this extension is denoted by $A \oplus \mathbb{C}$, where the algebraic operations and the norm on $A \oplus \mathbb{C}$ are previously defined (see p. 7).

Also, recall that $(0,1)$ is the identity in $A \oplus \mathbb{C}$. It is a straightforward exercise to prove that the norm defined by

$$\|(x,\alpha)\| = \|x\| + |\alpha|$$

satisfies the multiplicative norm condition for a Banach algebra. However, this norm does not make $A \oplus \mathbb{C}$ into a C^*-algebra. But, if we define a norm $|||\cdot|||$ on $A \oplus \mathbb{C}$ by

$$|||(a,\lambda)||| = \sup_{\|x\|\le 1} \{\|ax + \lambda x\| : x \in A\}$$

and the involution $* : A \to A$ by $(a,\lambda)^* = (a^*, \overline{\lambda})$ then $A \oplus \mathbb{C}$ is also a C^*-algebra, see ([12], Proposition 1.9 or [28], Proposition 1.1.3).

Spectral Theory in Banach Algebras

Throughout this thesis, we will only consider complex Banach algebras with unit $\mathbf{1}$. Recall that the invertible group A^{-1} is an open subset of A containing the unit $\mathbf{1}$. The *spectrum* of an element a in A will be denoted by $\sigma(a, A)$ and defined by the set

$$\sigma(a, A) = \{\lambda \in \mathbb{C} : \lambda\mathbf{1} - a \notin A^{-1}\}.$$

The *spectral radius* of an element a in A will be denoted by $r(a, A)$ and defined by

$$r(a, A) = \sup_{\lambda \in \sigma(a)} |\lambda|.$$

Note that $\sigma(a, A)$ is a non-empty compact subset of the complex plane $\mathbb{C}$ and for all $a \in A$, we have $r(a, A) \leq \|a\|$. Moreover, it is remarkable and well known that

$$r(a, A) = \lim_{n \to \infty} \|a^n\|^{\frac{1}{n}}.$$

If the algebra is clear from the context, then A will be dropped from σ and r, i.e.,

$$\sigma(a) = \sigma(a, A) \text{ and } r(a) = r(a, A).$$

If K is a subset of a topological space, we designate the boundary of K by ∂K. Note also that if $B \subset A$ is a subalgebra with $a \in B$ then $\sigma(a, A) \subset \sigma(a, B)$. However, if B is a closed subalgebra of A one can say more.

Theorem 1.1.2 ([6], Theorem 3.2.13) *Let A be a Banach algebra and let B be a closed subalgebra containing the unit $\mathbf{1}$. Then:*

(i) *B^{-1} is the union of some components of $B \cap A^{-1}$ and furthermore the set $\partial B^{-1} \cap B \cap A^{-1}$ is empty;*

(ii) *If $a \in B$ then $\sigma(a, B)$ is the union of $\sigma(a, A)$ and a (possibly empty) collection of bounded components of $\mathbb{C} \backslash \sigma(a, A)$, in particular $\partial\sigma(a, B) \subset \partial\sigma(a, A)$.*

Corollary 1.1.3 ([6], Corollary 3.2.14) *With the same hypotheses suppose that $\sigma(a, A)$ does not separate the complex plane. Then $\sigma(a, B) = \sigma(a, A)$.*

The above corollary will be used later to prove Theorem 3.5.1 which is one of our main results in the last section.

Corollary 1.1.4 ([18], Corollary 2.4(c)) *Let A and B be Banach algebras with $\mathbf{1} \in B \subset A$ and $b \in B$. If $\sigma(b, B)$ is totally disconnected then $\sigma(b, B) = \sigma(b, A)$.*

Note that in Corollary 1.1.3 we assume that B is a closed subalgebra of A. However, in Corollary 1.1.4 we merely assume B is a subalgebra of A.

A linear map $T : A \to B$, with A and B Banach algebras, is called a *homomorphism* if $T\mathbf{1} = \mathbf{1}$ and $Txy = TxTy$, for all $x, y \in A$. If in addition T is one-to-one then T will be called an *isomorphism*. The following proposition is a well-known result which is going be used later.

Proposition 1.1.5 *Let A and B be Banach algebras and let $T : A \to B$ be a homomorphism. Then $\sigma(Ta) \subset \sigma(a)$, for all $a \in A$.*

Proof:
Since T is a homomorphism, $TA^{-1} \subset B^{-1}$. Hence we have the following:

$$\begin{aligned} \lambda \notin \sigma(a) &\Rightarrow \lambda\mathbf{1} - a \in A^{-1} \\ &\Rightarrow T(\lambda\mathbf{1} - a) \in T(A^{-1}) \\ &\Rightarrow T(\lambda\mathbf{1} - a) \in B^{-1} \\ &\Rightarrow (\lambda\mathbf{1} - Ta) \in B^{-1} \\ &\Rightarrow \lambda \notin \sigma(Ta). \end{aligned}$$

■

Let $a \in A$ be fixed in the Banach algebra A and let Ω be an arbitrary open set containing $\sigma(a)$. Consider Γ to be a smooth oriented contour contained in $\Omega \backslash \sigma(a)$ and surrounding $\sigma(a)$. For $f \in H(\Omega)$, the algebra of holomorphic functions on Ω, the integral

$$\frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda\mathbf{1} - a)^{-1} d\lambda$$

is well-defined because $\lambda \mapsto (\lambda\mathbf{1} - a)^{-1}$ is defined and continuous on Γ. Moreover this integral is independent of the contour Γ surrounding $\sigma(a)$. The next result is called the Holomorphic Functional Calculus. It plays a crucial role in spectral theory of Banach algebras.

Theorem 1.1.6 ([6], Theorem 3.3.3) *Let A be a Banach algebra and let $a \in A$. If we consider the above assumptions then the mapping : $H(\Omega) \to A$ defined by*

$$f \mapsto f(a) = \frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda\mathbf{1} - a)^{-1} d\lambda$$

has the following properties:

(i) $(f_1 + f_2)(a) = f_1(a) + f_2(a)$;

(ii) $(f_1 \cdot f_2)(a) = f_1(a) \cdot f_2(a) = f_2(a) \cdot f_1(a)$;

(iii) $1(a) = \mathbf{1}$ *and* $I(a) = a$, *where* $I(\lambda) = \lambda$;

(iv) if (f_n) *converges to* f *uniformly on compact subsets of* Ω, *then* $f(a) = \lim f_n(a)$;

(v) $\sigma(f(a)) = f(\sigma(a))$.

The last property of the above theorem is called the Spectral Mapping Theorem. Note that if A is a C^*-algebra and $a \in A$ is normal then it is always possible to extend the holomorphic functional calculus to a continuous functional calculus, see ([6], Theorem 6.2.7). An element a in a C^*-algebra is said to be positive, denoted $a \geq 0$, if it is self-adjoint and if its spectrum contains only positive real numbers. The positive elements of a C^*-algebra play an important role in Chapter 3. The following result shows the monotonicity of the norm in C^*-algebras.

Proposition 1.1.7 ([28], Proposition 1.3.5) *Let A be a C^*-algebra and let $0 \leq x \leq y$. Then $a^*xa \leq a^*ya$, for each $a \in A$ and $\|x\| \leq \|y\|$.*

Theorem 1.1.8 ([6], Theorem 6.4.5) *Let A be a C^*-algebra and let I be a closed two-sided ideal of A. Then I is stable under involution and A/I is a C^*-algebra with the standard involution and the quotient norm.*

Recall that an element p in a Banach algebra is called a *non-trivial idempotent* if $p^2 = p$ and $0 \neq p \neq 1$. One can easily prove that the spectrum of a non-trivial idempotent consists only of zero and one, i.e., $\sigma(p) = \{0, 1\}$. For this fact we refer the reader to the remark following Corollary 3.2.9 in [6]. The next result is an application of Theorem 1.1.6.

Theorem 1.1.9 ([6], Theorem 3.3.4) *Let A be a Banach algebra. Suppose that $a \in A$ has a disconnected spectrum. Let U_0, U_1 be two disjoint open sets such that*

$$\sigma(a) \subset U_0 \cup U_1,\ \sigma(a) \cap U_0 \neq \emptyset \text{ and } \sigma(a) \cap U_1 \neq \emptyset.$$

Then there exists a non-trivial idempotent p commuting with a such that

$$\sigma(pa) = (\sigma(a) \cap U_1) \cup \{0\} \text{ and } \sigma(a - pa) = (\sigma(a) \cap U_0) \cup \{0\}.$$

The idempotent p in the above theorem is called a *spectral idempotent*. If for instance λ_o is an isolated point in $\sigma(a)$ then the spectral idempotent associated with a and λ_o is

$$p = p(\lambda_o, a) = \frac{1}{2\pi i} \int_\Gamma (\lambda \mathbf{1} - a)^{-1} d\lambda,$$

where Γ is a small circle that surrounds λ_o and avoids the rest of the spectrum of a.

Let I be an ideal of a Banach algebra A and let $a \in A$. Recall that any $\lambda \in \mathbb{C}$ is said to be a *Riesz point of $\sigma(a)$ relative to I* if and only if λ is an isolated point of $\sigma(a)$ and $p(\lambda, a) \in I$. Next, we characterize Riesz elements in a subalgebra of a Banach algebra in terms of Riesz elements in the Banach algebra. A two-sided ideal in a Banach algebra will be called *inessential* whenever the spectrum of every element in the ideal is either finite or a sequence converging to zero, see ([6], p. 106). If I is an inessential ideal in A, then the closure of I in A, denoted by I_A, is also an inessential ideal in A. An element a in A is called a *Riesz element relative to I_A* if the spectrum of the coset $a + I_A$ in the quotient algebra A/I_A consists of zero. We denote the set of Riesz elements in A relative to I_A by $\mathcal{R}(A, I_A)$.

Proposition 1.1.10 ([6], Corollary 5.7.5) *Let A be a Banach algebra and I an inessential ideal of A. If $a \in \mathcal{R}(A, I_A)$ then $\sigma(a)$ is either finite or a sequence converging to zero and, for every non-zero $\lambda \in \sigma(a)$ the spectral idempotent $p(\lambda, a)$ lies in I.*

Following ([8], Theorem O.3.5, p.11 and Corollary R.2.5, p.57) we call the next result the Ruston Characterization Theorem. It is a very important result because it provides us with another way of characterizing Riesz elements. We shall make use of this characterization to prove Theorem 3.5.1 which is one of our results in Section 3.5.

Theorem 1.1.11 ([27], Theorem 1.1) *Let A be a Banach algebra, I a closed inessential ideal of A and $a \in A$. Then $a \in \mathcal{R}(A, I)$ if and only if $\sigma(a)$ is finite or a sequence converging to zero and for every non-zero $\lambda \in \sigma(a)$ the spectral idempotent $p(\lambda, a)$ lies in I.*

Our next result relates Riesz elements in a Banach algebra and Riesz elements in a subalgebra.

Theorem 1.1.12 ([32], Theorem 2.1) *Let A and B be Banach algebras such that $1 \in B \subset A$ and $b \in B$. Suppose I is an inessential ideal in both A and B. Then $b \in \mathcal{R}(B, I_B)$ if and only if $b \in \mathcal{R}(A, I_A)$ and $\sigma(b, B) = \sigma(b, A)$.*

Note that in general $\mathcal{R}(B, I_B) \subset \mathcal{R}(A, I_A) \cap B$ whenever A and B are Banach algebras with $1 \in B \subset A$ and I is an inessential ideal in both A and B. However, if B is a closed subalgebra of A and I is a closed inessential ideal in both A and B then $\mathcal{R}(B, I) = \mathcal{R}(A, I) \cap B$ holds.

Let I be a closed inessential ideal in A. We call an element $a \in A$ *quasi-inessential* relative to I if there exists $k \in I$ and $n \in \mathbb{N}$ such that $\|a^n - k\| < 1$. We characterize quasi-inessential elements as follows.

Proposition 1.1.13 ([27], Proposition 5.1) *Let A be a Banach algebra, I a closed inessential ideal of A and $a \in A$. Then the following assertions are equivalent:*

(i) *a is quasi-inessential relative to* I*;*

(ii) $\lim_{n\to\infty} \|a^n + I\| = 0$*;*

(iii) $r(a + I, A/I) < 1$*;*

(iv) $a = u + k$ *with* $k \in I$ *and* $u \in A$ *with* $r(u, A) < 1$.

In the Banach algebra $\mathcal{L}(X)$ the quasi-inessential elements relative to the closed ideal $\mathcal{K}(X)$ of compact operators in $\mathcal{L}(X)$, are the quasi-compact operators on X, see section 1.4.4.

1.2 Direct Sums and Quotient Spaces

Let X be a vector space. Given two vector subspaces M and N of X, we let $M + N$ denote the set of all vectors of the form $m + n$, where $m \in M$ and $n \in N$, i.e.,

$$M + N = \{m + n : m \in M, n \in N\}.$$

The set $M + N$ is called the *sum* of M and N; it is the smallest vector subspace of X containing both M and N. If in addition $M \cap N = \{0\}$, that is, when M and N have only the zero vector in common, we write $M \oplus N$ in place of $M + N$, and we call $M \oplus N$ the *direct sum* of M and N.

Theorem 1.2.1 ([39], Theorem I.6.1) *Let M and N be subspaces of a vector space X. Then $X = M \oplus N$ if and only if each $x \in X$ may be written in the form $x = m+n$, with $m \in M$ and $n \in N$, in one and only one way.*

Let X be a Banach space and let M be a closed subspace of X. If $\pi : X \to X/M$ is the canonical mapping, i.e., $\pi(x) = x+M$, then π is a continuous mapping. It can also be shown that for each open set U in X, the image set $\pi(U)$ is open in X/M. However, it is not true that $\pi(F)$ is closed in X/M whenever F is closed in X. This follows from the next theorem and the fact that in general there exist closed sets F and M in X such that $F + M$ is not closed in X, where $F + M = \{x + m : x \in F, m \in M\}$. Observe that $\pi^{-1}(\pi(F)) = F + M$.

The next result will play a crucial role when proving Theorem 2.1.3 and Theorem 2.1.2 which are our main results in Section 2.1.

Theorem 1.2.2 ([39], Theorem 5.2) *Let M be a closed subspace of a normed vector space X. Let F be a (not necessarily closed) subset of X. Then $\pi(F)$ is closed in X/M if and only if $F + M$ is closed in X.*

Theorem 1.2.3 ([39], Theorem 5.3) *Let M be a closed subspace of a normed vector space X and let F be a finite-dimensional subspace of X. Then $F + M$ is closed in X.*

1.3 Basic Operator Theory

This section aims to provide basic knowledge needed for the study of operators in Chapters 2 and 3. Note that in this thesis operators will be considered to be linear and bounded. If X and Y are Banach spaces, we denote by $\mathcal{L}(X, Y)$ the collection of bounded linear operators from X to Y. We write $\mathcal{L}(X)$ instead of $\mathcal{L}(X, X)$. For an operator $T \in \mathcal{L}(X, Y)$, we will denote the *null space* of T by

$$N(T) = \{x \in X : Tx = 0\}$$

and the *range* of T by

$$R(T) = \{Tx : x \in X\}.$$

Recall that an operator $T : X \to Y$ is said to be *surjective* (or *onto*) whenever its range $R(T)$ coincides with Y, i.e., whenever for each $y \in Y$ there exists some $x \in X$ such that $y = Tx$. The operator $T \in \mathcal{L}(X, Y)$ is said to be *injective* (or *one-to-one*) if $N(T) = \{0\}$. And $T \in \mathcal{L}(X, Y)$ is called *bijective* if and only if T is both onto and one-to-one. Now, considering an operator $T : X \to Y$ between normed spaces, its *(operator) norm* is defined by

$$\|T\| = \sup_{\|x\| \leq 1} \|Tx\| = \sup_{\|x\| = 1} \|Tx\|.$$

If $\|T\| < \infty$, then the operator T is called a *bounded* operator but if $\|T\| = \infty$, the operator T is called an *unbounded* operator. Note that T is bounded if and only if there exists $k > 0$ such that $\|Tx\| \leq k\|x\|$, for all $x \in X$. Also note that a linear operator $T : X \to Y$ is bounded if and only if T is continuous. An operator $T : X \to Y$ is said to be *bounded below* if there exists $k > 0$ such that

$$k\|x\| \leq \|Tx\|, \text{ for all } x \in X$$

and T is called an *isometry* if $\|Tx\| = \|x\|$, for each $x \in X$. If X is a Banach space, then $\mathcal{L}(X)$ with the operator norm is a Banach algebra because $\|ST\| \leq \|S\|\|T\|$, for all $S, T \in \mathcal{L}(X)$. The identity in this Banach algebra is the identity operator I.

Theorem 1.3.1 ([1], Corollary 2.15) *Let X and Y be Banach spaces. For an operator $T \in \mathcal{L}(X,Y)$ the following conditions are equivalent.*

(i) *$R(T)$ is closed;*

(ii) *There exists a constant $c > 0$ such that for each $y \in R(T)$ there exists some $x \in X$ satisfying $y = Tx$ and $\|x\| \leq c\|y\|$;*

(iii) *For any sequence $(y_n) \in R(T)$ satisfying $\|y_n\| \to 0$, there exists a sequence $(x_n) \subset X$ such that $\|x_n\| \to 0$ and $y_n = Tx_n$, for each n.*

Corollary 1.3.2 ([1], Corollary 2.17) *Let X and Y be Banach spaces. For an operator $T \in \mathcal{L}(X,Y)$ the range $R(T)$ is closed whenever the quotient space $Y/R(T)$ is finite dimensional.*

Lemma 1.3.3 ([9], Lemma 3.2.4) *Let X and Y be Banach spaces and let $T \in \mathcal{L}(X,Y)$. If $N(T)$ is a closed subspace such that $R(T) \oplus N(T)$ is closed then $R(T)$ is closed.*

Restriction and Induced operators

Let X be a Banach space and let $T \in \mathcal{L}(X)$. A subspace M of X is said to be a *closed invariant subspace* under the operator T if $T(M) \subset M$. Obviously the trivial subspaces $\{0\}$ and X are always invariant for any bounded operator T. So the interesting closed invariant subspaces are the non-trivial ones. If M is an invariant subspace of X under T then the operator $(T|_M)$ is called the *restriction* operator of T to M and it is defined in $\mathcal{L}(M)$ by

$$(T|_M)x = Tx, \text{ for all } x \in M.$$

The operator T_M is called the *induced* operator of T on the quotient space X/M and is defined in $\mathcal{L}(X/M)$ by

$$T_M(x + M) = Tx + M, \text{ for all } x + M \in X/M.$$

Note that the operator T_M is a well-defined operator. Recall that an operator $P \in \mathcal{L}(X)$ is called a *projection* if $P^2 = P$. Each projection $P \in \mathcal{L}(X)$ determines a direct sum decomposition of X that is written as $X = R(P) \oplus N(P)$. It is also true that every direct sum decomposition of X determines a projection. An interesting alternative way of studying any operator T could be to look for projections that commute with the operator T and map X into itself. The next definition is taken from ([15], p.25) and it has the following statement: Let $T \in \mathcal{L}(X)$. Closed subspaces M_1 and M_2 of X are said to *reduce* T or to be *reducing subspaces for* T if we have the following:

(i) $X = M_1 \oplus M_2$;

(ii) $T(M_k) \subseteq M_k$, for all $k = 1, 2$.

If an operator is reduced by a pair of subspaces of X, the operator may be studied by studying the restrictions of it to these subspaces. If these restrictions can also be reduced, the study of the operator may be simplified further.

Ascent and Descent of an operator

Now, let us introduce the notions of ascent and descent of any linear operator. These two notions will be crucial in identifying finite rank Riesz operators in Chapter 2. Relationships between the ascent of an operator T, its restriction $T|_M$ and its induced operator T_M will be discussed. We also outline existing relationships between the descent of the operators above. Note that for any operator T defined on a Banach space X, we always have:

$$\{0\} \subset N(T) \subset N(T^2) \subset N(T^3) \subset \cdots \text{ and } X \supset R(T) \supset R(T^2) \supset R(T^3) \supset \cdots$$

For all $n \geq k$, it can be shown that $N(T^n) = N(T^k)$ whenever $N(T^k) = N(T^{k+1})$, for some positive integer k, see ([15], Proposition 1.43). The smallest integer n such that $N(T^n) = N(T^{n+1})$ is called the *ascent* of T and is denoted by $\alpha(T)$. Similarly, if $R(T^k) = R(T^{k+1})$, for some positive integer k, then $R(T^n) = R(T^k)$, for all $n \geq k$, see ([15], Proposition 1.45). The *descent* of T is the smallest integer n such that $R(T^n) = R(T^{n+1})$ is and it is denoted by $\delta(T)$. We will put $\alpha(T) = \infty$ and $\delta(T) = \infty$

if the ascent of T and the descent of T do not exist, respectively. The dimension of the null-space $N(T)$ is called the *nullity* of T and is denoted by $n(T)$. The dimension of the quotient space $X/R(T)$ is called the *defect* of T and is denoted $d(T)$. Next, we give an important result which shall be used in our study.

Proposition 1.3.4 ([15], Proposition 1.49) *Let X be a Banach space and $T \in \mathcal{L}(X)$. If $\alpha(T)$ and $\delta(T)$ are both finite then $\alpha(T) = \delta(T)$.*

If we let X be a Banach space and $T \in \mathcal{L}(X)$ be a nilpotent operator, then there exists a positive integer p with $\alpha(T) = \delta(T) = p$. Also for all non-zero $\lambda \in \mathbb{C}$, it follows that $\alpha(\lambda - T) = \delta(\lambda - T) = 0$. To verify it, one may first observe that since T is a nilpotent operator, we have $T^p = 0$, for some power p to be the least such natural number. Therefore, it follows that

$$R(T^p) = \{0\} \tag{1.1}$$

and

$$N(T^p) = X. \tag{1.2}$$

So from the above equations (1.1) and (1.2), we can deduce that $\delta(T) \leq p$ and $\alpha(T) \leq p$, respectively. Since both the ascent and the descent of T are finite, it follows from Proposition 1.3.4, that $\alpha(T) = \delta(T) \leq p$. To complete our argument, it only remains to show $p \leq \alpha(T) = \delta(T)$. If we let $\delta(T) = m$ and assume that $m < p$, then we have that $R(T^m) = R(T^p)$. But since T is nilpotent of order $p > m$, it then implies that for some $x \in X$ we have $T^m x \neq 0$ and $T^p x = 0$. Hence, we have $R(T^m) \neq R(T^p)$, which is a contradiction. So we have $p \leq m$ and can therefore conclude that

$$\alpha(T) = \delta(T) = p.$$

Using the spectral mapping theorem, it follows that

$$\begin{aligned} \sigma(T^p) = \{0\} \quad &\Leftrightarrow \quad \sigma^p(T) = \{0\} \\ &\Leftrightarrow \quad \sigma(T) = \{0\}. \end{aligned}$$

Hence, for all non-zero $\lambda \in \mathbb{C}$, the operator $\lambda - T$ is invertible and its inverse $(\lambda - T)^{-1}$ exists in $\mathcal{L}(X)$. For any given $y \in X$, we can find $x \in X$ such that $x = (\lambda - T)^{-1}y$. This implies that $y = (\lambda - T)x$ and therefore it follows that $\lambda - T$ is a bijective operator. Finally we can conclude that $\alpha(\lambda - T) = \delta(\lambda - T) = 0$.

It is interesting to point out that the finiteness of the ascent and descent of an operator can lead to the decomposition of a Banach space into a direct sum of invariant subspaces.

Proposition 1.3.5 ([15], Proposition 1.51) *Let X be a Banach space and $T \in \mathcal{L}(X)$. Suppose that $\alpha(T)$, $\delta(T)$ are both finite and hence equal. If $\alpha(T) = \delta(T) = p$ then*

$$X = R(T^p) \oplus N(T^p).$$

Moreover $T|_{R(T^p)}$ is bijective.

Note that the restriction of T to $N(T^p)$ in Proposition 1.3.5 is clearly nilpotent. The above proposition can also be related to Lemma 3.4.2 in [9]. The next result shows ways of decomposing the null-space and the range of a reduced operator.

Theorem 1.3.6 ([39], Theorem V.5.2) *Let X be a Banach space. If $T \in \mathcal{L}(X)$ is reduced by a pair (M_1, M_2) of complementary subspaces of X, then*

(i) $N(T) = N(T|_{M_1}) \oplus N(T|_{M_2})$;

(ii) $R(T) = R(T|_{M_1}) \oplus R(T|_{M_2})$.

The next result provide relationship between the ascent and descent of an operator, its restrictions and its induced operator.

Proposition 1.3.7 ([39], problem 6, p.293) *Let X be a Banach space. If $T \in \mathcal{L}(X)$ is reduced by a pair (M_1, M_2) of complementary subspaces of X then*

(i) $\alpha(T) = \sup\{\alpha(T|_{M_1}), \alpha(T|_{M_2})\}$;

(ii) $\delta(T) = \sup\{\delta(T|_{M_1}), \delta(T|_{M_2})\}$.

Proof:

(i) Let p be a positive integer such that $\alpha(T) = p$. Then for all $i = 1, 2$ it follows that

$$\begin{aligned} x \in N(T^{p+1}|_{M_i}) &\Leftrightarrow T^{p+1}|_{M_i}x = 0 \\ &\Leftrightarrow T^{p+1}x = 0, \quad \text{for all } x \in M_i \\ &\Leftrightarrow x \in N(T^{p+1}) \\ &\Leftrightarrow x \in N(T^p), \quad \text{since } \alpha(T) = p \\ &\Leftrightarrow T^p x = 0 \\ &\Leftrightarrow T^p|_{M_i}x = 0 \\ &\Leftrightarrow x \in N(T^p|_{M_i}). \end{aligned}$$

So $N(T^p|_{M_i}) = N(T^{p+1}|_{M_i})$ and this implies that $\alpha(T|_{M_i}) \leq p$, for all $i = 1, 2$. Therefore we have

$$\sup\{\alpha(T|_{M_1}), \alpha(T|_{M_2})\} \leq \alpha(T). \tag{1.3}$$

Now suppose $\sup \alpha(T|_{M_1}), \alpha(T|_{M_2})\} = m$. So the subspaces M_1 and M_2 are invariant under the operator T^m since T is reduced by a pair (M_1, M_2). Also we can consider the operator $T^m|_{M_i}$ to be the restriction of T^m to M_i. Hence it follows that

$$\begin{aligned} N(T^{m+1}) &= N(T^{m+1}|_{M_1}) \oplus N(T^{m+1}|_{M_2}), \quad \text{by part } (i) \text{ of Theorem 1.3.6} \\ &= N(T^m|_{M_1}) \oplus N(T^m|_{M_2}), \quad \text{from our assumption} \\ &= N(T^m), \quad \text{by part } (i) \text{ of Theorem 1.3.6.} \end{aligned}$$

Thus we have

$$\alpha(T) \leq m = \sup \alpha(T|_{M_1}), \alpha(T|_{M_2})\}. \tag{1.4}$$

Using equations (1.3) and (1.4), we finally can conclude that

$$\alpha(T) = \sup \alpha(T|_{M_1}), \alpha(T|_{M_2})\}.$$

Now if we assume that $\alpha(T) = p = \infty$, then the null spaces of the iterates of T form an increasing chain of subspaces of X such that

$$\{0\} \subset N(T) \subset N(T^2) \subset \cdots \subset N(T^k) \subset N(T^{k+1}) \subset \cdots$$

But since for each positive integer k, we have $N(T^k|_{M_i}) \subseteq N(T^k)$, then we can deduce another ascending chain of null spaces of the iterates of $T|_{M_i}$ with

$$\{0\} \subset N(T|_{M_i}) \subset N(T^2|_{M_i}) \subset \cdots \subset N(T^k|_{M_i}) \subset N(T^{k+1}|_{M_i}) \subset \cdots$$

meaning that $\alpha(T|_{M_i}) = \infty$. Clearly, we have

$$\sup_{i=1,2} \alpha(T|_{M_i}) = \infty$$

and can finally conclude that

$$\alpha(T) = \sup\{\alpha(T|_{M_1}), \alpha(T|_{M_2})\}.$$

(ii) Let assume that $\delta(T) = p$ and first prove that $\sup\{\delta(T|_{M_1}), \delta(T|_{M_2})\} \leq \delta(T)$. In order to do so, it suffices to show that $R(T^p|_{M_i}) \subseteq R(T^{p+1}|_{M_i})$. Then it follows that

$$\begin{aligned}
y \in R(T^p|_{M_1}) \quad &\Rightarrow \quad y = T^p|_{M_1} x_1, \quad \text{for some } x_1 \in M_1 \\
&\Rightarrow \quad y = T^p x_1, \quad \text{by definition} \\
&\Rightarrow \quad y \in R(T^p) = R(T^{p+1}), \quad \text{by assumption} \\
&\Rightarrow \quad \text{there exists } z \in X \ ; \ y = T^{p+1} z \\
&\Rightarrow \quad y = T^{p+1}(z_1 + z_2), \ \text{where } z_i \in M_i \ \text{for all i's} \\
&\Rightarrow \quad y = T^{p+1} z_1 + T^{p+1} z_2 \\
&\Rightarrow \quad y = T^{p+1}|_{M_1} z_1 + T^{p+1}|_{M_2} z_2 \\
&\Rightarrow \quad y - T^{p+1}|_{M_1} z_1 = T^{p+1}|_{M_2} z_2 \\
&\Rightarrow \quad y - T^{p+1}|_{M_1} z_1 \in M_2, \quad \text{since } T^{p+1}|_{M_2} z_2 \in M_2 \\
&\Rightarrow \quad y - T^{p+1}|_{M_1} z_1 = 0 \\
&\Rightarrow \quad y \in R(T^{p+1}|_{M_1}).
\end{aligned}$$

So $R(T^p|_{M_1}) \subseteq R(T^{p+1}|_{M_1})$ and it follows likewise for M_2. So, this implies that $\delta(T|_{M_i}) \leq p$, for all $i = 1, 2$. Therefore we have

$$\sup\{\delta(T|_{M_1}), \delta(T|_{M_2})\} \leq \delta(T). \tag{1.5}$$

Now in order for us to prove that $\delta(T) \leq \sup\{\delta(T|_{M_1}), \delta(T|_{M_2})\}$, we will assume that $\sup\{\delta(T|_{M_1}), \delta(T|_{M_2})\} = m$. Again the subspaces M_1 and M_2 are invariant under the operator T^m since T is reduced by a pair (M_1, M_2). Let consider the operator $T^m|_{M_i}$ to be the restriction of T^m to M_i. Then we have

$$\begin{aligned} R(T^{m+1}) &= R(T^{m+1}|_{M_1}) \oplus R(T^{m+1}|_{M_2}), \quad \text{by part } (ii) \text{ of Theorem 1.3.6} \\ &= R(T^{m}|_{M_1}) \oplus R(T^{m}|_{M_2}), \quad \text{from our assumption} \\ &= R(T^m), \quad \text{by part } (ii) \text{ of Theorem 1.3.6.} \end{aligned}$$

Thus, we have

$$\delta(T) \leq m = \sup\{\delta(T|_{M_1}), \delta(T|_{M_2})\}. \tag{1.6}$$

By the use of equations (1.5) and (1.6), we can therefore conclude that

$$\delta(T) = \sup\{\delta(T|_{M_1}), \delta(T|_{M_2})\}.$$

If we assume that $\delta(T) = \infty$, then the ranges $R(T^k)$ of the iterates of T form a nested chain of subspaces of X such that

$$X \supset R(T) \supset R(T^2) \supset \cdots \supset R(T^k) \supset R(T^{k+1}) \supset \cdots$$

But since for each positive integer k we have $R(T^k|_{M_i}) \subseteq R(T^k)$, then it follows that the ranges $R(T^k_{M_i})$ of the iterates of $T|_{M_i}$ form also a descending chain of subspaces of X such that

$$X \supset R(T|_{M_i}) \supset R(T^2|_{M_i}) \supset \cdots \supset R(T^k|_{M_i}) \supset R(T^{k+1}|_{M_i}) \supset \cdots$$

This shows that $\delta(T|_{M_i}) = \infty$, and therefore implies that

$$\sup_{i=1,2} \delta(T|_{M_i}) = \infty.$$

By symmetry from above, we still have $\delta(T) = \sup\{\delta(T|_{M_1}), \delta(T|_{M_2})\}$. ■

Moreover we have a relationship between the ascent of an operator, its restriction and its induced operator. In a similarly way, we also provide in part (ii) the case for the descent of an operator however its proof will be omitted.

Proposition 1.3.8 ([39], problem 7, p.293) *Let X be a Banach space. If a subspace $M \subseteq X$ is invariant under T then we have*

(i) $\alpha(T|_M) \leq \alpha(T) \leq \alpha(T|_M) + \alpha(T_M)$*;*

(ii) $\delta(T_M) \leq \delta(T) \leq \delta(T|_M) + \delta(T_M)$*.*

Proof:
(i) It follows from part (i) of Proposition 1.3.7 that $\alpha(T|_M) \leq \alpha(T)$. Now in order to prove the right-hand side inequality, we let $\alpha(T|_M) = r$ and $\alpha(T_M) = s$. Thus, we have

$$\begin{aligned} x \in N(T^{r+s+1}) &\Rightarrow T^{r+s+1}x = 0 \\ &\Rightarrow T^{r+s+1}x + M = M \\ &\Rightarrow T_M^{r+s+1}(x + M) = M, \quad \text{by definition} \\ &\Rightarrow T_M^{s}(x + M) = M, \quad \text{by assumption} \\ &\Rightarrow T^{s}x + M = M, \quad \text{by definition} \\ &\Rightarrow T^{s}x \in M \\ &\Rightarrow T^{r+1}|_M(T^{s}x) = 0, \quad \text{since } T^{r+s+1}x = 0 \\ &\Rightarrow T^{r}|_M(T^{s}x) = 0, \quad \text{by assumption} \\ &\Rightarrow T^{r}(T^{s}x) = 0, \quad \text{by definition} \\ &\Rightarrow T^{r+s}x = 0 \\ &\Rightarrow x \in N(T^{r+s}). \end{aligned}$$

So we can deduce that $N(T^{r+s}) = N(T^{r+s+1})$. Hence, $\alpha(T) \leq r + s$ and consequently $\alpha(T|_M) \leq \alpha(T) \leq \alpha(T|_M) + \alpha(T_M)$. ∎

Our next result provides conditions for the ascent of an operator T to dominate the ascent of its induced operator.

Theorem 1.3.9 *Let X be a Banach space and let $T \in \mathcal{L}(X)$ with $\alpha(T) = p$. If $N = N(T^p)$ then the induced operator T_N defined on the Banach space X/N has also finite ascent with $\alpha(T_N) \leq \alpha(T)$.*

Proof:

Suppose that $\alpha(T) = p$ and let $N = N(T^p)$. Thus, $N = N(T^{p+1})$ and it follows that

$$\begin{aligned} x + N \in N(T_N^{p+1}) &\Rightarrow T_N^{p+1}(x+N) = N \\ &\Rightarrow T^{p+1}x + N = N \\ &\Rightarrow T^{p+1}x \in N \\ &\Rightarrow T^p(T^{p+1}x) = 0 \\ &\Rightarrow T^{p+1}(T^p x) = 0 \\ &\Rightarrow T^p x \in N(T^{p+1}) \\ &\Rightarrow T^p x \in N \\ &\Rightarrow T^p x + N = N \\ &\Rightarrow T_N^p(x+N) = N \\ &\Rightarrow x + N \in N(T_N^p). \end{aligned}$$

But since $N(T_N^p) \subseteq N(T_N^{p+1})$, we can conclude that $N(T_N^p) = N(T_N^{p+1})$. Consequently, $\alpha(T_N) \leq \alpha(T)$. ■

The statements in the next proposition follow from [9], Lemma 3.4.2.

Proposition 1.3.10 *Let X be a Banach space and let $T \in \mathcal{L}(X)$. If we have that $\alpha(T) = \delta(T) = n \neq 0$ then*

(i) $\lambda = 0$ is an isolated point of $\sigma(T)$;

(ii) $\lambda = 0$ is a pole of $R(\lambda, T) = (\lambda I - T)^{-1}$ of order n;

(iii) the spectral idempotent belonging to T and $\lambda = 0$ has $N(T^n)$ as its range and $R(T^n)$ as its null space.

The next result is a consequence of Theorem 1.3.6.

Proposition 1.3.11 ([15], Proposition 1.37) *Let X be a Banach space and let $T \in \mathcal{L}(X)$. If closed subspaces M_1 and M_2 of X reduce T, then*

$$\sigma(T) = \sigma(T|_{M_1}) \cup \sigma(T|_{M_2}).$$

1.4 Special Classes of Operators

We discuss here basic properties of several important classes of operators such as: finite rank operators, compact operators, weakly compact operators, Riesz operators and finally quasi-compact operators.

Finite Rank operators

Let X be a Banach space and X^* the dual space of X. We denote by $\mathcal{F}(X)$ the collection of all finite rank operators in $\mathcal{L}(X)$ and by $\overline{\mathcal{F}(X)}$ its closure in $\mathcal{L}(X)$. An element $T \in \mathcal{L}(X)$ is a *finite rank operator* if its range is finite dimensional. Let y and f be two non-zero elements in X and X^*, respectively. Then the operator $y \otimes f$ defined on X by $(y \otimes f)(x) = f(x)y$ is called a *rank one operator*. It can be shown that every finite rank operator can be expressed as a finite sum of operators of rank one.

Theorem 1.4.1 ([15], Theorem 2.5) *Let X be a Banach space. Then $\mathcal{F}(X)$ is a two-sided ideal in the Banach algebra $\mathcal{L}(X)$.*

Observe that $\overline{\mathcal{F}(X)}$ is also a closed two-sided ideal in the Banach algebra $\mathcal{L}(X)$. Therefore the quotient algebra $\mathcal{L}(X)/\overline{\mathcal{F}(X)}$ is a Banach algebra too.

Compact and Weakly Compact operators

Let X be a Banach space. We call an operator $T \in \mathcal{L}(X)$ *compact* if for every bounded subset M of X the closure of $T(M)$ is compact in X. One can also prove that a linear operator $T \in \mathcal{L}(X)$ is compact if and only if for each bounded sequence $(x_n) \subset X$, the sequence (Tx_n) has a convergent subsequence. Recall that $\mathcal{K}(X)$ denote the collection of all compact operators in $\mathcal{L}(X)$. It can be shown that $\mathcal{K}(X)$ forms a closed two-sided ideal in the Banach algebra $\mathcal{L}(X)$. Note that finite rank operators on X are always compact. If T is compact and X is infinite dimensional then $0 \in \sigma(T)$. We refer the reader to Dowson ([15], Part 2) for further details on spectral theory of compact operators.

The next result is well known and it is one of the results motivating the question: When is a Riesz operator defined on a Banach space a finite rank operator?

Theorem 1.4.2 ([6], Theorem 2.2.5) *Let X be a Banach space and $T \in \mathcal{L}(X)$ be a compact operator. If the range of T is closed, then T is a finite rank operator.*

Let X be a Banach space. We call an operator $T \in \mathcal{L}(X)$ *weakly compact* if T carries norm bounded subsets of X to relatively weakly compact subsets of X. It is well-known that an operator $T \in \mathcal{L}(X)$ is weakly compact if and only if for every norm bounded sequence (x_n) in X, the sequence (Tx_n) has a subsequence that converges weakly. We denote the collection of all weakly compact operators in $\mathcal{L}(X)$ by $\mathcal{W}(X)$. It can be shown that every compact operator in $\mathcal{L}(X)$ is weakly compact and that $\mathcal{W}(X)$ is a closed two-sided ideal in $\mathcal{L}(X)$.

Riesz operators

In this subsection X is a Banach space and we introduce the collection of Riesz operators. We call $T \in \mathcal{L}(X)$ a *Riesz operator* if the coset $T + \mathcal{K}(X)$ in the quotient algebra $\mathcal{L}(X)/\mathcal{K}(X)$ is quasinilpotent, i.e., $\sigma(T + \mathcal{K}(X), \mathcal{L}(X)/\mathcal{K}(X)) = \{0\}$. Note that quasi-nilpotent operators and nilpotent operators on a Banach space are Riesz operators. Note also that a Riesz operator is a Riesz element in some Banach algebra (see p. 12). The Riesz operators do not share the same algebraic properties of the compact operators. In general, the collection of Riesz operators is not closed under addition and the product of a Riesz operator with a bounded operator need not be Riesz operator. The collection of Riesz operators on a Banach space X will be denoted by $\mathcal{R}(X)$. For more properties on Riesz operators, we refer the reader to Dowson ([15], Chapter 3).

Theorem 1.4.3 ([15], Theorem 3.13) *Let X be a Banach space and $T \in \mathcal{L}(X)$. Then T is a Riesz operator if and only if $T + \overline{\mathcal{F}(X)}$ is quasinilpotent in the quotient algebra $\mathcal{L}(X)/\overline{\mathcal{F}(X)}$.*

Our next result follows from the previous theorem and summarizes the spectral theory of Riesz operators.

Theorem 1.4.4 ([15], Theorem 3.14) *Let X be a Banach space and $T \in \mathcal{L}(X)$ be a Riesz operator.*

(i) $\sigma(T)$ *is countable and has no cluster point except possibly 0. Every non-zero $\lambda \in \sigma(T)$ is an eigenvalue of T and moreover a pole of the resolvent of T;*

Let λ be a non-zero point in $\sigma(T)$, and $\nu(\lambda)$ be the order of the pole at λ.

(ii) For each positive integer n, $N((\lambda I - T)^n)$ is finite dimensional. Also

$$N((\lambda I - T)^m) = N((\lambda I - T)^{m+1}), \text{ for all } m \geq \nu(\lambda)$$

and $\nu(\lambda)$is the smallest positive integer with this property;

(iii) For each positive integer n, $R((\lambda I - T)^n)$ is closed. Also

$$R((\lambda I - T)^m) = R((\lambda I - T)^{m+1}), \text{ for all } m \geq \nu(\lambda)$$

and $\nu(\lambda)$is the smallest positive integer with this property;

(iv) The spectral projection $p(\lambda, T)$ has a non-zero finite dimensional range given by

$$R(p(\lambda, T)) = N((\lambda I - T)^{\nu(\lambda)}).$$

The null space of $p(\lambda, T)$ is $R((\lambda I - T)^{\nu(\lambda)})$;

(v) $1 \leq \nu(\lambda) \leq dim\ R(p(\lambda, T))$.

Observe from condition (iv) that the space X can be decomposed as a direct sum: $X = R((\lambda I - T)^{\nu(\lambda)}) \oplus N((\lambda I - T)^{\nu(\lambda)}),$ where $\nu(\lambda) = \alpha(\lambda I - T) = \delta(\lambda I - T)$.

Theorem 1.4.5 ([15], Theorem 3.21) *Let X be a Banach space and $T \in \mathcal{L}(X)$ be a Riesz operator. If M is a closed invariant subspace of X under T then $T|_M$ is a Riesz operator.*

Corollary 1.4.6 *Let X be a Banach space. If $T \in \mathcal{L}(X)$ is a Riesz operator with $\alpha(T)$ and $\delta(T)$ finite, then $\sigma(T)$ is a finite set.*

Proof:
Using Proposition 1.3.4, one can let $\alpha(T) = \delta(T) = q$. Then by Proposition 1.3.5, it follows that

$$X = M \oplus N, \text{ where } M = R(T^q) \text{ and } N = N(T^q).$$

From Lemma 1.3.3, M is a closed subspace. So using Proposition 1.3.11 we have that $\sigma(T) = \sigma(T|_M) \cup \{0\}$, since $T|_N$ is nilpotent. But since $T|_M$ is a bijective Riesz operator, see Theorem 1.4.5 and Proposition 1.3.5, $0 \notin \sigma(T|_M)$. Hence 0 is not an accumulated point of $\sigma(T)$. Therefore $\sigma(T)$ is a finite set. ∎

We will show later that we can prove a much stronger result (Corollary 2.1.3). We conclude with a very important result which shall be used extensively in Chapter 2.

Theorem 1.4.7 ([15], Theorem 3.23) *Let X be a Banach space and $T \in \mathcal{L}(X)$ be a Riesz operator. If M is a closed invariant subspace of X under T then T_M is a Riesz operator.*

Quasi-Compact Operators

Let X be a Banach space. An operator $T \in \mathcal{L}(X)$ is called *quasi-compact* if there exists a compact operator K and $n \in \mathbb{N}$ such that $\|T^n - K\| < 1$. It follows from the definition of a Riesz operator and Proposition 1.4.8 below that every Riesz operator is quasi-compact. However, a quasi-compact operator need not be a Riesz operator. The following characterization of quasi-compact operators is well known.

Proposition 1.4.8 ([22], Proposition 2.2) *Let X be a Banach space. For any operator $T \in \mathcal{L}(X)$, the following assertions are equivalent.*

(i) T *is quasi-compact;*

(ii) $\lim_{n\to\infty} \|T^n + \mathcal{K}(X)\| = 0;$

(iii) $r(T+\mathcal{K}(X), \mathcal{L}(X)/\mathcal{K}(X)) < 1$;

(iv) $T = U + K$, *where* $K \in \mathcal{K}(X)$ *is of finite rank and* $U \in \mathcal{L}(X)$ *has spectral radius* $r(U) < 1$.

The collection of quasi-compact operators on a Banach space X will be denoted by $\mathcal{Q}(X)$. The operators discussed hitherto are related as follows:

$$\mathcal{F}(X) \subset \overline{\mathcal{F}(X)} \subset \mathcal{K}(X) \subset \mathcal{R}(X) \subset \mathcal{Q}(X)$$

also,

$$\mathcal{F}(X) \subset \overline{\mathcal{F}(X)} \subset \mathcal{K}(X) \subset \mathcal{W}(X).$$

There are examples to show that the above inclusions may be strict. In general, Riesz operators are not weakly compact and weakly compact operators need not be Riesz operators: To verify the above statement, one may consider the identity operator I defined on an infinite dimensional reflexive space X. Then $I : X \to X$ is a weakly compact operator but it is not a Riesz operator since X is not finite dimensional. On the other hand, if we let A be a non-unital C^*-algebra and $S : A \times A \to A \times A$ the operator defined by $S(x, y) = (y, 0)$ then S is a Riesz operator and a homomorphism but S not a finite rank operator (see proof of Example 2.2.1). In view of this it follows from Theorem 1.7.5 that S is not a weakly compact operator.

1.5 Ordered Vector Spaces

The objective of this section is to discuss a few elementary notions of Banach lattices and ordered Banach algebras. It lays the foundation for positive operators to which Chapter 3 will mostly be devoted.

Banach lattices

Here we briefly review some basic properties of ordered vector spaces, Banach lattices and positive operators. Special attention will be given to the complexification of real Banach spaces since the work in this thesis is in a complex Banach space setting. However for more details and extensive treatments of the material in this section we refer the reader to the books [34] and [1]. Recall that a reflexive, antisymmetric and transitive relation on a set is known as an order relation. That is, a binary relation $\leq$ on a set is an *order relation* whenever it satisfies the properties:

(i) $x \leq x$, for each x (relexivity);

(ii) $x \leq y$ and $y \leq x$ imply $x = y$ (antisymmetry);

(iii) $z \leq x$ and $x \leq y$ imply $x \leq y$ (transitivity).

A *vector space order* on a real vector space E is an order relation that is compatible with the algebraic structure of E in the sense that if $x, y \in E$ then

(i) $x \leq y \Rightarrow x + z \leq y + z$, for all $z \in E$;

(ii) $x \leq y \Rightarrow \lambda x \leq \lambda y$, for each $\lambda \geq 0$.

An *ordered vector space* is a real vector space equipped with a vector space order. If $(E, \leq)$ is an ordered vector space and $x, y \in E$ with $x \leq y$, then the set $[x, y]$ defined by

$$[x, y] = \{z \in E : x \leq z \leq y\}$$

is called an *order interval.* If a subset of E is contained in some order interval it is called *order bounded.* If E is an ordered vector space, the set $E^+ = \{x \in E : x \geq 0\}$ is referred to as the *positive cone* or just the *cone* of E. The cone E^+ has the following properties:

(i) $E^+ + E^+ \subset E^+$;

(ii) $\lambda E^+ \subset E^+$, for all $\lambda \geq 0$;

(iii) $E^+ \cap (-E^+) = \{0\}$.

Any subset C of a real vector space satisfying the above three conditions is referred to as a cone in E. It gives rise to a vector space order $\leq$ if one defines $x \leq y$ if and only if $y - x \in C$. Then $(E, \leq)$ is an ordered vector space.

An operator $T : E \to F$ between ordered vector spaces is called *positive*, denoted by $T \geq 0$ or $0 \leq T$, if $TE^+ \subset F^+$, i.e., $Tx \geq 0$ whenever $x \geq 0$. If E is an ordered vector space and $A \subset E$ then an element $x \in E$ is called an *upper bound* for A if $a \leq x$, for all $a \in A$. An element $y \in E$ is called an *lower bound* for A if $y \leq a$, for all $a \in A$. An ordered vector space E is called a *vector lattice* (*Riesz space*) if the supremum (least upper bound) $x \vee y$ and infimum (greatest lower bound) $x \wedge y$ of any two elements $x, y \in E$ exist. For any element x in a vector lattice E, its *positive part*, *negative part* and *absolute value* are defined by $x^+ := x \vee 0$, $x^- := (-x) \vee 0$ and $|x| := x \vee (-x)$, respectively. Note that $x = x^+ - x^-$ and $|x| = x^+ + x^-$ and that $x \leq y$ if and only if $x^+ \leq y^+$ and $y^- \leq x^-$, see ([34], Corollary 2, p.52). The functions $(x, y) \mapsto x \wedge y, x \mapsto x^+, x \mapsto x^-$ and $x \mapsto |x|$ are called the lattice operations on a vector lattice. Elements $x, y \in E$ are called *disjoint* if $|x| \wedge |y| = 0$. A non-zero element u of a vector lattice E is called an *atom* whenever $0 \leq x \leq u$, $0 \leq y \leq u$ and $x \wedge y = 0$ imply that either $x = 0$ or $y = 0$. E is said to be *non-atomic* if it does not contain any atoms.

We will say that a vector lattice E is *Dedekind complete* if for any non-empty subset A of E that is bounded above, $\sup(A)$ exists in E. Equivalently, a vector lattice E is *Dedekind complete* if for each non-empty subset A of E that is bounded below, inf(A) exists in E. If E and F are vector lattices, then an operator $T : E \to F$ is called a *lattice homomorphism* if $T(x \vee y) = Tx \vee Ty$ and $T(x \wedge y) = Tx \wedge Ty$, for all $x, y \in E$. It is easy to see that lattice homomorphisms are positive operators. Indeed, if we consider a positive element $x \in E$ it follows that

$$\begin{aligned} x = x \vee 0 \Longrightarrow Tx &= T(x \vee 0) \\ &= Tx \vee T0 \\ &= Tx \vee 0. \end{aligned}$$

Hence, Tx is a positive element in E. A positive operator $T : E \to F$ is said to be *interval preserving* if $T[0, x] = [0, Tx]$, for each $x \in E^+$.

A *normed vector lattice* E is a vector lattice equipped with a norm $\| \cdot \|$ such that if $x, y \in E$ and $|x| \leq |y|$, then $\|x\| \leq \|y\|$. In a normed vector lattice the norm is monotone because it follows from $0 \leq x \leq y$ that $\|x\| \leq \|y\|$. One can also show in a normed vector lattice that $\|x\| = \||x|\|$, for all $x \in E$. Also, in a normed vector lattice E the lattice operations are uniformly continuous and the positive cone E^+ is closed. If a normed vector lattice is complete with respect to its norm it is called a *Banach lattice.*

Two important classes of Banach lattices are the AL- and AM-spaces. A Banach lattice E is said to be an *AL-space* if

$$\|x + y\| = \|x\| + \|y\|, \text{ for all } x, y \in E^+ \text{ with } x \wedge y = 0,$$

and E is said to be an *AM-space* if

$$\|x \vee y\| = \max\{\|x\|, \|y\|\}, \text{ for all } x, y \in E^+ \text{ with } x \wedge y = 0.$$

These Banach lattices play a fundamental role in the study of positive operators. Recall that a sequence (x_n) in a Banach lattice E is called *summable* if $\lim_H \sum_{n \in H} x_n$ exists in E, where H runs through the family of all finite subsets of $\mathbb{N}$ directed under inclusion; (x_n) is called *absolutely summable* if $\sum_n \|x_n\| < +\infty$. Let E and F be two normed vector lattices and G and H two normed vector spaces. A linear map $T : E \to H$ is called *cone absolutely summing* if for every positive summable sequence (x_n) in E, the sequence (Tx_n) is absolutely summable in H. A linear map $T : G \to F$ is called *majorizing* if for every null sequence (x_n) in G, the sequence $(|Tx_n|)$ is a majorized sequence in F, i.e., the sequence $(|Tx_n|)$ is bounded above.

If X is a real vector space, then the $complexification$ of X is the complex vector space $X_{\mathbf{c}}$ defined by

$$X_{\mathbf{c}} = X \oplus iX = \{x + iy : x, y \in X\},$$

whose vector space operations are defined by

$$\begin{aligned} (x_1 + iy_1) + (x_2 + iy_2) &= x_1 + x_2 + i(y_1 + y_2) \quad \text{and} \\ (\lambda + i\gamma)(x + iy) &= \lambda x - \gamma y + i(\gamma x + \lambda y). \end{aligned}$$

The real vector space X itself is usually identified with the (real) subspace $X + i\{0\}$ of $X_{\mathbf{c}}$. That is, by identifying every vector $x \in X$ with $x + i0 \in X_{\mathbf{c}}$, we can consider the vectors of X as vectors of $X_{\mathbf{c}}$. If X is also a normed space with norm $\|\cdot\|$, then we can extend the norm $\|\cdot\|$ to a norm on $X_{\mathbf{c}}$ via the formula

$$\|z\| = \sup_{\theta \in [0,2\pi]} \|x \cos\theta + y \sin\theta\|,$$

for each $z = x + iy \in X_{\mathbf{c}}$. Clearly, $\|x\| = \|x + i0\|$, for each $x \in X$. Note that

$$\frac{1}{2}(\|x\| + \|y\|) \le \|z\| \le \|x\| + \|y\|, \text{ for all } z = x + iy \in X_{\mathbf{c}}.$$

Hence, $z_n = x_n + iy_n \to x + iy$ in $X_{\mathbf{c}}$ if and only if $x_n \to x$ and $y_n \to y$ in X. Therefore, if X is a Banach space, then $X_{\mathbf{c}}$ is also a Banach space. Note also that if X is a Banach space, then every operator $T \in \mathcal{L}(X)$ gives rise naturally to a complex operator $T_{\mathbf{c}} \in \mathcal{L}(X_{\mathbf{c}})$ via the formula

$$T_{\mathbf{c}}(x + iy) = Tx + iTy\ , \text{ for all } x, y \in X$$

and satisfies $\|T_{\mathbf{c}}\| = \|T\|$. Quite often, we shall identify T with $T_{\mathbf{c}}$ without any mention.

Note also that every operator $T_{\mathbf{c}} \in \mathcal{L}(X_{\mathbf{c}})$ can be identified with the vector $T + iT$ in $\mathcal{L}(X) \oplus i\mathcal{L}(X)$. Moreover $T + iT \in \mathcal{L}(X_{\mathbf{c}})$ can be computed by the following formula: For all $x + iy \in X_c$,

$$(T + iT)(x + iy) = (Tx - Ty) + i(Tx + Ty).$$

Any complex Banach space of the form $E_{\mathbb{C}} = E + iE$ where E is a real Banach lattice will be called a *complex Banach lattice*. The *modulus* of $z = x + iy \in E_{\mathbb{C}}$ is defined by

$$|z| = \sup_{\theta \in [0,2\pi]} (x \cos\theta + y \sin\theta)$$

and the norm of $z \in E_{\mathbb{C}}$ is defined by $\|z\| = \||z|\|$.

Let E be a Dedekind complete complex Banach lattice. We call an operator $T \in \mathcal{L}(E)$ *regular* if it can be written as a linear combination over $\mathbb{C}$ of positive operators. We denote the collection of regular operators in $\mathcal{L}(E)$ by $\mathcal{L}^r(E)$. We equip $\mathcal{L}^r(E)$ with the *r-norm* defined by

$$\|T\|_r = \||T|\|, \text{ for all } T \in \mathcal{L}^r(E),$$

and note that

$$\|T\|_r = \inf\{\|S\| : \pm T \leq S\}.$$

It can be shown that $\mathcal{L}^r(E)$ with the r-norm is a Banach lattice, see ([34], Corollary 1, p.235). Since the r-norm also satisfies $\|ST\|_r \leq \|S\|_r\|T\|_r$, for all $S, T \in \mathcal{L}^r(E)$ then $\mathcal{L}^r(E)$ with the r-norm is also a Banach algebra. We will see in Chapter 3 that one can say more. Note that if $T \in \mathcal{L}^r(E)$ then in general $\|T\| \leq \|T\|_r$ and if T is a positive operator then $\|T\| = \|T\|_r$.

Ordered Banach Algebras

The study of the spectral theory of positive operators on ordered Banach spaces (or on Banach lattices) yields some interesting results. Most of the well-known spectral theoretical results in ordered structures have been proved in the operator algebra $\mathcal{L}(X)$ of bounded linear operators on an ordered Banach space X, or even on a Banach lattice X. In this section, we introduce the notion of ordered Banach algebra A, the ordering which is induced by the cone C of A. This ordering is compatible with the algebraic structure of A. A non-empty subset C of a Banach algebra A is called a *cone* of A if it satisfies the following:

(i) $C + C \subset C$;

(ii) $\lambda C \subset C$, for all $\lambda \geq 0$.

If in addition C satisfies $C \cap -C = \{0\}$, then C is called a *proper cone*. Any cone C on A induces an ordering " $\leq$ " on A in the following way:

$$a \leq b \Longleftrightarrow b - a \in C, \text{ for all } a, b \in A.$$

It can be shown that this ordering is a partial ordering on A, i.e., the ordering $\leq$ is reflexive and transitive. Moreover, C is proper if and only if this ordering is *antisymmetric*, i.e., $a \leq b$ and $b \leq a$, then $a = b$. Since C induces a partial order on A, we find that $C = \{a \in A : a \geq 0\}$ and therefore the elements of C are called *positive*. A cone C of a Banach algebra A is called an *algebra cone* if C satisfies the following conditions:

(i) $C \cdot C \subset C$;

(ii) $\mathbf{1} \in C$.

We call the two tuple (pair) (A, C) an *ordered Banach algebra* (OBA) if C is an algebra cone in A. Conversely, if $\leq$ is a partial ordering on A such that for every $a, b, c \in A$ and $\lambda \in \mathbb{C}$

(i) $a, b \geq 0 \Rightarrow a + b \geq 0$;

(ii) $a \geq 0, \lambda \geq 0 \Rightarrow \lambda a \geq 0$;

(iii) $a, b \geq 0 \Rightarrow ab \geq 0$;

(iv) $\mathbf{1} \geq 0$

and $C = \{a \in A : a \geq 0\}$ then (A, C) is an OBA. An algebra cone C in A is said to be *normal* if there exists a constant $\alpha > 0$ such that if follows from $0 \leq a \leq b$ in A that $\|a\| \leq \alpha \|b\|$. It is well known that if C is a normal algebra cone then it is proper.

Proposition 1.5.1 *Let (A, C) be an OBA and let B be a Banach algebra. If the mapping $T : A \to B$ is a homomorphism then (B, TC) is an OBA.*

Proof:
Since T is linear, TC is a cone in B. Since $T : A \to B$ is a homomorphism, see the definition following Corollary 1.1.3, it follows that TC is closed under multiplication, i.e. $TC \cdot TC = T(C \cdot C) \subset TC$. Note that $\mathbf{1} = T\mathbf{1} \in TC$ since C is an algebra cone. Hence TC is an algebra cone. ■

In particular, if F is a closed ideal in the OBA (A, C) and $\pi : A \to A/F$ is the canonical homomorphism then πC is an algebra cone of A/F, although we cannot deduce normality or closedness of πC from the corresponding properties of C. Hence, if (A, C) is an OBA, then $(A/F, \pi C)$ is an OBA too.

Let E be a complex Banach lattice with positive cone $E^+ = \{x \in E : x \geq 0\}$ and let $K = \{T \in \mathcal{L}(E) : TE^+ \subset E^+\}$. Then $(\mathcal{L}(E), K)$ is an OBA with K a closed and normal algebra cone. If E is a Dedekind complete Banach lattice then $(\mathcal{L}^r(E), K)$ is also an OBA with K a closed and normal algebra cone.

1.6 Ultrapowers

The purpose of this section is to present a method that allows one to enlarge an infinite dimensional complex Banach space X via the concept of filtration. We specially include this section for the treatment of the domination problem which is done at the last stage of the thesis.

A non-empty collection $\mathfrak{F}$ of subsets of an infinite set Δ is called a $filter$ if $\mathfrak{F}$ is closed under finite intersections, i.e., $A, B \in \mathfrak{F}$ implies $A \cap B \in \mathfrak{F}$, and closed under supersets, i.e., $A \in \mathfrak{F}$ and $A \subseteq B$ implies $B \in \mathfrak{F}$. A filter $\mathfrak{F}$ is said to be an $ultrafilter$ if it is maximal with respect to inclusion, i.e., if $\mathfrak{F} \subseteq \mathfrak{G}$ and $\mathfrak{G}$ is a filter, then $\mathfrak{G} = \mathfrak{F}$. Note that if $\mathcal{U}$ is an ultrafilter and $\cap_{A \in \mathcal{U}} A \neq \emptyset$, then one can find a unique point $\delta \in \Delta$ such that

$$\mathcal{U} = \{A \subset \Delta : \delta \in A\}.$$

Any ultrafilter $\mathcal{U}$ of the above form is called a $fixed\ ultrafilter$ and one that is not fixed is referred to as a $free\ ultrafilter$. A sequence (x_n) in a topological space Ω *converges to some* $x \in \Omega$ along the ultrafilter $\mathcal{U}$, in symbols, $x = \lim_{\mathcal{U}} x_n$, if for each neighbourhood V of x the set $\{n \in \mathbb{N} : x_n \in V\}$ belongs to $\mathcal{U}$. The point x is called a *$\mathcal{U}$-limit* of the sequence (x_n).

It should also be noted that our basic notions on ultrafilters and ultrapowers of Banach spaces are taken from the book of Abramovich and Aliprantis [1]. Let X be a Banach space and $\mathcal{U}$ a free ultrafilter on $\mathbb{N}$. We shall denote the vector space of all bounded sequences in X by $\ell^\infty(X)$. That is, $x = (x_1, x_2, \cdots)$ belongs to $\ell^\infty(X)$ if and only if (x_n) is a bounded sequence. Under the sup-norm

$$\|x\|_\infty = \sup_{n \in \mathbb{N}} \|x_n\|,$$

the vector space $\ell^\infty(X)$ is a Banach space. Also, if E a Banach lattice then $\ell^\infty(E)$ is a Banach lattice with pointwise lattice operations.

We shall denote by $\mathcal{N}_U$ the vector space of all sequences in $\ell^\infty(X)$ that converge to zero along the free ultrafilter $\mathcal{U}$, i.e.,

$$\mathcal{N}_\mathcal{U} = \{x = (x_1, x_2, \cdots) \in \ell^\infty(X) : \lim_{\mathcal{U}} x_n = 0\}.$$

One can prove that $\mathcal{N}_\mathcal{U}$ is a closed subspace of $\ell^\infty(X)$. The quotient Banach space $\ell^\infty(X)/\mathcal{N}_\mathcal{U}$ is called the *ultrapower* of X with respect to the free ultrafilter $\mathcal{U}$ and it is denoted by $X_\mathcal{U}$, i.e.,

$$X_\mathcal{U} = \ell^\infty(X)/\mathcal{N}_\mathcal{U}.$$

For each $x = (x_1, x_2, \cdots) \in \ell^\infty(X)$ we will denote its equivalent class in $X_\mathcal{U}$ by $[x]$. Note that if E is a Banach lattice then $\mathcal{N}_\mathcal{U}$ is a closed Banach lattice ideal in $\ell^\infty(E)$ and therefore the quotient space $E_\mathcal{U} = \ell^\infty(E)/\mathcal{N}_\mathcal{U}$ is a Banach lattice with the quotient norm. The natural embedding of a Banach space X into its ultrapowers, namely, $x \mapsto [(x, x, \cdots)]$ is a linear isometry ([1], Lemma 1.62) and in the case that E is a Banach lattice it is also a lattice isometry. Hence, identifying X with it image in $X_\mathcal{U}$, we can assume without loss of generality that $X_\mathcal{U}$ contains X as a closed vector subspace ([1], Theorem 1.63). Given a Banach space X, the importance of ultrapowers to operator theory comes from the existence of a natural extension of an operator $T \in \mathcal{L}(X)$ to an operator $T_\mathcal{U}$ on $X_\mathcal{U}$ defined by

$$\begin{aligned} T_\mathcal{U}[(x_1, x_2, \cdots)] &= [(Tx_1, Tx_2, \cdots)] \\ &= (Tx_1, Tx_2, \cdots) + \mathcal{N}_\mathcal{U}, \end{aligned}$$

for each $(x_1, x_2, \cdots) \in \ell^\infty(X)$. Furthermore, the mapping $\phi : \mathcal{L}(X) \to \mathcal{L}(X_\mathcal{U})$ defined by

$$\phi(T) = T_\mathcal{U}$$

is a homomorphism and an isometry ([1], Corollary 1.66). The next result shows the relationship between the spectrum of the operator T and its extension $T_\mathcal{U}$ along the ultrafilter $\mathcal{U}$.

Theorem 1.6.1 ([1], Theorem 7.18 (1)) *If X is a Banach space and $T \in \mathcal{L}(X)$ then for every ultrafilter $\mathcal{U}$ on $\mathbb{N}$ we have $\sigma(T, \mathcal{L}(X)) = \sigma(T_{\mathcal{U}}, \mathcal{L}(X_{\mathcal{U}}))$.*

Note that because X is a closed vector subspace of $X_{\mathcal{U}}$, the quotient space $\widetilde{X}$ defined by

$$\widetilde{X} = X_{\mathcal{U}}/X$$

is a Banach space. The vectors of $\widetilde{X}$ are equivalence classes and we will denote them by $\widetilde{x}$ or $[x_{\mathcal{U}}]$, for each $x = (x_1, x_2, \cdots) \in \ell^\infty(X)$. Thus, it follows that each sequence $x = (x_1, x_2, \cdots) \in \ell^\infty(X)$ satisfies

$$\widetilde{x} = 0 \Longleftrightarrow \lim_{\mathcal{U}} x_n \text{ exists in } X.$$

In particular, notice that for a sequence $y = (y_1, y_2, \cdots) \in \ell^\infty(X)$ we have that $y_{\mathcal{U}} \in [x_{\mathcal{U}}]$ if and only if there is some $v \in X$ with

$$\lim_{\mathcal{U}}(x_n - y_n - v) = 0.$$

Each operator $T \in \mathcal{L}(X)$ gives rise to a bounded linear extension $\widetilde{T} \in \mathcal{L}(\widetilde{X})$ via the well-defined formula

$$\widetilde{T}\widetilde{x} = (T_{\mathcal{U}}x_{\mathcal{U}})^{\sim} = [(Tx_1, Tx_2, \cdots)_{\mathcal{U}}], \text{ for each } (x_1, x_2, \cdots) \in \ell^\infty(X).$$

Lemma 1.6.2 ([1], Lemma 7.54 (1)) *For any operator $T \in \mathcal{L}(X)$, we have $\widetilde{T} = 0$ if and only if T is a compact operator.*

If A is a bounded subset of a metric space, then the *measure of non-compactness* $\chi(A)$ of A is the infimum of all $r > 0$ such that A can be covered by a finite number of open balls of radius r. That is,

$$\chi(A) = \inf\{r > 0 : \text{ there exist } x_1, \ldots, x_n \text{ in } A \text{ such that } A \subset \bigcup_{i=1}^{n} B(x_i, r)\}.$$

For an unbounded subset A of a metric space we let $\chi(A) = \infty$. If X is a Banach space then the *measure of non-compactness* $\chi(T)$ of an operator $T \in \mathcal{L}(X)$ is defined by

$$\chi(T) = \chi(T(B(0,1))),$$

where $B(0,1) = \{x \in X : \|x\| \leq 1\}$. In the next result, V.G. Troitsky in [40] defined the norm of an operator $\widetilde{T} \in \mathcal{L}(\widetilde{X})$ via the measure of non-compactness of $T \in \mathcal{L}(X)$ provided that X is a separable Banach space. The importance of the measure of non-compactness of an operator is highlighted by the following result.

Corollary 1.6.3 ([1], Corollary 7.58) *If X is a separable Banach space then for each operator $T \in \mathcal{L}(X)$ we have $\|\widetilde{T}\| = \chi(T)$.*

If $\mathcal{K}(X)$ is the collection of compact operators defined on a Banach space X, then $\mathcal{K}(X)$ is a closed two-sided ideal in the Banach algebra $\mathcal{L}(X)$. Moreover, the quotient space $\mathcal{L}(X)/\mathcal{K}(X)$ under the quotient norm is a Banach algebra called the *Calkin algebra* and the canonical mapping $\pi : \mathcal{L}(X) \to \mathcal{L}(X)/\mathcal{K}(X)$ defined by

$$\pi(T) = T + \mathcal{K}(X)$$

is an onto homomorphism, see ([1], Theorem 7.37). As a direct consequence of Corollary 1.6.3, note that $\chi(\cdot) : \mathcal{L}(X) \to \mathbb{R}$ is a semi-norm that induces a norm on the Calkin algebra $\mathcal{L}(X)/\mathcal{K}(X)$, provided X is a separable Banach space, see ([1], Lemma 7.59). Hence, $\mathcal{L}(X)/\mathcal{K}(X)$ can be identified as an subalgebra of bounded operators on a Banach space as stated by Troitsky in the following theorem.

Theorem 1.6.4 ([1], Theorem 7.60) *If X is a separable Banach space then the mapping $\Phi : \mathcal{L}(X)/\mathcal{K}(X) \to \mathcal{L}(\widetilde{X})$ defined by*

$$\Phi(T + \mathcal{K}(X)) = \widetilde{T},$$

is a well-defined isomorphism and so $\mathcal{L}(X)/\mathcal{K}(X)$ can be considered as an algebra of bounded operators on a Banach space $\widetilde{X}$. In addition, if $\mathcal{L}(X)/\mathcal{K}(X)$ is equipped with the norm $\chi(\cdot)$ then Φ is a linear isometry from $\mathcal{L}(X)/\mathcal{K}(X)$ into $\mathcal{L}(\widetilde{X})$.

The last result shows an interesting way of computing the spectrum of an operator $\widetilde{T} \in \mathcal{L}(\widetilde{X})$.

Theorem 1.6.5 ([1], Theorem 7.61) *If X is a Banach space then for each operator $T \in \mathcal{L}(X)$ we have*

$$\sigma(\widetilde{T}) = \sigma(T + \mathcal{K}(X), \mathcal{L}(X)/\mathcal{K}(X)).$$

As an immediate consequence of the above theorem, one may use Corollary 1.6.3 to deduce that

$$r(T + \mathcal{K}(X), \mathcal{L}(X)/\mathcal{K}(X)) = r(\widetilde{T}) = \lim_{n\to\infty} \sqrt[n]{\chi(T^n)}.$$

1.7 Finite Rank Weakly Compact Operators

It is well known that if a compact operator on a Banach space has closed range, then it is a finite rank operator (Theorem 1.4.2). Here, we illustrate the fact that compact homomorphisms on a C^*-algebra are finite rank operators. This was proved by Ghahramani [17]. Martin Mathieu in [23], generalized this result by proving that a weakly compact homomorphism defined on a C^*-algebra into a normed algebra is a finite rank operator. For an analytic proof of this fact, we refer the reader to Galé, Ransford and White in ([16], Theorem 3.1). We include a different approach to prove both results since the same ideas will be employed in Section 2 of Chapter 2. To prove that a compact homomorphism and a weakly compact homomorphism defined on C^*-algebras are of finite rank, we need some preparations.

Let X be a Banach space and $T \in \mathcal{L}(X)$. If $\widetilde{X} = X/N$ with $N = N(T)$, then the operator $\widetilde{T} : \widetilde{X} \to X$ defined by

$$\widetilde{T}(x + N) = Tx, \text{ for all } x \in X$$

is a well-defined injective operator. Note that $T = \widetilde{T} \circ \pi$ with $\pi : X \to \widetilde{X}$ the natural mapping.

Lemma 1.7.1 *Every bounded set in $\widetilde{X}$ is the image under π of a bounded set in X.*

Proof:
From $\|\pi(x)\| = \inf_{y \in N} \|x + y\|$ it easily follows that, for every $x + N \in \widetilde{X}$, there is $x \in X$ with $\pi(x) = x + N$ and $\|x\| \leq 2\|x + N\|$. Therefore, for any $r > 0$, $B_{\widetilde{X}}(N, r) \subseteq \pi(B_X(0, 2r))$, where $B(\cdot, r)$ stands for the open ball with radius r. ■

If the operator $\widetilde{T}$ is defined as above, see discussion preceding Lemma 1.7.1 then we have the following two results.

Proposition 1.7.2 *Let X be a Banach space and let $T \in \mathcal{L}(X)$ be compact. Then $\widetilde{T} : \widetilde{X} \to X$ is compact too.*

Proof:

Let $M + N$ be a bounded subset of $\widetilde{X}$. By Lemma 1.7.1, there is a bounded subset B in X such that $M + N = \pi(B)$. But the closure $\overline{T(B)}$ is compact since T is compact, and $\widetilde{T}(M + N) = \widetilde{T}(\pi(B)) = T(B)$. Thus, $\overline{\widetilde{T}(M + N)}$ is a compact set and consequently $\widetilde{T}$ is compact. ■

Proposition 1.7.3 *Let X be a Banach space and let $T \in \mathcal{L}(X)$ be weakly compact. Then $\widetilde{T} : \widetilde{X} \to X$ is weakly compact too.*

Proof:

Let $(y_n + N)$ be a bounded sequence in $\widetilde{X}$. By Lemma 1.7.1, we can find a bounded sequence (x_n) in X such that $y_n + N = \pi(x_n)$. But (Tx_n) has a weakly converging subsequence since T is weakly compact. We also have $\widetilde{T}(y_n + N) = \widetilde{T} \circ \pi(x_n) = Tx_n$ So, $\widetilde{T}(y_n + N)$ has a weakly converging subsequence and consequently $\widetilde{T}$ is weakly compact. ■

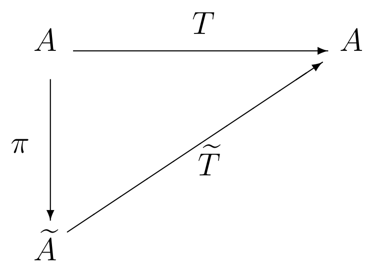
The next theorem is the result of Ghahramani. We prove it by different means. A key step in our proof is the fact that an isomorphism (by which we mean an injective homomorphism) defined on a C^*-algebra has closed range. This is a result of Sandra B. Cleveland, see ([11], Theorem 5.4).

Theorem 1.7.4 ([17], Theorem 1.) *Let A be a C^*-algebra and let $T \in \mathcal{L}(A)$ be a compact homomorphism. Then T is a finite rank operator.*

Proof:

Denote by $\widetilde{A}$ the quotient space $A/N(T)$ and note that $\widetilde{A}$ is a C^*-algebra, see Theorem 1.1.8. Let us consider the diagram defined below by the factorization of $T = \widetilde{T} \circ \pi$, where $\pi : A \to \widetilde{A}$ is the canonical (natural) mapping and $\widetilde{T} : \widetilde{A} \to A$ the mapping defined by

$$\widetilde{T}(x + N) = Tx, \quad \text{where } N = N(T). \tag{1.7}$$



Note from Proposition 1.7.2 that $\widetilde{T}$ is a compact operator since T is compact. Also, note that the operator $\widetilde{T}$ is an isomorphism, since $T : A \to A$ is a homomorphism.

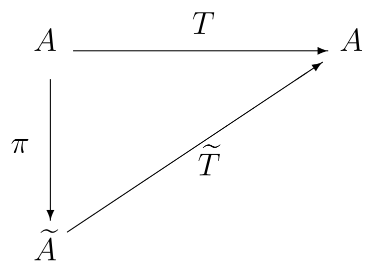
Then by Cleveland, see ([11], Theorem 5.4), it follows that $\widetilde{T}$ has closed range. But since $\widetilde{T}$ is a one-to-one operator, its inverse $\widetilde{T}^{-1}$ exists as a mapping from $\widetilde{T}\widetilde{A}$ to $\widetilde{A}$ and is therefore bounded. Then, the identity $I = \widetilde{T}^{-1} \circ \widetilde{T} : \widetilde{A} \to \widetilde{A}$ is a compact operator. This implies that the quotient space $\widetilde{A}$ is finite dimensional and $\widetilde{T}$ is a finite rank operator. But since $R(T) = R(\widetilde{T})$, our proof is complete. ∎

Alternatively, in the above proof note that $\widetilde{T}$ has closed range ([11], Theorem 5.4). Since $\widetilde{T}$ is compact, it follows from Theorem 1.4.2 that $\widetilde{T}$ is a finite rank operator and so T is a finite rank operator because $R(T) = R(\widetilde{T})$. As mentioned previously, Mathieu in [23], generalized the result of Ghahramani. He proved that a weakly compact homomorphism from a C^*-algebra into a normed algebra is a finite rank operator. We are going to prove a weaker result.

Theorem 1.7.5 *Let A be a C^*-algebra and let $T \in \mathcal{L}(A)$ be a weakly compact homomorphism. Then T is a finite rank operator.*

Proof:
Denote by $\widetilde{A}$ the quotient space $A/N(T)$ which is a C^*-algebra since $N(T)$ is a closed ideal (Theorem 1.1.8). Let us consider the diagram defined below by the factorization of $T = \widetilde{T} \circ \pi$, where $\pi : A \to \widetilde{A}$ is the canonical mapping and $\widetilde{T} : \widetilde{A} \to A$ the mapping defined by

$$\widetilde{T}(x + N) = Tx, \quad \text{where } N = N(T). \tag{1.8}$$



By Proposition 1.7.3, $\widetilde{T}$ is a weakly compact operator since T is weakly compact. Similarly to the previous proof note that $\widetilde{T}$ has closed range, because $\widetilde{T}$ is an isomor
phism, see ([11], Theorem 5.4). But since $\widetilde{T}$ is a one-to-one operator, its inverse $\widetilde{T}^{-1} : \widetilde{T}\widetilde{A} \to \widetilde{A}$ exists and is bounded, and the identity $I = \widetilde{T}^{-1} \circ \widetilde{T} : \widetilde{A} \to \widetilde{A}$ is weakly compact. Thus, $\widetilde{A}$ is a reflexive C^*-algebra. This implies that $\widetilde{A}$ is finite dimensional, see ([16], Corollary 3.4). Hence, $\widetilde{T}$ is a finite rank operator and since $R(T) = R(\widetilde{T})$, we can conclude that $R(T)$ is also finite dimensional. ■

Chapter 2

Finite Rank Riesz Operators

2.1 Banach spaces

In this section we investigate conditions for a Riesz operator defined on a Banach space to be of finite rank. Our main results in this chapter are Theorem 2.1.2 and Theorem 2.2.3. As previously mentioned, for the case of compact operators, Theorem 1.4.2 is a well-known result and a motivation for our study. The example below, shows that for Riesz operators, Theorem 1.4.2 is not necessarily true.

Example 2.1.1 *Let X be an infinite dimensional Banach space and $Y = X \times X$. Define the operator $T : Y \to Y$ by*

$$T(x_1, x_2) = (x_2, 0). \tag{2.1}$$

Then T is a Riesz operator that has closed range, but T is not a finite rank operator.

Proof:
Note that T is a nilpotent operator in $\mathcal{L}(Y)$ since $T^2 = 0$. This makes the coset

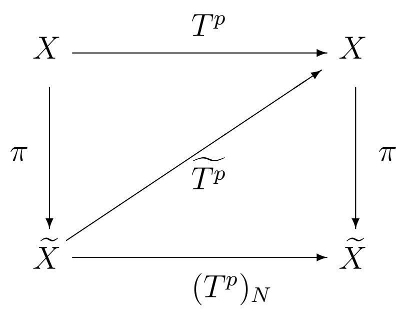
$T + \mathcal{K}(Y)$ nilpotent in the quotient space $\mathcal{L}(Y)/\mathcal{K}(Y)$. Since any nilpotent element is quasi-nilpotent, it follows that T is a Riesz operator. Note also that $R(T) = X \times \{0\}$. So $R(T)$ is closed in Y since it is the cross product of two closed subspaces of X. Therefore T is a Riesz operator with closed range. However, T is not a finite rank operator because X is infinite dimensional. ■

We are now ready to provide conditions under which a Riesz operator is a finite rank operator. As mentioned before, the next theorem is one of our main results in this chapter.

Theorem 2.1.2 *Let X be a Banach space and let $T \in \mathcal{L}(X)$ be a Riesz operator with $\alpha(T) = p$. If $R(T^p) + N(T^p)$ is closed in X, then T^p is a finite rank operator.*

Proof:
Since $\alpha(T) = p$, then we have $N(T^p) = N(T^{p+1})$. Note that $T^p \in \mathcal{L}(X)$ is a Riesz operator because it is the composition of Riesz operators and the quotient space $\widetilde{X} = X/N(T^p)$ is also a Banach space since $N(T^p)$ is a closed subspace of X. Let us consider the diagram defined by the factorization of $T^p = \widetilde{T^p} \circ \pi$, where $\pi : X \to \widetilde{X}$ is the canonical mapping and $\widetilde{T^p} : \widetilde{X} \to X$ is the natural mapping defined by

$$\widetilde{T^p}(x + N) = T^p x, \quad \text{where } N = N(T^p). \tag{2.2}$$



Since N is a closed invariant subspace of X under T^p (i.e., $T^p N \subseteq N$), the induced operator $(T^p)_N : \widetilde{X} \to \widetilde{X}$ defined by

$$(T^p)_N(x + N) = T^p x + N \tag{2.3}$$

is also a well-defined Riesz operator, see Subsection 1.3.1 and Theorem 1.4.7. We now claim that $(T^p)_N$ is also a one-to-one operator. Since the above diagram is commutative, one can notice that $\pi \circ T^p = (T^p)_N \circ \pi$ and have the following:

$$\begin{aligned} x + N \in N((T^p)_N) &\Leftrightarrow (T^p)_N(x+N) = N \\ &\Leftrightarrow T^p x + N = N \\ &\Leftrightarrow T^p x \in N \\ &\Leftrightarrow T^{2p} x = 0 \\ &\Leftrightarrow x \in N(T^{2p}) = N(T^p) \\ &\Leftrightarrow x + N = N. \end{aligned}$$

Hence $N((T^p)_N) = \{N\}$, meaning that $(T^p)_N$ is one-to-one. So the inverse operator $((T^p)_N)^{-1}$ of $(T^p)_N$ exists and is defined on $R((T^p)_N)$. Since $R(T^p) + N$ is also closed in X, it follows from Theorem 1.2.2 that $\pi(R(T^p))$ is closed in $\widetilde{X}$. This together with $R((T^p)_N) = \pi(R(T^p))$ gives that $(T^p)_N$ has closed range. Thus, $((T^p)_N)^{-1}$ is bounded and the identity $I_N = ((T^p)_N)^{-1} \circ (T^p)_N : \widetilde{X} \to \widetilde{X}$ is also a Riesz operator on $\widetilde{X}$. This implies that $\widetilde{X}$ is finite dimensional. Since $R(T^p) = R(\widetilde{T^p})$, it follows that $R(T^p)$ is finite dimensional. ■

Note that in Example 2.1.1 it is shown that $T^2 = 0$. Hence, from remarks following Proposition 1.3.4, one can deduce that T has finite ascent and descent with $\alpha(T) = \delta(T) = 2$. Now, observe in Example 2.1.1 that although T is not a finite rank operator, T^2 is a finite rank operator. The next result is a generalization of this fact.

Corollary 2.1.3 *Let X be a Banach space and let $T \in \mathcal{L}(X)$ be a Riesz operator with $\alpha(T) = \delta(T) = p$. Then T^p is a finite rank operator.*

Proof:
Note that $X = R(T^p) \oplus N(T^p)$ whenever $\alpha(T) = \delta(T) = p$, see Proposition 1.3.5. Since $R(T^p) \oplus N(T^p)$ is closed in X, it follows from Theorem 2.1.2 that T^p is a finite rank operator. ■

An alternative proof of the above corollary can be done by using the restriction operator instead of the induced operator. Indeed, if we let $\alpha(T) = \delta(T) = p$ then one can decompose X as a direct sum defined below by

$$X = M \oplus N, \text{ where } M = R(T^p) \text{ and } N = N(T^p). \tag{2.4}$$

So for all $x \in X$, we have a unique representation $x = x_1 + x_2$ with $x_1 \in M$ and $x_2 \in N$. Now if we let $T|_M$ and $T|_N$ be the restrictions of T to M and N respectively, i.e.,

$$T|_M(x) = Tx, \text{ for all } x \in M \quad \text{and} \quad T|_N(x) = Tx, \text{ for all } x \in N,$$

then by Proposition 1.3.5, $T|_M$ is a bijection and since N is a closed $T-$invariant subspace of X, one can see that $T|_N$ is nilpotent. From equation (2.4), we can define two projections $P_M : X \to M$ and $P_N : X \to N$ by

$$P_M(x) = P_M(x_1 + x_2) = x_1 \quad \text{and} \quad P_N(x) = P_N(x_1 + x_2) = x_2$$

with $P_M P_N = P_N P_M = 0$. If the operators $T^{(M)} : X \to M$ and $T^{(N)} : X \to N$ are respectively defined by $T^{(M)} = T|_M P_M$ and $T^{(N)} = T|_N P_N$, it follows that

$$\begin{aligned} Tx &= T(x_1 + x_2) \\ &= Tx_1 + Tx_2 \\ &= T|_M x_1 + T|_N x_2 \\ &= T|_M P_M(x) + T|_N P_N(x) \\ &= T^{(M)}(x) + T^{(N)}(x) \\ &= (T^{(M)} + T^{(N)})(x), \quad \text{for all } x \in X. \end{aligned}$$

Thus $T = T^{(M)} + T^{(N)}$ and since T, P_M and P_N commute, we can also deduce that $T^{(M)}T^{(N)} = T^{(N)}T^{(M)} = 0$. This together with the binomial theorem gives

$$\begin{aligned} T^p &= (T^{(M)} + T^{(N)})^p \\ &= \sum_{r=0}^{p} \binom{p}{r} (T^{(M)})^r (T^{(N)})^{p-r} \\ &= (T^{(M)})^p, \quad \text{since } T^{(N)} \text{ is nilpotent.} \end{aligned}$$

From Lemma 1.3.3, M is a closed subspace. Since $T|_M$ is a Riesz operator, see Theorem 1.4.5 and a bijection, then $T|_M$ is a finite rank operator on M. Hence

$T^{(M)} = T|_M P_M$ is a finite rank operator on X and consequently $T^p = (T^{(M)})^p$ is a finite rank operator.

Corollary 2.1.4 *Let X be a Banach space and let $T \in \mathcal{L}(X)$ be a Riesz operator. If T is a projection then T is a finite rank operator.*

Proof:
Since T is a projection, $\alpha(T) = \delta(T) = 1$. Therefore it suffices to apply Corollary 2.1.3 to complete our proof. ■

2.2 C^*-algebras

In this section we are investigating conditions for Riesz operators on a C^*-algebra A to be finite rank operators. From the definition of Riesz operator, one can easily see that every compact operator on a Banach space is a Riesz operator. Next, we are going to investigate to what extent the theorem of Ghahramani (Theorem 1.7.4) can be generalized to Riesz operators. In our following example, we will indicate that in general, a homomorphism on a C^*-algebra that is also a Riesz operator need not be a finite rank operator.

Example 2.2.1 *Let A be a non-unital C^*-algebra and B the C^*-algebra defined by $B = A \times A$. The operator $S : B \to B$ defined by*

$$S(x, y) = (y, 0), \quad \textit{for all } (x, y) \in B$$

is a Riesz operator and also a homomorphism. But S is not a finite rank operator.

Proof:

We have already shown in Example 2.1.1 that S is a Riesz operator which is not a finite rank operator. However, it is easy to show that S is also a homomorphism. ■

Given a non-unital Banach algebra A and a homomorphism $T : A \to A$, one can extend the operator T to the unitalization of A so that T remains a homomorphism. We can define $\overline{T}$ to be the extension of T on $A \oplus \mathbb{C}$ by $\overline{T}(x, \lambda) = (Tx, \lambda)$, for all $(\lambda, x) \in A \oplus \mathbb{C}$ Note that $\overline{T}$ is a homomorphism and $\overline{T}(0, 1) = (T0, 1) = (0, 1)$.

If we adjoin an identity to the C^*-algebra B in Example 2.2.1 and denote it by $B \oplus \mathbb{C}$, then $B \oplus \mathbb{C}$ is a unital C^*-algebra. As mentioned above, S can be extended to the operator $\overline{S}$ defined on $B \oplus \mathbb{C}$ by

$$\overline{S}((x, y), \lambda) = (S(x, y), \lambda), \quad \text{for all } ((x, y), \lambda) \in B \oplus \mathbb{C}.$$

Note that it is straightforward to show that $\overline{S}$ is a homomorphism. Also, note that $\overline{S}$ is a Riesz operator. Indeed, we can consider a projection $P : B \oplus \mathbb{C} \to B \oplus \mathbb{C}$ defined by

$$P(x, \lambda) = (0, \lambda), \quad \text{for all } (x, \lambda) \in B \oplus \mathbb{C}$$

and consider two operators S_1 and S_2 defined by $S_1 = \overline{S}P$ and $S_2 = \overline{S}(I - P)$, respectively. Note that the operators $\overline{S}$ and P commute and $\overline{S} = S_1 + S_2$. Also, note that S_1 is a rank one operator on $B \oplus \mathbb{C}$ and $S_2 = S$ on B with $S_2^2 = 0$ on B. Hence it follows from the definition of Riesz operator that $\overline{S}$ is a Riesz operator too, but $\overline{S}$ is not of finite rank. In our next result, we are going to provide conditions under which a homomorphism that is also a Riesz operator defined on a C^*-algebra is a finite rank operator.

Proposition 2.2.2 *Let A be a C^*-algebra and let $T \in \mathcal{L}(A)$ be a Riesz operator. If T is a monomorphism then T is a finite rank operator.*

Proof:

Suppose that T is a Riesz operator and a monomorphism defined on a C^*-algebra A. By Cleveland, it follows that T has closed range, see ([11], Theorem 5.4). Since T is a one-to-one operator, its inverse T^{-1} exists, it is defined on $T(A)$ and it is also bounded. Therefore, the identity operator $I = T^{-1} \circ T : A \to A$ is also a Riesz operator. So A is finite dimensional and consequently T is a finite rank operator. ■

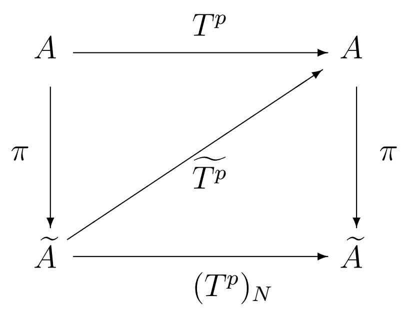
The above proposition indicates that the existence of ascent and the notion of homomorphism are of great importance in the C^*-algebra setting. The following theorem is the main result of this section.

Theorem 2.2.3 *Let A be a C^*-algebra and let $T \in \mathcal{L}(A)$ be a Riesz operator. If T is a homomorphism with $\alpha(T) = p$, then T^p is a finite rank operator.*

Proof:

Suppose that T is a Riesz operator and a homomorphism defined on a C^*-algebra A with $\alpha(T) = p$. Then as an immediate consequence, T^p is also a Riesz operator and a homomorphism. If $N = N(T^p)$ then the quotient space $\widetilde{A} = A/N$ is also a C^*-algebra since N is a closed ideal of A, see Theorem 1.1.8. Again, let us consider the diagram defined by the factorization of $T^p = \widetilde{T^p} \circ \pi$, where $\pi : A \to \widetilde{A}$ is the canonical mapping and $\widetilde{T^p} : \widetilde{A} \to A$ a natural mapping defined by

$$\widetilde{T^p}(x + N) = T^p x, \text{ for all } x \in A.$$



Since the above diagram is commutative, one can notice that $\pi \circ T^p = (T^p)_N \circ \pi$. In a similar manner done by proving Theorem 2.1.2, one can easily see that $(T^p)_N$ is a one-to-one Riesz operator. Note also that $(T^p)_N = \pi \circ \widetilde{T^p}$ is a homomorphism since it is the composition of two homomorphisms. Therefore, the operator $(T^p)_N$ has closed range, see ([11], Theorem 5.4). This implies that the inverse $((T^p)_N)^{-1}$ of $(T^p)_N$ exists as a mapping from $(T^p)_N\widetilde{A}$ to $\widetilde{A}$ and is bounded, and that the identity $I = ((T^p)_N)^{-1} \circ (T^p)_N : \widetilde{A} \to \widetilde{A}$ is also a Riesz operator. Thus, $\widetilde{A}$ is finite dimensional and $\widetilde{T^p}$ is of finite rank. But since $R(\widetilde{T^p}) = R(T^p)$, our proof is complete. ■

From the relationship between ascent of an operator and its induced operator provided in Theorem 1.3.9, one can say more.

Corollary 2.2.4 *Let A be a C^*-algebra and let $T \in \mathcal{L}(A)$ be a Riesz operator. If T is a homomorphism with $\alpha(T) = p$ and $N = N(T^p)$, then the induced operators $(T^r)_N$ are finite rank operators for all $r = 1, 2, \ldots$*

Proof:
Note that since $(T^r)_N = (T_N)^r$ and the collection of finite rank operators is an ideal, it suffices to prove the statement for $r = 1$. Henceforth, T_N is a Riesz operator, see Theorem 1.4.7 and the quotient space $\widetilde{A} = A/N$ is also a C^*-algebra since N is a closed ideal of A, see Theorem 1.1.8. One can easily observe that T_N is also a monomorphism and use Proposition 2.2.2 to complete our proof. ■

let A be a C^*-algebra and recall an operator $T \in \mathcal{L}(A)$ to be a *-*homomorphism* if T is a homomorphism and the involution * is preserved, i.e., $(Tx)^* = T(x^*)$, for all

$x \in A$. Although we have formulated and proved our two main results, Theorem 2.1.2 and Theorem 2.2.3 as seemingly independent results, Theorem 2.2.3 actually follows from Theorem 2.1.2. Indeed, if we let a Riesz operator $T \in \mathcal{L}(A)$ be a homomorphism with $\alpha(T) = p$ and suppose that $N = N(T^p)$ then one can easily see that the natural mapping $\widetilde{T^p} : \widetilde{A} \to A$ defined by

$$\widetilde{T^p}(x + N) = T^p x, \text{ for all } x \in A$$

is an isomorphism. This fact follows immediately from the discussion preceding Lemma 1.7.1 and T being a homomorphism. Hence by Cleveland, $\widetilde{T^p}$ has closed range, see ([11], Theorem 5.4). But since $R(T^p) = R(\widetilde{T^p})$, it follows that $R(T^p)$ is also closed and consequently, $\pi(R(T^p))$ is closed since $\pi : A \to \widetilde{A}$ is a *-homomorphism, see ([12], Theorem VIII.4.8). From Theorem 1.2.2, one can deduce that $R(T^p) + N$ is closed in A, and employ Theorem 2.1.2 to conclude that T^p is a finite rank operator.

Chapter 3

Domination Problem for Riesz Operators

In this chapter we consider two operators S and T defined on a Banach lattice E and satisfying $0 \leq S \leq T$. If T is a Riesz operator, when is it true that S is a Riesz operator? Since 1979, this type of question has received a lot of attention. For instance, if T is a compact operator, when is it true that S is a compact operator? (see [2] and [14]). If T is a weakly compact operator, when is it true that S is a weakly compact operator? (see [41]). Also, if T has certain spectral properties, does S inherit these properties? (see [10] and [29]).

Sometimes it will be convenient to consider the above question in the context of OBA's: Let (A, C) be an OBA and let $a, b \in A$ satisfy $0 \leq a \leq b$. If b is Riesz relative to some closed ideal, when is it true that a is Riesz relative to the same closed ideal? This question was first investigated by Raubenheimer and Rode in [33]. If b belongs to the radical of A, when does a belong to the radical of A? (see [26]). This chapter is devoted to shed some lights on the domination problem for Riesz elements and Riesz operators. Sections 3.2, 3.3, 3.4 and 3.5 contain new results.

3.1 Positive Riesz Elements

Let E be a Banach lattice and let S and T be operators defined on E and satisfy $0 \leq S \leq T$. We are going to investigate the question that if T is a Riesz operator, when does it follow that S is a Riesz operator? Firstly we mention some special cases of this question. If T is compact (then it is a Riesz operator), then S^3 is a compact operator (see [2]) and so S is a Riesz operator. If T is quasinilpotent (then T is a Riesz operator), then S is quasinilpotent because the positive cone in the OBA $\mathcal{L}(E)$ is normal, see ([33], Theorem 4.1(2)), and so S is a Riesz operator. If T is a Riesz operator such that $T + \mathcal{K}(E)$ is nilpotent in the quotient algebra $\mathcal{L}(E)/\mathcal{K}(E)$, say $T^n + \mathcal{K}(E) = \mathcal{K}(E)$, for some $n \in \mathbb{N}$ then by the remarks above S^{3n} is compact and so S is a Riesz operator.

We are now ready to gather some well-known results that shed light on our question under investigation. For all unexplained terminology in the next theorem we refer the reader to Section 1.5.

Theorem 3.1.1 ([32], Theorem 3.3) *Let E be a Banach lattice and let the operators S and T defined on E satisfy $0 \leq S \leq T$. If T is a Riesz operator then S is a Riesz operator if any one of the following conditions is satisfied:*

(i) E is an AL-space or an AM-space;

(ii) E is a Dedekind complete Banach lattice and T is majorising or cone absolutely summing;

(iii) S is an AM-compact operator;

(iv) T is either a lattice homomorphism or interval preserving;

(v) T is disjointness preserving and E has non-atomic dual space.

Proposition 3.1.2 ([32], Proposition 3.4) *Let E be a Banach lattice and let the operators S and T defined on E satisfy $0 \leq S \leq T$. If T is a Riesz operator and $r(T) \leq 1$ then S is quasi-compact.*

The key step to prove the above proposition is ([22], Proposition 2.5). One can use the above proposition to identify positive operators that cannot be dominated by a Riesz operator.

Example 3.1.3 Let $S : \ell^p \to \ell^p$, where $1 < p < \infty$, be the forward shift defined by

$$S(x_1, x_2, x_3, \cdots) = (0, x_1, x_2, x_3, \cdots).$$

Note that S is a positive operator. Then there does not exist a Riesz operator T with $r(T) \leq 1$ such that $0 \leq S \leq T$. If there exists a Riesz operator T with $r(T) \leq 1$ and $0 \leq S \leq T$, then by the proposition above S is quasi-compact. In view of Proposition 1.4.8 (iii), this is a contradiction because

$$\sigma(S + \mathcal{K}(\ell^p), \mathcal{L}(\ell^p)/\mathcal{K}(\ell^p)) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}.$$

This implies that $r(S + \mathcal{K}(\ell^p), \mathcal{L}(\ell^p)/\mathcal{K}(\ell^p)) = 1$ and consequently means that S is not quasi-compact by Proposition 1.4.8.

Let E be a Banach lattice and let $S, T \in \mathcal{L}(E)$ satisfy $0 \leq S \leq T$. We investigate the question that if T is a Riesz operator, does it follow that S is a Riesz operator? One can instead investigate the following weaker question: Let E be a Banach lattice and let $S, T \in \mathcal{L}(E)$ satisfy $S \leq T$. If T is a Riesz operator, does it follow that S is a Riesz operator? We provide an example to show that this weaker question has a negative answer.

Example 3.1.4 Let E be a Banach lattice and P be any projection on E such that $I - P$ is of finite rank. Then for all $\lambda \in (0, 1)$ we have $\lambda I - P \leq I - P$ and $I - P$ is a Riesz operator but $\lambda I - P$ is not.

Note that $I - P$ is a projection since P is a projection. Since $I - P$ is of finite rank, it is a Riesz operator, see Theorem 1.4.3. It is clear that if $\lambda \in (0, 1)$ then $\lambda I - P \leq I - P$. By the Spectral Mapping Theorem and the fact that $P + \mathcal{K}(E)$ is an idempotent in the quotient algebra $\mathcal{L}(E)/\mathcal{K}(E)$, it follows that

$$\begin{aligned}\sigma((\lambda I - P) + \mathcal{K}(E), \mathcal{L}(E)/\mathcal{K}(E)) &= \sigma(\lambda I - (P + \mathcal{K}(E)), \mathcal{L}(E)/\mathcal{K}(E)) \\ &= \lambda\mathbf{1} - \sigma(P + \mathcal{K}(E), \mathcal{L}(E)/\mathcal{K}(E)) \\ &= \lambda\mathbf{1} - \{0, 1\} \\ &\neq \{0\}.\end{aligned}$$

Hence, the operator $\lambda I - P$ is not a Riesz operator. In the above example, if we consider E to be Dedekind complete with $T = I - P$ and $S = \lambda I - P$ then although S is not a Riesz operator, S^+ the positive part of S is also a Riesz operator. Indeed, note that $0 \leq S^+ \leq T^+ \leq |T|$ since $S \leq T$ (Subsection 1.5.1, page 28). Since T is a finite rank operator, the modulus $|T|$ of T is compact, see ([1], Theorem 4.14). So using Corollary 2.35 in [1], it follows that $(S^+)^3$ is compact. Hence, S^+ is a Riesz operator because $S^+ + \mathcal{K}(E)$ is nilpotent in the quotient Banach algebra $\mathcal{L}(E)/\mathcal{K}(E)$. Recall that if I is a closed ideal in a Banach algebra A then the collection of Riesz elements relative to I is denoted by $\mathcal{R}(A, I)$. See remarks preceding Proposition 1.1.10.

Theorem 3.1.5 ([33], Theorem 6.2) *Let (A, C) be an OBA and F a closed ideal in A such that the spectral radius in the OBA $(A/F, \pi C)$ is monotone. If $a, b \in A$ is such that $0 \leq a \leq b$ with respect to C and $b \in \mathcal{R}(A, F)$ then $a \in \mathcal{R}(A, F)$.*

Let E be a complex Banach lattice. An operator $T : E \to E$ is called *regular* if it can be written as a linear combination over $\mathbb{C}$ of positive operators. The space of all regular operators on E is denoted by $\mathcal{L}^r(E)$ and it is a subspace of $\mathcal{L}(E)$. When $\mathcal{L}^r(E)$ is provided with the r-norm $\|T\|_r = \inf\{\|S\| : S \in \mathcal{L}(E), |Tx| \leq S|x|, \text{ for all } x \in E\}$ it becomes a Banach algebra which contains the unit of $\mathcal{L}(E)$, see ([34], IV §1). If in addition E is Dedekind complete, then $\mathcal{L}^r(E)$ is Banach lattice under the r-norm $\|T\|_r = \||T|\|$. This concept was introduced by Schaefer in [34]. Let $\mathcal{K}^r(E)$ denote the closure in $\mathcal{L}^r(E)$ of the ideal of finite rank operators on E. Recall that if E is a

Banach lattice then $T \in \mathcal{L}^r(E)$ is called *r-asymptotically quasi finite rank* if

$$\sigma(T + \mathcal{K}^r(E), \mathcal{L}^r(E)/\mathcal{K}^r(E)) = \{0\}.$$

Observe that an r-asymptotically quasi finite rank operator is a Riesz operator (see remark following definition 1.2 of [31]).

Corollary 3.1.6 ([33], Corollary 6.3) *Let E be a Dedekind complete Banach lattice and let the operators S and T defined on E satisfy $0 \leq S \leq T$. If T is a Riesz operator which is r-asymptotically quasi finite rank then S is r-asymptotically quasi finite rank and hence a Riesz operator.*

A key observation to prove the above corollary is the fact that the spectral radius in the OBA $(\mathcal{L}^r(E)/\mathcal{K}^r(E), \pi\mathcal{K})$ is monotone, see ([22], Theorem 2.8). In order to prove our next results, we need to investigate how Riesz elements behave with respect to subalgebras. Recall that I_A is a closed inessential ideal in A whenever I is an inessential ideal in the Banach algebra A, see remarks preceding Proposition 1.1.10.

Recall that the spectrum of T in $\mathcal{L}(E)$ is denoted by $\sigma(T) = \sigma(T, \mathcal{L}(E))$ and if T is regular, then the spectrum of T in $\mathcal{L}^r(E)$ is denoted by $\sigma_o(T) = \sigma(T, \mathcal{L}^r(E))$ and it is called the *o-spectrum* of T. This notion was introduced by Schaefer in [35].

Corollary 3.1.7 ([32], Corollary 2.2) *Let T be a regular operator on a Banach lattice E. Then T is r-asymptotically quasi finite rank if and only if T is a Riesz operator and $\sigma_o(T) = \sigma(T)$.*

Theorem 3.1.8 ([32], Theorem 3.2) *Let E be a Dedekind complete Banach lattice and let the operators S and T defined on E satisfy $0 \leq S \leq T$. If T is a Riesz operator such that $\sigma_o(T) = \sigma(T)$ then S is a Riesz operator.*

3.2 Monotonicity of the Spectral Radius

Recall that if (A, C) is an OBA then we say that the spectral radius r is *monotone with respect to* C if it follows from $x, y \in A$ and $0 \leq x \leq y$ that $r(x) \leq r(y)$. If E is a Banach lattice and $K = \{T \in \mathcal{L}(E) : TE^+ \subset E^+\}$, then in the OBA $(\mathcal{L}(E), K)$ it is well known that the spectral radius is monotone. This is because the algebra cone K is normal (Section 1.5). We have seen in the previous section what consequence is if in the domination problem the spectral radius is monotone in a quotient algebra (Theorem 3.1.5).

In this section, we are going to introduce two more notions of monotonicity of the spectral radius. We will show how these notions are related and we will illustrate these notions with examples.

Let (A, C) be an OBA and $x, y, z \in A$. If it follows from $-y \leq x \leq y$ that $r(x) \leq r(y)$, then we say that the spectral radius is *absolute monotone with respect to* C. If it follows from $x \leq y \leq z$ that $r(y) \leq \max\{r(x), r(z)\}$, then we say that the spectral radius is *strong monotone with respect to* C. These notions are related as follows:

$$r \text{ is strong monotone } \Longrightarrow r \text{ is absolute monotone } \Longrightarrow r \text{ is monotone.}$$

To verify the first implication, assume r is strong monotone and let $-y \leq x \leq y$, for all $x, y \in A$. Then it follows that

$$\begin{aligned} r(x) &\leq \max\{r(-y), r(y)\}, \text{ by assumption} \\ &= \max\{r(y), r(y)\} \\ &= r(y). \end{aligned}$$

Hence r is absolute monotone. For the other implication, let r be absolute monotone and $0 \leq x \leq y$, for all $x, y \in A$. This implies that $-y \leq x \leq y$ and from our assumption we get $r(x) \leq r(y)$, i.e., r is monotone.

A is said to be a *Banach lattice algebra* if it is both a Banach algebra and a Banach lattice such that $|ab| \leq |a||b|$, for all $a, b \in A$. For basic properties of Banach lattice algebras we refer the reader to [37].

Proposition 3.2.1 *If A is a Banach lattice algebra then the spectral radius is monotone.*

Proof:
If we consider two positive elements a, b in A then by definition we obtain that $|ab| \leq |a||b| = ab$. But since $ab \leq |ab|$, for all $a, b \in A$ it follows that $ab = |ab|$. This means that ab is positive. Therefore for all $n \in \mathbb{N}$ we get

$$\begin{aligned} 0 \leq a \leq b &\implies 0 \leq a^n \leq b^n, \text{ by induction} \\ &\implies \|a^n\| \leq \|b^n\|, \text{ since } A \text{ is a Banach lattice} \\ &\implies \|a^n\|^{1/n} \leq \|b^n\|^{1/n} \\ &\implies r(a) \leq r(b). \end{aligned}$$

Thus, r is monotone in A. ■

Our next result shows that we can actually say more.

Proposition 3.2.2 *If A is a Banach lattice algebra then the spectral radius is absolute monotone.*

Proof:
Consider two elements a, b in A such that $-b \leq a \leq b$. This implies that $\pm a \leq b$. Note that $|a| := \sup\{-a, a\}$ exists in A since A is a Banach lattice and so $0 \leq |a| \leq b$. Hence, by Proposition 3.2.1 we obtain that $r(|a|) \leq r(b)$. Since A is a Banach lattice algebra, for all $n \in \mathbb{N}$ we have

$$\begin{aligned} 0 \leq |a^n| \leq |a|^n &\implies \||a^n|\| \leq \||a|^n\| \\ &\implies \|a^n\| \leq \||a|^n\| \\ &\implies \|a^n\|^{1/n} \leq \||a|^n\|^{1/n} \\ &\implies r(a) \leq r(|a|). \end{aligned}$$

If we combine the arguments above, we get that $r(a) \leq r(b)$. ■

Recall that if E is a Banach lattice and $T \in \mathcal{L}^r(E)$ then the spectral radius of T in the Banach algebra $(\mathcal{L}^r(E), \|\cdot\|_r)$ is denoted by $r_o(T)$.

Corollary 3.2.3 *If E is a Dedekind complete Banach lattice, then the spectral radius $r_o(\cdot)$ in $\mathcal{L}^r(E)$ is absolute monotone.*

Proof:
Note that since $\mathcal{L}^r(E)$ is a Banach lattice algebra, see ([34], Exercise 4(c) p. 297), we are done. ■

Let X be a compact Hausdorff space. Denote by $C(X)$ the collection of all continuous complex-valued functions defined on X. With the pointwise addition, multiplication, scalar multiplication and the supremum norm, $C(X)$ is a unital Banach algebra. But with pointwise ordering, i.e., $f \leq g$ if and only if $f(x) \leq g(x)$, for all $x \in X$, $C(X)$ is also a Banach lattice. Since f and g are complex-valued functions, we have the following:

$$\begin{aligned} |f \cdot g|(x) &= |f(x) \cdot g(x)| \\ &= |f(x)| \cdot |g(x)| \\ &= (|f| \cdot |g|)(x), \text{ for all } x \in X, \end{aligned}$$

i.e., $|f \cdot g| = |f| \cdot |g|$. Thus, $C(X)$ is a Banach lattice algebra and in view of Proposition 3.2.2 the spectral radius in $C(X)$ is absolute monotone. Note that $C(X)$ is also an AM-space. Our next result shows that one can say more.

Proposition 3.2.4 *In the Banach lattice algebra $C(X)$ the spectral radius is strong monotone.*

Proof:
Let $f, g, h \in C(X)$ be such that $f \leq g \leq h$. Then, for each $x \in X$, $g(x) \leq h(x)$ and $-g(x) \leq -f(x)$. Therefore, $|g(x)| \leq \max\{|h(x)|, |f(x)|\}$, for all $x \in X$. This

implies that $\|g\| \leq \max\{\|f\|, \|h\|\}$ and thus $r(g) \leq \max\{r(f), r(h)\}$ since $r(\cdot)$ and $\|\cdot\|$ coincide in $C(X)$. ■

3.3 Spectral Radius in Quotient Algebras

Let I be a closed ideal in a Banach algebra A. If $x \in A$ and $x + I$ denotes the coset in A/I that contains x, then by the use of the quotient mapping $\pi : A \to A/I$ one can see that $x + I$ is invertible in A/I whenever x is invertible in A. Thus, $\sigma(x + I, A/I) \subset \sigma(x, A)$. Hence, $r(x + I, A/I) \leq r(x, A)$. Consequently

$$r(x + I, A/I) \leq \inf_{y \in I} r(x + y, A). \tag{3.1}$$

Smyth and West in [38] examined Banach algebras for which equality holds in equation (3.1). For our purposes we mention

Theorem 3.3.1 ([38], Example 3) *If X is a Banach space and $T \in \mathcal{L}(X)$, then*

$$r(T + \mathcal{K}(X), \mathcal{L}(X)/\mathcal{K}(X)) = \inf_{K \in \mathcal{K}(X)} r(T + K, \mathcal{L}(X)).$$

A key step in their proof of this result is the use of the Punctured Neighbourhood Theorem, see ([9], Theorem 3.2.10). We provide a Banach algebra proof of this fact without using the Punctured Neighbourhood Theorem.

Theorem 3.3.2 *Let A be a Banach algebra and I a closed inessential ideal in A. Then for all $x \in A$,*

$$r(x + I, A/I) = \inf_{y \in I} r(x + y, A). \tag{3.2}$$

Proof:

For all $x \in A$, we always have $\sigma(x + I) \subseteq \sigma(x)$ and so

$$\begin{aligned} \sigma(x)\backslash\sigma(x + I) &= \{\lambda \in \sigma(x) : \lambda \notin \sigma(x + I)\} \\ &= \{\lambda \in \sigma(x) : \lambda\mathbf{1} - (x + I) \in (A/I)^{-1}\}, \end{aligned}$$

where $\mathbf{1}$ is the identity in A/I. Put

$$\Delta := \{\lambda \in \sigma(x) : r(x + I) < |\lambda|\}.$$

We are going to consider two cases. Firstly,

$$\begin{aligned}\Delta = \emptyset \implies & |\lambda| \leq r(x+I), \text{ for all } \lambda \in \sigma(x) \\ \implies & r(x) \leq r(x+I) \\ \implies & \inf_{y \in I} r(x+y) \leq r(x+I).\end{aligned}$$

This together with equation (3.1) gives

$$r(x+I) = \inf_{y \in I} r(x+y).$$

Secondly, if $\Delta \neq \emptyset$, then by ([18], Corollary 6.2) Δ is a finite set and every $\lambda_i \in \Delta$ is a Riesz point of $\sigma(x)$. Let $\varepsilon > 0$ and let $F = \{\lambda_1, \cdots, \lambda_n\}$ be the finite set of points in Δ such that

$$r(x+I) + \varepsilon \leq |\lambda_i|, \tag{3.3}$$

for $1 \leq i \leq m$. If p is the spectral idempotent belonging to F and x, then $p = p(\lambda_1, x) + \cdots + p(\lambda_m, x) \in I$ and by Theorem 1.1.9, it follows from $x = (1-p)x + px$ that $r((1-p)x) < r(x+I) + \varepsilon$. Since I is an ideal, $px \in I$. Hence, if $y = -px$ then $r(x+y) < r(x+I) + \varepsilon$. Consequently,

$$r(x+I) = \inf_{y \in I} r(x+y).$$

■

Let (A, C) be an OBA and let I be a closed ideal in A. Many results in spectral theory of Banach algebras are proved under the assumption that the spectral radius in the quotient algebra A/I is monotone, see for instance [33] and [27]. In our main result in this section we provide conditions under which the spectral radius in a quotient OBA is monotone. To prove this result we need some preparation. Firstly, we define an ideal I in a Banach lattice algebra A to be called an *m-order ideal* if

(i) $x \in I \Rightarrow |x| \in I$;

(ii) $0 \leq x \leq y$ and $y \in I \Rightarrow x^m \in I$, for some $m \in \mathbb{N}$

and finally we will need the following lemma.

Lemma 3.3.3 *Let I be a closed m-order ideal in a Banach lattice algebra A. If $x \in A$ and $|x| + I$ is nilpotent in A/I then $x + I$ is nilpotent in A/I.*

Proof:

Say $|x|^k + I = I$, for some $k \in \mathbb{N}$. Since A is a Banach lattice algebra it follows that

$$\begin{aligned} |x^k| \le |x|^k &\implies -|x|^k \le x^k \le |x|^k \\ &\implies -|x|^k + I \le x^k + I \le |x|^k + I \\ &\implies I \le x^k + I \le I. \end{aligned}$$

Hence, there are $c_1, c_2 \in A^+$ such that $x^k + I = c_1 + I = -c_2 + I$. Therefore, $c_1 + c_2 \in I$ and $0 \le c_1 \le c_1 + c_2$. Since I is an m-order ideal, $c_1^m \in I$. Consequently, $(x + I)^{km} = (x^k + I)^m = (c_1 + I)^m = c_1^m + I = I$. ■

We are now ready to prove our main theorem in this section. Its proof is a modification of the proof of Theorem 2.8 in [22].

Theorem 3.3.4 *Let A be a Banach lattice algebra and let I be a closed inessential m-order ideal in A. If $a, b \in A$ is such that $0 \le a \le b$ then*

$$r(a + I, A/I) \le r(b + I, A/I).$$

Proof:

Suppose $r(b + I, A/I) < 1$. We are going to show that $r(a + I, A/I) < 1$ since this would imply that $r(a+I, A/I) \le r(b+I, A/I)$. Since b is quasi-inessential relative to I then by Proposition 1.1.13 we can find $k \in I$ and $d \in A$ such that $b = k + d$ with $r(d) < 1$. So, it follows that $\|d^n\| < 1$ for some $n \in \mathbb{N}$ and

$$\begin{aligned} 0 \le a \le b &\implies 0 \le a^n \le b^n \\ &\implies 0 \le a^n \le |k_n| + |d^n|. \end{aligned}$$

Note that the Riesz Decomposition Property holds in A because A is a Banach lattice, see ([24], Theorem 1.1.1(viii)). Therefore we can find $f, u \in A$ such that

$$a^n = f + u,$$

with $0 \leq f \leq |k_n|$ and $0 \leq u \leq |d^n|$. This implies that $\|u\| \leq \|d^n\| < 1$ because A is a Banach lattice algebra. Given $\lambda \in \mathbb{C}$ with $|\lambda| \geq 1$, it follows that $\lambda \notin \sigma(u)$ and

$$\begin{aligned} \lambda \mathbf{1} - a^n &= \lambda \mathbf{1} - u - f \\ &= (\lambda \mathbf{1} - u)(\mathbf{1} - R(\lambda, u) f), \end{aligned}$$

where $R(\lambda, u) = (\lambda \mathbf{1} - u)^{-1}$. Then,

$$\lambda \mathbf{1} - a^n + I = (\lambda \mathbf{1} - u + I)(\mathbf{1} - R(\lambda, u) f + I). \tag{3.4}$$

Since A is a Banach lattice algebra and since the modulus is continuous in A, we obtain the following inequalities $0 \leq |R(\lambda, u) f| \leq |R(\lambda, u)| f \leq R(|\lambda|, u) f \leq R(|\lambda|, u)|k_n|$. Note that $k_n \in I$ since I is an ideal in A. Thus, $|k_n| \in I$ since I is an m-order ideal. This together with I being an m-order ideal implies $|R(\lambda, u) f|^m \in I$. Hence $|R(\lambda, u) f| + I$ is nilpotent in A/I. By Lemma 3.3.3, $R(\lambda, u) f + I$ is nilpotent in A/I and therefore $r(R(\lambda, u) f + I) = 0$. Consequently, $(\mathbf{1} - R(\lambda, u) f) + I$ is invertible in A/I. Since $\lambda \notin \sigma(u)$, it follows that $\lambda \notin \sigma(u + I)$. By combining both arguments it follows from equation (3.4) that $\lambda \mathbf{1} - a^n + I$ is invertible in A/I and therefore $\lambda \notin \sigma(a^n + I)$. Hence, $r(a^n + I, A/I) < 1$ and by the Spectral Mapping Theorem we can conclude that $r(a + I, A/I) < 1$. ■

Corollary 3.3.5 *Let E be a Dedekind complete Banach lattice and let $S, T \in \mathcal{L}(E)$ be such that $0 \leq S \leq T$. Then*

$$r(S + \mathcal{K}^r(E), \mathcal{L}(E)/\mathcal{K}^r(E)) \leq r(T + \mathcal{K}^r(E), \mathcal{L}(E)/\mathcal{K}^r(E)).$$

Proof:

In view of Theorem 1.3 in [4] and Lemma 2.7 in [22], $\mathcal{K}^r(E)$ is a 3-order ideal in the Banach lattice algebra $\mathcal{L}^r(E)$. ■

Corollary 3.3.6 *Let A be a Banach lattice algebra and let I be a closed inessential m-order ideal in A. If $a, b \in A$ is such that $0 \leq a \leq b$, and $b \in \mathcal{R}(A, I)$ then $a \in \mathcal{R}(A, I)$.*

Proof:

The proof follows immediately from Theorem 3.3.4. ■

3.4 Monotonicity of the Spectral Radius in C^*-Algebras

Let H be a Hilbert space and $T \in \mathcal{L}(H)$. We call T *positive*, and write $T \geq 0$, if $\langle Tx, x\rangle \geq 0$ for every $x \in H$. It can be shown that $T \geq 0$ if and only if $T^* = T$ and $\sigma(T) \subset [0, \infty)$. The next result was proved by Raubenheimer and Rode in [33].

Corollary 3.4.1 ([33], Corollary 6.6) *Let H be a Hilbert space and let $S, T \in \mathcal{L}(H)$ satisfy $0 \leq S \leq T$ and $ST = TS$. If T is a Riesz operator, then S is a Riesz operator.*

We want to show that the commutativity condition in the above corollary can be dropped and that one can actually say more.

An element x of a C^*-algebra A is said to be *positive*, denoted by $x \geq 0$, if it is self-adjoint and if its spectrum contains only positive real numbers. This gives rise to an order relation $\leq$ between elements of A by defining $a \leq b$ if and only if $b - a \geq 0$. We define a positive cone C in A by

$$C := \{x \in A : x = x^* \text{ and } \sigma(x) \subset [0, +\infty)\}.$$

Note that (A, C) is not an OBA because C is not closed under multiplication. However, if A is a commutative then (A, C) is an OBA. Note that every class of OBAs are in included in a general structure called commutatively ordered Banach algebra (COBA). Hence, every C^*-algebra is a COBA, see [25]. Our next result improves the above corollary.

Proposition 3.4.2 *Let A be a C^*-algebra and I a closed inessential ideal in A. If $a, b \in A$ is such that $0 \leq a \leq b$ and $b \in \mathcal{R}(A, I)$ then $a \in I$.*

Proof:

Let A be a C^*-algebra and I a closed inessential ideal in A. It follows that

$$C_I := \{x + I \in A/I : x + I = x^* + I \text{ and } \sigma(x + I) \subset [0, +\infty)\}$$

is a cone in the quotient algebra A/I. Since A/I is a C^*-algebra, it follows that

$$\begin{aligned} 0 \le a \le b &\implies I \le a + I \le b + I \\ &\implies \|a + I\| \le \|b + I\| \\ &\implies 0 \le \|a + I\| \le \|b + I\| = r(b + I) = 0. \end{aligned}$$

The last implication above holds because $b + I$ is self-adjoint and $b \in \mathcal{R}(A, I)$. Hence $a \in I$. ■

Corollary 3.4.3 *Let H be a Hilbert space. There does not exist any non-compact positive Riesz operators on H.*

Proof:
Note that if H is a Hilbert space then $\mathcal{L}(H)$ is a C^*-algebra with T^* the adjoint of T for every $T \in \mathcal{L}(H)$ and $\|T\| = \|T^*\| = \|TT^*\|^{\frac{1}{2}}$. Suppose $T \in \mathcal{L}(H)$ is a positive Riesz operator. Then it follows from $0 \le T \le T$ and the above proposition that T is compact because $T \in \mathcal{R}(\mathcal{L}(H), \mathcal{K}(H))$, where $\mathcal{K}(H)$ denotes the closed ideal of compact operators in $\mathcal{L}(H)$. ■

3.5 Riesz Operators on Ultrapowers

In this section, we will make use of our basic notions on ultrapowers discussed in Section 1.6. Our main objectives in this final section are firstly to characterize Riesz operators $T \in \mathcal{L}(X)$ in terms of Riesz operators $T_{\mathcal{U}} \in \mathcal{L}(X_{\mathcal{U}})$ (Theorem 3.5.1), and secondly to simplify the domination problem: If $0 \leq S \leq T$ with S and T operators defined on a Banach lattice and T a Riesz operator, then there are operators $\widetilde{S}$ and $\widetilde{T}$ defined on an ordered Banach space $\widetilde{X}$ such that $\widetilde{T}$ is a quasinilpotent Riesz operator and $0 \leq \widetilde{S} \leq \widetilde{T}$ (Proposition 3.5.4). To give our characterization of Riesz operators in terms of Riesz operators defined on ultrapowers, note that in the case of a Banach space X, the spectrum of both operators $T \in \mathcal{L}(X)$ and $T_{\mathcal{U}} \in \mathcal{L}(X_{\mathcal{U}})$ coincide (Theorem 1.6.1), and we shall now make use of it to prove one of our main results in this section.

Theorem 3.5.1 *Let X be a Banach space. Then $T \in \mathcal{L}(X)$ is a Riesz operator if and only if $T_{\mathcal{U}} \in \mathcal{L}(X_{\mathcal{U}})$ is a Riesz operator.*

Proof:
$\Rightarrow$: Suppose $T \in \mathcal{L}(X)$ is a Riesz operator. Then in view of Theorem 1.1.11 or Theorem 1.4.4, $\sigma(T, \mathcal{L}(X))$ is either a finite set or a sequence converging to zero and for any non-zero $\lambda \in \sigma(T, \mathcal{L}(X))$, the spectral idempotent $p(\lambda, T)$ lies in $\mathcal{K}(X)$. By Theorem 1.6.1, $\sigma(T_{\mathcal{U}}, \mathcal{L}(X_{\mathcal{U}}))$ is therefore either a finite set or a sequence converging to zero. Now, for us to prove that $p(\lambda, T_{\mathcal{U}})$ lies in $\mathcal{K}(X_{\mathcal{U}})$, we first claim that $\sigma(T_{\mathcal{U}}, \phi(\mathcal{L}(X))) = \sigma(T, \mathcal{L}(X))$:
Since the homomorphism $\phi : \mathcal{L}(X) \to \phi(\mathcal{L}(X)) \subset \mathcal{L}(X_{\mathcal{U}})$ defined by

$$\phi(T) = T_{\mathcal{U}},$$

is also an isometry (see remark preceding Theorem 1.6.1), it follows that $\phi(\mathcal{L}(X))$ is a closed subalgebra of $\mathcal{L}(X_{\mathcal{U}})$. Since $T_{\mathcal{U}} \in \phi(\mathcal{L}(X)) \subset \mathcal{L}(X_{\mathcal{U}})$ and $\mathbb{C} \backslash \sigma(T_{\mathcal{U}}, \mathcal{L}(X_{\mathcal{U}}))$ is connected, one can use Corollary 1.1.3 to deduce that $\sigma(T_{\mathcal{U}}, \phi(\mathcal{L}(X))) = \sigma(T_{\mathcal{U}}, \mathcal{L}(X_{\mathcal{U}}))$. This together with Theorem 1.6.1 proves our claim. Secondly, note that $p(\lambda, T)$ is a finite rank operator (Theorem 1.4.4 (iv)). Since ϕ is onto its range $\phi(\mathcal{L}(X))$ and since it is spectrum preserving then $p(\lambda, T_{\mathcal{U}}) = \phi(p(\lambda, T))$ is also a finite rank operator, see ([7], Proposition 3.1) and therefore $p(\lambda, T_{\mathcal{U}})$ lies in $\mathcal{K}(X_{\mathcal{U}})$, see ([15], Proposition 3.1). If we employ the Ruston characterization Theorem (Theorem 1.1.11) again, it follows that $T_{\mathcal{U}}$ is a Riesz operator.

$\Leftarrow$: Suppose that $T_{\mathcal{U}} \in \mathcal{L}(X_{\mathcal{U}})$ is a Riesz operator. Recall that there is a natural embedding of X into its ultrapowers $X_{\mathcal{U}}$ by the mapping

$$x \mapsto [(x, x, \cdots)] = (x, x, \cdots) + \mathcal{N}_{\mathcal{U}}$$

which is an isometry (Section 1.6). This means that if X is identified with its image in $X_{\mathcal{U}}$, then X becomes a closed subspace of $X_{\mathcal{U}}$. By definition of $T_{\mathcal{U}}$ (Section 1.6), it follows that

$$T_{\mathcal{U}}((x, x, \cdots) + \mathcal{N}_{\mathcal{U}}) = (Tx, Tx, \cdots) + \mathcal{N}_{\mathcal{U}}.$$

If we remember that Tx is identified with $(Tx, Tx, \cdots) + \mathcal{N}_{\mathcal{U}}$, we get that $T_{\mathcal{U}}$ leaves X invariant. Therefore by Theorem 1.4.5, T is a Riesz operator because it is the restriction of $T_{\mathcal{U}}$ to X. ■

Backwards implication in the previous theorem can also be proved in the same way as forward implication by using the fact that the inverse mapping ϕ^{-1} of ϕ exists. It is defined on $\phi(\mathcal{L}(X))$ and it is a homomorphism as well as an isometry.

Proposition 3.5.2 *Let X be a Banach space. If $T \in \mathcal{L}(X)$ is a Riesz operator then $\widetilde{T} \in \mathcal{L}(\widetilde{X})$ is a Riesz operator.*

Proof:
Since $T \in \mathcal{L}(X)$ is a Riesz operator, it follows from Theorem 3.5.1 that $T_{\mathcal{U}} \in \mathcal{L}(X_{\mathcal{U}})$ is

also a Riesz operator. By the remarks in Section 1.6, note that X is a closed invariant subspace of $X_{\mathcal{U}}$ under $T_{\mathcal{U}}$. Now, since we have the following

$$\begin{aligned}\widetilde{T}[x_{\mathcal{U}}] &= \widetilde{T}\widetilde{x} \\ &= (T_{\mathcal{U}}x_{\mathcal{U}})^{\sim} \\ &= [(Tx)_{\mathcal{U}}] \\ &= [T_{\mathcal{U}}x_{\mathcal{U}}],\end{aligned}$$

$\widetilde{T}$ is the induced operator of $T_{\mathcal{U}}$ (Section 1.3.1). Hence using Theorem 1.4.7, it follows that $\widetilde{T}$ is a Riesz operator. ∎

However one can say more. If X is a Banach space then it is well known that the operator $T \in \mathcal{L}(X)$ is compact if and only if $\widetilde{T} = 0$, see Lemma 1.6.2. For a Riesz operator $T \in \mathcal{L}(X)$, this statement takes the following form:

Theorem 3.5.3 *Let X be a Banach space. Then $T \in \mathcal{L}(X)$ is a Riesz operator if and only if $\widetilde{T} \in \mathcal{L}(\widetilde{X})$ is a quasinilpotent (Riesz) operator.*

Proof:
If X is a Banach space then it follows that

$$\begin{aligned}T \text{ is a Riesz operator} &\iff \sigma(T + \mathcal{K}(X), \mathcal{L}(X)/\mathcal{K}(X)) = \{0\} \\ &\iff \sigma(\widetilde{T}, \mathcal{L}(\widetilde{X})) = \{0\}, \text{ by Theorem 1.6.5} \\ &\iff \widetilde{T} \text{ is quasinilpotent,}\end{aligned}$$

and so $\widetilde{T}$ is a Riesz operator. ∎

Let E be a separable Banach lattice. Since the Calkin algebra $\mathcal{L}(E)/\mathcal{K}(E)$ can be represented as a subalgebra of bounded operators on some Banach space (Theorem 1.6.4), our next result shows that the domination problem for a positive Riesz operator can in some sense be simplified. There is a natural way to define the order on $\mathcal{L}(\widetilde{E})$: Recall that $(\mathcal{L}(E), K)$ is an OBA with $K = \{T \in \mathcal{L}(E) : TC \subset C\}$, where

$C = \{x \in E : x \geq 0\}$. Since the natural map $\pi : \mathcal{L}(E) \to \mathcal{L}(E)/\mathcal{K}(E)$ defined by

$$\pi(T) = T + \mathcal{K}(E) \quad (T \in \mathcal{L}(E)),$$

is a homomorphism then $(\mathcal{L}(E)/\mathcal{K}(E), \pi K)$ is an OBA, see Proposition 1.5.1. Hence, if $S \in \mathcal{L}(E)$ and $S \geq 0$, i.e., $S \in K$, then $\pi(S) = S + \mathcal{K}(E) \geq 0$ in the OBA $(\mathcal{L}(E)/\mathcal{K}(E), \pi K)$. Recall that the mapping $\Phi : \mathcal{L}(E)/\mathcal{K}(E) \to \mathcal{L}(\widetilde{E})$ defined by

$$\Phi(S + \mathcal{K}(E)) = \widetilde{S},$$

is a homomorphism (Theorem 1.6.4). Thus, $\Phi(S + \mathcal{K}(E)) = \widetilde{S}$ is positive in the OBA $(\mathcal{L}(\widetilde{E}), \Phi\pi K)$ since $\Phi\pi$ is a homomorphism. Hence, if $0 \leq S \leq T$, we have $T - S \geq 0$ and since $\Phi\pi$ is linear, it follows that

$$\Phi\pi(T - S) = \Phi\pi(T) - \Phi\pi(S) = \widetilde{T} - \widetilde{S} \geq 0 \text{ in } \mathcal{L}(\widetilde{E}).$$

Corollary 3.5.4 *Let E be a separable Banach lattice and let $S, T \in \mathcal{L}(E)$ satisfy $0 \leq S \leq T$. If T is a Riesz operator then $0 \leq \widetilde{S} \leq \widetilde{T}$ with $\widetilde{T}$ a quasinilpotent Riesz operator.*

Proof:
The proof follows from Theorem 3.5.3 and the remark above. ■

Although we have simplified the domination problem for Riesz operators in the above result, the limitation of the above result lies in the fact that the holomorphic image of an algebra cone may loose some of the desirable properties of the algebra cone, see for instance Example 4.2 in [33] and the remarks following it. The interest in this last section came from the work done by B. de Pagter and A. R. Schep in [13] and V.G. Troitsky in [40] to provide some results on certain questions on the domination problem, such as: Given a Banach lattice E and two operators $S, T \in \mathcal{L}(E)$ with $0 \leq S \leq T$, when does $r_{ess}(S) \leq r_{ess}(T)$ hold? In his paper [40], V.G. Troitsky used a certain representation space technique based on the nonstandard hull construction of Nonstandard Analysis which is closely connected with the study of ultrapowers of Banach spaces.

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