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Introduction

This dissertation is divided into two chapters. In the first chapter we state some basic definitions and theorems which sets the scene for the main chapter on finite rank elements. As an introduction to this dissertation, we will give a brief description of what each section entails and also state some of the main results of each section. We start off (in section 1.1) by introducing a few concepts such as Banach algebras, the exponential function, ideals, the radical and many more well known concepts and results in the field of Functional Analysis. The main result in this section is most definitely Theorem 1.1.10., which is used in many proofs throughout this dissertation. The concept of semisimplicity is also introduced in this section. Section 1.2 deals with the spectrum and the concepts and results surrounding it. Perhaps the most important (and familiar) statement in this section is Lemma 1.2.4. which says that for x, y in a unital algebra (defined in section 1.1)

$$\sigma'(xy) = \sigma'(yx)$$

The reader is then introduced to the Holomorphic Functional Calculus (section 1.3). We state (in Theorem 1.3.1.) the most important properties of the Holomorphic Functional Calculus, including the well known Spectral Mapping Theorem

$$\sigma\big(f(x)\big) = f\big(\sigma(x)\big)$$

valid for functions holomorphic in a neighborhood of $\sigma(x)$. It can be said that this section forms the basis of our work in chapter 2 and is followed by a section on idempotents in Banach algebras. We define here the concept of an idempotent and look at the spectrum of idempotents. We also give (in Theorem 1.4.4.) a complete proof and extend a result which Robin Harte uses in the proof of Lemma 3 ([9]). In the last section of chapter 1, we deal with semiprime Banach algebras and minimal ideals. One of the most important statements in this dissertation is made in this section - Theorem 1.5.5. Chapter 2 (Finite Rank Elements) forms the core of this dissertation and is started off by section 2.1 which deals with spatially and spectrally rank one elements. Throughout this dissertation we use Robin Harte's and J. Puhl's definitions for spatially rank one elements. According to Harte, an element $0 \neq a$ in a semiprime Banach algebra A is spatially rank one if $aAa \subseteq \mathbb{C}a$. Whereas Puhl's definition states that $a \in A$ is called one-dimensional if there exists a linear functional, f_a, such that $axa = f_a(x)a \ \forall \ x \in A$. These definitions are of course equivalent. As done by Robin Harte (in [9]), we also define a spectrally rank one element as an $0 \neq a \in A$ such that $x \in A \Longrightarrow \#\sigma'(xa) \leq 1$ where $\#K$ denotes the number of elements in a set K. This section also deals with many a characteristic of spatially rank one and spectrally rank one elements, as well as the relationship between these elements. The structure of rank one elements is then discussed in section 2.2. Finally, we take a look at the socle of a Banach algebra. The socle is defined as the smallest left ideal containing all minimal left ideals. We also define the rank of an element (as in [3]) $a \in A$ by

$$\operatorname{rank}(a) = \sup_{x \in A} \#\sigma'(xa) \leq \infty$$

and denote by F_n the set $F_n = \{a \in A : \operatorname{rank}(a) = n\}$. We proceed to show that the set of all finite rank elements equals the socle:

$$\operatorname{Soc}(A) = \bigcup_{n=0}^{\infty} F_n$$

as Aupetit and Mouton did in ([3]), but without using the Scarcity Theorem. What then follows is a simplified approach to rank - we provide alternative (and much simpler) proofs for rank subadditivity and various other results which Aupetit and Mouton proved by using the theory of analytic multifunctions. We are certain that the reader will appreciate the beauty (in the simplicity) of our approach to rank. In Mathematics, as with many other obsessions, the solution often lies in simplicity!

Chapter 1 - Preliminary Information

In this chapter we state the most basic definitions and results which will be used in forthcoming chapters.

1.1 Banach Algebras

Definitions 1.1.1. A *complex algebra* is a vector space A over the field $\mathbb{C}$ such that for each ordered pair of elements $x, y \in A$ a unique product $xy \in A$ is defined with properties:

(1) $(xy)z = x(yz)$

(2a) $x(y+z) = xy + xz$

(2b) $(x+y)z = xz + yz$

(3) $\alpha(xy) = (\alpha x)y = x(\alpha y)$

where $x, y, z \in A$ and $\alpha \in \mathbb{C}$.

A *normed algebra* is a normed space which is an algebra such that

$$||xy|| \leq ||x|| \cdot ||y|| \qquad x, y \in A$$

A *Banach algebra* is a normed algebra which is complete in the norm. A is called a Banach algebra with *identity* if $\exists\ 1 \in A$ such that $1x = x1 = x \quad \forall\ x \in A$. Such A are called *unital Banach algebras*. For ease of writing, we shall not always write the symbol 1; if $\lambda \in \mathbb{C}$ then we only write λ to indicate $\lambda 1$. We can also always assume that $\| 1 \| = 1$, since otherwise ([1], p. 66) $\| \cdot \|$ can be replaced by and equivalent norm $\| \cdot \|_1$ such that

$$||1||_1 = 1 \qquad \text{and} \qquad ||xy||_1 \leq ||x||_1 \cdot ||y||_1 \qquad x, y \in A$$

Thus, the underlying topological structure does not change with the replacement of $|| \cdot ||$ by $|| \cdot ||_1$. Definitions 1.1.1. is the result of an abstraction of the prototype Banach algebra $B(X)$: let X be any Banach space and denote by $B(X)$ the Banach space of all bounded linear operators from X into X. With multiplication defined by composition, $B(X)$ becomes a Banach algebra under the canonical sup norm $\left(||T|| = \sup\limits_{||x||=1} ||Tx|| \right)$.

The existence of an identity in A suggests the concept of *invertibility* in A; $x \in A$ is said to be *invertible* (*regular*) if there exists $x^{-1} \in A$, called the *inverse* of x, such that $xx^{-1} = x^{-1}x = 1$. We shall denote the set of invertible elements of A by A^{-1}. An element of A which is not invertible in A is said to be *singular* in A.

Definitions 1.1.2. If A and B are topological spaces, then a *homeomorphism* is a continuous bijective mapping $T : A \longrightarrow B$ whose inverse is also continuous. Let A and B be unital Banach algebras. A *homomorphism* is a linear operator $T : A \longrightarrow B$ such that $T(1) = 1$ and $T(ab) = T(a)T(b)$. An *isomorphism* is a bijective homomorphism $T : A \longrightarrow B$. If $\phi : A \longrightarrow \mathbb{C}$ is a homomorphism then ϕ is called a *character* of A. We also denote by A' the dual space of A.

The concept of invertibility is not purely algebraic; the presence of a complete norm implies the existence of infinite series in certain circumstances (just think of geometric series). As such we have the following topological result which is well known in the literature:

Theorem 1.1.3. ([1], Theorem 3.2.3). *Suppose A is a Banach algebra and that $a \in A$ is invertible. If $\|x - a\| < \frac{1}{\|a^{-1}\|}$ then x is invertible and hence A^{-1} is open. Moreover, the mapping $x \to x^{-1}$ is a homeomorphism from A^{-1} onto A^{-1}.*

As a special case of Theorem 1.1.3. we state the following:

Theorem 1.1.4. *Let A be a complex Banach algebra with identity 1. If $x \in A$ satisfies $\|x\| < 1$ then $1 - x$ is invertible and*

$$(1 - x)^{-1} = 1 + \sum_{j=1}^{\infty} x^j \tag{1.1.4.1}$$

Definitions 1.1.5. If A is a Banach algebra, we say that Π is a *representation* of A on a complex vector space X (with $\dim X \geq 1$) if Π is a non-trivial homomorphism from A into the algebra of linear operators on X. If a linear subspace Y of X satisfies $\Pi(x)Y \subset Y$ for all $x \in A$ then Y is called *invariant* under $\Pi(x)$. A representation Π is said to be *irreducible* if the only linear subspaces of X invariant under $\Pi(x)$ are $\{0\}$ and X. By a $faithful\ representation$ of A we mean a one-to-one homomorphism $\Pi : A \longrightarrow B(X)$ where X is some Banach space. A representation is called *bounded* when X is a Banach space and $\Pi(x)$ a bounded linear operator on X $\forall\ x \in A$. It is said to be *continuous* if it is bounded and if there exists a constant $C > 0$ such that $\|\Pi(x)\| \leq C\|x\|\ \ \forall\ x \in A$.

Definitions 1.1.6. If A is a Banach algebra and $x \in A$, then we define the *exponential function*, denoted by e^x, by the power series

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

where, by definition, $x^0 = 1$. The existence of e^x follows easily from the triangle inequality together with the multiplicative inequality in 1.1.1. One may also show that $e^{x+y} = e^x e^y$ whenever $xy = yx$ and hence that e^x belongs to A^{-1} with inverse e^{-x}. We use $\exp(A)$ to denote the set of all exponentials of A. In other words $\exp(A) = \{e^x : x \in A\}$. Of particular importance for us is the group generated by $\exp(A)$, this set is called the *generalized exponentials*, denoted by $\text{Exp}(A)$, and it consists of all finite products of elements belonging to $\exp(A)$. The set $\text{Exp}(A)$ is a normal open and closed subgroup of A^{-1} and therefore equals the connected component of A^{-1} containing the identity element ([1], Theorem 3.3.7.).

Definitions 1.1.7. Important amongst Banach algebras are the commutative ones; an algebra A is said to be *commutative* if for every $x, y \in A$ we have $xy = yx$. An element a of a Banach algebra is called *central* in A if a commutes with every element of A, that is, $xa = ax \quad \forall\ x \in A$. The set of all central elements of A is aptly named the *centre* of A and denoted by $Z(A)$. Notice that $Z(A)$ is a commutative Banach algebra containing the identity of A. We also define the *commutant* of an element a of Banach algebra A, denoted by $\text{comm}(a)$, as the set containing all the elements of A commuting with a. In other words $\text{comm}(a) = \{x \in A : xa = ax\}$. Even more important and useful is the *bicommutant* of a, denoted by $\text{comm}^2(a)$; it consists of all those elements of A which commutes with every element of $\text{comm}(a)$ i.e. $\text{comm}^2(a)=\{z \in A : zx = xz$ whenever $xa = ax\}$. The bicommutant is a commutative Banach algebra containing the identity of A. Notice that it is of no consequence to take these constructions any further; $\text{comm}^2(a)=\text{comm}^3(a)=\text{comm}^4(a)$ and so on.

Definitions 1.1.8. A subalgebra of an algebra A is a vector subspace B of A such that B is closed under multiplication, i.e. for every a and b in B, the product ab is also in B. Note that the unit of A does not necessarily belong to the subalgebra B and that B might have a unit different from the one in A.

Definitions 1.1.9. A *left ideal* J of an algebra A is a subset J of A such that:

(1) J is a vector subspace of A

(2) for any $r \in J$ and $a \in A$ we have $ar \in J$.

In other words, a left ideal J of an algebra A is a subalgebra of A with the additional property that J absorbs products from the left. Similarly, a *right ideal* is a subalgebra which absorbs products from the right. A $two - sided\ ideal$ is a subalgebra J of A which is both a left and a right ideal. The ideal $J = \{0\}$ is called the *trivial ideal*. An ideal J

is called *proper* if $J \subsetneq A$ and *maximal* if it is proper and not contained in a larger proper ideal of the same kind. Using Zorn's Lemma, one can easily prove that every proper ideal is contained in a maximal ideal of the same kind. Also, one may prove that a maximal ideal of a Banach algebra A is always closed in A ([1] Corollary 3.3.3.). By the *radical* of A, denoted $\mathrm{Rad}(A)$, we mean the subset of elements $x \in A$ belonging to every maximal left (right) ideal of A. Amongst the many characterizations for the radical of a Banach algebra, the following appears most frequently:

Theorem 1.1.10. ([1], Theorem 3.1.3.). *In a Banach algebra A the following sets are equivalent:*

(1) the intersection of all maximal left ideals of A

(2) the intersection of all maximal right ideals of A

(3) $\{x \in A : 1 - xz$ is invertible in $A, \ \forall\ z \in A\}$

(4) $\{x \in A : 1 - zx$ is invertible in $A, \ \forall\ z \in A\}$

It is clear from (1) and (2) that $\mathrm{Rad}(A)$ is a closed two-sided ideal of A. We call A *semisimple* when $\mathrm{Rad}(A) = \{0\}$. In general, a non-zero radical is a rather pathological mathematical object; radical elements behave very much like the zero element of A, but they are not necessarily equal to zero. However, if A is not semisimple then it is always possible to turn A into a semisimple Banach algebra by 'factorizing out' the radical. For this we need to define the notion of a quotient algebra. First we look at what is meant by a quotient space: If X is a vector space and Y a subspace of X then the coset of $x \in X$ with respect to Y is $x + Y = \{v : v = x + y,\ y \in Y\}$. Under the operations $(w+Y)+(x+Y) = (w+x)+Y$ and $\alpha(x+Y) = \alpha x+Y$ the cosets constitute the elements of a vector space, called the quotient space of X by Y and is denoted by X/Y. Using the above together with the fact that J is a closed two-sided ideal of Banach algebra A, we see that A/J is a Banach space under the canonical quotient norm. If we define coset multiplication by $(x + J)(y + J) = xy + J$ then, under the same norm, A/J becomes a Banach algebra called the *quotient of A by J*. With multiplication defined as above, we have the identity in X/Y as $1 + Y = \{1 + y,\ y \in Y\}$ and also the zero element as $0 + Y = \{0 + y,\ y \in Y\} = Y$. It is also not hard to verify ([1], Theorem 3.1.5.) that the coset $x + \mathrm{Rad}(A) \in \big(A/\mathrm{Rad}(A)\big)^{-1} \iff x \in A^{-1}$ and that $A/\mathrm{Rad}(A)$ is a semisimple Banach algebra. Another interesting fact about the radical of A is that it is precisely the intersection of the kernels of all continuous irreducible representations of A, for more on this see ([1], Theorem 4.2.1).

To conclude this section, we define the following: An ideal I of a Banach algebra A is said to be *primitive* if I is the kernel of some continuous irreducible representation of A.

1.2 The Spectrum

Definitions 1.2.1. If A is a Banach algebra with identity and $x \in A$, then we define the *spectrum* of x to be the set

$$\sigma_A(x) = \{\lambda \in \mathbb{C} : \lambda - x \notin A^{-1}\}.$$

The set $\sigma_A(x)\backslash\{0\}$ is called the *non − zero spectrum* of x and we shall denote it by $\sigma'_A(x)$. We also agree to denote $\sigma_A(x)$ by $\sigma(x)$ if the algebra under discussion is clear from context. The *spectral radius* of x is defined to be the number

$$r_\sigma(x) = \sup_{\lambda \in \sigma(x)} |\lambda|$$

and since the spectrum is always a nonempty compact subset of $\mathbb{C}$ ([1], Theorem 3.2.8.), the spectral radius exists as a finite positive number. The Gelfand-Beurling formula gives an expression for the spectral radius in terms of the norm:

$$r_\sigma(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}}.$$

If $x \in A$ satisfies $r_\sigma(x) = 0$ then x is called *quasinilpotent*, and if $x^k = 0$ for some $k \in \mathbb{N}$ then x is called *nilpotent*. Theorem 1.1.10. implies that every element of Rad(A) is quasinilpotent. If A is commutative then the converse of this is also true, but in general quasinilpotents are not radical elements.

If A is a commutative Banach algebra, then there is a bijective correspondence between the set of maximal ideals of A and the set of the characters of A in the sense that the maximal ideals of A are precisely the kernels of the characters of A (see [16]). That is

$$J \text{ is a maximal ideal of } A \iff J = \ker\phi \text{ for some } \phi \text{ a character} \qquad (1.2.1.1)$$

As a result of (1.2.1.1) we have:

Theorem 1.2.2. *If A is a commutative Banach algebra then*

$$\sigma(x) = \{\phi(x) : \phi \text{ is a character of } A\}.$$

Proof : Consider $\phi(x) - x$ and suppose $\phi(x) - x$ is invertible. Then for some $y \in A$ we have $[\phi(x) - x]y = 1$. So, $\phi\big[\big(\phi(x) - x\big)y\big] = \phi(1) = 1$. Therefore,

$$\phi\big(\phi(x) - x\big)\phi(y) = 1$$

$$\therefore \big[\phi\big(\phi(x)\big) - \phi(x)\big]\phi(y) = 1 \text{ since } \phi \text{ is linear}$$

$$\therefore \big[\phi(x)\phi(1) - \phi(x)\big]\phi(y) = 1 \text{ since } \phi \text{ is multiplicative}$$

$$\therefore \big[\phi(x) - \phi(x)\big]\phi(y) = 1 \text{ since } \phi(1) = 1$$

which leads to a contradiction. Thus $\phi(x) - x$ is not invertible, which means that $\phi(x) \in \sigma(x)$. Suppose on the other hand that $\lambda \in \sigma(x)$. We want to prove that for some character ϕ, $\lambda = \phi(x)$. But $\lambda \in \sigma(x) \implies \lambda - x \notin A^{-1}$. So, $J = A(\lambda - x) \neq A$ (since $1 \notin J$) is a proper ideal of A and hence J is contained in some maximal ideal. But $\lambda - x \in J$ and hence belongs to the same maximal ideal. So $\lambda - x \in \ker \phi$ for some ϕ a character (using 1.2.1.1). Hence $0 = \phi(\lambda - x) = \phi(\lambda.1) - \phi(x) = \lambda\phi(1) - \phi(x) = \lambda - \phi(x)$ and thus $\phi(x) = \lambda$. We have thus shown that for $\lambda \in \sigma(x)$, $\lambda = \phi(x)$ where $\phi(x)$ is some character of A. ■

The following result is well known in Spectral Theory and will be very useful to us.

Theorem 1.2.3. *If A is a Banach algebra and $x_1, x_2, ..., x_n \in A$ are mutually orthogonal, that is, $x_i x_j = 0$ for $i \neq j$, then*

$$\sigma'(x_1 + \cdots + x_n) = \bigcup_{j=1}^{n} \sigma'(x_j).$$

Proof : The proof follows by induction on n. We first prove the case when $n = 2$. If $x_1 x_2 = x_2 x_1 = 0$ then for $\lambda \neq 0$,

$$\begin{aligned} \lambda - (x_1 + x_2) &= \frac{1}{\lambda}(\lambda - x_1)(\lambda - x_2) \\ &= \frac{1}{\lambda}(\lambda - x_2)(\lambda - x_1). \end{aligned}$$

So, $\lambda - (x_1 + x_2)$ is invertible if and only if $(\lambda - x_1)$ and $(\lambda - x_2)$ are both invertible. Therefore,

$$\lambda - (x_1 + x_2) \notin A^{-1} \iff (\lambda - x_1) \notin A^{-1} \text{ or } (\lambda - x_2) \notin A^{-1}.$$

Hence $\sigma'(x_1 + x_2) = \sigma'(x_1) \cup \sigma'(x_2)$. Suppose now that the theorem holds for $n = k$, that is

$$\sigma'(x_1 + \cdots + x_k) = \sigma'(x_1) \cup \cdots \cup \sigma'(x_k).$$

Consider the case $n = k + 1$ i.e. the case where $x_1, x_2, ..., x_{k+1}$ are mutually orthogonal. Then

$$\lambda - (x_1 + \cdots + x_k + x_{k+1}) \text{ can be written as } \lambda - \big[(x_1 + x_2 + \cdots + x_k) + x_{k+1}\big]$$

which is equal to $\frac{1}{\lambda}\big(\lambda - (x_1 + \cdots + x_k)\big)\big(\lambda - x_{k+1}\big)$, and shows that $\lambda - (x_1 + \cdots + x_k + x_{k+1})$ can be written as the product of two elements. Using this together with the fact that the result holds for $n = 2$ and $n = k$, we have that

$$\begin{aligned}\sigma'\big((x_1 + \cdots + x_k) + x_{k+1}\big) &= \big(\sigma'(x_1) \cup \ldots \cup \sigma'(x_k)\big) \cup \big(\sigma'(x_{k+1})\big)\\ &= \sigma'(x_1) \cup \ldots \cup \sigma'(x_k) \cup \sigma'(x_{k+1}).\end{aligned}$$

■

Lemma 1.2.4. ([1], Lemma 3.1.2). *Let A be an algebra with identity and let x and y be elements of A and $0 \neq \lambda \in \mathbb{C}$. Then $\lambda - xy$ is invertible in A if and only if $\lambda - yx$ is invertible in A.*

Proof : Suppose $\lambda - xy$ has inverse z in A. Then

$$\begin{aligned}(\lambda - xy)z &= z(\lambda - xy) = 1\\ \therefore (\lambda - yx)(yzx + 1) &= \lambda yzx + \lambda - y(xyz)x - yx\\ &= \lambda yzx + \lambda - y(\lambda z - 1)x - yx\\ &= \lambda\\ \text{and } (yzx + 1)(\lambda - yx) &= \lambda yzx + \lambda - y(zxy)x - yx\\ &= \lambda yzx + \lambda - y(\lambda z - 1)x - yx\\ &= \lambda.\end{aligned}$$

Thus $(\lambda - yx)$ has inverse $\frac{(yzx+1)}{\lambda}$ in A. The converse obviously follows by symmetry. ■

This means that for a Banach algebra A and $x, y \in A$ we have $\sigma'(xy) = \sigma'(yx)$. Note that we must remove 0 from both the spectra for this to hold, since it is easy to find examples of Banach algebras where xy is invertible whilst yx is not (consider left and right shift operators on the l-spaces). The above result will be used throughout this dissertation and without any specific reference to it.

We now proceed to state two results (due to J.D. Newburgh) which will be used at a later stage in this dissertation. To fully understand these two results, however, we need to look at what is meant by the 'continuity of the spectrum function'. For this we introduce a metric on the collection of compact subsets of $\mathbb{C}$.

Definitions 1.2.5. The *Hausdorff distance*, on the set of compact subsets of $\mathbb{C}$, is defined by

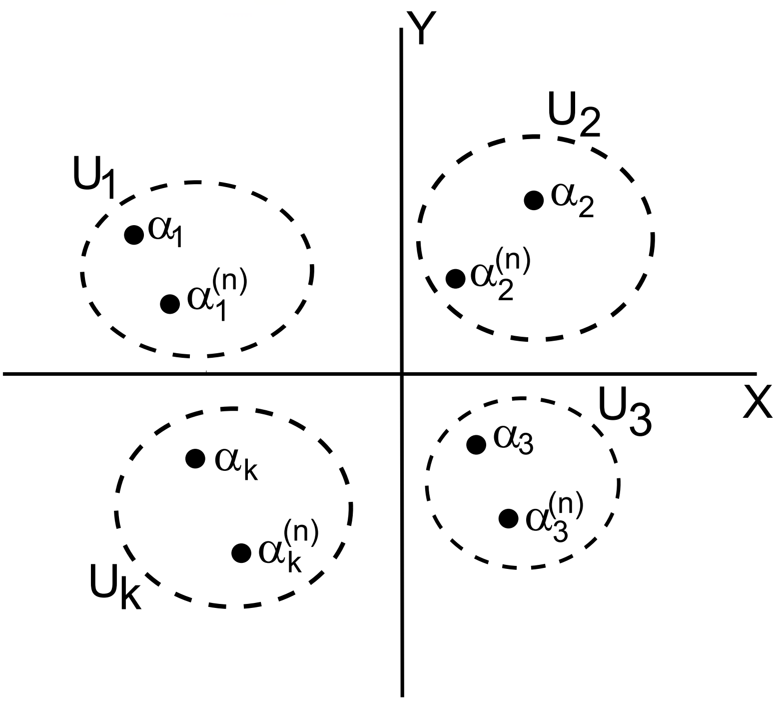
$$\triangle(K_1, K_2) = \max\big(\sup_{z\in K_2} \operatorname{dist}(z, K_1), \sup_{z\in K_1} \operatorname{dist}(z, K_2)\big)$$

where K_1, K_2 are compact subsets of $\mathbb{C}$. It is easy to check that $\triangle$ is a metric on the set of compact subsets of $\mathbb{C}$. We say that $x \mapsto \sigma(x)$ is *continuous at* $a \in A$ if given any $\epsilon > 0$, $\exists\ \delta > 0$ such that $\|x - a\| < \delta \Longrightarrow \triangle\big(\sigma(x), \sigma(a)\big) < \epsilon$. Also, $x \mapsto \sigma(x)$ is *continuous on a subset of* A if it is continuous at every point of that set. We are now ready to state Newburgh's results.

Theorem 1.2.6. ([1], Theorem 3.4.4). *Let A be a Banach algebra and $x \in A$. Suppose that U and V are two disjoint open sets such that $\sigma(x) \subset U \cup V$ and $\sigma(x) \cap U \neq \emptyset$, then there exists $r > 0$ such that $\|x - y\| < r$ implies $\sigma(y) \cap U \neq \emptyset$.*

Corollary 1.2.7. ([1], Corollary 3.4.5). *Suppose $\sigma(a)$ is totally disconnected, then $x \mapsto \sigma(x)$ is continuous at a.*

In particular if $\sigma(a)$ is finite, then Corollary 1.2.7. applies. That is, if $a_n \longrightarrow a$ then $\sigma(a_n) \longrightarrow \sigma(a)$ in terms of Definitions 1.2.5. We illustrate this by means of the following diagram: suppose $\sigma(a) = \{\alpha_1, ..., \alpha_k\}$, then for n sufficiently large there are $\alpha_i^{(n)}$'s such that the $\alpha_i^{(n)}$'s are in $\sigma(a_n)$ and $\alpha_i^{(n)} \in U_i$ for each $i \in \{1, ..., k\}$.



1.3 Holomorphic Functional Calculus

In Definitions 1.1.6. we defined the Banach algebra analogue of the complex exponential function. The Holomorphic Functional Calculus deals with a much broader class of functions in Banach algebras and is one of the main tools we use throughout this dissertation. If

$$f(\lambda) = \alpha_0 + \alpha_1\lambda + \cdots + \alpha_n\lambda^n$$

is a complex polynomial, then for a Banach algebra A and $x \in A$, the Banach algebra version of this polynomial is given by

$$f(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n$$

which is obviously well-defined in A. Note that $f(x)$ exists as an element of A whenever f is holomorphic on an open set containing $\sigma(x)$ (see Theorem 1.3.2.). This means that although $f(x)$ belongs to A, some of its properties may depend on the behaviour of its complex counterpart $f(\lambda)$ on an open set containing $\sigma(x)$. In the following definition we combine the definitions and notations as found in ([1]) and ([16]).

Definitions 1.3.1. Suppose K is compact in $\mathbb{C}$ and μ is a Borel measure on K and that $f : K \longrightarrow A$ is a continuous function from K into the Banach algebra A such that the scalar functions $\phi\big(f(\lambda)\big)$ are integrable with respect to μ, for every $\phi \in A'$ and $\lambda \in K$. The element,

$$y = \int_K f(\lambda)d\mu\,,$$

is the unique element of A which satisfies

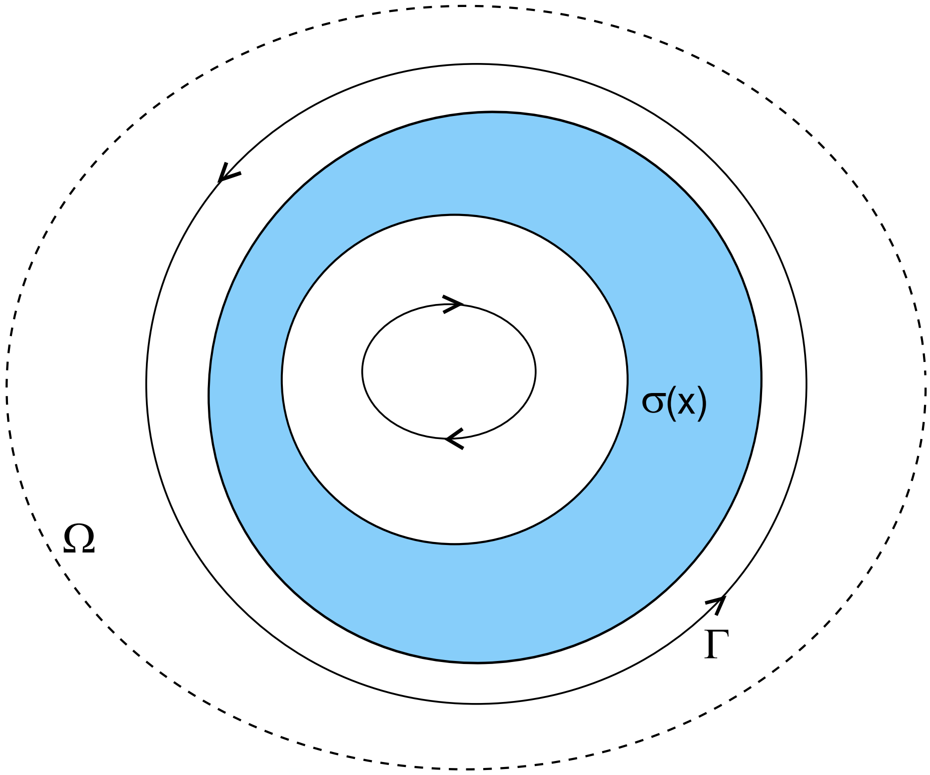
$$\phi(y) = \int_K \phi\big(f(\lambda)\big)d\mu \;\text{ for every } \phi \in A'\,.$$

For the existence and uniqueness of this integral, see ([16]. Theorem 3.27). The Holomorphic Functional Calculus, defined hereafter, may be viewed as a generalization of Cauchy's Theorem for complex functions:

Theorem 1.3.2. ([1], Theorem 3.3.3). *Let A be a Banach algebra and let $x \in A$. Suppose Ω is an open set containing $\sigma(x)$ and that Γ is an arbitrary smooth contour in Ω, surrounding $\sigma(x)$. Then for an analytic function f on Ω the element*

$$f(x) = \frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda - x)^{-1}d\lambda$$

is a well-defined element of A. This idea is illustrated by the diagram below:



The mapping $f \mapsto f(x)$ also has the following properties:

(1) $(f_1 + f_2)(x) = f_1(x) + f_2(x)$

(2) $(f_1 \cdot f_2)(x) = f_1(x) \cdot f_2(x) = f_2(x) \cdot f_1(x)$

(3) $f(x) = 1$ *for the function* $f(\lambda) = 1$ *and* $f(x) = x$ *for the function* $f(\lambda) = \lambda$

(4) if (f_n) *converges to* f *uniformly on compact subsets of* Ω*, then* $f(x) = \lim_{n\to\infty} f_n(x)$

(5) $\sigma\big(f(x)\big) = f\big(\sigma(x)\big)$.

Property (5) is known as the *Spectral Mapping Theorem* whilst (1) and (2) express the fact that the mapping $f \mapsto f(x)$ is a homomorphism from the algebra of holomorphic functions on Ω into A.

For an alternative formulation of the Holomorphic Functional Calculus, in terms of power series see ([6], 4.7, p.201).

An immediate consequence of Theorem 1.3.2. is the following: if $x \in A$ and $\lambda > r_\sigma(x)$ then $\sigma(\lambda - x)$ does not separate 0 from infinity so that by ([1], Theorem 3.3.6), we have $\lambda - x \in \exp(A)$. But for all $\lambda \neq 0$, we have

$$\lambda 1 = e^{\alpha} 1 = 1 \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} = \sum_{n=0}^{\infty} \frac{(1\alpha)^n}{n!} = e^{\alpha 1}$$

which means $\lambda 1 \in \exp(A)$. Since $x = \lambda - (\lambda - x) = \lambda 1 - (\lambda - x)$, we have that A can be written as $A = \exp A + \exp A$.

1.4 Idempotents In Banach Algebras

Definitions 1.4.1. If A is a Banach algebra, then an element $p \in A$ is called an *idempotent* or a *projection* if $p = p^2$. The zero and identity elements of A are called the *trivial idempotents* of A. Unless explicitely stated, we assume all idempotents to be non-trivial.

Lemma 1.4.2. *If A is a Banach algebra and p is a (non-trivial) idempotent in A then*

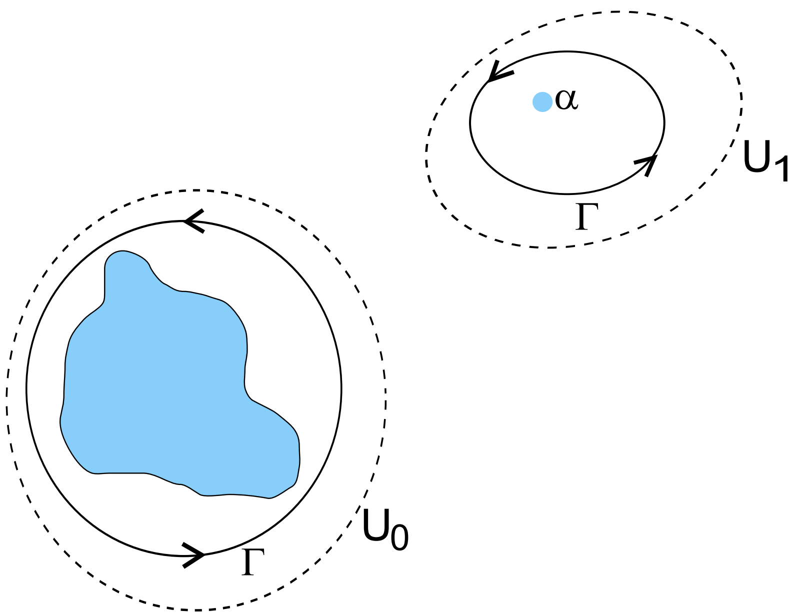
$$\sigma(p) = \{0, 1\}.$$

Proof : Since p is an idempotent, $p - p^2 = 0$. This implies $\sigma(p - p^2) = \{0\}$, but, by the Spectral Mapping Theorem, $\sigma(p-p^2) = \{\lambda - \lambda^2 : \lambda \in \sigma(p)\}$ which implies that $\lambda - \lambda^2 = 0$ for all $\lambda \in \sigma(p)$. Hence $\lambda \in \{0, 1\}$. What is left to show is that $\{0, 1\} \subseteq \sigma(p)$ when $p \neq 0$ and $p \neq 1$. Since $p^2 = p$ we have $p(1 - p) = (1 - p)p = 0$. This gives $1 - p \notin A^{-1}$ and $p \notin A^{-1}$ since, if $1-p$ is invertible, then $p = 0$ and we have a contradiction and similarly, if p is invertible then $p = 1$, which is also a contradiction. Hence $\{0, 1\} \subseteq \sigma(p)$. ■

Definitions 1.4.3. Let x be in A and let α be isolated in $\sigma(x)$. The *idempotent corre− sponding to x and α* is defined by

$$p = \frac{1}{2\pi i} \int_{\Gamma_\alpha} (\lambda - x)^{-1} d\lambda$$

where Γ_α is a small circle with center at α and separating α from the remaining spectrum of x. To motivate the terminology we consider the following diagram:



To see that p is indeed an idempotent, write p as $p = \frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda - x)^{-1}d\lambda$ where

$$f(\lambda) = \begin{cases} 1, & \lambda \in U_1 \\ 0, & \lambda \in U_0 \end{cases}$$

Then $f^2(\lambda) = f(\lambda)\ \forall\ \lambda \in U_0 \cup U_1$. By the Holomorphic Functional Calculus we get

$$\begin{aligned} p^2 &= \left(\frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda - x)^{-1}d\lambda\right)\left(\frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda - x)^{-1}d\lambda\right) \\ &= \frac{1}{2\pi i}\int_\Gamma f^2(\lambda)(\lambda - x)^{-1}d\lambda \\ &= \frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda - x)^{-1}d\lambda \\ &= p. \end{aligned}$$

We show that p belongs to the bicommutant of x:
If $zx = xz$ then

$$z(\lambda - x) = (\lambda - x)z\ \ \forall\ \lambda \Longrightarrow (\lambda - x)^{-1}z = z(\lambda - x)^{-1}\ \forall\ \lambda \notin \sigma(x)$$

and thus

$$\begin{aligned} zp &= \frac{1}{2\pi i}\int_\Gamma z\,(\lambda - x)^{-1}d\lambda \\ &= \frac{1}{2\pi i}\int_\Gamma (\lambda - x)^{-1}z d\lambda \\ &= \Big[\frac{1}{2\pi i}\int_\Gamma (\lambda - x)^{-1}d\lambda\Big]z \\ &= p\,z. \end{aligned}$$

This means that the idempotent, as defined above, commutes with every element which commutes with x, i.e. p is an element of the bicommutant of x.

Another key result (a part of which appears in ([2])), which we will use throughout this dissertation, is as follows:

Theorem 1.4.4. *If A is a semisimple Banach algebra with identity and $p \in A$ is an idempotent, then $p\,Ap$ is a semisimple Banach algebra with identity p. Moreover, for every $x \in A$ we have that $\sigma'_A(p\,xp) = \sigma'_{p\,Ap}(p\,xp)$.*

Proof : We first prove that p is the identity of $p\,Ap$. For every $p\,xp$ in $p\,Ap$ we have $p(p\,xp) = p^2xp = p\,xp = p\,xp^2 = (p\,xp)p$ and hence p is the identity of $p\,Ap$. We now proceed to prove that $p\,Ap$ is an algebra, after which we will prove that it is closed in A and hence complete.

- <u>for $pxp, pyp \in pAp,\ \alpha pxp + \beta pyp \in pAp$</u> :

$$\begin{aligned}\alpha p\,xp + \beta p\,yp &= \alpha p^2xp^2 + \beta p^2yp^2 \ \text{ since } p \text{ is an idempotent}\\ &= p(\alpha p\,xp + \beta p\,yp)p \in p\,Ap \ \text{ since } \alpha p\,xp + \beta p\,yp \in A\end{aligned}$$

- <u>for $pxp, pyp \in pAp,\ pxppyp \in pAp$</u> :

$$pxppyp = pxpyp \in pAp.$$

Therefore, pAp is a vector space and $p\,Ap$ is closed under multiplication. Note here that all the other characteristics such as the associative and distributive laws are inherited from the fact that A is an algebra. Hence $p\,Ap$ is an algebra.

Since $pAp \subset A$, we only need to show that $p\,Ap$ is closed in order to show that it is complete. Let $p\,x_np \to x$. Then, by continuity of multiplication, we have $p^2x_np^2 \to p\,xp$. But $p^2 = p$ implies that $p\,x_np \to p\,xp$. By uniqueness of limits $x = p\,xp$. Therefore, $p\,Ap$ is closed in A and hence complete.

We shall now prove that $\sigma'_A(p\,xp) = \sigma'_{pAp}(p\,xp)$ from which the semisimplicity of $p\,Ap$ readily follows. If $\lambda p - pap$ has inverse pbp in $p\,Ap$ then

$$\begin{aligned}\big(\lambda - p\,ap\big)\big(p\,bp + \frac{1}{\lambda}(1-p)\big) &= (\lambda - p\,ap)p\,bp + (1-p) - \frac{1}{\lambda}(1-p)p\,ap\\ &= (\lambda - p\,ap)p\,bp + (1-p)\\ &= (\lambda - p\,ap)p^2bp + (1-p) \ \text{ since } p \text{ is an idempotent}\\ &= (\lambda p - p\,ap)p\,bp + (1-p)\\ &= p + 1 - p \ \text{ since } \lambda p - p\,ap \text{ has inverse } p\,bp \text{ in } p\,Ap\\ &= 1.\end{aligned}$$

Similarly $\big(p\,bp + \frac{1}{\lambda}(1-p)\big)\big(\lambda - p\,ap\big) = 1$. Thus $\mathbb{C}\backslash\sigma'_{p\,Ap}(p\,xp) \subseteq \mathbb{C}\backslash\sigma'_A(p\,xp)$ which means $\sigma'_A(p\,xp) \subseteq \sigma'_{p\,Ap}(p\,xp)$. On the other hand, if $\lambda - p\,ap \in A^{-1}$ with inverse b then $(\lambda - p\,ap)b = b(\lambda - p\,ap) = 1$.

Consider $(\lambda p - p\,ap)pbp$:

$$\begin{aligned}(\lambda p - p\,ap)p\,bp &= (\lambda p^2 - p\,ap^2)bp\\ &= (\lambda p - p\,ap)bp \ \text{ since } p \text{ is an idempotent}\\ &= (\lambda p - p^2ap)bp \ \text{ since } p \text{ is an idempotent}\\ &= p(\lambda - p\,ap)bp\\ &= p1p\\ &= p^2\\ &= p.\end{aligned}$$

Similarly $p\,bp(\lambda p - p\,ap) = p$ and hence $(\lambda p - p\,ap)$ has inverse $p\,bp$ in $p\,Ap$. This gives $\mathbb{C}\backslash\sigma'_A(p\,xp) \subseteq \mathbb{C}\backslash\sigma'_{p\,Ap}(p\,xp)$ or, in other words, $\sigma'_{p\,Ap}(p\,xp) \subseteq \sigma'_A(p\,xp)$. Hence $\sigma'_{p\,Ap}(p\,xp) = \sigma'_A(p\,xp)$ which completes the proof. It now remains to show that $p\,Ap$ is semisimple whenever A is semisimple. Suppose $p\,ap$ belongs to the radical of $p\,Ap$. Then $\sigma'_{p\,Ap}(p\,xpp\,ap) = \emptyset$ for all $x \in A$. From the above result we have that $\sigma'_A(p\,xpp\,ap) = \emptyset$ for all $x \in A$. By Lemma 1.2.4 we get $\sigma'_A(xpp\,ap) = \emptyset$ for all $x \in A$. But this says that $pp\,ap = p\,ap$ belongs to the radical of A. So, since A is semisimple, $p\,ap = 0$ which implies that $p\,Ap$ is semisimple. ■

We shall denote by $A^\bullet$ the collection of idempotents in a Banach algebra A. Note that $A^\bullet$ is closed in A, since if $x_n \in A^\bullet$ and x_n converges to x then $(x_n)^2$ converges to x^2. So $(x_n)^2 = x_n$ implies x_n converges to x and x^2, and hence, by uniqueness of the limit, we have $x = x^2$ and thus $x \in A^\bullet$. If $p \in A^\bullet$ then we denote by K_p the connected component of $A^\bullet$ containing the element p. It is shown, in ([17], Theorem 3.3.), that there is a relationship between the sets K_p and $\mathrm{Exp}(A)$:

$$K_p = \{vpv^{-1} : v \in \mathrm{Exp}(A)\}.$$

Also, if $p, q \in A^\bullet$ and the spectral distance between p and q is less than 1, then p and q belong to the same connected component of $A^\bullet$. That is to say $r_\sigma(p-q) < 1 \Rightarrow K_p = K_q$. This result will be used, in Chapter 2, with respect to so–called rank one elements. For more interesting results on idempotents one may consult ([10], [11] and [17]).

1.5 Minimal Ideals In Banach Algebras

Definitions 1.5.1. A Banach algebra A is called *semiprime* if the only two-sided ideal $J \subseteq A$ with the property that $J^2 = \{ab : a, b \in J\} = \{0\}$ is $J = \{0\}$. The following characterization of this is sometimes given as the definition: A is a semiprime Banach algebra if and only if

$$xAx = \{0\} \Longrightarrow x = 0 \tag{1.5.1.1}$$

To prove this, suppose A is semiprime but for some $0 \neq x \in A$ we have $xAx = \{0\}$. Define

$$J = \Big\{ \sum_{i=1}^{n} a_i x b_i : a_i, b_i \in A, n \in \mathbb{N} \Big\}.$$

We show J is a two-sided ideal:

(1) From the definition of J, the zero element of A belongs to J.

(2) If $a, b \in J$ and $\alpha, \beta \in \mathbb{C}$ then

$$\alpha a + \beta b = \sum_{i=1}^{n_1} \alpha a_i x b_i + \sum_{j=1}^{n_2} \beta c_j x d_j \quad \text{which, by definition, belongs to } J.$$

(3) Finally, if $r \in J$ and $c \in A$ then

$$\begin{aligned} rc &= (\sum_{i=1}^{n} a_i x b_i) c \\ &= \sum_{i=1}^{n} a_i x b_i c \\ &= \sum_{i=1}^{n} a_i x d_i, \quad d_i \in A. \end{aligned}$$

This proves that J is a right ideal of A. Similarly, we may show J is a left ideal.
Now,

$$\begin{aligned} J^2 &= \{ \sum_{i=1}^{n} a_i x b_i \sum_{j=1}^{m} c_j x d_j, \quad a_i, b_i, c_j, d_j \in A \} \\ &= \sum_{i=1}^{n} \Big[\sum_{j=1}^{m} a_i x b_i c_j x d_j \Big] \\ &= \{0\} \text{ since } xAx = \{0\}. \end{aligned}$$

But $J \neq \{0\}$ since $x \neq 0$ and not all a_i, b_i are zero. We have therefore a contradiction to A semiprime, as defined in Definition 1.5.1. and consequently we must have $x = 0$.

For the converse, suppose for some two–sided ideal $J \neq \{0\}$ we have $J^2 = \{0\}$. Take $0 \neq x \in J$ arbitrary and $a \in A$ arbitrary and let $\lambda \notin \sigma(a)$. We show that $xax = 0$. Now,

$$\begin{aligned} & x(\lambda - a) \in J \;\text{ since } J \text{ is a two-sided ideal} \\ \Longrightarrow\; & x(\lambda - a)x(\lambda - a) = 0 \;\text{ since } J^2 = \{0\} \\ \Longrightarrow\; & x(\lambda - a)x = 0 \;\text{ since } (\lambda - a) \in A^{-1} \\ \Longrightarrow\; & \lambda x^2 - xax = 0 \\ \Longrightarrow\; & xax = 0 \;\text{ because } x^2 = 0. \end{aligned}$$

But a was chosen arbitrarily. Hence $xAx = \{0\}$. Note that, since J is two–sided we may immediately conclude that $xax = 0$, but the foregoing arguments show that, in the definition of semiprime, one may replace the definition of J^2 by the 'weaker definition', $J^2 = \{a^2 : a \in J\}$.

If A is semisimple but not semiprime i.e. A is semisimple and for some $0 \neq x \in A$ we have $xAx = \{0\}$ then for arbitrary $z \in A$, $1 - (zx)^2 = 1 - z(xzx) = 1 - z0 = 1$ and thus $1 - (zx)^2 \in A^{-1}$. But, by the Spectral Mapping Theorem, $\sigma\big((zx)^2\big) = \sigma\big((zx)\big)^2$. So if $1 \notin \sigma\big((zx)^2\big)$ then $1 \notin \big[\sigma(zx)\big]^2$ so that $1 \notin \sigma(zx)$. Thus $1 - zx \in A^{-1}\ \forall\ z \in A$. This, together with the fact that $\text{Rad}(A) = \{0\}$, (since A is semisimple) implies that $x = 0$, contradicting the assumption that $x \neq 0$. Hence all semisimple Banach algebras are semiprime. It is, generally speaking, not true that a semiprime Banach algebra is also semisimple. However, as we show now, in these cases the dimension of the radical must be infinite dimensional. We start with:

Theorem 1.5.2. *If* $\dim \text{Rad}(A) = n$ *then the product of* $n+1$ *radical elements equals zero.*

Proof : Consider the product $a_1 a_2 ... a_{n+1}$ where each $a_i \in \text{Rad}(A)$ and consider the $n+1$ elements $a_1,\ a_1a_2,\ a_1a_2a_3,\ \dots\ , a_1a_2a_3 \dots a_{n+1}$. Since $\dim \text{Rad}(A) = n$ and $\text{Rad}(A)$ is a vector space, we can write $a_1a_2 \dots a_{n+1} = \alpha_1 a_1 + \alpha_2 a_1 a_2 + \cdots + \alpha_n a_1 a_2 \dots a_n$ for some scalars α_i, $i \in \{1, ..., n\}$. Then $\alpha_1 a_1 + \alpha_2 a_1 a_2 + \cdots + \alpha_n a_1 a_2 \dots a_n - a_1 a_2 \dots a_{n+1} = 0$ and thus

$$a_1\big[\alpha_1 + \alpha_2 a_2 + \cdots + \alpha_n(a_2 \dots a_n) - (a_2 \dots a_{n+1})\big] = 0.$$

If $\alpha_1 \neq 0$ then the term in the bracket is invertible (since all the $a_i \in \text{Rad}(A)$), which implies $a_1 = 0$ and the result follows immediately. So, we may assume that $\alpha_1 = 0$. Applying this concept inductively, we see that all the $\alpha_i = 0$ and hence $a_1a_2 \dots a_{n+1} = 0$ and the theorem is proved. ■

Corollary 1.5.3. *If* $\dim \mathrm{Rad}(A) = n > 0$ *then* A *cannot be semiprime.*

Proof : Let m be the largest number such that there exist $a_1, a_2, ..., a_m \in \mathrm{Rad}(A)$ with $a_1 a_2 ... a_m \neq 0$ but $a_1 a_2 ... a_m a = 0$ for each $a \in \mathrm{Rad}(A)$. Note that this number exists by the previous theorem. By putting $x = a_1 a_2 ... a_m$, we have $xAx = \{0\}$, $x \neq 0$ and hence A is not semiprime. ■

Definitions 1.5.4. Let A be a semiprime algebra. Then a *minimal left* (*right*) *ideal* J of A is a left(right) ideal such that $\{0\}$ and J are the only left(right) ideals contained in J. We shall also exclude the trivial ideal, $J = \{0\}$, from the aforementioned definitions. A *minimal idempotent* is a non-zero idempotent $p \in A$ such that pAp is one–dimensional, that is, $pAp = \mathbb{C}p$. Minimal idempotents and minimal ideals are related in the following way:

Theorem 1.5.5. ([4]) *Let* A *be semiprime. Then*

(1) L *is a minimal left ideal of* A *if and only if* $L = Ap$ *where* p *is a minimal idempotent in* A.

(2) R *is a minimal right ideal of* A *if and only if* $R = pA$ *where* p *is a minimal idempotent in* A.

We now show that, for any Banach space $X \neq \{0\}$, the Banach algebra $A = B(X)$ always contains minimal idempotents. By the Hahn-Banach Theorem, given $x_0 \neq 0$ there is $f \in X'$ such that $f(x_0) = \|x_0\|$. Defining $Tx = f(x)x_0$ we have $P = \frac{1}{\|x_0\|}T$ is a minimal idempotent: from our definition we have $Px = \frac{1}{\|x_0\|} f(x)x_0$, so that

$$\begin{aligned} P^2 x &= \frac{1}{\|x_0\|} T\Big(\frac{1}{\|x_0\|} f(x)x_0\Big) \\ &= \frac{1}{\|x_0\|} f\Big(\frac{1}{\|x_0\|} f(x)x_0\Big) x_0 \\ &= \frac{\|x_0\|}{\|x_0\|^2} f(x)x_0 \quad \text{since } f \in X' \text{ and } f(x_0) = \|x_0\| \\ &= \frac{1}{\|x_0\|} f(x)x_0 \\ &= Px. \end{aligned}$$

Hence P is an idempotent. To show that it is minimal, we need to show $PB(X)P = \mathbb{C}P$, that is, $(PT_2P)(x) = \alpha Px$ for any $T_2 \in B(X)$ where $\alpha \in \mathbb{C}$ depends only on T_2 and not on x. Now,

$$\begin{aligned}
(PT_2P)(x) &= PT_2(Px) \\
&= PT_2\Big(\frac{1}{\|x_0\|}f(x)x_0\Big) \\
&= P\Big[T_2\Big(\frac{1}{\|x_0\|}f(x)x_0\Big)\Big] \\
&= P\Big[\frac{1}{\|x_0\|}f(x)T_2x_0\Big] \quad \text{since } T_2 \in B(X) \\
&= \frac{1}{\|x_0\|}f(x)P(T_2x_0) \quad \text{since } P \in B(X) \\
&= \frac{1}{\|x_0\|}f(x)\frac{1}{\|x_0\|}f(T_2x_0)x_0 \\
&= \alpha\frac{1}{\|x_0\|}f(x)x_0 \quad \text{where } \alpha = \frac{1}{\|x_0\|}f(T_2x_0) \\
&= \alpha Px.
\end{aligned}$$

Hence P is a minimal idempotent. Using this result together with Theorem 1.5.5., we have that $B(X)$ always contains minimal ideals.

Chapter 2 - Finite Rank Elements

2.1 Rank One Elements In Banach Algebras

Definitions 2.1.1. Let A be a semiprime Banach algebra. We call an element $0 \neq a \in A$ *spatially rank one* if

$$aAa \subseteq \mathbb{C}a \tag{2.1.1.1}$$

We will denote the set of spatially rank one elements by F_1. It is important to note that F_1 absorbs non-zero products from both sides in A. To see this, let $a \in F_1$ and let $0 \neq x \in A$. For any $z \in A$ we get

$$\begin{aligned}(xa)z(xa) &= x(azxa)\\ &= x\lambda a \;\text{ since } azxa \in \mathbb{C}a\\ &= \lambda xa.\end{aligned}$$

A similar argument holds for products from the right. Another important observation is that the containment in (2.1.1.1) is actually equality; if A is semiprime and $aAa \subseteq \mathbb{C}a$ then there is $x \in A$ such that $axa \neq 0$. Thus $axa = \lambda a$ for some $\lambda \in \mathbb{C}$. Take $\alpha \in \mathbb{C}$ arbitrary and consider

$$\begin{aligned}a\left(\frac{\alpha x}{\lambda}\right)a &= \frac{\alpha}{\lambda}axa\\ &= \frac{\alpha}{\lambda}\lambda a\\ &= \alpha a.\end{aligned}$$

Note that, by definition, minimal idempotents are spatially rank one idempotents and that spatially rank one idempotents are minimal. Since aAa is a vector space, definition (2.1.1.1) is equivalent to the definition of a one-dimensional element given by J. Puhl ([14], p. 656): $a \in A$ is called *one-dimensional* if there exists a linear functional, f_a, such that

$$axa = f_a(x)a \text{ for all } x \in A \tag{2.1.1.2}$$

This definition and terminology is almost certainly due to the fact that, for $T \in B(X)$, we have that $\dim R(T) = 1$ if and only if (2.1.1.1) holds: we show that if $T \in B(X)$ and $\dim R(T) = 1$ then T is of rank 1. If $\dim R(T) = 1$ then $Tx = \alpha_x z_0$ where $z_0 \in X$ and $\alpha_x \in \mathbb{C}$ depending on x. Define f in the following manner: $f(x) = \alpha_x$ where α_x is the element in $\mathbb{C}$ such that $Tx = \alpha_x z_0$.

We need to show $f \in X'$. Now,

$$\begin{aligned} f(x+y) &= \alpha_{(x+y)} \quad \text{where } T(x+y) = \alpha_{(x+y)}z_0 \\ &= (\alpha_x + \alpha_y) \quad \text{since } T(x+y) = T(x) + T(y) = \alpha_x z_0 + \alpha_y z_0 \\ &= f(x) + f(y) \end{aligned}$$

and

$$\begin{aligned} f(\beta x) &= \alpha_{(\beta x)} \\ &= \beta \alpha_x \quad \text{since } T(\beta x) = \beta T(x) = \beta(\alpha_x) z_0 \\ &= \beta f(x). \end{aligned}$$

Hence f is linear. To show that f is bounded, we show that it is continuous i.e. if $x_n \longrightarrow x$ then $f(x_n) \longrightarrow f(x)$. Suppose therefore that $x_n \longrightarrow x$. Now,

$$\begin{aligned} &Tx_n \longrightarrow Tx \quad \text{since } T \text{ is bounded} \\ &\therefore f(x_n)z_0 \longrightarrow f(x)z_0 \quad \text{since } Tx = f(x)z_0 \\ &\therefore \lim_{n\to\infty} \|\big(f(x_n) - f(x)\big)z_0\| = 0 \\ &\therefore \lim_{n\to\infty} |f(x_n) - f(x)| \cdot \|z_0\| = 0 \\ \Longrightarrow &\lim_{n\to\infty} |f(x_n) - f(x)| = 0 \quad \text{since } \|z_0\| \neq 0. \end{aligned}$$

Hence $f(x_n) \longrightarrow f(x)$ and f is bounded. Since f is also linear, $f \in X'$. Now, for an arbitrary $T_2 \in B(X)$, we have:

$$\begin{aligned} (TT_2T)(x) &= TT_2(Tx) \\ &= TT_2(f(x)z_0) \\ &= T\big(T_2(f(x)z_0)\big) \\ &= T\big(f(x)T_2(z_0)\big) \quad \text{since } T_2 \in B(X) \\ &= f(x)T\big(T_2(z_0)\big) \quad \text{since } T \in B(X) \\ &= f(x)f\big(T_2(z_0)\big)z_0 \\ &= f\big(T_2(z_0)\big)f(x)z_0 \\ &= f\big(T_2(z_0)\big)Tx. \end{aligned}$$

Hence T has spatial rank 1 and thus $\dim R(T) = 1 \Longrightarrow T$ is spatially rank 1.
The linear functional in (2.1.1.2) is continuous and also uniquely determined by a;
Suppose $x_n \longrightarrow x$. Then by the continuity of multiplication, $ax_na \longrightarrow axa$. That is, $f_a(x_n)a \longrightarrow f_a(x)a$, which implies $\|f_a(x_n)a - f_a(x)a\| \longrightarrow 0$.

Hence $|f_a(x_n) - f_a(x)| \cdot \|a\| \longrightarrow 0$, with $a \neq 0$, implies $|f_a(x_n) - f_a(x)| \longrightarrow 0$. Thus $f_a(x_n) \longrightarrow f_a(x)$, and we have the continuity of f_a. Suppose now that f_a and f'_a are two functionals such that $axa = f_a(x)a \,\forall\, x$ and $axa = f'_a(x)a \,\forall\, x$. So, $f_a(x)a - f'_a(x)a = 0 \,\forall\, x$ and hence $|f_a(x) - f'_a(x)| \cdot \|a\| = 0$. Since $a \neq 0$, we have $|f_a(x) - f'_a(x)| = 0$ and thus $f_a(x) = f'_a(x)$. So, f_a is uniquely determined by a.
Spatially rank one elements are not invertible, except for the trivial case where $A = \mathbb{C}$; consider $a \in F_1 \cap A^{-1}$. This means there exists $a^{-1} \in A$ such that $aa^{-1} = a^{-1}a = 1$ and $axa = f_a(x)a \;\forall\; x \in A$. We then have that $a^{-1}axaa^{-1} = a^{-1}f_a(x)aa^{-1}$, so that $x = a^{-1}f_a(x)$. If $x \neq 0$ then $f_a(x) \neq 0$, (since $axa = f_a(x)a \Longrightarrow x = f_a(x)a^{-1}$), which means x is invertible for all $0 \neq x \in A$. By the Gelfand-Mazur Theorem ([1], p. 39) we then have A isometrically isomorphic to $\mathbb{C}$. So, in the remainder of our work we assume that $A \neq \mathbb{C}$.

Corollary 2.1.2. *If $a \in F_1$ then $\sigma(a) = \{0, f_a(1)\}$.*

Proof : Let $a \in F_1$. If $f_a(1) = 0$ then $a^2 = 0$ which implies that $r_\sigma(a) = 0$ and hence that $\sigma(a) = \{0\}$. We can thus assume $f_a(1) \neq 0$. For $a \in F_1$, we have $axa = f_a(x)a$ so that $axax = f_a(x)ax$ and $(ax)^2 = f_a(x)ax$. Now,

$$\left(\frac{a}{f_a(1)}\right)^2 = \frac{a^2}{f_a(1)f_a(1)} = \frac{f_a(1)a}{f_a(1)f_a(1)} = \frac{a}{f_a(1)}$$

and therefore, $\frac{a}{f_a(1)}$ is an idempotent. This means $\sigma\big(\frac{a}{f_a(1)}\big) = \{0, 1\}$ and we thus have $\sigma(a) = \{0, f_a(1)\}$. ■

The complex number $f_a(1)$ is called the *trace* of a and is denoted by $t_r(a)$. For $a \in F_1$, it might happen that $t_r(a) = 0$. In fact, as we illustrate now, every quasinilpotent $a \in F_1$ is nilpotent with $a^2 = 0$; if $a \in F_1$ is quasinilpotent then $\sigma(a) = \{0\}$, and hence, by Corollary 2.1.2, $f_a(1) = 0$. But then $a^2 = a1a = f_a(1)a = 0$.

Corollary 2.1.3. *If $a \in F_1$ is nilpotent, then there is no homomorphism of aAa onto $\mathbb{C}$.*

Proof : Let $a \in F_1$ be nilpotent. From the previous result, $aa = f_a(1)a = 0$. Suppose now that $T : aAa \longrightarrow \mathbb{C}$ is a homomorphism. Take axa such that $T(axa) \neq 0$. What we then have is that $T(axa)T(axa) \neq 0$, whilst $T\big((axa)(axa)\big) = T\big((axaaxa)\big) = 0$
since $aa = 0$. This means that T fails to be multiplicative, and thus aAa is not homomorphic onto $\mathbb{C}$. ■

We will give the precise algebraic structure of aAa when $a \in F_1$ is nilpotent in paragraph 2.3 (Theorem 2.3.14). Consider now $a \in F_1$ such that a is not nilpotent and let $x \in A$ be arbitrary. We have,

$$\begin{aligned} axa\Big(\frac{a}{f_a(1)}\Big) &= \frac{axa^2}{f_a(1)} \\ &= \frac{axf_a(1)a}{f_a(1)} \quad \text{since } a^2 = f_a(1)a \\ &= axa. \end{aligned}$$

Similarly $\big(\frac{a}{f_a(1)}\big)axa = axa$. Also

$$\frac{a}{f_a(1)} = \frac{a1a}{(f_a(1))^2} \quad \text{since } a^2 = f_a(1)a$$

and thus $\frac{a}{f_a(1)} \in aAa$. So, we have just shown that if $a \in F_1$ is not nilpotent, then $\frac{a}{f_a(1)}$ is an identity element for aAa.

The functional $\phi : aAa \longrightarrow \mathbb{C}$ given by $\phi(axa) = f_a(1)f_a(x)$ is linear (since f_a is), multiplicative and maps $\frac{a}{f_a(1)}$ onto $1 \in \mathbb{C}$:

Suppose $axa = \lambda a$ and $aya = \alpha a$ then

$$\begin{aligned} \phi\big((axa)(aya)\big) &= \phi(\lambda\alpha a^2) \\ &= \lambda\alpha\phi(a1a) \\ &= \lambda\alpha f_a(1)f_a(1) \\ &= \big(\lambda f_a(1)\big)\big(\alpha f_a(1)\big) \end{aligned}$$

and

$$\begin{aligned} \phi(axa) &= \phi(\lambda a) \\ &= \lambda\phi(a) \\ &= \lambda\phi\Big(\frac{a^2}{f_a(1)}\Big) \\ &= \frac{\lambda}{f_a(1)}\phi(a1a) \\ &= \lambda f_a(1) \end{aligned}$$

and similarly $\phi(aya) = \alpha f_a(1)$.

Hence

$$\phi(axa)\phi(aya) = \lambda f_a(1)\alpha f_a(1) = \phi\big((axa)(aya)\big)$$

which proves that ϕ is multiplicative. Using the fact that ϕ is linear, we have:

$$\begin{aligned}
\phi\Big(\frac{a}{f_a(1)}\Big) &= \frac{1}{f_a(1)}\phi(a) \\
&= \frac{1}{f_a(1)}\phi\Big(\frac{a^2}{f_a(1)}\Big) \\
&= \frac{1}{f_a(1)} \cdot \frac{1}{f_a(1)}\phi(a1a) \\
&= \frac{1}{f_a(1)} \cdot \frac{1}{f_a(1)} \cdot f_a(1) \cdot f_a(1) \\
&= 1
\end{aligned}$$

which proves the statement. Hence, if $a \in F_1$ is not nilpotent then aAa is isomorphic to $\mathbb{C}$. Furthermore, if $a \in F_1$ is central then a cannot be nilpotent; suppose a is central and A is semiprime. Then $axa = \lambda a$ for some $\lambda \neq 0, x \in A$. This implies $xa^2 = \lambda a$ (since a is central). So, if $a^2 = 0$ then $\lambda a = 0$ which gives $a = 0$ since $\lambda \neq 0$. Thus, we are lead to a contradiction and hence a cannot be nilpotent.

Definitions 2.1.4. A second definition for rank one elements in a semiprime Banach algebra is given by Robin Harte in ([9], p. 74): an element $0 \neq a \in A$ is said to be *spectrally rank one* if

$$x \in A \Longrightarrow \#\sigma'(xa) \leq 1 \tag{2.1.4.1}$$

where $\#K$ denotes the number of elements in a set K. We will denote the set of spectrally rank one elements by S_1. Note also that if $x \in A \Longrightarrow \#\sigma'(xa) \leq 1$ then for $b \in A$, xb is also in A and thus $\#\sigma'(xba) \leq 1$. Therefore, S_1 absorbs products from the left. Similarly, since $\sigma'((xa)b) = \sigma'(bxa)$, S_1 also absorbs products from the right.

Corollary 2.1.5. *Every spatially rank one element is spectrally rank one.*

Proof : Suppose $a \in F_1$. Then $\sigma(a) = \{0, f_a(1)\}$. For every $x \in A$, we also know that $xa \in F_1$ since F_1 absorbs products from the left. By Corollary 2.1.2. we have that $\sigma(xa) = \{0, f_{xa}(1)\}$, which means $\#\sigma'(xa) \leq 1$. Hence $F_1 \subseteq S_1$. ■

If A is semiprime but not semisimple then $xAx = \{0\} \Longrightarrow x = 0$, but $\mathrm{Rad}(A) \neq \{0\}$. Let $a \in \mathrm{Rad}(A)$. Then for every $x \in A$, $xa \in \mathrm{Rad}(A)$ since the radical is an ideal. This implies $\sigma(xa) = \{0\}$ and hence $\#\sigma'(xa) \leq 1$. So $\mathrm{Rad}(A) \subseteq S_1$. If, on the other hand, $a \in \mathrm{Rad}(A) \cap F_1$ then for some $x \in A$ we have $axa = \lambda a$ with $\lambda \neq 0$ (since A is semiprime). As before, this means $\frac{ax}{\lambda}$ is a non-zero idempotent. But, again by the fact that $\mathrm{Rad}A$

is an ideal, we now have a non-zero idempotent belonging to $\text{Rad}(A)$. So, in view of Theorem 1.1.10 and Lemma 1.4.2 we get a contradiction. Hence $\text{Rad}(A) \cap F_1 = \emptyset$.

If A is semisimple then, as R. Harte shows in ([9]), the above mentioned notions of rank are identical. Harte's result was actually (implicitly) discovered earlier in a paper ([12]) by T. Mouton and H. Raubenheimer. The aforementioned authors give in ([12]) an alternative description of F_1 in terms of the spectrum where A is a semisimple Banach algebra: $a \in F_1$ if and only if for every $b \in A$ and any choice of scalars $\lambda_1, \lambda_2 \in \mathbb{C}$ with $0 \neq \lambda_1 \neq \lambda_2 \neq 0$ we have that

$$\sigma(b + \lambda_1 a) \cap \sigma(b + \lambda_2 a) \subseteq \sigma(b) \tag{2.1.5.1}$$

The point of Harte's article, however, was to show that this spectral characterization could be obtained without the use of 'heavy machinery' *viz*, one could get away by using elementary arguments as opposed to arguments involving analytical properties of the spectrum. As we shall later see, in section 2.3, this is more or less also the main topic of this dissertation.

To conclude this section, we also mention another definition of rank one elements given by J.A. Erdos, S. Giotopoulos and M.S. Lambrou ([8]), published around the same time as Puhl's article: a non-zero element a of a semisimple Banach algebra A is defined to be rank one if a has an image of rank one in some continuous faithful representation of A. This means the image of a under the representation, say T_a, has one-dimensional range in the space it acts on. Erdos *et al* then proceeds to give a characterization of this definition in terms of the original algebra A. In other words, they give an equivalent definition in A that does not refer to representation theoretical ideas. We need the following brief definitions:

Definitions 2.1.6. Let A be a Banach algebra. A non-zero element $s \in A$ is defined to be *single* if, whenever $asb = 0$ for some $a, b \in A$, then either $as = 0$ or $sb = 0$. An element $u \in A$ is said to *act compactly* on A if the map $a \mapsto uau$ is compact. One observes immediately that single compactly acting elements, as with spatially and spectrally rank one elements, also absorb products from the left and the right. We show, in the next section, the relationship between single compactly acting elements and our definitions of spatially and spectrally rank one elements. The main result in ([8]) is given by:

Theorem 2.1.7. *Let A be a complex semisimple Banach algebra. Then there exists a continuous faithful representation $a \mapsto T_a$ of A on Banach space such that $T_s \neq 0$ has rank one, if and only if s is single and acts compactly on A.*

Erdos shows, as his main result in ([7]), that Theorem 2.1.7 holds in the special cases of $C^\star$-algebras with s being only a single element. The fact that s is also compactly acting is, in these instances, part of the conclusion rather than the assumption.

2.2 The Structure Of Rank One Elements

We investigate, in this section, the topological structure of rank one elements; in particular we shall try to illuminate, in some detail, the difference between spatially and spectrally rank one elements. Along the way we shall also prove that the notions of rank one elements as put forward by Puhl, on the one hand, and that of Erdos *et al* on the other, in fact coincide in the semiprime setting. Let $A \neq \mathbb{C}$ be a semiprime Banach algebra with $F_1 \neq \emptyset$. It will be to some advantage, at this early stage, to distinguish between nilpotent and non–nilpotent elements of F_1: F_1^0 will denote the nilpotent elements of F_1 and F_1^1 the non-nilpotent elements of F_1. Notice first that $F_1^1 \neq \emptyset$; let $a \in F_1$. Since A is semiprime, there is $x \in A$ such that $axa \neq 0$. Thus $axa = \lambda a,\ \lambda \neq 0$ which implies $(ax)(ax) = \lambda ax \neq 0$. So, clearly $ax \in F_1^1$. We now proceed by stating a lemma which plays a fundamental role throughout this section.

Lemma 2.2.1. ([15], Lemma 2.1.) *Let A be a semiprime Banach algebra with $F_1 \neq \emptyset$.*

(1) If $u \in F_1^1$, $v \in A$ with $uv \in F_1^1$ then uv can be written as $uv = \alpha e^x u e^{-x}$ where $0 \neq \alpha \in \mathbb{C}$ and $x \in A$. A similar statement holds for $vu \in F_1^1$.

(2) If $u \in F_1^0$ then there exists $v \in exp(A)$ such that uv and $vu \in F_1^1$.

Proof : (1) Remember that the spatially rank 1 elements $\frac{u}{t_r(u)}$ and $\frac{uv}{t_r(uv)}$ are idempotents. Now,

$$\frac{u^2v}{t_r(u)t_r(uv)} = \frac{t_r(u)uv}{t_r(u)t_r(uv)} = \frac{uv}{t_r(uv)} \ .$$

We would now like to show that

$$\frac{uvu}{t_r(uv)t_r(u)} = \frac{u}{t_r(u)} \ .$$

If we use the notation in (2.1.1.2), we have $uvu = f_u(v)u$ and $\frac{u}{t_r(u)} = \frac{u}{f_u(1)}$. Thus,

$$\frac{uvu}{f_{uv}(1)f_u(1)} = \frac{f_u(v)u}{f_{uv}(1)f_u(1)} \tag{2.2.1.1}$$

So, we only need to prove that $f_u(v) = f_{uv}(1)$. But we know $(uv)1(uv) = f_{uv}(1)uv$ and also $(uvu)v = f_u(v)uv$. This implies $f_{uv}(1) = f_u(v)$, which proves (from (2.2.1.1)) the result. Now,

$$\begin{aligned}\left(\frac{u}{t_r(u)}-\frac{uv}{t_r(uv)}\right)^2 &= \left(\frac{u}{t_r(u)}-\frac{uv}{t_r(uv)}\right)\left(\frac{u}{t_r(u)}-\frac{uv}{t_r(uv)}\right)\\ &= \left(\frac{u}{t_r(u)}\right)^2-\frac{uvu}{t_r(uv)t_r(u)}-\frac{u^2v}{t_r(u)t_r(uv)}+\left(\frac{uv}{t_r(uv)}\right)^2\\ &= \frac{u}{t_r(u)}-\frac{u}{t_r(u)}-\frac{uv}{t_r(uv)}+\frac{uv}{t_r(uv)}\\ &= 0.\end{aligned}$$

Thus $\frac{u}{t_r(u)}-\frac{uv}{t_r(uv)}$ is nilpotent, from which the Spectral Mapping Theorem implies that

$$\sigma\left(\left(\frac{u}{t_r(u)}-\frac{uv}{t_r(uv)}\right)^2\right)=\{0\}$$

and hence

$$r_\sigma\left(\left(\frac{u}{t_r(u)}-\frac{uv}{t_r(uv)}\right)\right)=0<1.$$

So, by ([17], Lemma 3.1), there exists $a \in \exp(A)$ such that $\frac{uv}{t_r(uv)}=a\left(\frac{u}{t_r(u)}\right)a^{-1}$. Thus $uv=\frac{t_r(uv)}{t_r(u)}aua^{-1}=\alpha(e^x ue^{-x})$ for some $x \in A$ (where $a=e^x$) and $\alpha=\frac{t_r(uv)}{t_r(u)}$.

(2) If $0 \neq u \in F_1^0$ then $u^2=0$. Since A is semiprime (and $u \in F_1$), there exists $x \in A$ such that $uxu=u$. Choose $\lambda \in \mathbb{C}$ such that $|\lambda|>r_\sigma(x)$. Then $\lambda - x \in A^{-1}$ and, moreover, $\sigma(x)$ does not separate λ from infinity. By([1], Theorem 3.3.6) it follows that $\lambda - x$, and hence also $-\lambda + x$, belong to $\exp(A)$. Calculating

$$\begin{aligned}\big(u(-\lambda+x)\big)^2 &= u(-\lambda+x)u(-\lambda+x)\\ &= u(-\lambda+x)\neq 0 \ \text{ since} -\lambda+x \in \exp A \subseteq A^{-1} \text{ and } \ u\neq 0.\end{aligned}$$

Hence $u(-\lambda+x) \in F_1^1$ which means that (2) follows with $v=-\lambda+x$. ■

With respect to the above lemma, note that for $u \in F_1$ the set

$$\begin{aligned}N(u) &= \{x \in A : ux \in F_1^0\}\\ &= \{x \in A : (ux)^2=0\}\\ &= \{x \in A : (uxu)x=0\}\\ &= \{x \in A : uxu=0\} \ \text{ since } uxu=\lambda u\\ &= \{x \in A : x(uxu)=0\}\\ &= \{x \in A : xu \in F_1^0\}.\end{aligned}$$

Taking α and β as scalars and $x, y \in N(u)$ and considering $u(\alpha x + \beta y)u$ we have: $u(\alpha x + \beta y)u = u\alpha x u + u\beta y u = \alpha u x u + \beta u y u = 0$ which means that $N(u)$ is a vector subspace of A. If $N(u)$ was equal to A then we would have, from the definition of semiprime, that $u = 0$ which is a contradiction. Hence $N(u)$ is a proper vector subspace of A. Moreover, $N(u)$ is closed in A; let $x_n \in N(u)$ such that $x_n \longrightarrow x \in A$. We then have $ux_nu = 0$ and by the continuity of multiplication, $ux_nu \longrightarrow uxu$ which implies that $uxu = 0$ and hence that $x \in N(u)$. Thus, the complement of $N(u)$, namely, $E(u) = \{x \in A : ux \in F_1^1\} = \{x \in A : xu \in F_1^1\}$ is an open and dense subset of A (since $N(u)$ is a proper vector subspace of A).

Corollary 2.2.2. *Let A be a semiprime Banach algebra and let $L = Ap$ be a minimal left ideal of A. Then every element of L can be written as a linear combination of minimal idempotents belonging to the same connected component of $A^\bullet$. In particular,*

$$xp = \alpha p - \lambda e^z p e^{-z}$$

for some $\alpha, \lambda \in \mathbb{C}$ and $z \in A$ where either $\alpha = \lambda$ (when xp is nilpotent) or at least one of α or λ equals zero (when xp is not nilpotent). Similarly, this result holds for minimal right ideals.

Proof : We first note that p is a minimal idempotent (by Theorem 1.5.5.) and hence $p \in F_1^1$. If xp is not nilpotent then, by Lemma 2.2.1(1),

$$xp = \lambda e^z p e^{-z}.$$

If xp is nilpotent then, following Lemma 2.2.1(2), we may find $\lambda \in \mathbb{C}$ such that $(\lambda - x)p$ is not nilpotent. Again by Lemma 2.2.1(1) there is $\alpha \in \mathbb{C}$ such that $(\lambda - x)p = \alpha e^z p e^{-z}$ i.e. $\lambda p - xp = \alpha e^z p e^{-z}$ and thus $xp = \lambda p - \alpha e^z p e^{-z}$. But

$$\begin{aligned}\sigma'\big((\lambda - x)p\big) &= \sigma'\big((\lambda - x)p^2\big) \\ &= \sigma'\big((p(\lambda - x)p\big) \text{ since in general } \sigma'(ab) = \sigma'(ba) \\ &= \sigma'(\lambda p) \text{ since } pxp = 0 \\ &= \{\lambda\} \text{ since } p \text{ is an idempotent .}\end{aligned}$$

Since $(\lambda - x)p = \alpha e^z p e^{-z}$, we have

$$\begin{aligned}\sigma\big((\lambda - x)p\big) &= \sigma\big(\alpha e^z p e^{-z}\big) \\ &= \alpha\sigma(p_1) \text{ where } p_1 = e^z p e^{-z} \text{ idempotent} \\ &= \{0, \alpha\}.\end{aligned}$$

But we have just shown that $\sigma(\lambda p - xp) = \{0, \lambda\}$ and hence $\alpha = \lambda$. So, it follows that xp has the form $xp = \lambda p - \lambda e^z p e^{-z}$. ■

We now state Lemma 2.2.3. which will be used in the proof of Theorem 2.2.4.

Lemma 2.2.3. *An element u of Banach algebra A is in the center of A if and only if u commutes with every element of* $\exp(A)$.

Proof : If $u \in Z(A)$ i.e. $xu = ux \ \forall \ x \in A$, then obviously u commutes with every element of $\exp(A)$. On the other hand, if u commutes with every element of $\exp(A)$, then let $x \in A$ be arbitrary. We can choose $\lambda > r_\sigma(x)$ such that $\lambda - x \in \exp(A)$. Thus $u(\lambda - x) = (\lambda - x)u$ so that $\lambda u - ux = \lambda u - xu$ and hence $xu = ux$. ■

Theorem 2.2.4. ([15], Theorem 2.2.) *Let A be a semiprime Banach algebra with $F_1 \neq \emptyset$.*

(1) If $u \in F_1$ then the connected component of F_1 containing u is the set

$$K_u^{F_1} = \mathrm{Exp}(A) u \mathrm{Exp}(A).$$

(2) If $u \in F_1$ then u belongs to the center of A if and only if

$$K_u^{F_1} = \mathbb{C}u \backslash \{0\}.$$

Proof : (1) If $a, b \in \mathrm{Exp}(A)$ with $a = \prod_{i=1}^{k} e^{x_i}$ and $b = \prod_{i=1}^{n} e^{y_i}$ with $n \geq k$ then $g(t) = \prod_{i=1}^{n} e^{tx_i + (1-t)y_i}, x_i = 0$ for $i > k; t \in [0, 1]$ satisfies:

(a) $g(0) = \prod_{i=1}^{n} e^{y_i} = b$

(b) $g(1) = \prod_{i=1}^{n} e^{x_i} = \prod_{i=1}^{k} e^{x_i} = a$ since $x_i = 0$ when $i > k$

(c) $g(t) = \prod_{i=1}^{n} e^{tx_i + (1-t)y_i} \in \mathrm{Exp}(A) \ \forall \ t \in (0, 1)$

(d) If $t_k \longrightarrow t$ then

$$\begin{aligned} g(t_k) &= \prod_{i=1}^{n} e^{t_k x_i + (1-t_k)y_i} \\ &= \prod_{i=1}^{n} [\sum_{j=1}^{\infty} \frac{1}{j!} \big(t_k x_i + (1 - t_k) y_i\big)^j] \\ &\longrightarrow \prod_{i=1}^{n} [\sum_{j=1}^{\infty} \frac{1}{j!} \big(t x_i + (1 - t) y_i\big)^j] \quad \text{by the continuity of the operations} \\ &= g(t) \text{ which implies the continuity of } g(t). \end{aligned}$$

Since (a) to (d) holds, $g(t)$ is a path in $\operatorname{Exp}(A)$ connecting a and b. For $c, d \in \operatorname{Exp}(A)$ let $h(t),\ t \in [0,1]$ be a path connecting c and d. If we define $f(t) := g(t)uh(t),\ t \in [0,1]$ then $f(0) = g(0)uh(0) = bud$ and $f(1) = g(1)uh(1) = auc$. Also f is continuous since both g and h are continuous and finally $f(t) = g(t)uh(t)$ is in $\operatorname{Exp}(A)u\operatorname{Exp}(A)\ \forall\ t \in (0,1)$. Hence $f(t) = g(t)uh(t),\ t \in [0,1]$ is a path in $K_u^{F_1}$ $=\operatorname{Exp}(A)u\operatorname{Exp}(A)$, connecting bud and auc. Thus $K_u^{F_1}$ is arcwise connected. We show $K_u^{F_1}$ is maximal connected in F_1. Let $u \in F_1$ and $(r_n u s_n)$ a sequence in $K_u^{F_1}$ such that $r_n u s_n \longrightarrow v \in F_1$. If $v \in F_1^1$ then $t_r(v) \neq 0$ since we chose $v \in F_1^1$. Now, $r_n u s_n \longrightarrow v$ implies that $(r_n u s_n)^2 \longrightarrow v^2$. This gives $(r_n u s_n)^2 \neq 0$ for n sufficiently large. Hence $r_n u s_n \subseteq F_1^1$. Now, $r_n u s_n \longrightarrow v$, that is $\lim\limits_{n\to\infty} \|r_n u s_n - v\| = 0$ and so $\lim\limits_{n\to\infty} \big\|\frac{r_n u s_n}{t_r(r_n u s_n)} - \frac{v}{t_r(v)}\big\| = 0$. So, for n large enough, $\big\|\frac{r_n u s_n}{t_r(r_n u s_n)} - \frac{v}{t_r(v)}\big\| < 1$. But,

$$r_\sigma(r_n u s_n - v) \leq \Big\|\frac{r_n u s_n}{t_r(r_n u s_n)} - \frac{v}{t_r(v)}\Big\|$$

which means that $r_\sigma(r_n u s_n - v) < 1$. Since $\frac{v}{t_r(v)}$ and $\frac{r_n u s_n}{t_r(r_n u s_n)}$ are idempotents and $r_n u s_n \in F_1^1$, it follows from Lemma 3.1 ([17]) that $\frac{v}{t_r(v)} = z^{-1}\frac{r_n u s_n}{t_r(r_n u s_n)}z$ (for n large enough) where $z \in \exp(A)$. Thus $v \in \operatorname{Exp}(A)u\operatorname{Exp}(A)$ since $t_r(v),\ t_r(r_n u s_n),\ r_n$ and s_n belong to $\operatorname{Exp}(A)$. Hence $v \in K_u^{F_1}$. If $v \in F_1^0$ then by lemma 2.2.1(2) there exists an $a \in \exp(A)$ such that $va \in F_1^1$. Since $r_n u s_n a \longrightarrow va$ (by the continuity of multiplication), we have (using the same method as above, since va is the limit of $r_n u s_n a$, and $va \in F_1^1$) $va \in K_u^{F_1}$. But $a \in \exp(A)$ and thus $v \in K_u^{F_1}$. This proves $K_u^{F_1}$ is closed in F_1. On the other hand, if (v_n) is a sequence in $F_1 \backslash K_u^{F_1}$ and $v_n \longrightarrow rus \in K_u^{F_1}$ then $r^{-1}v_n s^{-1} \longrightarrow u$. As before, considering the two cases $u \in F_1^0$ and $u \in F_1^1$, $r_\sigma\big(-\frac{r^{-1}v_n s^{-1}}{t_r(r^{-1}v_n s^{-1})} + \frac{u}{t_r(u)}\big) < 1$. Thus $-\frac{r^{-1}v_n s^{-1}}{t_r(r^{-1}v_n s^{-1})} = z\frac{u}{t_r(u)}z^{-1}$ where $z \in \exp(A)$. This gives $v_n = \frac{t_r(r^{-1}v_n s^{-1})}{t_r(u)} rzuz^{-1}s \in K_u^{F_1}$ for n large enough which contradicts $v_n \in F_1 \backslash K_u^{F_1}$. These arguments prove that $K_u^{F_1}$ is both open and closed in F_1 and thus maximal connected in F_1.

(2) Suppose $u \in F_1$ and $u \in Z(A)$. Then, from the first part of this theorem,

$$\begin{aligned} u \in K_u^{F_1} &= \operatorname{Exp}(A)\, u \operatorname{Exp}(A) \\ &= u \operatorname{Exp}(A)\operatorname{Exp}(A) \quad \text{since } u \in Z(A) \\ &= u \operatorname{Exp}(A). \end{aligned}$$

Also, since A is semiprime and $u \in Z(A)$, u cannot be nilpotent (see the discussion following Corollary 2.1.3.). Thus $u^2 \neq 0$ and $u \in F_1^1$. Suppose for $v \in \operatorname{Exp}(A)$ we have

$(uv)^2 = 0$. Then (since $v \in \text{Exp}(A)$ and therefore invertible) we would have that $u^2 = 0$, which would be a contradiction. Hence for uv as above i.e. for $uv \in u\,\text{Exp}(A)$ we have $uv \in F_1^1$. In view of Lemma 2.2.1(1), $uv = \alpha w^{-1}uw,\ w \in \ \text{Exp}(A)$. So,

$$\begin{aligned} uv &= \alpha w^{-1}uw \quad \text{where } w \in \text{Exp}(A) \text{ and } 0 \neq \alpha \in \mathbb{C} \\ &= \alpha u w^{-1} w \quad \text{since } u \in Z(A) \\ &= \alpha u. \end{aligned}$$

Therefore, $K_u^{F_1} = u\,\text{Exp}(A) = \mathbb{C}u\backslash\{0\}$.
Suppose on the other hand, that $K_u^{F_1}$ reduces to $\mathbb{C}u\backslash\{0\}$. If $u \in F_1^0$ then by Lemma 2.2.1(2) there exists $v \in \exp(A)$ such that $uv \in F_1^1$. By Lemma 2.2.1(1), $uv = \alpha e^x u e^{-x}$ where $0 \neq \alpha \in \mathbb{C},\ x \in A$. If $u \in F_1^0$ then choose $v \in \text{Exp}(A)$ such that $uv \in F_1^1$. So $uv \in K_u^{F_1}$. By assumption $uv = \alpha u$ with $\alpha \neq 0$. Squaring both sides we have $(uv)^2 = \alpha^2 u^2 = 0$ (since $u \in F_1^0$) which contradicts $uv \in F_1^1$. Hence $u \in F_1^1$. For $w \in \text{Exp}(A)$ we have that $\frac{w^{-1}uw}{t_r(u)} \in \text{Exp}(A)\,u\,\text{Exp}(A) = K_u^{F_1}$. It follows that for each $w \in \text{Exp}(A)$ there is $0 \neq \alpha \in \mathbb{C}$ such that $\frac{w^{-1}uw}{t_r(u)} = \alpha u$. But we also know that

$$\left(\frac{w^{-1}uw}{t_r(u)}\right)^2 = \frac{w^{-1}uw}{t_r(u)}\ .$$

Hence $\frac{w^{-1}uw}{t_r(u)}$ is an idempotent. Using this, coupled with the fact that $\frac{w^{-1}uw}{t_r(u)} = \alpha u$, we must have that $(\alpha u)^2 = \alpha u$. So, $u^2 = \frac{1}{\alpha}u$. But $u1u = t_r(u)u$ so that $\alpha = \frac{1}{t_r(u)}$. This gives $\frac{w^{-1}uw}{t_r(u)} = \frac{1}{t_r(u)}u$ and thus $w^{-1}uw = u$. It follows that $uw = wu$ for all $w \in \text{Exp}(A)$, and hence, by Lemma 2.2.3, $u \in Z(A)$. ■

Lemma 2.2.5. *Let A be a semiprime Banach algebra. If $L = Au$ with $u \in F_1$ then $Au = Ap$ with $p \in F_1$ an idempotent.*

Proof : Suppose $L = Au$ with $u \in F_1$. Then for some $x \in A$ we have $uxu = \lambda u,\ \lambda \neq 0$. So that $\frac{xu}{\lambda}$ is an idempotent. We now show that $Au = A\frac{xu}{\lambda}$. Since $\frac{xu}{\lambda} \in Au$, we have $A\frac{xu}{\lambda} \subseteq Au$. So what's left to show is that $Au \subseteq A\frac{xu}{\lambda}$. Let $y \in A$ be arbitrary, then $yu = y(x^{-1}\lambda)\frac{x}{\lambda}u$, provided $x \in A^{-1}$. It thus suffices to find an invertible x such that $\frac{xu}{\lambda} \in A^\bullet$. Suppose for all invertible $x \in A$, we have $uxu = 0$. Since $A = \exp(A) + \exp(A)$ we have for $x \in A$ arbitrary, that $x = x_1 + x_2$, where x_1 and x_2 are elements of $\exp(A)$. Therefore since $\exp(A) \subseteq A$ and all the elements of $\exp(A)$ are invertible in A, we have $uxu = u(x_1 + x_2)u = ux_1u + ux_2u = 0$. But uxu is not zero for all $x \in A$, since if this was the case then u would be zero which would contradict $u \in F_1$. Therefore, there is at least one invertible $x \in A$ such that $uxu \neq 0$. Hence, there exists $x \in A^{-1}$ such that $uxu = \lambda u$, $\lambda \neq 0$ and thus (as above) $\frac{xu}{\lambda} \in A^{\bullet}$. ■

Corollary 2.2.6. ([15], Corollary 2.3.) *Let A be a semiprime Banach algebra with $F_1 \neq \emptyset$. Every minimal left ideal J of A has the form*

$$J = \mathrm{Exp}(A)u \cup \{0\} \text{ with } u \in F_1.$$

Similarly, every minimal right ideal has the form

$$J = u\mathrm{Exp}(A) \cup \{0\} \text{ with } u \in F_1.$$

Proof : By Lemma 2.2.5., we have that if $J = Au$ with $u \in F_1$ then $Au = Ap$ where $p \in F_1$ is an idempotent. Therefore, by Theorem 1.5.5., J is minimal if and only if $J = Au$ with $u \in F_1$. So, to prove the theorem we need to prove that $Au = \mathrm{Exp}(A)u \cup \{0\}$, $u \in F_1$ where u is also an idempotent. We show that $\mathrm{Exp}(A)u\mathrm{Exp}(A) = AuA - \{0\}$. Let $aub \neq 0$. Then, since the spectra of au and ub consist of at most two points, one of which is $0 \in \mathbb{C}$, we can find a sequence $\lambda_n \to 0$ such that both $\lambda_n + au$ and $\lambda_n + ub$ belong to $\exp(A)$ for each n. Thus $(\lambda_n + au)u(\lambda_n + ub)$ is a sequence in $\mathrm{Exp}(A)u\mathrm{Exp}(A)$ converging to aub. Since $\mathrm{Exp}(A)u\mathrm{Exp}(A)$ is the connected component of F_1 containing the element u, and $aub \in F_1$, we must have that aub belongs to $\mathrm{Exp}(A)u\mathrm{Exp}(A)$ and our result follows. Of course this implies that $\mathrm{Exp}(A)u\mathrm{Exp}(A) \cup \{0\} = AuA$ and hence $\mathrm{Exp}(A)u\mathrm{Exp}(A)u \cup \{0\} = AuAu$. Now, $AuAu = A\mathbb{C}u$ since $uAu = \mathbb{C}u$ which means $AuAu = \mathbb{C}Au = Au$. We thus get

$$\begin{aligned} AuAu &= \{0\} \cup \mathrm{Exp}(A)u\mathrm{Exp}(A)u \\ &= \{0\} \cup \mathrm{Exp}(A)\mathbb{C}u \quad \text{using } u\,\mathrm{Exp}(A)\,u = \mathbb{C}u \\ &= \mathbb{C}\mathrm{Exp}(A)u \cup \{0\} \\ &= \mathrm{Exp}(A)u \cup \{0\}. \end{aligned}$$

Therefore, since $AuAu = Au$, we obtain $J = Au = \mathrm{Exp}(A)u \cup \{0\}$. ■

It is shown in ([8], Theorem 4) that if A is semisimple and $u \in A$ is single and compactly acting, then u belongs to F_1. As promised, we show that the converse also holds, and that these two rank 1 definitions are equivalent under the weaker premise of semiprimeness. For this, one must go back to the original proof in the semisimple case: in ([8], Theorem 4) semisimplicity is only used in the argument in the first line of

the proof and in the sense that, if A is semisimple then the left ideal Au contains an element which is not quasinilpotent. So, for the analogous result in the semiprime case, it suffices to prove that a single, compactly acting u does not belong to $\mathrm{Rad}(A)$. Once this is established the proof of Theorem 4 in ([8]) applies as is.
Before we state the next corollary, recall the following:

(1) Bolzano-Weierstrass Theorem: Every bounded sequence of complex numbers has a convergent subsequence.

(2) An operator $T : X \longrightarrow Y$ is compact if and only if the image of a bounded sequence (x_n) in X has a convergent subsequence in Y.

(3) If $T : X \longrightarrow X$ is compact and T has closed range then $T(X)$ is finite-dimensional.

Corollary 2.2.7. ([15], Corollary 2.4.) *Let A be a semiprime Banach algebra with $F_1 \neq \emptyset$. Then $u \in F_1$ if and only if u is a single element of A and u acts compactly on A with closed range.*

Proof : Let $u \in F_1$. We show that u is a single compactly acting element of A. We need to prove that if (a_n) is a bounded sequence in A then ua_nu has a convergent subsequence. But $ua_nu = \lambda_n u$ since $u \in F_1$. Thus

$$|\lambda_n| \cdot \|u\| = \|ua_nu\| \leq \|u\| \cdot \|a_n\| \cdot \|u\| \leq M$$

for some M, since (a_n) is bounded. Therefore, (λ_n) is bounded and has a convergent subsequence (Bolzano-Weierstrass Theorem), say $\lambda_{n_k} \longrightarrow \lambda$. Then $\lambda_{n_k} u \longrightarrow \lambda u$. So, since $ua_{n_k}u = \lambda_{n_k} u$, we have $ua_{n_k}u \longrightarrow \lambda u$ and hence a convergent subsequence of ua_nu. This gives $T : A \longrightarrow \mathbb{C}u$ where $Ta = uau$ is compact and thus u acts compactly on A.
We now show that $u \in F_1$ is single, that is, if $aub = 0$ for some $a, b \in A$ and $au \neq 0$ then by Corollary 2.2.6., $au = vu$ for some $v \in \mathrm{Exp}(A)$. But $0 = aub = vub$, therefore $vub = 0$. This gives $ub = 0$ since $v \in \mathrm{Exp}(A)$ is invertible. Hence u is single.
For the converse, suppose $0 \neq u \in \mathrm{Rad}(A)$ is single and compactly acting. We consider two cases.

• $u^2 \neq 0$:

Let B be the Banach algebra generated by $\{1, u^2\}$. By assumption Bu^2 is closed in B. So, since the operator $T : B \longrightarrow Bu^2$ given by $Tz = uzu = zu^2$ (B commutative) is compact with closed range (since Bu^2 is closed and u compactly acting), it follows that Bu^2 is finite-dimensional (by result (3) from the statements preceding Corollary 2.2.7.). Now, suppose $(zu^2)(yu^2) = 0$ where $z, y \in B$. Then, since u is single, we have $(zu)u(yu^2) = 0 \Longrightarrow zu^2 = 0$ or $uyu^2 = 0$. Therefore, since u is single, we have $uyu = 0$ or $u^2 = 0$. Since, by assumption, $u^2 \neq 0$, we get $uyu = yu^2 = 0$ (since B is commutative). Thus $(zu^2)(yu^2) = 0 \Longrightarrow zu^2 = 0$ or $yu^2 = 0$. We have now shown that Bu^2 is finite dimensional with no zero divisors. It follows from ([8], Lemma 2) that Bu^2 is one-dimensional. Using the fact that u and hence also u^2 are single, we get $u^4 \neq 0$; since if $u^4 = 0$ then $u^2(u)u = 0 \Longrightarrow u^2u = 0$ or $u^2 = 0$. Then (again since u is single) $u(u)u = 0 \Longrightarrow u^2 = 0$ which contradicts our hypothesis. Using $\dim(Bu^2) = 1$, we can take $\{u^2\}$ as the basis for Bu^2 so that $u^4 = \alpha u^2$ for some $\alpha \neq 0$. Therefore $u^2(u^2 - \alpha) = 0$ for some $\alpha \neq 0$. If $u^2 - \alpha$ is invertible then $u^2 = 0$ which is a contradiction. Hence $u^2 - \alpha \notin B^{-1}$, that is $\alpha \in \sigma(u^2)$. But since $u^2 \in \mathrm{Rad}(A)$, we have $\sigma(u^2) = \{0\}$ which gives a contradiction. Hence a single compactly acting element, u, with $u^2 \neq 0$, cannot be in $\mathrm{Rad}(A)$.

- $\underline{u^2 = 0:}$

For A is semiprime we have that if $uxu = 0\ \forall\ x \in A$ then $u = 0$. But since u is single, we have $u \neq 0$ and therefore $uxu \neq 0$ for some $x \in A$. Choose $\lambda \in \mathbb{C}$ sufficiently large so that $\lambda + x \in A^{-1}$ i.e. choose λ such that $|\lambda| > r_\sigma(x)$. Then

$$\begin{aligned}
\big(u(\lambda + x)\big)^2 &= u(\lambda + x)u(\lambda + x) \\
&= \lambda^2u^2 + \lambda uxu + \lambda u^2x + uxux \\
&= 0 + \lambda uxu + 0 + (uxu)x \\
&= (uxu)(\lambda + x) \\
&\neq 0 \ \text{ since } (\lambda + x) \in A^{-1}.
\end{aligned}$$

Also note that $u(\lambda + x)$ belongs to $\mathrm{Rad}(A)$ since $\mathrm{Rad}(A)$ is an ideal and $u \in \mathrm{Rad}(A)$ by hypothesis. Using the facts that:

(1) $u(\lambda + x) \in \mathrm{Rad}(A)$ is single and compactly acting

(2) $\big(u(\lambda + x)\big)^2 \neq 0$

it follows by consideration of the first part of the proof that $u(\lambda+x)$ cannot be in $\mathrm{Rad}(A)$. Hence $u \notin \mathrm{Rad}(A)$. This means radical elements can never be single and compactly acting in a semiprime Banach algebra. The conclusion follows from the remarks preceding Lemma 2.2.7. ■

Let A be a semiprime Banach algebra and let $p \neq 0$ be a central minimal idempotent belonging to A. By Theorem 2.2.4(2) $\text{Exp}(A)p\,\text{Exp}(A) = \mathbb{C}p\backslash\{0\}$. But since p is central, we have $\text{Exp}(A)\text{Exp}(A)p = \mathbb{C}p\backslash\{0\}$ and therefore $\text{Exp}(A)p = \mathbb{C}p\backslash\{0\}$. If we define $\Pi : \text{Exp}(A) \longrightarrow \mathbb{C}p\backslash\{0\}$ by $\Pi(a) = \lambda$ where $ap = \lambda p$, then Π is a surjective group homomorphism from $\text{Exp}(A)$ onto $\mathbb{C}\backslash\{0\}$; suppose $\Pi(ab) = \gamma$ where $(ab)p = \gamma p$, $\Pi(a) = \alpha$ where $ap = \alpha p$ and $\Pi(b) = \beta$ where $bp = \beta p$. Then $apbp = \alpha p\beta p = \alpha\beta p^2 = \alpha\beta p$. But $apbp = abp^2 = abp$, since p is central and therefore (since $(ab)p = \gamma p$) we have $\alpha\beta p = \gamma p$, implying that $\alpha\beta = \gamma$. Hence $\Pi(a)\Pi(b) = \Pi(ab)$. Also $\Pi(1) = \lambda$ where $1p = \lambda p$. But this holds if and only if $\lambda = 1$, hence $\Pi(1) = 1$ and Π is a homomorphism. Π is also surjective since $\lambda 1 \in \text{Exp}(A)$, so that $\lambda 1 \in \text{Exp}(A)$ maps to $\lambda \in \mathbb{C}\backslash\{0\}$. By considering a suitable subgroup the group $\text{Exp}(A)$ can be factorized in such a manner that the resultant quotient group is isomorphically identifiable with the multiplicative group $\mathbb{C}' = \mathbb{C}\backslash\{0\}$. Recall that a subgroup H of G is called normal if $aH = Ha$. That is, H is normal if for every $x \in H$ there is a $y \in H$ such that $ax = ya$.

If we define

$$H_p = \{a \in \text{Exp}(A) : \ ap = p\}$$

we see that H_p is a group and we claim that H_p is a normal subgroup of $\text{Exp}(A)$;
let $b \in \text{Exp}(A)$ arbitrary. We want to show $bH_p = H_pb$. Therefore if $a \in H_p$ then to prove is that there is $c \in H_p$ such that $ba = cb$. This is true if $bab^{-1} \in H_p$. Now,

$$\begin{aligned}(bab^{-1})p &= b(ap)b^{-1} \ \text{ since } p \text{ central}\\ &= bpb^{-1} \ \text{ since } a \in H_p\\ &= bb^{-1}p \ \text{ since } p \text{ central}\\ &= p.\end{aligned}$$

Therefore $bab^{-1} \in H_p$ by the definition of H_p and hence H_p is a normal subgroup of $\text{Exp}(A)$. The mapping

$$\phi : \text{Exp}(A)/H_p \longrightarrow \mathbb{C}'$$

defined by $\phi(yH_p) = t_r(yp)$ is a homeomorphic isomorphism from $\text{Exp}(A)/H_p$ onto $\mathbb{C}'$ with the topology on $\text{Exp}(A)/H_p$ being the usual quotient space topology. We first show that ϕ is an isomorphism. To prove is:

(1) $\phi\big((y_1H_p)(y_2H_p)\big) = \phi(y_1H_p)\phi(y_2H_p)$

(2) $\phi(H_p) = 1$

(3) ϕ is bijective

- $\underline{(1)\ \phi\big((y_1H_p)(y_2H_p)\big) = \phi(y_1H_p)\phi(y_2H_p):}$

$$\begin{aligned}\phi\big((y_1H_p)(y_2H_p)\big) &= \phi(y_1y_2H_p)\\ &= t_r(y_1y_2p)\\ &= t_r(py_1ppy_2p) \text{ since } p \text{ is central}\\ &= t_r(\lambda_1p\lambda_2p) \text{ since } p \text{ is minimal}\\ &= \lambda_1\lambda_2t_r(p)\\ &= \lambda_1\lambda_2 \text{ since } t_r(p) = 1.\end{aligned}$$

$$\begin{aligned}\phi(y_1H_p)\phi(y_2H_p) &= t_r(y_1p)t_r(y_2p)\\ &= t_r(py_1p)t_r(py_2p) \text{ since } p \text{ is central and } p = p^2\\ &= t_r(\lambda_1p)t_r(\lambda_2p) \text{ since } p \text{ is minimal}\\ &= \lambda_1\lambda_2t_r(p)t_r(p)\\ &= \lambda_1\lambda_2 \text{ since } t_r(p) = 1.\end{aligned}$$

- $\underline{(2)\ \phi(H_p) = 1:}$

This is true since $\phi(H_p) = t_r(p) = 1$.

- $\underline{(3)\ \phi \text{ is bijective}:}$

To prove ϕ is injective, we need to show $\phi(y_1H_p) = \phi(y_2H_p) \Longrightarrow y_1H_p = y_2H_p$. Suppose $\phi(y_1H_p) = \phi(y_2H_p)$. Then

$$\begin{aligned}&t_r(y_1p) = t_r(y_2p)\\ &\therefore t_r(py_1p) = t_r(py_2p) \text{ since } p \text{ is central}\\ &\therefore t_r(\lambda_1p) = t_r(\lambda_2p) \text{ since } p \text{ is minimal}\\ &\therefore \lambda_1t_r(p) = \lambda_2t_r(p)\\ &\Longrightarrow \lambda_1 = \lambda_2 \text{ since } t_r(p) = 1.\end{aligned}$$

Now, $y_1H_p = \{y_1a : a \in H_p\}$. Let $y_1a \in y_1H_p,\ a \in H_p$. We need to show $y_1a = y_2b$ for some $b \in H_p$, that is, we need $y_2^{-1}y_1 \in H_p$. Now,

$$\begin{aligned}
y_2^{-1}y_1ap &= py_2^{-1}ppy_1pap \text{ since } p \text{ is central and } p^2 = p \\
&= \frac{1}{\lambda_2}\lambda_1 ap \text{ since } p \text{ is minimal} \\
&= \frac{1}{\lambda_1}\lambda_1 ap \text{ since } \lambda_1 = \lambda_2 \\
&= ap \\
&= p \text{ since } a \in H_p.
\end{aligned}$$

Therefore $y_2^{-1}y_1ap \in H_p$ and thus $y_1H_p = y_2H_p$ so that ϕ is injective. Also for every $\lambda \in \mathbb{C}'$, we have $\lambda 1 \in \text{Exp}(A)$. This gives

$$\begin{aligned}
\phi(\lambda 1 H_p) &= t_r(\lambda 1 p) \\
&= \lambda t_r(p) \\
&= \lambda \text{ since the trace of an idempotent is 1.}
\end{aligned}$$

This means that ϕ is surjective. Hence ϕ is bijective.

To show that ϕ is homeomorphic, we need to show that $\phi : \text{Exp}(A)/H_p \longrightarrow \mathbb{C}'$ and $\phi^{-1} : \mathbb{C}' \longrightarrow \text{Exp}(A)/H_p$ are continuous. In this order we define the following mappings

(a) $T : A \longrightarrow \mathbb{C}$ by $Tx = t_r(xp)$

(b) $q : \text{Exp}(A) \longrightarrow \text{Exp}(A)/H_p$ by $q(y) = yH_p$

(c) $f : \text{Exp}(A) \longrightarrow \mathbb{C}'$ by $f(y) = t_r(yp)$ and

(d) $\phi : \text{Exp}(A)/H_p \longrightarrow \mathbb{C}'$ by $\phi(yH_p) = t_r(yp)$.

It is obvious that T is linear, bounded and onto. Hence T is an open mapping, that is T maps open sets in A onto open sets in $\mathbb{C}$. But, $\text{Exp}(A)$ is open in A and therefore every open set in $\text{Exp}(A)$ is also open in A. Hence $f : \text{Exp}(A) \longrightarrow \mathbb{C}'$ is an open mapping. Recall that the topology on $\text{Exp}(A)/H_p$ works as follows: a set V in $\text{Exp}(A)/H_p$ is open if $V = q(U)$ where U is open in $\text{Exp}(A)$ and $H_p \subseteq U$ or $U \subseteq \text{Exp}(A) - H_p$. We now proceed to show that ϕ is continuous. Let $B = B(\alpha, \epsilon)$ be an open ball in $\mathbb{C}'$. We consider two cases, the first where B does not contain the element $1 \in \mathbb{C}$ and the second where B contains $1 \in \mathbb{C}$.

- <u>B does not contain the element $1 \in \mathbb{C}$</u>

In this case

$$\begin{aligned}
f^{-1}(B) &= \{y \in \text{Exp}(A) : f(y) \in B\} \\
&= \{y \in \text{Exp}(A) : t_r(yp) \in B\}.
\end{aligned}$$

Since f is continuous, we have that $f^{-1}(B)$ is open in $\text{Exp}(A)$. If $f^{-1}(B) \cap H_p \neq \emptyset$ then there is $y \in H_p$ such that $t_r(yp) \in B$, that is $1 \in B$ (because $t_r(yp) = t_r(p) = 1$) which is a contradiction. So, $B \subseteq \text{Exp}(A) - H_p$ and hence $q\big(f^{-1}(B)\big)$ is open in $\text{Exp}(A)/H_p$. But $q\big(f^{-1}(B)\big) = \phi^{-1}(B)$ and hence $\phi^{-1}(B)$ is open in $\text{Exp}(A)/H_p$.

• <u>B contains $1 \in \mathbb{C}$</u>

Here,

$$f^{-1}(B) = \{y \in \text{Exp}(A) : t_r(yp) \in B\}$$

implies that $H_p \subseteq f^{-1}(B)$. So, this says $f^{-1}(B)$ is open in $\text{Exp}(A)$ and $q\big(f^{-1}(B)\big)$ is open in $\text{Exp}(A)/H_p$ and hence also $\phi^{-1}(B)$ is open in $\text{Exp}(A)/H_p$.

We thus have that $\phi^{-1}(B)$ is open in $\text{Exp}(A)/H_p$ in both of the above cases. We now proceed to prove that ϕ^{-1} is continuous. To do this, we take any open set $V \subseteq \text{Exp}(A)/H_p$ and show that $\big(\phi^{-1}\big)^{-1}\big(V\big) = \phi\big(V\big)$ is open in $\mathbb{C}'$. But $\phi(V) = f\big(q^{-1}(V)\big)$ and q is continuous, so $q^{-1}(V)$ is open in $\text{Exp}(A)$. We also know that f is an open mapping and therefore, we have that $f\big(q^{-1}(V)\big)$ is open in $\mathbb{C}'$. Hence ϕ is homeomorphic isomorphism from $\text{Exp}(A)/H_p$ onto $\mathbb{C}'$. We now show that each coset of the quotient space is a convex subset of $\text{Exp}(A)$; let y_1H_p be any coset. We want to show y_1H_p is convex i.e. for any two points in y_1H_p, the line segment joining the two points also belongs to y_1H_p. Let y_1a and y_1b be any two elements of y_1H_p. Define $f(t) = ty_1a + (1-t)y_1b, \quad t \in [0,1]$. Then $f(0) = y_1b$ and $f(1) = y_1a$ and f is continuous, so that f is a path joining y_1a and y_1b. We now need to show that $f(t)$ is in y_1H_p. But $f(t) = ty_1a+(1-t)y_1b = y_1\big(ta+(1-t)b\big)$ therefore, what's left to prove is that $ta + (1-t)b \in H_p$. Now,

$$\begin{aligned}\big(ta + (1-t)b\big)p &= tap + p - tbp \\ &= tp + p - tp \ \text{ since } a, b \in H_p \\ &= p.\end{aligned}$$

Therefore $\big(ta + (1-t)b\big) \in H_p$ and thus $f(t) \in y_1H_p \ \forall \ t \in [0,1]$. Since y_1H_p was arbitrary, each coset of the quotient space is a convex subset of $\text{Exp}(A)$. We now proceed to show that no such factorization of $\text{Exp}(A)$ is possible when p is not central (hence proving the converse). In particular, we will show that if $p \in A^\bullet$ is minimal and H is any normal subgroup of $\text{Exp}(A)$ such that the formula $\phi(yH) = t_r(yp)$ defines a group homomorphism of $\text{Exp}(A)$ into $\mathbb{C}$, then p must be central. The result follows readily once we have the next:

Lemma 2.2.8. *If A is a semisimple Banach algebra and $p \in A^{\bullet}$ satisfies $pxyp = pyxp$ for all $x, y \in \operatorname{Exp}(A)$ then p is central.*

Proof : We first show that if $pxyp = pyxp \; \forall \; x, y \in \operatorname{Exp}(A)$ then $pxyp = pyxp \; \forall \; x, y \in A$. Let $x \in A$ be arbitrary. Choose λ such that $|\lambda| > r_{\sigma}(x)$, that is, choose λ such that $\lambda - x$ belongs to $\operatorname{Exp}(A)$. Now,

$$\begin{aligned} p(\lambda - x)yp &= py(\lambda - x)p \quad \forall \; y \in \operatorname{Exp}(A) \\ \therefore \lambda pyp - pxyp &= \lambda pyp - pyxp \quad \forall \; y \in \operatorname{Exp}(A) \\ \therefore pxyp &= pyxp \quad \forall \; y \in \operatorname{Exp}(A), \; \forall \; x \in A. \end{aligned}$$

Now, let $y \in A$ be arbitrary. Choose λ such that $|\lambda| > r_{\sigma}(y)$, that is (as above), choose λ such that $\lambda - y \in \operatorname{Exp}(A)$. Then, similarly, we obtain

$$\begin{aligned} p(\lambda - y)xp &= px(\lambda - y)p \quad \forall \; x \in A \\ \implies pyxp &= pxyp \quad \forall \; x \in A. \end{aligned}$$

Since y was arbitrary $pyxp = pxyp \; \forall x, y \in A$. We now show $pxy = pyx \; \forall \; x, y \in A$. If x, y are fixed in A and $z \in A$ arbitrary then

$$\begin{aligned} pzp(xy - yx)p &= pz(pxyp - pyxp) \\ &= pz(0) \quad \text{since } pxyp = pyxp \\ &= 0. \end{aligned}$$

Now, $p\big(p(xy - yx)\big)zp = pz\big(p(xy - yx)\big)p = 0$. Therefore,

$$\begin{aligned} \sigma_A\big(p(xy - yx)z\big) &= \sigma_A\big(p(xy - yx)zp\big) \quad \text{since } p \in A^{\bullet} \\ &= \{0\} \quad \text{since } p(xy - yx)zp = 0. \end{aligned}$$

Hence $p(xy - yx) \in \operatorname{Rad}(A) = \{0\}$ since A semisimple. Therefore

$$\begin{aligned} pxy - pyx &= p(xy - yx) \\ &= 0 \end{aligned}$$

and

$$pxy = pyx \tag{2.2.8.1}$$

Similarly we may prove

$$xyp = yxp \tag{2.2.8.2}$$

Replacing y in (2.2.8.1) and (2.2.8.2) with p, we get

$$pxp = ppx = px$$

and

$$xpp = xp = pxp$$

and hence $px = xp$. But since x arbitrary, we have $px = xp \ \forall \ x \in A$. Therefore p is central and the lemma proved. ■

<u>**Note :**</u> Since $F_1 \cap \text{Rad}(A) = \emptyset$, the proof of the above lemma shows that the result also holds for semiprime A, provided that p is minimal.

Theorem 2.2.9. *Let A be a semiprime Banach algebra and let $p \in A^{\bullet}$ be a non-trivial, minimal idempotent. If there exists a normal subgroup H of* $\text{Exp}(A)$ *such that $\phi : \text{Exp}(A)/H \longrightarrow \mathbb{C}$ defined by $\phi(yH) = t_r(yp)$ is a multiplicative homomorphism, then $p \in Z(A)$.*

Proof : Let $x, y \in \text{Exp}(A)$ be arbitrary. Then

$$\begin{aligned} \phi(xyH) &= \phi(xH)\phi(yH) \quad \text{since } \phi \text{ is a homomorphism} \\ &= \phi(yH)\phi(xH) \quad \text{since } \phi \text{ takes values in } \mathbb{C} \\ &= \phi(yxH). \end{aligned}$$

Therefore, by the definition of ϕ, we have $t_r(xyp) = t_r(yxp)$. But p is minimal which means $p \in F_1$ and therefore that $yxp \in F_1$ (since F_1 absorbs products). This implies

$$\begin{aligned} \sigma(yxp) &= \{0, f_{yxp}(1)\} \quad \text{by Corollary 2.1.2.} \\ &= \{0, t_r(yxp)\}. \end{aligned}$$

Which gives $\sigma'(yxp) = \{t_r(yxp)\}$ and (similarly) $\sigma'(xyp) = \{t_r(xyp)\}$. Since p is an idempotent, we have $\sigma'(pyxp) = \sigma'(yxp)$, from this follows that $t_r(pyxp) = t_r(yxp)$. Similarly $t_r(pxyp) = t_r(xyp)$. But we have shown that $t_r(yxp) = t_r(xyp)$, therefore, $t_r(pyxp) = t_r(pxyp)$. Now,

$$\begin{aligned} pxyp &= t_r(pxyp)p \\ &= t_r(pyxp)p \\ &= pyxp. \end{aligned}$$

So, by Lemma 2.2.8., $p \in Z(A)$. ■

With respect to the foregoing discussion, the problem in the case of a non-central p seems to lie in the fact that then, Ap always contains non-zero nilpotent elements:

Theorem 2.2.10. *Let A be a semiprime Banach algebra and let $p \in A^{\bullet}$ be minimal. Then Ap contains no non-trivial nilpotent elements if and only if $p \in Z(A)$.*

Proof :

$\Longleftarrow$: Let p be central and suppose $xp \neq 0$ but $(xp)^2 = 0$. Then,

$$\begin{aligned} xp \neq 0 &\Longrightarrow xpp \neq 0 \\ &\Longrightarrow pxp \neq 0 \;\; \text{since } p \text{ is central} \\ &\Longrightarrow pxp = \lambda p, \;\text{with } \lambda \neq 0 \;\; \text{since } p \text{ is minimal} \\ &\Longrightarrow xpxp = \lambda xp \neq 0 \end{aligned}$$

which contradicts the assumption that $(xp)^2 = 0$. Thus, if p is central then Ap contains no non-trivial nilpotents.

$\Longrightarrow$: Let $v \in \text{Exp}(A)$ be arbitrary. By assumption Ap has no non-trivial nilpotent elements, therefore vp is not nilpotent in Ap and hence $\frac{vp}{t_r(vp)} \in A^{\bullet}$. Following the proof of Lemma 2.2.1.(1), we have $(p - \frac{vp}{t_r(vp)})^2 = 0$, in other words, $(p - \frac{vp}{t_r(vp)})$ is nilpotent. Now, $(p - \frac{vp}{t_r(vp)}) = (1 - \frac{v}{t_r(vp)})p \in Ap$. But, by assumption, Ap contains no non-trivial nilpotent elements and hence $p - \frac{vp}{t_r(vp)} = 0$. We now prove that if $v^{-1}p$ is not nilpotent then pv^{-1} is not nilpotent: observe that

$$\begin{aligned} &v^{-1}pv^{-1}p \neq 0 \\ \Longrightarrow\; &pv^{-1}p \neq 0 \\ \Longrightarrow\; &pv^{-1}pv^{-1} \neq 0. \end{aligned}$$

Following the same procedure as before, this implies $p - \frac{pv^{-1}}{t_r(pv^{-1})} = 0$. So, $p - \frac{vp}{t_r(vp)} = 0$ and $p - \frac{pv^{-1}}{t_r(pv^{-1})} = 0$. Therefore $pt_r(vp) = vp$ and $pt_r(pv^{-1}) = pv^{-1}$. Now,

$$\begin{aligned} vpv^{-1} &= vppv^{-1} \\ &= pt_r(vp)pt_r(pv^{-1}) \\ &= t_r(vp)t_r(pv^{-1})p^2 \;\; \text{since the trace is an element of } \mathbb{C} \\ &= t_r(vp)t_r(pv^{-1})p. \end{aligned}$$

Using the fact that vpv^{-1} is an idempotent, together with the Spectral Mapping Theorem, we must have that $t_r(vp)t_r(pv^{-1}) = 1$, so that $vpv^{-1} = t_r(vp)t_r(pv^{-1})p = p$ and therefore $pv = vp$ since $v \in \text{Exp}(A)$. Since $v \in \text{Exp}(A)$ was arbitrary we have that p commutes with every element of $\text{Exp}(A)$. It follows from $\text{Exp}(A) + \text{Exp}(A) = A$ that p commutes with every element of A. Hence $p \in Z(A)$. ■

In our next result, we show that there are exactly two semiprime Banach algebras, one commutative and one non-commutative, with the property that both $p \in A^\bullet$ as well as its orthogonal complement $1 - p$ belong to F_1. We start with the following lemma which extends, ([8], Lemma 3(ii)) to the semiprime case:

Lemma 2.2.11. *Let A be a semiprime Banach algebra and let g and f be in F_1. Then* $\dim gAf \leq 1$.

Proof : If $gvf = 0 \;\forall\; v \in \text{Exp}(A)$ then $gAf = \{0\}$ since $\text{Exp}(A) + \text{Exp}(A) = A$. This means that $\dim gAf = 0$ and the proof is complete.
So, assume for some $v \in \text{Exp}(A)$ we have $gvf \neq 0$. Since $g, f \in F_1$ and since elements of F_1 absorb products, we have $gvf \in F_1$. We can therefore apply Lemma 2.2.1. If gvf is nilpotent then we can use part (2) of this lemma to find $w \in \text{Exp}(A)$ such that $gvfw$ not nilpotent. Using part (1) of the same lemma, we have $gvfw = \alpha e^x g e^{-x}$ for $0 \neq \alpha \in \mathbb{C}, x \in A$ and $gvfw = \beta e^y f w e^{-y}$ for $0 \neq \beta \in \mathbb{C}, y \in A$. That is, $gvfw$ can be written in two ways. The idea is now to solve for f in terms of g :
Since w is invertible,

$$\alpha e^x g e^{-x} = \beta e^y f w e^{-y}$$

which implies that

$$\frac{\alpha}{\beta} e^{-y} e^x g e^{-x} e^y w^{-1} = f.$$

Hence, f is of the form $f = \kappa u g u^{-1} w^{-1}$ where $0 \neq \kappa \in \mathbb{C}, u \in \text{Exp}(A)$. Substituting for f in gAf we have

$$\begin{aligned} gAf &= gA\kappa u g u^{-1} w^{-1} \\ &= gAugu^{-1}w^{-1} \\ &= gu^{-1}w^{-1}Augu^{-1}w^{-1} \;\text{ since } u^{-1}w^{-1}A = A \\ &= gu^{-1}w^{-1}Agu^{-1}w^{-1} \text{ since } Au = A. \end{aligned}$$

This implies that gAf has the form aAa with $a \in F_1$. Thus $gAf = \mathbb{C}a$ where $a = gu^{-1}w^{-1} \in F_1$. Hence $\dim gAf = \dim \mathbb{C}a = 1$, which completes the proof. ■

Corollary 2.2.12. **([15], Theorem 2.7.)** *Let A be a semiprime Banach algebra. Then the following are equivalent:*

(1) A contains a non-trivial minimal idempotent p such that $1-p$ is also minimal.

(2) Every non-trivial idempotent $p \in A$ is minimal.

(3) A is isomorphic to $\mathbb{C}^2$ or A is isomorphic to $M_2(\mathbb{C})$.

Proof :

• $\mathbf{(2) \Longrightarrow (1)}$:

Since p is an idempotent, $1-p$ is also an idempotent. Also, if p is non-trivial then $1-p$ is non-trivial. Thus $1-p$ is minimal by hypothesis.

• $\mathbf{(3) \Longrightarrow (2)}$:

We need to prove that every non-trivial idempotent in $\mathbb{C}^2$ or $M_2(\mathbb{C})$ is minimal.

(a) The case for $\mathbb{C}^2$:

$A \simeq \mathbb{C}^2$ by assumption and the only non-trivial idempotents in $\mathbb{C}^2$ are $(0,1)$ and $(1,0)$. So, we only look at pAp where $p=(0,1)$ and $p=(1,0)$. For $p=(0,1)$ and $(a,b) \in \mathbb{C}^2$,

$$\begin{aligned}(0,1)(a,b)(0,1) &= (0,b)(0,1)\\ &= (0,b)\\ &= b(0,1)\\ &\in \mathbb{C}p \quad \text{since } b \in \mathbb{C}.\end{aligned}$$

The proof for $(1,0)$ is similar.

(b) The case for $M_2(\mathbb{C})$:

If xp is any matrix with p a non-trivial idempotent then xp is not invertible, because $\det(xp) = \det(x)\det(p) = \det(x).0$. Note that $\det(p)=0$ since p is an idempotent and therefore not invertible. So, xp not invertible implies $0 \in \sigma(xp)$. Let A be an arbitrary 2×2 matrix, say,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and consider the characteristic equation: $\det(\lambda I - A) = 0$ i.e.

$$\left|\begin{bmatrix} \lambda - a & -b \\ -c & \lambda - d \end{bmatrix}\right| = 0.$$

Then $(\lambda-a)(\lambda-d)-(bc)=0$, which gives $\lambda^2-(a+d)\lambda+ad-bc=0$. Thus, the spectrum (eigenvalues) of any 2×2 matrix cannot have more than 2 points (since the characteristic equation is quadratic). But we have shown that $0 \in \sigma(xp)$ which implies that $\#\sigma'(xp) \leq 1 \; \forall \; x$. Therefore p is spectrally rank one. Since $M_2(\mathbb{C})$ is semisimple the spatially and spectrally rank one elements coincide, and hence p has spatial rank one.

• $\underline{\mathbf{(1)} \Longrightarrow \mathbf{(3)} :}$

We show, if (1) is true, then the dimension of A must be less or equal to 4: note that for $z \in A$ arbitrary,

$$z = p\,zp + pz(1-p) + (1-p)zp + (1-p)z(1-p).$$

So

$$A = pAp + pA(1-p) + (1-p)Ap + (1-p)A(1-p).$$

Now, since p and $1-p$ are both in F_1, we have that each of the algebras on the right has $\dim \leq 1$ (Lemma 2.2.11.). Hence $\dim A \leq 4$. But A is semiprime and since $\dim A < \infty$, it is also semisimple, so that we are in a position to apply the Wedderburn-Artin Theorem: If A is a semisimple finite dimensional algebra over $\mathbb{C}$ then there exist positive integers greater or equal to one, $n_1, ..., n_k$ such that $A \simeq M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$. Notice here that $\dim M_{n_k}(\mathbb{C}) = (n_k)^2$ for each k. Therefore (since $\dim A \leq 4$) each $n_k \leq 2$. This means there are only five possibilities to consider, namely: $A \simeq M_2(\mathbb{C}),\ \ A \simeq \mathbb{C}^4,\ \ A \simeq \mathbb{C}^3,$
$A \simeq \mathbb{C}^2,\ \ A \simeq \mathbb{C}$. But for A isomorphic to $\mathbb{C}^4$ and $\mathbb{C}^3$, it is easy to find examples which will contradict our assumption whereas $\mathbb{C}$ contains no non-trivial idempotents. Thus $A \simeq \mathbb{C}^2$ or $A \simeq M_2(\mathbb{C})$. ■

<u>Note :</u> The semiprime condition in Corollary 2.2.12. cannot be omitted; consider the subalgebra B of $M_2(\mathbb{C})$ which consists of complex matrices with lower left entries equal to zero. For

$$a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and

$$b = \begin{bmatrix} \alpha & \beta \\ 0 & \gamma \end{bmatrix}$$

where $\alpha, \beta, \gamma \in \mathbb{C}$ are arbitrary we have $aba = 0$. Thus $aBa = \{0\}$ does not imply that $a = \{0\}$. Hence B is not semiprime. Note that $\dim B = 3$.

Lemma 2.2.11. has the following consequence:

Corollary 2.2.13. *Let A be semiprime Banach algebra and let e and f be elements of F_1. Then e and f belong to some connected component of F_1 if and only if* $\dim(eAf) = 1$. *If e and f belong to distinct components of F_1 then $eAf = \{0\}$.*

Proof : Since $e, f \in F_1$ we have by Lemma 2.2.11. that $\dim eAf \leq 1$. So, we either have $\dim eAf = 1$ or $eAf = \{0\}$. If for some x, $exf \neq 0$, then $exf \in \mathrm{Exp}(A)f\mathrm{Exp}(A)$ and $exf \in \mathrm{Exp}(A)e\mathrm{Exp}(A)$. But $\mathrm{Exp}(A)f\mathrm{Exp}(A) = K_f^{F_1}$ and $\mathrm{Exp}(A)e\mathrm{Exp}(A) = K_e^{F_1}$. Thus, since $K_f^{F_1}$ and $K_e^{F_1}$ contains a common element, they must be equal. If e and f belong to distinct components of F_1 then $\dim eAf \neq 1$ and thus $eAf = \{0\}$. ■

Lemma 2.2.14. ([15], Lemma 2.8.) *Let A be a semiprime Banach algebra which is not semisimple and suppose $a \notin \mathrm{Rad}(A)$. Then $a \in S_1$ if and only if given any $x \in A$ there is a unique $\lambda_x \in \mathbb{C}$ (depending on x) such that $axa - \lambda_x a \in \mathrm{Rad}(A)$.*

Proof : Let $a \in S_1$ with $a \notin \mathrm{Rad}(A)$. If $B = A/\mathrm{Rad}(A)$ and $\dot{x}$ denotes the coset elements of B under the canonical homomorphism, then by Lemma 3.1.5. ([1]), $\dot{x}$ is invertible in $A/\mathrm{Rad}(A)$ if and only if x is invertible in A. And thus $\sigma_A(x) = \sigma_B(\dot{x})$. Also, by the same lemma, we have that B is semisimple. We thus have that $F_1 = S_1$ in B (by Theorem 4 ([9] p. 75)). Therefore, $\dot{a} \in S_1 = F_1$ (in B) if and only if there is $f \in B'$ such that $\dot{a}\dot{x}\dot{a} = f_{\dot{a}}(\dot{x})\dot{a}$, that is $\dot{a}\dot{x}\dot{a} - f_{\dot{a}}(\dot{x})\dot{a} = \dot{0}$ implies (in terms of A) that $axa - f_a(x)a$ belongs to $\mathrm{Rad}(A)$. Putting $f_a(x) = \lambda_x$, we have the existence of λ_x.
If $\lambda_1 \neq \lambda_x$ also satisfies $axa - \lambda_1 a \in \mathrm{Rad}(A)$ then $axa - \lambda_x a$ and $axa - \lambda_1 a$ belong to $\mathrm{Rad}(A)$. Therefore $-(axa - \lambda_x a) + (axa - \lambda_1 a) \in \mathrm{Rad}(A)$ and hence $(\lambda_x - \lambda_1)a$ is in $\mathrm{Rad}(A)$. But $\lambda_x - \lambda_1 \in \mathbb{C}$ and $\lambda_x - \lambda_1 \neq 0$ which means $a \in \mathrm{Rad}(A)$. This contradicts our assumption and thus $\lambda_1 = \lambda_x$ which proves uniqueness.
On the other hand, if for $x \in A$ arbitrary there is a unique $\lambda_x \in \mathbb{C}$ such that $axa - \lambda_x a$ is in $\mathrm{Rad}(A)$ then $(xa)^2 - \lambda_x xa = x(axa - \lambda_x a)$ is in $\mathrm{Rad}(A)$ since the radical is an ideal. This means that $\sigma\big((xa)^2 - \lambda_x xa\big) = \{0\}$. The Spectral Mapping Theorem, applied to the element xa implies that for $\alpha \in \sigma(xa)$ we must have $0 = \alpha^2 - \lambda_x \alpha$. Therefore $\alpha(\alpha - \lambda_x) = 0$ which implies $\sigma(xa) \subseteq \{0, \lambda_x\}$. Hence $\#\sigma'(xa) \leq 1$ and $a \in S_1$. ■

Definitions 2.2.15. Lemma 2.2.14. says that elements of S_1 which are not in F_1 or in $\mathrm{Rad}(A)$ are those $a \in A$ such that given $x \in A$, there exists unique $\lambda_x \in \mathbb{C}$ such that $axa = \lambda_x a \in \mathrm{Rad}(A)$ with at least one $x_0 \in A$ satisfying $ax_0a - \lambda_{x_0} a \neq 0$. For a semiprime Banach algebra A, we call such an element of S_1 a *quasispatially rank one element* and denote the set of these elements by H_1. Since $\mathrm{Rad}(A) \subseteq S_1$, $F_1 \subseteq S_1$ and

$H_1 = S_1 \backslash (\mathrm{Rad}(A) \cup F_1)$ we have

$$S_1 = F_1 \cup H_1 \cup \mathrm{Rad}(A) \backslash \{0\} \tag{2.2.15.1}$$

If A is semiprime but not semisimple, let $0 \neq a \in \mathrm{Rad}(A)$. If u is any other element of S_1, then for $t \in [0,1]$ and $x \in A$ arbitrary we have from [18] p. 193 that

$$\sigma\Big(x\big((tu) + (1-t)a\big)\Big) = \sigma\Big(txu\Big).$$

Using the above equation and the fact that $u \in S_1$ we have

$$\#\sigma'\Big(x\big((tu) + (1-t)a\big)\Big) = \#\sigma'\Big(txu\Big) \leq 1.$$

Therefore $f(t) = tu + (1-t)a \in S_1$, which means the line segment with end points u and a belongs to S_1. We thus have that if A is semiprime but not semisimple then S_1 is path connected and hence connected. This gives: for any two elements u and v in S_1, there exists a path via a radical element connecting u and v, such that the path lies in S_1. We have the following:

Proposition 2.2.16. ([15], Proposition 2.9.) *If A is a semiprime Banach algebra which is not semisimple, then S_1 is a connected set.*

If $F_1 \neq \emptyset$ then the whole of S_1 can be obtained as the closure of H_1 where the boundary of H_1 consists of all the radical and spatially rank one elements:

Proposition 2.2.17. ([15], Theorem 2.10.) *Let A be a semiprime Banach algebra which is not semisimple and suppose $F_1 \neq \emptyset$. Then*

(1) $\bar{F}_1 \cap \mathrm{Rad}(A) = \{0\}$

(2) $\bar{F}_1 \cdot \mathrm{Rad}(A) = \mathrm{Rad}(A) \cdot \bar{F}_1 = \{0\}$

(3) F_1 *and* $\mathrm{Rad}(A)\backslash\{0\}$ *are closed in* S_1 *and hence* H_1 *is open in* S_1 *with* $H_1 \neq \emptyset$

(4) $\mathrm{Rad}(A)\backslash\{0\} + F_1 \subseteq H_1$

(5) H_1 *is dense in* S_1.

Proof : (1) We already know $F_1 \cap \mathrm{Rad}(A) = \emptyset$. Let u_n be a sequence in F_1 converging to an element u in $\mathrm{Rad}(A)$ and let x be arbitrary. We have $u_n(x)u_n = t_r(u_n x)u_n$, since $u_n \in F_1$. Now, $u_n \longrightarrow u$ so that $u_n x u_n \longrightarrow uxu$ and also $t_r(u_n x)u_n \longrightarrow uxu$. Since $u \in \mathrm{Rad}(A)$, we have $ux \in \mathrm{Rad}(A)$, which gives $\sigma(ux) = \{0\}$. Observe now that $\sigma(u_n x) = \{0, t_r(u_n x)\}$ for each n. Since the spectrum is continuous on $\mathrm{Rad}(A)$, we must have that $\sigma(u_n x) \longrightarrow \sigma(ux)$. This means that $t_r(u_n x) \longrightarrow 0$, from which we may conclude that $uxu = 0$. Since x was arbitrary, the semiprime condition implies that $u = 0$.

(2) Let $u \in F_1$ and $r \in \text{Rad}(A)$. Then $ur \in \text{Rad}(A)$ since $\text{Rad}(A)$ is an ideal. If $ur \neq 0$ then $ur \in F_1$ since elements of F_1 absorbs products from the left and the right. So,

$$\begin{aligned} ur \neq 0 &\Longrightarrow ur \in F_1 \text{ and } ur \in \text{ Rad}(A) \\ &\Longrightarrow ur \in \bar{F}_1 \cap \text{Rad}(A). \end{aligned}$$

But by (1), $\bar{F}_1 \cap \text{Rad}(A) = \{0\}$ and hence $ur = 0$. Similarly, for products from the left, $ru = 0$. Hence $\bar{F}_1 \cdot \text{ Rad}(A) = \text{Rad}(A) \cdot \bar{F}_1 = \{0\}$.

(3) Since $\text{Rad}(A)$ is closed in A and $\text{Rad}(A)\backslash\{0\} \subseteq S_1$, we have that $\text{Rad}(A)\backslash\{0\}$ is closed in S_1. Let u_n be a sequence in F_1 such that $u_n \longrightarrow u \in S_1$. It follows from (1) that $u \notin \text{Rad}(A)$, since $u \in \bar{F}_1$. So, if $u \notin F_1$ then by (2.2.15.1), $u \in H_1$. Thus by the comments following Definitions 2.2.15. there exist $x_0 \in A$ and $\lambda_0 \in \mathbb{C}$ such that $ux_0u - \lambda_0 u = r$ where $r \in \text{Rad}(A)$ and $r \neq 0$. Now, $u_n x_0 u_n = \lambda_n u_n$ where λ_n is a convergent sequence (since $u_n x_0 u_n$ converges) in $\mathbb{C}$; say $\lambda_n \longrightarrow \lambda \in \mathbb{C}$. But $u_n x_0 u_n \longrightarrow ux_0 u$ and $\lambda_n u_n \longrightarrow \lambda u$ and therefore (since $u_n x_o u_n = \lambda_n u_n$) $ux_0 u = \lambda u$. Thus $(\lambda - \lambda_0)u = \lambda u - \lambda_0 u = r$ with $r \neq 0$, which implies that $\lambda - \lambda_0 \neq 0$. Therefore, $(\lambda - \lambda_0)u \in \text{Rad}(A)$ which shows that $u \in \text{Rad}(A)$. But (by (1)), this is impossible and therefore $u \in F_1$ which proves F_1 closed in S_1. Using this fact, equation (2.2.15.1) and the fact that F_1 and $\text{Rad}(A)\backslash\{0\}$ are closed in S_1 (and thus $F_1 \cup \text{Rad}(A)\backslash\{0\}$ closed in S_1), we have H_1 is open in S_1. Moreover, if $H_1 = \emptyset$ then S_1 is the union of two disjoint closed sets in S_1, which means S_1 has a separation and is not connected. But this contradicts Proposition 2.2.16. so we must have $H_1 \neq \emptyset$.

(4) Let $r \in \text{Rad}(A)\backslash\{0\}$ and $u \in F_1$. Then, $u + r \notin \text{Rad}(A)$, since $F_1 \cap \text{Rad}(A) = \emptyset$. If $u + r \in F_1$ then

$$(u + r)x(u + r) = \lambda^{(x)}(u + r) \ \forall \ x \in A$$

therefore

$$\begin{aligned} uxu + rxu + uxr + rxr &= \lambda^{(x)}u + \lambda^{(x)}r \\ \therefore uxu + 0 + 0 + rxr &= \lambda^{(x)}u + \lambda^{(x)}r \quad \text{using (2)} \\ \therefore uxu + rxr &= \lambda^{(x)}u + \lambda^{(x)}r. \end{aligned}$$

But since $u \in F_1$, there exists $\lambda_x \in \mathbb{C}$ such that $uxu = \lambda_x u$. So, $\lambda_x u + rxr = \lambda^{(x)}u + \lambda^{(x)}r$.

• If $\lambda_x \neq \lambda^{(x)}$ then

$$\lambda_x u + rxr = \lambda^{(x)}(u + r)$$
$$\implies (\lambda_x - \lambda^{(x)})u = \lambda^{(x)} r - rxr$$
$$\implies u = \frac{1}{\lambda_x - \lambda^{(x)}}\left[\lambda^{(x)} r - rxr\right] \in \mathrm{Rad}(A)$$

which is impossible by (1). So, for each x, $\lambda_x = \lambda^{(x)}$.

• But if $\lambda_x = \lambda^{(x)}$ then $\lambda^{(x)} u + rxr = \lambda^{(x)} u + \lambda^{(x)} r$ and therefore $rxr = \lambda^{(x)} r$. But if this happens for every $x \in A$ then $r \in F_1$, which contradicts (1). So, we arrive at contradictions for both $\lambda_x \neq \lambda^{(x)}$ and $\lambda_x = \lambda^{(x)}$. Therefore, $u + r$ belongs to neither F_1 nor to $\mathrm{Rad}(A)\backslash\{0\}$. In view of (2.2.15.1), we get $u + r \in H_1$.

(5) If $u \in F_1$ and $r \in \mathrm{Rad}(A)\backslash\{0\}$ then $u+r \in H_1$, therefore, $f(t) = tu+(1-t)r,\ t \in (0,1)$, lies in H_1. This means that the interior of the line joining u and r lies in H_1. Hence $\bar{H}_1 = S_1$ and H_1 is dense in S_1. ■

2.3 The Socle Of A Banach Algebra

Following our discussion in paragraph 2.2, we shall from now on only consider the case where A is a semisimple Banach algebra.

Definitions 2.3.1. If A is a semisimple Banach algebra with minimal left ideals. Then let $L \subseteq A$ be one of these left ideals. Using Theorem 1.5.5(1) we have $L = Ap$ with $pAp = \mathbb{C}p$ and $p^2 = p$. Note that (using Theorem 1.5.5(2)) if A contains minimal left ideals then A also contains minimal right ideals and vice versa. It can be shown (see [4], p. 156) that the smallest left ideal containing all minimal left ideals is a two-sided ideal and moreover, that it equals the smallest right ideal containing all minimal right ideals. This two-sided ideal is called the *socle* of A and is denoted by $\mathrm{Soc}(A)$.

Theorem 2.3.2. *The socle of A, $Soc(A)$, is equal to J where J consists of all finite sums of minimal left ideals.*

Proof : Since the sum of two finite sums is a finite sum, J is closed under addition, and since a finite sum of left ideals is a left ideal we have that J absorbs products from the left. Thus J is a left ideal. By definition, J contains every minimal left ideal of A. So, J is a left ideal containing all minimal left ideals. Since $\mathrm{Soc}(A)$ is the smallest left ideal containing all minimal left ideals we have that $\mathrm{Soc}(A) \subseteq J$. On the other hand, if $\mathrm{Soc}(A)$ is a left ideal that contains every minimal left ideal then $\mathrm{Soc}(A)$ must also contain every finite sum of minimal left ideals (since itself is a left ideal). Hence $J \subseteq \mathrm{Soc}(A)$. So, $\mathrm{Soc}(A) = J$. ■

Definitions 2.3.3. As in ([3]) we define the *rank* of an element $a \in A$ by

$$\mathrm{rank}(a) = \sup_{x \in A} \#\sigma'(xa) \leq \infty.$$

For each non-negative n, we denote by F_n the set

$$F_n = \{a \in A : \mathrm{rank}(a) = n\}.$$

Note that $F_0 = \{a \in A : \mathrm{rank}(a) = \sup_{x \in A} \#\sigma'(xa) = 0\}$, which means that, given any x, no $\lambda \neq 0$ exists such that $\lambda - xa \notin A^{-1}$. But this says that $a \in \mathrm{Rad}(A) = \{0\}$. Hence

$$F_0 = \{0\}.$$

This definition is motivated by the fact that spectral rank coincides with the standard rank in $B(X)$ ([3] p.118). Aupetit and Mouton proceeds to show ([3], Corollary 2.9) that the set of all finite rank elements equals the socle:

$$\text{Soc}(A) = \bigcup_{n=0}^{\infty} F_n.$$

What is rather remarkable is the fact that if $a \in A$ is such that $\#\sigma'(xa) < \infty \ \forall \ x \in A$, then, by the Scarcity Theorem ([1] 3.4.25), finiteness occurs in a uniform manner. That is, $\#\sigma'(xa) < \infty \ \forall \ x \in A \implies \exists \ n \in \mathbb{N}$ such that $\#\sigma'(xa) \leq n \ \forall \ x \in A$. If a is in the socle of A then ([3], Theorem 2.2) the sets $E(a) = \{x \in A : \#\sigma'(xa) = \text{rank}(a)\}$ are dense open subsets of A. Amongst the elements of the socle, Aupetit and Mouton distinguish the *maximal finite rank elements* as those $a \in \text{Soc}(A)$ with the property that $\text{rank}(a) = \#\sigma'(a)$. Since the sets $E(a)$ are dense in A, the set of maximal finite rank elements is dense in $\text{Soc}(A)$. Note also that $\dim A < \infty \iff A = \text{Soc}(A)$ and that in general $\text{Soc}(A)$ is not necessarily closed in A and thus not necessarily finite dimensional.

We will need the following Theorem in the proof of Theorem 2.3.5.:

Theorem 2.3.4. *Let A be a Banach algebra, let $b \in A$ and denote $B = \text{comm}^2(b)$. Then for each $y \in B$ we have $\sigma_B(y) = \sigma_A(y)$.*

Proof : Let $y \in \text{comm}^2(b)$. Then $ya = ay$ whenever a commutes with b. But we know that $\sigma_A(y) \subseteq \sigma_B(y)$. So, we only need to show that $\sigma_B(y) \subseteq \sigma_A(y)$, but that is the same as proving $\mathbb{C} \backslash \sigma_A(y) \subseteq \mathbb{C} \backslash \sigma_B(y)$ and is what we proceed to prove. Let $\alpha - y \in A^{-1}$ and let z be the inverse of $\alpha - y$. Then $(\alpha - y)z = z(\alpha - y) = 1$. Thus $\alpha z - yz = z\alpha - zy = 1$ and hence $\alpha z - yz = \alpha z - zy$, which implies $yz = zy$. For
$a \in$ comm(b) (as above), we then have $(\alpha z - yz)a = a(z\alpha - zy)$ and therefore

$$\alpha za - yza = \alpha az - azy \tag{2.3.4.1}$$

But

$$\alpha za - yza = (\alpha - y)za$$

and

$$\begin{aligned} \alpha az - azy &= \alpha az - ayz \ \ \text{since } zy = yz \\ &= \alpha az - yaz \ \ \text{since } y \in \ \text{comm}^2(b) \text{ and } a \in \ \text{comm}(b) \\ &= (\alpha - y)az. \end{aligned}$$

Using these two equations and (2.3.4.1), we have $(\alpha - y)za = (\alpha - y)az$ which implies $za = az$ and thus (since $a \in \text{comm}(b)$ was arbitrary) $z \in \text{comm}^2(b)$. ■

Theorem 2.3.5. *If A is a semisimple Banach algebra then*

$$\operatorname{Soc}(A) = \bigcup_{n=0}^{\infty} F_n.$$

Proof: Let $a \in A$ with $\operatorname{rank}(a) = n$. Let $x_0 \in A$ such that $\sigma'(x_0 a) = \{\lambda_1, ..., \lambda_n\}$ where $\lambda_k \neq 0$ all distinct. For ease of writing let $b = x_0 a$. We first show that $b \in \operatorname{Soc}(A)$. For each k we denote the spectral idempotent corresponding to b and λ_k by p_k (recall Definitions 1.4.3.). That is,

$$p_k = \frac{1}{2\pi i} \int_{\Gamma_k} (\lambda - b)^{-1} d\lambda$$

where Γ_k is a small circle surrounding λ_k and separating λ_k from the remaining points in $\sigma(b)$. The p_k's are mutually orthogonal because of the following: If we cover $\sigma(b)$ by disjoint open balls $U_0, U_1, ..., U_n$ each of which has a corresponding center λ_k ($\lambda_0 = 0$) and Γ is the union of circles $\Gamma_0, \Gamma_1, ..., \Gamma_n$ where each Γ_k is a circle in U_k surrounding λ_k, then

$$p_k = \frac{1}{2\pi i} \int_{\Gamma} f_k(\lambda)(\lambda - b)^{-1} d\lambda$$

$$\text{where } f_k(\lambda) = \begin{cases} 1 & \lambda \in U_k \\ 0 & \lambda \in U_j \neq U_k \end{cases}$$

We also note that the idempotent p_0 is of no real importance to us, so we only consider p_k with $k \in \{1, ..., n\}$. Notice that the p_k's commute with b (see the discussion following Definitions 1.4.3.). We show that each p_k belongs to the left ideal Ab. Let Γ_k be a circle with center λ_k, separating λ_k from the remaining spectrum of b. Then, using

$$\begin{aligned}(\lambda - b)^{-1} - \frac{1}{\lambda} - \frac{1}{\lambda} b(\lambda - b)^{-1} &= \left(\lambda - b\right)^{-1}\left(1 - \frac{1}{\lambda} b - \frac{1}{\lambda}(\lambda - b)\right) \\ &= (\lambda - b)^{-1}(1 - \frac{1}{\lambda} b - 1 + \frac{1}{\lambda} b) \\ &= 0\end{aligned}$$

we get

$$(\lambda - b)^{-1} = \frac{1}{\lambda} + \frac{1}{\lambda} b(\lambda - b)^{-1} \tag{2.3.5.1}$$

It follows that

$$
\begin{aligned}
p_k &= \frac{1}{2\pi i}\int_{\Gamma_k}(\lambda - b)^{-1}d\lambda \\
&= \frac{1}{2\pi i}\int_{\Gamma_k}\left(\frac{1}{\lambda} + \frac{1}{\lambda}b(\lambda - b)^{-1}\right)d\lambda \\
&= \frac{1}{2\pi i}\int_{\Gamma_k}\frac{1}{\lambda}d\lambda + \frac{1}{2\pi i}\int_{\Gamma_k}\frac{1}{\lambda}b(\lambda - b)^{-1}d\lambda \\
&= 0 + \frac{1}{2\pi i}\int_{\Gamma_k}\frac{1}{\lambda}b(\lambda - b)^{-1}d\lambda \quad \text{by the Cauchy Integral Theorem.}
\end{aligned}
$$

Thus

$$
p_k = \left(\frac{1}{2\pi i}\int_{\Gamma_k}\frac{1}{\lambda}(\lambda - b)^{-1}d\lambda\right)b \tag{2.3.5.2}
$$

and hence $p_k \in Ab$. For each $k \in \{1, 2, ..., n\}$ and $x \in A$ we thus have that $p_k x p_k$ also belongs to Ab. Also, the left ideal Ab cannot contain more than n non-trivial, distinct orthogonal idempotents; if $\{q_1, q_2, ..., q_{n+1}\}$ is such a set then $y \in Ab$ where $y = q_1 + 2q_2 + \cdots + (n+1)q_{n+1}$, and by Theorem 1.2.3. we have

$$
\begin{aligned}
\sigma'(q_1 + 2q_2 + \cdots + (n+1)q_{n+1}) &= \bigcup_{j=1}^{n+1}\sigma'(jq_j) \\
&= \{1, 2, ..., n+1\} \quad \text{since each } q_j \text{ is an idempotent.}
\end{aligned}
$$

Thus $\#\sigma'(y) = n + 1$. But since $\operatorname{rank}(b) = n$, we have $\#\sigma'(xb) \leq n$ for all $x \in A$. That is, elements belonging to Ab cannot have more than n non-zero spectral points. This refutes the existence of y and hence the orthogonal set $\{q_1, ..., q_{n+1}\}$ in Ab. We now show that each of the p_k's are minimal. Since $p_k x p_k \in Ab$, we have $\sigma(p_k x p_k)$ is finite for each $x \in A$. But if $\#\sigma'(p_k x p_k) = m > 1$ for some $x \in A$ then, by the Holomorphic Functional Calculus, we can find m distinct and orthogonal Riesz-idempotents (spectral idempotents): $\sigma'(p_k x p_k) = \{\alpha_1, ..., \alpha_m\}$, $\alpha_i \neq \alpha_j$ if $i \neq j$. For each $j \in \{1, ..., m\}$ let Γ_j be a circle with center α_j, separating α_j from the remaining spectrum of $p_k x p_k$. Now,

$$
q_j = \frac{1}{2\pi i}\int_{\Gamma_j}(\lambda - p_k x p_k)^{-1}d\lambda.
$$

We therefore have m mutually orthogonal and distinct Riesz-idempotents. As before, using (2.3.5.1), we have

$$
q_j = \left(\frac{1}{2\pi i}\int_{\Gamma_j}\frac{1}{\lambda}(\lambda - p_k x p_k)^{-1}d\lambda\right)p_k x p_k
$$

for some suitable circle Γ_j. Notice that $q_j p_k = q_j$ since p_k is an idempotent and also that $q_j p_i = 0$ for $i \neq k$ since p_i is orthogonal to p_k. This shows that each q_j is orthogonal to each p_i where $i \neq k$. But there are at least two q_j's (since $m > 1$) and $q_j \in Ab$ so that Ab contains at least $n+1$ orthogonal idempotents, which is a contradiction (since Ab cannot contain more than n non-trivial distinct idempotents, as shown above). We conclude that $\sup_{x \in A} \#\sigma'(p_k x p_k) \leq 1$ and thus that p_k is spectrally rank one. By [9] Theorem 4, we have (since A is semisimple) that p_k is spatially rank one and hence a minimal idempotent. Using Theorem 1.5.5., together with the definition of $\mathrm{Soc}(A)$, we have $p_k \in \mathrm{Soc}(A)$.
By the Holomorphic Functional Calculus

$$b = \frac{1}{2\pi i} \int_{\Gamma} \lambda(\lambda - b)^{-1} d\lambda$$

where Γ a smooth contour surrounding the spectrum of b

$$\therefore b = \frac{1}{2\pi i} \int_{\Gamma_0} \lambda(\lambda - b)^{-1} d\lambda + \cdots + \frac{1}{2\pi i} \int_{\Gamma_n} \lambda(\lambda - b)^{-1} d\lambda.$$

Now,

$$\begin{aligned}
\sum_{k=1}^{n} bp_k &= bp_1 + \cdots + bp_n \\
&= \Big(\frac{1}{2\pi i} \int_{\Gamma} \lambda(\lambda - b)^{-1} d\lambda\Big)(p_1 + \cdots + p_n) \\
&= \Big(\sum_{j=0}^{n} \frac{1}{2\pi i} \int_{\Gamma_j} \lambda(\lambda - b)^{-1} d\lambda\Big)\Big(\sum_{k=1}^{n} \frac{1}{2\pi i} \int_{\Gamma_k} (\lambda - b)^{-1} d\lambda\Big) \\
&= \frac{1}{2\pi i} \int_{\Gamma_1} \lambda(\lambda - b)^{-1} d\lambda + \cdots + \frac{1}{2\pi i} \int_{\Gamma_n} \lambda(\lambda - b)^{-1} d\lambda.
\end{aligned}$$

Note that $\frac{1}{2\pi i} \int_{\Gamma_j} \lambda(\lambda - b)^{-1} d\lambda \frac{1}{2\pi i} \int_{\Gamma_k} (\lambda - b)^{-1} d\lambda = 0$ whenever $j \neq k$.
From the Holomorphic Functional Calculus, we thus obtain

$$b = \sum_{k=1}^{n} bp_k + \frac{1}{2\pi i} \int_{\Gamma_0} \lambda(\lambda - b)^{-1} d\lambda.$$

Writing $c = \frac{1}{2\pi i} \int_{\Gamma_0} \lambda(\lambda - b)^{-1} d\lambda$ we have

$$p_k c = c p_k = \frac{1}{2\pi i} \int_{\Gamma_0} \lambda(\lambda - b)^{-1} d\lambda \frac{1}{2\pi i} \int_{\Gamma_k} (\lambda - b)^{-1} d\lambda = 0$$

and hence c is orthogonal to p_k for all k.

By Theorem 1.3.2(2), we have that $c(\sum_{k=1}^{n} bp_k) = (\sum_{k=1}^{n} bp_k)c$. Since we also know that $b = \sum_{k=1}^{n} bp_k + c$, it follows that $cb = bc$. Writing c as,

$$c = \frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda - b)^{-1}d\lambda \ \text{ where } f(\lambda) = \begin{cases} \lambda \text{ on } U_0 \\ 0 \text{ for } \lambda \in U_j \neq U_0 \end{cases}$$

we have $\sigma(f(b)) = \sigma(c)$. Remembering that $\sigma'(b) = \{\lambda_1, \lambda_2, ..., \lambda_n\}$ we have $\sigma(b) = \{\lambda_1, \lambda_2, ..., \lambda_n, 0\}$. Now, using the Spectral Mapping Theorem, $(f(\sigma(b)) = \sigma(c))$ we have (by the definition of f) that $f(\sigma(b)) = \{0\}$ and thus $\sigma(c) = 0$ which gives $r_\sigma(c) = \{0\}$ and hence c is quasinilpotent. Furthermore, for each k we have

$$\begin{aligned} bp_k &= p_k b p_k \text{ since } p_k \text{ commutes with } b \\ &= \lambda_k p_k \text{ since } p_k \text{ is minimal} \end{aligned}$$

$$\therefore b = \sum_{k=1}^{n} \lambda_k p_k + c \ \text{ since } \ b = \sum_{k=1}^{n} bp_k + \frac{1}{2\pi i}\int_{\Gamma_0} \lambda(\lambda - b)^{-1}d\lambda.$$

Our next step is to show $c = 0$. Note first that

$$\begin{aligned} c &= \frac{1}{2\pi i}\int_{\Gamma_0} \lambda(\lambda - b)^{-1}d\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma_0} \lambda\left(\frac{1}{\lambda} + \frac{1}{\lambda}b(\lambda - b)^{-1}\right)d\lambda \quad \text{using (2.3.5.1)} \\ &= \frac{1}{2\pi i}\int_{\Gamma_0} \left(1 + b(\lambda - b)^{-1}\right)d\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma_0} 1 d\lambda + \frac{1}{2\pi i}\int_{\Gamma_0} b(\lambda - b)^{-1}d\lambda \\ &= \left(\frac{1}{2\pi i}\int_{\Gamma_0} (\lambda - b)^{-1}d\lambda\right)b \quad \text{by Cauchy's Integral Theorem.} \end{aligned}$$

So that $c \in Ab$, and hence of finite rank.
Let $x_1 \in A$ be such that $\text{rank}(c) = \#\sigma'(x_1 c) = m \leq n$ and suppose $m \neq 0$. As in the first part of the proof, we can write

$$x_1 c = \sum_{k=1}^{m} (x_1 c) r_k + \frac{1}{2\pi i}\int_{\Gamma_0} \lambda(\lambda - x_1 c)^{-1}d\lambda \tag{2.3.5.3}$$

where the r_k are orthogonal minimal idempotents and the integral in the equation is quasinilpotent. Since each r_k is a minimal idempotent we have $r_k x_1 c r_k \in \mathbb{C}r_k$, but, as before, x_1c commutes with r_k. Therefore $(x_1c)r_k \in \mathbb{C}r_k$ for each k. So for each $k \in \{1, ..., m\}$ we have $(x_1c)r_k = \alpha_k r_k$ with $\alpha_k \in \mathbb{C}$. We have also shown that p_j is orthogonal to c for each $j \in \{1, ..., n\}$, which implies $\alpha_k r_k p_j = r_k(x_1c)p_j = 0$ and hence $r_k p_j = 0$ for each $k \in \{1, ..., m\}$ and $j \in \{1, ..., n\}$. Now,

$$r_k\big[1 - (r_k + 2p_1 + 3p_2 + \cdots + (n+1)p_n)\big] = 0$$

and

$$\big[2(1 - p_1) - r_k - 3p_2 - \cdots - (n+1)p_n\big]p_1 = 0$$

$$\big[3(1 - p_2) - r_k - 2p_1 - 4p_3 - \cdots - (n+1)p_n\big]p_2 = 0$$

$$\vdots$$

$$\vdots$$

$$\big[(n+1)(1 - p_n) - r_k - 2p_1 - \cdots - np_{n-1}\big]p_n = 0$$

which means the expressions in the brackets are zero divisors since r_k and all the p_k's are non-zero. Therefore $r_k + 2p_1 + 3p_2 + \cdots + (n+1)p_n$ has $n+1$ non-zero spectral values, namely $\{1, 2, ..., n+1\}$. But $r_k + 2p_1 + \cdots + (n+1)p_n$ is in Ab and we thus have a contradiction to the rank of b; (rank$(b) = \sup\limits_{x \in A} \sigma'(xb) = n$). This implies that each r_k is some p_j and thus $\sum\limits_{k=1}^{m}(x_1c)r_k = 0$ since p_j is orthogonal to c. From (2.3.5.3) it follows that $\#\sigma'(x_1c) = \{0\}$ and hence that rank$(c) = \#\sigma'(x_1c) = \{0\}$. So, since $F_0 = \{0\}$, in a semisimple Banach algebra, $c = 0$. This gives $b = \sum\limits_{k=1}^{n}\lambda_k p_k$. But p_k is minimal, which implies (by Theorem 1.5.5.) that Ap_k is a minimal left ideal. And, since $\lambda_k p_k \in Ap_k \ \forall \ k$, we have b is a finite sum of elements belonging to minimal left ideals. This gives $b = x_0a \in \mathrm{Soc}(A)$. If x_0 has an inverse x_0^{-1} then $x_0^{-1}x_0a \in \mathrm{Soc}(A)$, since Soc$(A)$ is a two-sided ideal. So to prove $a \in \mathrm{Soc}(A)$, it suffices to prove that there exists $y_0 \in A^{-1}$ such that $\#\sigma'(y_0a) = \#\ \sigma'(x_0a) = \mathrm{rank}(a)$. This result is true and is a direct consequence of ([3] Theorem 2.2) but the proof there relies on the Scarcity Theorem. We will give an alternative argument, using Newburgh's Theorem ([1], Corollary 3.4.5).

Let $\beta \in \mathbb{C}$ and consider

$$\begin{aligned}(x_0+\beta)a &= b+\beta a\\ &= \sum_{k=1}^{n} bp_k + \beta a\\ &= \sum_{k=1}^{n} x_0 a p_k + \beta a\\ &= \big(\sum_{k=1}^{n} p_k x_0 + \beta\big)a.\end{aligned}$$

Using (2.3.5.2), we obtain

$$\begin{aligned}\sigma'\Big(\sum_{k=1}^{n} p_k x_0\Big) &= \sigma'\Big(\big(\sum_{k=1}^{n} p_k\big)x_0\Big)\\ &= \sigma'\Big(\sum_{k=1}^{n}\Big(\frac{1}{2\pi i}\int_{\Gamma_k}\frac{1}{\lambda}(\lambda-b)^{-1}d\lambda\Big)bx_0\Big)\\ &= \sigma'\Big(x_0\sum_{k=1}^{n}\Big(\frac{1}{2\pi i}\int_{\Gamma_k}\frac{1}{\lambda}(\lambda-b)^{-1}d\lambda\Big)b\Big)\end{aligned}$$

and thus, since b has finite rank, we have $\sigma'\Big(x_0\sum_{k=1}^{n}\big(\frac{1}{2\pi i}\int_{\Gamma_k}\frac{1}{\lambda}(\lambda-b)^{-1}d\lambda\big)b\Big)$ is finite from which we conclude that $\sum_{k=1}^{n} p_k x_0$ has finite spectrum. So if β_j is a complex sequence with $\beta_j \longrightarrow 0$ then there is an $N \in \mathbb{N}$ such that for $j \geq N$ we have $\big(\sum_{k=1}^{n} p_k x_0 + \beta_j\big) \in A^{-1}$, (from the fact that the spectrum of $\sum_{k=1}^{n} p_k x_0$ is finite). Also, since $\beta_j \longrightarrow 0$ we have

$$\big(\sum_{k=1}^{n} p_k x_0 + \beta_j\big)a \longrightarrow \sum_{k=1}^{n} p_k x_0 a = \sum_{k=1}^{n} p_k b = b = x_0 a.$$

So, by continuity of the spectrum function on elements with finite spectrum, we have that there exists an $M \in \mathbb{N}$ such that for $j \geq M$ we have

$$\#\sigma'\big(\big(\sum_{k=1}^{n} p_k x_0 + \beta_j\big)a\big) = \#\sigma'\big(x_0 a\big) = \operatorname{rank}(a) = n.$$

By taking $y_0 = \sum_{k=1}^{n} p_k x_0 + \beta_j$ with $j > \max\{M, N\}$. We get $y_0 \in A^{-1}$ and $\#\sigma'(y_0 a) = n$. Note that j must be bigger than both M and N to ensure the invertibility of y_0 and also to enable us to use the continuity of the spectrum function (see the diagram following Corollary 1.2.7.). We thus have that $\bigcup\limits_{n=0}^{\infty} F_n \subseteq \text{Soc}(A)$.

Suppose, on the other hand, $a \in \text{Soc}(A)$ and let $x \in A$ be such that $\text{rank}(a) = \#\sigma'(xa)$. Now, $a \in \text{Soc}(A)$ and thus $xa \in \text{Soc}(A)$ since the socle is a two-sided ideal of A. By Theorem 2.3.2. $xa \in \text{Soc}(A) \Longrightarrow xa$ belongs to a finite sum of minimal left ideals and therefore, by Theorem 1.5.5., $xa \in \sum\limits_{k=1}^{n} Ap_k$ where each p_i is a minimal idempotent. But we know that minimal idempotents have rank 1 and that rank 1 elements absorb products (Definitions 2.1.1.). Hence, xa is a finite sum of rank 1 elements. Putting $b = xa$ and taking $z \in A$ arbitrary we have,

$$\begin{aligned} bzb &= xazxa \\ &= (\sum_{j=1}^{n} v_j) z (\sum_{i=1}^{n} v_i) \\ &= (v_1 z v_1 + \cdots + v_1 z v_n) + (v_2 z v_1 + \cdots + v_2 z v_n) + \cdots + (v_n z v_1 + \cdots + v_n z v_n) \\ &= \sum_{i,j} v_i z v_j \quad \text{with } v_i, v_j \in F_1. \end{aligned}$$

By Lemma 2.2.11. $\dim(v_i A v_j) \leq 1 \ \forall \ i,j$ and thus $\dim(bAb) < \infty$. Say $\dim(bAb) = m$ where $m \in \mathbb{N}$. Notice that bAb is not only a linear space, but also an algebra. Denote by X the bicommutant of b and let $L_{b^2} : X \longrightarrow X$ be the operator defined by $L_{b^2}(y) = b^2 y$. Note that $L_{b^2}(y) = byb$ since $y \in \text{comm}^2(b)$. Also,

$$\begin{aligned} L_{b^2}(\alpha y + \beta x) &= b^2(\alpha y + \beta x) \\ &= b^2 \alpha y + b^2 \beta x \\ &= \alpha b^2 y + \beta b^2 x \\ &= \alpha L_{b^2}(y) + \beta L_{b^2}(x) \end{aligned}$$

so that L_{b^2} is linear. Now,

$$\begin{aligned} \|L_{b^2}\| &= \sup_{\|y\| \neq 0} \frac{\|L_{b^2}(y)\|}{\|y\|} \\ &= \sup_{\|y\| \neq 0} \frac{\|b^2 y\|}{\|y\|} \\ &\leq \sup_{\|y\| \neq 0} \frac{\|b^2\| \cdot \|y\|}{\|y\|} \\ &= \|b^2\| \end{aligned}$$

and hence L_{b^2} is also bounded. We now show $\text{rank}(L_{b^2})$ is finite.

$$\begin{aligned} \text{rank}(L_{b^2}) &= \dim L_{b^2}(X) \quad \text{see Definitions 2.3.3.} \\ &= \dim bXb \\ &\leq \dim bAb \quad \text{since } bXb \subseteq bAb \\ &< \infty \quad \text{as shown before.} \end{aligned}$$

Now,

$$\begin{aligned} \#\sigma'_{B(X)}(L_{b^2}) &\leq \sup_{S\in B(X)} \#\sigma'_{B(X)}(SL_{b^2}) \quad \text{see Definitions 2.3.3.} \\ &= \text{rank}(L_{b^2}) \\ &< \infty \quad \text{as shown above.} \end{aligned}$$

We proceed to show $\sigma_{B(X)}(L_{b^2}) = \sigma_X(b^2)$. Suppose $\lambda - b^2 \in X^{-1}$. Then to prove is $\lambda I - L_{b^2} \in \big(B(X)\big)^{-1}$. So, let the z be the inverse of $(\lambda - b^2)$ i.e. $(\lambda - b^2)z = z(\lambda - b^2) = 1$. We need an S such that $(\lambda I - L_{b^2})S = I = S(\lambda I - L_{b^2})$. Define $S : X \longrightarrow X$ by $Sx = zx$. One easily verifies that S has the desired property. On the other hand, suppose $\lambda I - L_{b^2} \in \big(B(X)\big)^{-1}$ then $(\lambda I - L_{b^2})Sx = x = S(\lambda I - L_{b^2})(x)$ for all $x \in X$. We then have $\lambda Sx - b^2 Sx = x \Longrightarrow (\lambda - b^2)Sx = x$. Setting $x = 1$, we get $(\lambda - b^2)S(1) = 1$. And thus $\lambda - b^2 \in X^{-1}$. Hence,

$$\{\lambda : \lambda - b^2 \in X^{-1}\} = \{\lambda : \lambda I - L_{b^2} \in \big(B(X)\big)^{-1}\}$$

and

$$\sigma_X(b^2) = \sigma_{B(X)}(L_{b^2}).$$

By Corollary 2.3.4. we have $\sigma_X(b^2) = \sigma_A(b^2)$ and therefore,

$$\sigma_{B(X)}(L_{b^2}) = \sigma_X(b^2) = \sigma_A(b^2).$$

By the Spectral Mapping Theorem we thus have

$$\sigma_{B(X)}(L_{b^2}) = \big(\sigma_A(b)\big)^2.$$

Since $\#\sigma_{B(X)}(L_{b^2}) < \infty$, we have $\#\sigma'_A(b) = \#\sigma'_A(xa) < \infty$. Hence $\text{rank}(a)$ is finite, which means (by the definition of the rank) that $a \in F_n$ for some $n \in \mathbb{N}$ and thus that $a \in \bigcup_{n=0}^{\infty} F_n$. So, $\text{Soc}(A) \subseteq \bigcup_{n=0}^{\infty} F_n$. ■

Notice that the first part of the proof of Theorem 2.3.5. contains a proof of Aupetit and Moutons's Diagonalisation Theorem ([3], Theorem 2.8), which gives the precise structure of maximal finite-rank elements: If $a \neq 0$ is a maximal finite-rank element and $\lambda_1, ..., \lambda_n$ are the non-zero and distinct spectral values of a, then there exist n orthogonal minimal idempotents $p_1, ..., p_n$ such that

$$a = \lambda_1 p_1 + \cdots + \lambda_n p_n.$$

Using the Diagonalisation Theorem, we give an alternative proof of Theorem 2.3.7. (which was obtained by Aupetit and Mouton ([3], Theorem 2.18) as well as Brešar and Šemrl ([5], p. 288). We need the following:

Theorem 2.3.6. *If A is a semisimple finite dimensional Banach algebra, then the rank is subadditive. That is, rank$(a+b) \leq$ rank$(a)+$ rank(b) for all $a, b \in A$.*

Proof : If A is semisimple and finite dimensional then A is isomorphic to

$$B = M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C}).$$

Since

$$\sigma_B\big((a_1, a_2, ..., a_k)\big) = \bigcup_{i=1}^{k} \sigma_{B_i}(a_i) \text{ with } B_i = M_{n_i}(\mathbb{C})$$

it follows that $\text{rank}_B\big((a_1, a_2, ..., a_k)\big) = \text{rank}_{B_1}(a_1) + \cdots + \text{rank}_{B_k}(a_k)$. Now, for a matrix algebra $B_i = M_{n_i}(\mathbb{C})$ and $a_i \in B_i$ the rank of a_i is also given by the dimension of the range of a_i where a_i is considered a linear operator on $\mathbb{C}^{n_i}$. From this point of view one directly deduces that $\text{rank}_{B_i}(a_i + b_i) \leq \text{rank}_{B_i}(a_i) + \text{rank}_{B_i}(b_i)$ for $a_i, b_i \in B_i$. So, the subadditivity of the rank on each B_i implies the subadditivity of the rank on B and hence also on A. ■

Theorem 2.3.7. (B. Aupetit, H. du T. Mouton). *Let $a \in Soc(A)$. The rank of a is the smallest integer n such that $a \in I_1 + \cdots + I_n$ where $I_1, ..., I_n$ are distinct minimal left ideals.*

Proof : Suppose rank$(a) = n$. From the proof of Theorem 2.3.5. there exists $y_0 \in A^{-1}$ such that $\#\sigma'(y_0 a) = n$. In other words, $b = y_0 a$ is maximal rank and hence can be written as

$$b = \lambda_1 q_1 + \cdots + \lambda_n q_n$$

where the $\lambda_i \in \mathbb{C}$ are distinct and non-zero, and the q_i are orthogonal minimal idempotents. So it follows that n is an integer such that b, and hence also a, belong to the sum of n distinct minimal ideals. Suppose now that $m < n$ is the smallest integer such that b belongs to the sum of m distinct minimal left ideals. Then

$$\begin{aligned} &\lambda_1 q_1 + \cdots + \lambda_n q_n \in Ap_1 + \cdots + Ap_m \quad \text{with } p_1, ..., p_m \text{ minimal, distinct idempotents} \\ &\Longrightarrow \big(\frac{1}{\lambda_1}q_1 + \cdots + \frac{1}{\lambda_n}q_n\big)\big(\lambda_1 q_1 + \cdots + \lambda_n q_n\big) \in Ap_1 + \cdots + Ap_m \\ &\Longrightarrow q_1 + \cdots + q_n \in Ap_1 + \cdots + Ap_m. \end{aligned}$$

We can therefore write

$$q_1 + \cdots + q_n = r_1 + \cdots + r_m \tag{2.3.7.1}$$

where $r_1, ..., r_m$ belong to $Ap_1, ..., Ap_m$ respectively. Now, writing $q = q_1 + \cdots + q_n$ we observe that q is an idempotent. If $q = 1$ then $1 \in \mathrm{Soc}(A)$ and hence every $x \in A$ belongs to $\mathrm{Soc}(A)$, implying that $A = \mathrm{Soc}(A)$, which gives $\dim A \leq \infty$. So,

$$\begin{aligned} q_1 + \cdots + q_n &= r_1 + \cdots + r_m \\ \Longrightarrow \mathrm{rank}(q_1 + \cdots + q_n) &= \mathrm{rank}(r_1 + \cdots + r_m) \\ &\leq \mathrm{rank}(r_1) + \cdots + \mathrm{rank}(r_m) \\ &= m \end{aligned}$$

which is a contradiction. Hence we only consider the case where $q \neq 1$. In this case,

$$(1-q)(r_1 + r_2 + \cdots + r_m) = 0.$$

So we have that

$$(1-q)r_1 + (1-q)r_2 + \cdots + (1-q)r_m = 0.$$

Now if some $(1-q)r_i \neq 0$ then there exists $r_{l_1}, ..., r_{l_k}$ such that

$$(1-q)r_i = (1-q)r_{l_1} + \cdots + (1-q)r_{l_k}.$$

But since $\mathrm{Exp}(A)r_i = Ar_i - \{0\}$, we have

$$r_i = x_{l_1}r_{l_1} + \cdots + x_{l_k}r_{l_k}, \quad x_{l_j} \in A.$$

So we may substitute for r_i in (2.3.7.1). This then shows that $q_1 + \cdots + q_n$ can be written as the sum of less than m rank 1 elements and hence that b belongs to the sum of less than m minimal left ideals. So we conclude that, for each i, $(1-q)r_i = 0$. That is, $r_i = qr_i$ for each i. But we also have that

$$(r_1 + r_2 + \cdots + r_m)(1-q) = 0$$
$$\Longrightarrow r_1(1-q) + \cdots + r_m(1-q) = 0.$$

So if some $r_i(1-q) \neq 0$ then we have, as before, that r_i can be written as

$$r_i = r_{l_1}x_{l_1} + \cdots + r_{l_k}x_{l_k}.$$

So, substitution in (2.3.7.1) again implies that $q_1 + \cdots + q_n$ can be written as the sum of less than m rank 1 elements. Hence b belongs to the sum of less than m minimal left ideals which contradicts our assumptions about m. So we conclude that $r_i(1-q) = 0$ for each i. Thus $r_i = r_i q$ for each i. We have thus shown that each r_i commutes with q. In particular, we have that $qr_iq = r_i \in qAq$ for each i. But $B = qAq$ is a semisimple finite dimensional Banach algebra (with identity q) and $\text{rank}_B(q) = \text{rank}_A(q) = n$. Moreover, for each i we also have, $\text{rank}_B(r_i) = \text{rank}_A(r_i) = 1$. Hence, in view of Theorem 2.3.6., we get a contradiction. Thus, b cannot be written as belonging to the sum of less than n distinct minimal left ideals. ■

Corollary 2.3.8. *If* $a, b \in A$ *then* $\text{rank}(a+b) \leq \text{rank}(a) + \text{rank}(b)$.

Proof : If $\text{rank}(a) = \infty$ or $\text{rank}(b) = \infty$ then $\text{rank}(a+b) \leq \infty$ which means that in this case $\text{rank}(a+b) \leq \text{rank}(a) + \text{rank}(b)$. We consider now the case where $\text{rank}(a) \neq \infty$ and $\text{rank}(b) \neq \infty$. Let n be the smallest integer such that a belongs to the sum of n distinct minimal left ideals and m the smallest integer such that b belongs to the sum of m distinct minimal left ideals. By Theorem 2.3.7., $\text{rank}(a) = n$ and $\text{rank}(b) = m$. But now, $n+m$ is an integer such that $a+b$ belongs to $n+m$ (not necessarily distinct) minimal left ideals. Hence (by Theorem 2.3.7.) there exists an integer r such that $r \leq n+m$ and $a+b$ belongs to r distinct minimal left ideals. And thus $\text{rank}(a+b) \leq r \leq n+m = \text{rank}(a) + \text{rank}(b)$. ■

If $a \in \text{Soc}(A)$ has $\text{rank}(a) = n$ then by Theorem 2.3.7., $a = x_1p_1 + \cdots + x_np_n$. So, by using Corollary 2.2.2., we have that $a = (\alpha_1 p_1 - \lambda_1 e^z p_1 e^{-z}) + \cdots + (\alpha_n p_n - \lambda_n e^z p_n e^{-z})$ where the p_i's are minimal idempotents (from Theorem 1.5.5.) for $i \in \{1, ..., n\}$. This means that a can be written as a linear combination of $2n$ rank 1 idempotents (since the p_i and the $e^z p_i e^{-z}$ are idempotents), some of which may belong to the same minimal left ideal. This does not contradict Theorem 2.3.7., since if $\operatorname{rank}(a) = n$ then a is the sum of n rank 1 elements, each of which has the form $x_i p_i$ where p_i is a rank 1 idempotent for each i, but $x_i \in A$ not necessarily in $\mathbb{C}$.

We now prove that the Frobenius inequality, known to hold for matrices, can be extended to arbitrary semisimple Banach algebras. We shall use a characterization of rank, given by M. Brešar and P. Šemrl in [5], which is (of course) equivalent to the spectral defintion of rank: An element a of semisimple Banach algebra has rank n if and only if a satisfies:

(1) there exists finitely many distinct primitive ideals $P_1, \ldots, P_k$ of A such that $a \in P$ for every primitive ideal $P \neq P_i$ where $i = 1, \ldots, k$

(2) if π_i, $i = 1, \ldots, k$ are continuous irreducible representations of A on Banach spaces such that $\operatorname{Ker} \pi_i = P_i$, then $\pi_i(a)$ are finite rank operators and $n = \sum_{j=i}^{k} \operatorname{rank} \pi_j(a)$

We first prove the Frobenius inequality for the case $A = B(X)$.

Theorem 2.3.9. (Frobenius' inequality for the operator case). *For the Banach algebra $B(X)$ we have, for all $T_a, T_b, T_c \in B(X)$ that*

$$\operatorname{rank}(T_a T_b) + \operatorname{rank}(T_b T_c) \leq \operatorname{rank}(T_b) + \operatorname{rank}(T_a T_b T_c).$$

Proof: If $\operatorname{rank}(T_b) = \infty$ then obviously the result is true. So, suppose T_b has rank equal to n in $B(X)$, that is, $\dim T_b(X) = n$. From this follows the fact that $\operatorname{rank}(T_b T_c) = m \leq n$. So we may write $T_b T_c(X) = \operatorname{span}\{y_1, \ldots, y_m\}$ where the set $\{y_i\}_{i=1}^m$ is linearly independent. Since $T_b T_c(X)$ is a vector subspace of $T_b(X)$ we can adjoin to $\{y_i\}_{i=1}^m$ $n - m$ elements $y_{m+1}, \ldots, y_n$ so that $\{y_i\}_{i=1}^n$ is linearly independent and $T_b(X) = \operatorname{span}\{y_1, \ldots, y_n\}$. It now follows that $T_a T_b T_c(X) = \operatorname{span}\{T_a y_1, \ldots, T_a y_m\}$ and $T_a T_b(X) = \operatorname{span}\{T_a y_1, \ldots, T_a y_n\}$. Note that the sets $\{T_a y_i\}_{i=1}^m$ and $\{T_a y_i\}_{i=1}^n$ are not necessarily linearly independent. If $\operatorname{rank}(T_a T_b T_c) = k \leq m$ then we may remove $m - k$ elements from $\{T_a y_1, \ldots, T_a y_m\}$ so that the span of the resultant set is equal to $T_a T_b T_c(X)$. But that means that $\operatorname{rank}(T_a T_b) \leq n - k$ since if we remove the same set from $\{T_a y_1, \ldots, T_a y_n\}$ span of the resultant set equals $T_a T_b(X)$. ■

Corollary 2.3.10. (Generalized Frobenius inequality). *For any semisimple Banach algebra we have, for all $a, b, c \in A$, that*

$$\text{rank}(ab) + \text{rank}(bc) \leq \text{rank}(b) + \text{rank}(abc).$$

Proof: If $\text{rank}(b) = \infty$ then the result obviously holds. So assume that $\text{rank}(b) = n$. By the rank characterization preceding Theorem 2.3.9. we have that b belongs to every primitive ideal P where $P \neq P_1, ..., P_k$. Furthermore $n = \text{rank}(\pi_1(b)) + \cdots + \text{rank}(\pi_k(b))$ where the π_i's are continuous irreducible representations on Banach spaces. So clearly, since ideals absorb products, we have that ab, bc and abc also belong to every primitive ideal $P \neq P_1, ..., P_k$. Thus,

$$\begin{aligned} \text{rank}(ab) &= \text{rank}(\pi_1(ab)) + \cdots + \text{rank}(\pi_k(ab)), \\ \text{rank}(bc) &= \text{rank}(\pi_1(bc)) + \cdots + \text{rank}(\pi_k(bc)) \\ \text{and } \text{rank}(abc) &= \text{rank}(\pi_1(abc)) + \cdots + \text{rank}(\pi_k(abc)). \end{aligned}$$

The fact that the π_i's are homomorphisms enables us to apply Theorem 2.3.9. We thus have that

$$\text{rank}(\pi_i(ab)) + \text{rank}(\pi_i(bc)) \leq \text{rank}(\pi_i(b)) + \text{rank}(\pi_i(abc)) \text{for each } i.$$

So,

$$\sum_{i=1}^{k} \Big[\text{rank}(\pi_i(ab)) + \text{rank}(\pi_i(bc))\Big] \leq \sum_{i=1}^{k} \Big[\text{rank}(\pi_i(b)) + \text{rank}(\pi_i(abc))\Big]$$

from which the result follows. ■

Theorem 2.3.11. *Let A be a semisimple Banach algebra and let $p_1, p_2, ..., p_n$ be orthogonal finite rank idempotents in A. Then for any choice of non-zero scalars $\alpha_1, \alpha_2, ..., \alpha_n$ we have*

$$\text{rank}_A(\alpha_1 p_1 + \cdots + \alpha_n p_n) = \text{rank}_A(p_1) + \cdots + \text{rank}_A(p_n).$$

Proof : Since the rank is subadditive we have

$$\begin{aligned} \text{rank}_A(\alpha_1 p_1 + \cdots + \alpha_n p_n) &\leq \text{rank}_A(\alpha_1 p_1) + \cdots + \text{rank}_A(\alpha_n p_n) \\ &= \text{rank}_A(p_1) + \cdots + \text{rank}_A(p_n). \end{aligned}$$

So it remains to prove the reverse inequality. The idea is more or less to show that the conclusion in Theorem 2.3.11. holds in a subalgebra of A. Let $C = p_1 A p_1 + \cdots + p_n A p_n$.

Notice that C has unit $p_1 + \cdots + p_n$ and that C is semisimple since each p_iAp_i is semisimple. Also $\text{rank}_C(\alpha_1 p_1 + \cdots + \alpha_n p_n) = \text{rank}_C(p_1 + \cdots + p_n)$ since invertible elements in C, which is finite dimensional, must have the same rank. Now,

$$\begin{aligned}
\text{rank}_C(p_1 + \cdots + p_n)& \\
&= \sup\Big\{\#\sigma'_C\big((p_1 + \cdots + p_n)[p_1x_1p_1 + \cdots + p_nx_np_n]\big) : x_1, ..., x_n \in A\Big\} \\
&= \sup\Big\{\#\sigma'_C\big(p_1x_1p_1 + \cdots + p_nx_np_n\big) : x_1, ..., x_n \in A\Big\} \\
&= \sup\Big\{\#\big[\sigma'_C(p_1x_1p_1) \cup \cdots \cup \sigma'_C(p_nx_np_n)\big] : x_1, ..., x_n \in A\Big\} \\
&= \sup_{x\in A} \#\sigma'_C(p_1xp_1) + \cdots + \sup_{x\in A} \#\sigma'_C(p_nxp_n) \\
&= \sup_{x\in A} \#\sigma'_A(p_1xp_1) + \cdots + \sup_{x\in A} \#\sigma'_A(p_nxp_n) \\
&= \sup_{x\in A} \#\sigma'_A(p_1x) + \cdots + \sup_{x\in A} \#\sigma'_A(p_nx) \\
&= \text{rank}_A(p_1) + \cdots + \text{rank}_A(p_n).
\end{aligned}$$

We now demonstrate (non-trivially) that

$$\text{rank}_C(\alpha_1 p_1 + \cdots + \alpha_n p_n) \leq \text{rank}_A(\alpha_1 p_1 + \cdots + \alpha_n p_n).$$

We introduce the algebra

$$\begin{aligned}
B &= (p_1 + \cdots + p_n)A(p_1 + \cdots + p_n) \\
&= pAp \quad \text{where the idempotent } p = p_1 + \cdots + p_n.
\end{aligned}$$

We know that for $z \in B$

$$\sigma'_B(z) = \sigma'_A(z) \tag{2.3.11.1}$$

Now notice, since the p_i's are orthogonal, that $C = p_1Ap_1 + \cdots + p_nAp_n$ is a subalgebra of B which has the same identity as B. Hence

$$\begin{aligned}
\text{rank}_C(\alpha_1 p_1 + \cdots + \alpha_n p_n) &\leq \text{rank}_B(\alpha_1 p_1 + \cdots + \alpha_n p_n) \\
&= \sup_{x\in A} \#\sigma'_B\big((\alpha_1 p_1 + \cdots + \alpha_n p_n)pxp\big) \\
&= \sup_{x\in A} \#\sigma'_A\big((\alpha_1 p_1 + \cdots + \alpha_n p_n)pxp\big) \quad \text{using (2.3.11.1)} \\
&= \sup_{x\in A} \#\sigma'_A\big((\alpha_1 p_1 + \cdots + \alpha_n p_n)x\big) \\
&= \text{rank}_A(\alpha_1 p_1 + \cdots + \alpha_n p_n).
\end{aligned}$$

So it follows that

$$\begin{aligned}\operatorname{rank}_A(\alpha_1 p_1+\cdots+\alpha_n p_n) &\geq \operatorname{rank}_C(\alpha_1 p_1+\cdots+\alpha_n p_n)\\ &= \operatorname{rank}_A(p_1)+\cdots+\operatorname{rank}_A(p_n).\end{aligned}$$

■

Definitions 2.3.12. ([5]) An element of an algebra A is *indecomposable* if it cannot be written as a sum $a=b+c$ where $b, c \in A$ are non-zero elements satisfying $bAc=0$. We denote by $\mathrm{M}_{r,n}$ with $r \leq n \leq 2r$ the algebra of all $n \times n$ complex matrices $[\, a_{i,j} \,]$ satisfying $a_{i,j}=0$ whenever $i>r$ or $j \leq n-r$.

Theorem 2.3.13. ([5]) *Let A be a semisimple unital complex Banach algebra and let $n \in \mathbb{N}$. Then $a \in A$ has $\operatorname{rank}(a)=n \iff \exists\, a_1, ..., a_k \in A$ such that:*

(1) $a=a_1+\cdots+a_k$

(2) Each a_i is indecomposable

(3) $a_iAa_j=0$ *whenever* $i \neq j$

(4) $a_iAa_i \simeq M_{r_i,n_i}$ *for some* $r_i, n_i \in \mathbb{N}, r_i \leq n_i \leq 2r_i$

(5) $n=r_1+\cdots+r_k$.

Moreover $a_1, ..., a_k$ are unique non-zero elements in $\operatorname{Soc}(A)$, satisfying (1) to (3).

As an application of Theorem 2.3.13., we give an alternative proof of ([3], Theorem 2.12) which proves a bit more and is perhaps a bit easier.

Theorem 2.3.14. (B. Aupetit, H. du T. Mouton).

For $a \in A$

$$\operatorname{rank}(a) \leq \dim(aAa) \leq (\operatorname{rank}(a))^2.$$

Moreover, if $a \in \operatorname{Soc}(A)$ then $\operatorname{rank}(a)=\dim(aAa) \iff aAa \simeq M_{1,n_1} \oplus \cdots \oplus M_{1,n_k}$ where $n_i \in \{1,2\}$ and $k=\operatorname{rank}(a)$.

Proof : If $\dim(aAa)$ is finite then we have

$$\begin{aligned}\dim\Big((xa)A(xa)\Big) &= \dim\Big(x\big(aA(xa)\big)\Big)\\ &= \dim\Big(x\big(a(Ax)a\big)\Big)\\ &\leq \dim\Big(x(aAa)\Big) \quad \text{since } Ax \subseteq A.\end{aligned}$$

But if we let $\{z_1, ..., z_n\}$ be a basis for aAa and take $xz \in x(aAa)$ arbitrary then

$xz = x(\alpha_1 z_1 + \cdots + \alpha_n z_n) = \alpha_1(xz_1) + \cdots + \alpha_n(xz_n)$ so that xz is a linear combination of $\{xz_1, ..., xz_n\}$. Hence $\dim\big(x(aAa)\big) \leq \dim(aAa) < \infty$. Thus, from the above, $\dim\big((xa)A(xa)\big) < \infty$. By the Wedderburn-Artin Theorem, this implies that xa is algebraic i.e. there exists some polynomial p (of say, degree n) such that $p(xa) = 0$. If p is of degree n then p has exactly n roots, so that, by the Spectral Mapping Theorem, $\sigma\big(p(xa)\big) = p\big(\sigma(xa)\big)$ and therefore $\#\sigma(xa) \leq n < \infty$. Hence $\#\sigma'(xa) < \infty$ for all $x \in A$ and thus rank(a) is finite. We have thus proved that $\dim(aAa)$ finite $\Longrightarrow$ rank(a) finite. Now, if $a \notin \mathrm{Soc}(A)$ then rank(a) infinite $\Longrightarrow$ $\dim(aAa)$ infinite and hence the inequalities hold. So let $a \in \mathrm{Soc(A)}$ with rank structure decomposition $a = a_1 + \cdots + a_k$ and rank$(a) = r_1 + \cdots + r_k$ as in Theorem 2.3.13. For each i we first prove

$$a_iAa_i \bigcap (a_1Aa_1 + \cdots + a_{i-1}Aa_{i-1} + a_{i+1}Aa_{i+1} + \cdots + a_kAa_k) = \{0\}.$$

If for some set $\{x_1, ..., x_k\} \in A$ we have that

$$a_ix_ia_i \;= \sum_{\substack{j=1 \\ j\neq i}}^{k} a_jx_ja_j$$

then let $z \in A$ be such that rank$(a_ix_ia_i) = \#\sigma'(a_ix_ia_iz) < \infty$. Using $a_iAa_j = \{0\}$ (Theorem 2.3.13(3)) we obtain

$$\begin{aligned}\#\sigma'\big((a_ix_ia_iz)^2\big) &= \#\sigma'\big(a_ix_ia_iz(\sum_{\substack{j=1 \\ j\neq i}}^{k} a_jx_ja_j)z\big) \\ &= \#\sigma'(0) \\ &= 0.\end{aligned}$$

But, by the Spectral Mapping Theorem, $\sigma'\big((a_ix_ia_iz)^2\big) = \big(\sigma'(a_ix_ia_iz)\big)^2$ and therefore $\#\sigma'(a_ix_ia_iz) = 0$. Thus rank$(a_ix_ia_i) = 0$ and $a_ix_ia_i = 0$.

Using Theorem 2.3.13 (3) and (4) it follows that

$$aAa = (a_1 + \cdots + a_k)A(a_1 + \cdots + a_k) \subseteq a_1Aa_1 + \cdots a_kAa_k$$

where $\dim a_iAa_i = (r_i)^2$ for each i.

Let $V_i = \{e_1^{(i)}, ..., e_{(r_i)^2}^{(i)}\}$ be a basis for a_iAa_i. We now proceed to prove that

$$B_i = \{x \in A : a_ixa_i = \alpha_1 e_1^{(i)} + \cdots + \alpha_{(r_i)^2} e_{(r_i)^2}^{(i)}\ ,\ \alpha_j \neq 0\ \forall j\}$$

is a dense and open subset of A. Let

$$C_k = \{x \in A : a_i x a_i = \beta_1 e_1^{(i)} + \cdots + \beta_{(r_i)^2} e_{(r_i)^2}^{(i)} \ , \ \beta_k = 0\}.$$

For ease of writing set $a_i = a$ and $e_j^{(i)} = e_j$. We show that C_k is a vector subspace of A. Let $x, y \in C_k$ then

$$axa = \beta_1 e_1 + \cdots + \beta_k e_k + \cdots + \beta_{(r_i)^2} e_{(r_i)^2} \text{ with } \beta_k = 0$$

and

$$aya = \gamma_1 e_1 + \cdots + \gamma_k e_k + \cdots + \gamma_{(r_i)^2} e_{(r_i)^2} \text{ with } \gamma_k = 0$$

hence

$$\begin{aligned} \alpha(axa) + \nu(aya) &= \alpha\big(\sum_{l=1}^{k-1} \beta_l e_l + \beta_k e_k + \sum_{l=k+1}^{(r_i)^2} \beta_l e_l\big) + \nu\big(\sum_{l=1}^{k-1} \gamma_l e_l + \gamma_k e_k + \sum_{l=k+1}^{(r_i)^2} \gamma_l e_l\big) \\ &= \sum_{l=1}^{k-1} (\alpha\beta_l + \nu\gamma_l) e_l + (\alpha\beta_k + \nu\gamma_k) e_k + \sum_{l=k+1}^{(r_i)^2} (\alpha\beta_l + \nu\gamma_l) e_l \\ &\text{with } \beta_k = \gamma_k = 0 \text{ so that } \alpha x + \nu y \in C_k. \end{aligned}$$

Hence C_k is a vector subspace of A. Notice that C_k is also proper in A since otherwise every element in C_k can be written as a linear combination of $(r_i)^2 - 1$ basis vectors, contradicting $\dim a_i A a_i = (r_i)^2$. We proceed to prove that C_k is closed in A: let $x_s \in C_k$ such that $x_s \to x$.

$$ax_s a = \beta_1^{(s)} e_1 + \cdots + 0.e_k + \cdots + \beta_{(r_i)^2}^{(s)} e_{(r_i)^2}$$

$$axa = \alpha_1 e_1 + \cdots + \alpha_k . e_k + \cdots + \alpha_{(r_i)^2} e_{(r_i)^2} \ .$$

So $ax_s a \to axa$ by continuity of multiplication. Now,

$$\lim_{s\to\infty} \Big\| \sum_{j=1}^{k-1} (\beta_j^{(s)} - \alpha_j) e_j + (0 - \alpha_k) e_k + \sum_{j=k+1}^{(r_i)^2} (\beta_j^{(s)} - \alpha_j) e_j \Big\| = 0$$

But, for all s,

$$\Big\| \sum_{j=1}^{k-1} (\beta_j^{(s)} - \alpha_j) e_j \ + (0 - \alpha_k) e_k \ + \sum_{j=k+1}^{(r_i)^2} (\beta_j^{(s)} - \alpha_j) e_j \Big\|$$

is greater or equal to

$$K\Big(\sum_{j=1}^{k-1}|\beta_j^{(s)}-\alpha_j| + |0-\alpha_k| + \sum_{j=k+1}^{(r_i)^2}|\beta_j^{(s)}-\alpha_j)e_j^2|\Big)$$

for some positive constant K, which implies that

$$\sum_{j=1}^{k-1}|(\beta_j^{(s)}-\alpha_j)| + |(0-\alpha_k)| + \sum_{j=k+1}^{(r_i)^2}|(\beta_j^{(s)}-\alpha_j)|$$

tends to zero as s tends to infinity. This means $\alpha_k = 0$ so that $x \in C_k$ and hence C_k is closed. Since C_k is a proper vector subspace of A it has empty interior and hence the complement of C_k in A, C_k^c, is dense and open in A. But $B_i = \bigcap_{k=1}^{(r_i)^2} C_k^c$ so, by Baire's Theorem B_i is open and dense. Applying Baire's Theorem again, it follows that

$$\bigcap_{i=1}^{k} B_i$$

is also dense and open in A. We thus see that

$$aAa = a_1Aa_1 + \cdots + a_kAa_k \tag{2.3.13.1}$$

and hence $\dim(aAa) = \dim(a_1Aa_1 + \cdots + a_kAa_k)$. So,

$$\begin{aligned}\operatorname{rank}(a) &= r_1 + \cdots + r_k\\ &\le r_1^2 + \cdots + r_k^2\\ &= \dim(a_1Aa_1) + \cdots + \dim(a_kAa_k)\\ &= \dim(a_1Aa_1 + \cdots + a_kAa_k)\\ &= \dim(aAa)\end{aligned}$$

and also

$$\begin{aligned}\dim(aAa) &= \dim(a_1Aa_1) + \cdots + \dim(a_kAa_k)\\ &= r_1^2 + \cdots + r_k^2\\ &\le (r_1 + \cdots + r_k)^2\\ &= (\operatorname{rank}(a))^2.\end{aligned}$$

Therefore

$$\operatorname{rank}(a) \le \dim(aAa) \le (\operatorname{rank}(a))^2).$$

For the second part of the proof; if $\text{rank}(a) = \dim(aAa)$ then the above estimate forces $r_1 + \cdots + r_k = r_1^2 + \cdots + r_k^2$ and hence $r_i = 1$ for each i. Therefore $\dim(a_iAa_i) = 1$ for each i. What we have is thus that $\text{rank}(a_i) \leq 1$ which gives $\text{rank}(a_i) = 1$. Since A is semisimple, we have by ([9] Theorem 4), that a_i is spatially rank one i.e. $a_iAa_i = \mathbb{C}a_i$. But $a_iAa_i \simeq \text{M}_{r_i,r}$ where $r_i \leq r \leq 2r_i$, by Theorem 2.3.13. With $r_i = 1$ (as shown) this becomes $a_iAa_i \simeq \text{M}_{1,r}$ where $1 \leq r \leq 2$. Thus $a_iAa_i \simeq \text{M}_{1,r}$ where

$$r = \begin{cases} 2 \text{ if } a_i \text{ is nilpotent} \\ 1 \text{ if } a_i \text{ is not nilpotent} \end{cases}$$

since if it is nilpotent then it has nilpotency-index of 2 and if it is not nilpotent then it can only be a 1×1 matrix when it comes to matrices of this form. Noticing that (2.3.14.2) is a direct sum (since each a_iAa_i is a vector subspace of aAa) and remembering that $a_iAa_i \simeq \text{M}_{1,r}$, we have that $aAa = a_1Aa_1 + \cdots + a_kAa_k \simeq \text{M}_{1,n_1} \oplus \cdots \oplus \text{M}_{1,n_k}$ and the first part holds.

On the other hand, if $aAa \simeq \text{M}_{1,n_1} \oplus \cdots \oplus \text{M}_{1,n_k}$ where $n_i \in \{1, 2\}$ and $k = \text{rank}(a)$ then

$$\begin{aligned} \dim(aAa) &= \dim(\text{M}_{1,n_1} \oplus \cdots \oplus \text{M}_{1,n_k}) \\ &= \dim(\text{M}_{1,n_1}) + \cdots + \dim(\text{M}_{1,n_k}) \\ &= 1 + \cdots + 1 \\ &= k \\ &= \text{rank}(a) \end{aligned}$$

which proves the theorem. ■

Note that, in Theorem 2.3.14.,

$$a \in \text{Soc}(A) \text{ has } \text{rank}(a) = \dim(aAa) \Longrightarrow aAa \text{ is commutative.}$$

We will now provide a counterexample as proof that aAa commutative does not imply that $a \in \text{Soc}(A)$ has $\text{rank}(a) = \dim(aAa)$;

Take $A = M_4(\mathbb{C})$, that is elements in A are 4×4 complex matrices. Let $a \in A$ where

$$a = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We have thus chosen a to be of rank 2 (because it has 2 linear independent columns). Also, since $a^2 = 0$, it follows that aAa is commutative.

Now, writing $A = (a_{i,j})$, consider aAa;

$$aAa = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \left(a_{i,j} \right) \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (a_{3,1} + a_{4,1}) & (a_{3,2} + a_{4,2}) & (a_{3,3} + a_{4,3}) & (a_{3,4} + a_{4,4}) \\ (-a_{3,1} + a_{4,1}) & (-a_{3,2} + a_{4,2}) & (-a_{3,3} + a_{4,3}) & (-a_{3,4} + a_{4,4}) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & (a_{3,1} + a_{4,1}) - (a_{3,2} + a_{4,2}) & (a_{3,1} + a_{4,1}) + (a_{3,2} + a_{4,2}) \\ 0 & 0 & (-a_{3,1} + a_{4,1}) - (-a_{3,2} + a_{4,2}) & (-a_{3,1} + a_{4,1}) + (-a_{3,2} + a_{4,2}) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

To prove that $\dim(aAa) = 4$, one only needs to consider the coefficient matrix

$$C = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}$$

It is clear that if this matrix is invertible then $\dim(aAa) = 4$; since $\det(C) \neq 0$ this is indeed the case and our claim is proved.

List Of Symbols

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