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Acquisition of geometric thought: A case study of technical vocational Education and Training college learners

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1.1 BACKGROUND TO THE STUDY

Knowledge of Geometry in all levels of schooling is critical because it was regarded as a “reflective tool” that enabled humans to resolve numerous problems, and understand the world that presented comprehensive geometric forms. In Engineering, Science, Art and Architecture, Geometry is present and connected with everyday living. Bahadir (2016) mentioned that Geometry provides a systematic way of understanding geometric world we inherited. He stated that the ability to understand Geometry was the beginning of the development of spatial sense, which is the instinctive sense of the environment we live in and of the objects in it. Geometry afforded learners with the chance of mind activation through mental exercises, nurtured problem solving skills, generalisation and comparison skills (Bahadir, 2016). In essence, Geometry is an essential instrument through which learners were enabled to assign meaning to the environment.

Learning of Geometry began with learners’ encounter of seeing, knowing and interpreting the physical space around them; and further advanced with higher levels of geometric thinking that matured within deductive and inductive systems. This suggested that geometry provides a platform for advancement of learners’ proof and inference skills. Cueva (2015) emphasised that learning of Euclidean Geometry prepared learners for extreme Mathematics training because it involved well-known day to day objects that could be created verbally and virtually; and the logical methods employed were usually more precise, exact and distinct than other introductory parts of Mathematics. It became plausible to make significant mathematical learning in this area even without precise knowledge of axiomatic systems and principles to work with (Cueva, 2015).

Learning of Euclidean Geometry in pre-university Mathematics Education is the best way in which learners learned about proofs and demonstrations because they were normally short, needed only a few concepts were supported by visual properties and were formal in structure (Cueva, 2015). Halat (2003) defined proof as a deductive argument from mathematical statements. For instance, when the statement is posited, one had to evaluate whether the statement is a logical consequence of the preceding statements by identifying a theorem or logical principles which the statement is derived from (Halat, 2003). This suggested that in the argument, other previously established statements such as theorems were used through logical steps towards the conclusion. However, this view of demonstration has been questioned by many mathematicians and philosophers (Cueva, 2015).
1.2 THE RESEARCH PROBLEM STATEMENT

Euclidean Geometry was always been viewed as a significant component of the South African Mathematics syllabus. Not having gone through it during formal schooling years would present dire consequences for learners who intended to pursue courses related to Mathematics at universities, universities of technologies and technical vocational education and training colleges. In 2006 when the NCS was launched, Euclidean Geometry was not made compulsory in the FET curriculum (Dhlamini, 2012); and it was further made an optional assessment of Mathematics Paper 3, which a quite number of schools did not teach at that time (Van Putten et al., 2010). Dhlamini (2012) and Bowie (2012) attributed reasons for making Euclidean Geometry optional in the South African Mathematics curriculum to the fact that it was not taught accordingly at schools and that teachers were not well accustomed to the content. This led to poor performance by learners. Even with the introduction of CAPS as a revised syllabus for grades 10 – 12, only few learners completed grade 12 Euclidean Geometry as a compulsory part of their Mathematics.

Dhlamini (2012) believed that Euclidean Geometry did not make it because it was not presented well and it was camouflaged by assessments which examined rote learning, from which learners could be drilled to perform well without acquiring procedural and conceptual understanding of the content. Govender (2015), also noticed that educators focused on the lowest levels of geometric knowledge with their learners and on routine problems without considering higher levels of geometric activities and questions. These findings were likely to have an undesirable influence on learners’ understanding and achievement in Euclidean geometry and Mathematics as a subject. In addition, research (Walsh, 2015), reported that inefficacious teaching at previous grades may have been the root cause of the aforementioned issues. For example, in a study (Van Putten, 2008) that was conducted on why geometry was taught at school level, prospective teachers gave comments such as “know how to prove theorems”, “know which theorem to apply”, “memorise proofs”. These views indicated that teachers perceived geometry as a part in Mathematics that dealt with proof and theorems only, without any connection to real life situations. In addition, Walsh (2015) claimed that there is a growing concern with regard to the qualifications and qualities of mathematics teachers; with the majority of them having little or no exposure to Euclidean geometry in their post school training.

However, Dhlamini reported that even teachers who have had training in Euclidean geometry still regarded it as a “difficult section” to teach. In another previous study (Pournara, 2004) that was conducted on a group of teachers at Witwatersrand on why Euclidean geometry was taught at schools, some of their perceptions included “I do not know why Euclidean geometry is taught at schools because the vast majority of it cannot be applied to everyday life, nor does it have any meaning to learners’
lives”. These perspectives might well been the indicators of teachers’ insufficient essential capabilities to put into context the content to relevant cognitive levels of learners; taking into consideration learners’ different learning styles, cognitive abilities, inadequate procedural and conceptual knowledge, conceptions and misconceptions about Euclidean geometry, as well as instructional approaches during teaching and learning.

1.3 MOTIVATION FOR THE STUDY

I have been a Mathematics teacher at a TVET College for the past five years. Often enough I had come to a realisation that there are challenges to the mitigation of the problem of poor performance in Mathematics, with specific reference to Euclidean geometry at a college where I am based. Throughout the informal discussions I had with colleagues teaching the same subject, I could deduce that colleagues experienced difficulties in correcting errors that learners made when solving Euclidean geometry problems. Instead, these errors were discarded with the hope that learners would find their way out during examinations and tests. It appeared as though teachers did not realise the need to be pro-active, instead of being reactive, in identifying learning difficulties when it comes to Euclidean geometry so that learners can be helped.

Teachers lacked the courage to determine methods that learners used and why those methods were used when solving Euclidean geometry problems and errors that occurred as a result of such methods. As I continued with the informal talks with colleagues in trying to determine reasons for poor performance in Mathematics, some of the comments which came to the fore were “how do you expect them to pass if they are forever absent?”, “They lack geometry basics from lower grades, hence this performance”; “Once they fail from the schools where they come from, their teachers were quick to write them testimonials and send them to colleges, so that they become a burden here”; “You tell them about remedial lessons, but end up teaching tables and chairs”. From the comments above, I realised that my colleagues and I were quick to pass judgements without actually trying to figure out the main reasons why learners showed feelings of frustrations and disinterest towards Geometry lessons.

I started to think that if matters remained in the status quo, schools, and colleges would be faced with a vicious cycle of poor performance, resulting in the high drop-out of learners. Taking into consideration that South Africa produces fewer graduates in Mathematics related disciplines when compared to countries like Brazil, United States of America and China (BusinessTech, 2015; Statistics on Post School Education & Training in SA, 2011). This resulted in trying to find ways and means to address the errors that lead to poor performance in circle geometry. Research (Luneta, 2015) showed that even if teachers are successful in identifying errors
learners make, it does not mean that they have corrected them, and if left unattended, these errors lead to misconceptions which hamper learning.

1.4 RATIONALE

Although evaluating learners’ errors and misconceptions in building improved geometric understanding was used in teaching and research, there are fewer studies that solely focused on areas of learners looking at their own errors and misconceptions as a way forward for learning (Kembitzku, 2009; Drake & Armspaugh, 1994; Eggleton & Molayar, 2001). Traditionally, it is teachers who diagnosed learners’ work to determine causes of errors and not learners themselves. The notion of having learners reflecting on their individual errors and misconceptions might help learners’ awareness and self-regulation (Kembitzku, 2009). Teachers might have used various techniques to correct errors and misconceptions in Mathematics. Some of the techniques that might have been used are analogies, giving learners contradictions with regard to their thinking in order to create cognitive conflict, showing learners their common mistakes with the hope that they will be aware of them if they take part in Mathematics activities in future, and re-teaching concepts that were relative to learners’ inconsistencies while giving them more practice exercises regarding the concepts (Kembitzku, 2009).

Despite these efforts, instructional methods led to more errors and misconceptions. Research (Dhlamini, 2012; Pournara, 2004; Bowie, 2012) showed that learners were unable to understand important concepts in Geometry and leave left classrooms without basic Geometry knowledge. This was despite the view that Geometry played a significant part in the South African Mathematics curriculum. Sriphai et al., (2011) concurred that the quality of the curriculum influenced the instruction in a positive way, which yields positive results on learner performance and understanding in Mathematics. Furthermore, Burger and Shaughnessy (1986) revealed that sequencing the instruction activities played an important role on learners’ achievement, self-confidence, the topic and skills. For example, during the instruction, if preliminary Geometry activities were frustrating, the learning process would be affected in a negative way because learners would not be interested in what the teacher might have trying to teach. At the same time, if activities were too simple, they might also not draw the needed attention of learners to the topic and fail to bring about feelings of success in learning. Geometry activities should be designed in a way that they contain different learning needs in a class (Kariyana & Sonn, 2016).

Research (Kariyana & Sonn, 2016) suggested that it is not possible for one type of instruction to cater for a Mathematics class that is composed of learners with various interests, learning styles, talents and backgrounds. In the same breadth, not only one type of instruction is used everytime during Geometry learning. This is why there
are various instruction approaches such as discovery learning, learner-centered approaches, and cooperative learning approaches that should be applied (Kariyana & Sonn, 2016). These instructional approaches should also not be applied 100 percent of the time (Kariyana & Sonn, 2016). Fuys, Geddes and Tischler (1988) supported the notion that not only one type of instruction can support the needs to attain higher levels of geometric thought. “It is possible that certain methods of teaching do not permit the attainment of higher levels so that students cannot gain the methods of thought at those levels. It is also possible to face the same phenomenon that would take place between a student and a textbook” (Fuys, Geddes, & Tischler, 1988: 76; Luneta, 2015). What can be deduced from the aforementioned statement is that each learner possesses inherent desire to learn, distinct intelligence, distinct level of ability and learning style. It was apparent that learners in any Mathematics classes may show different interests, abilities and intelligences. All of these should be related to a corresponding variety of modes of learning.

Robichaux-Davis and Guarino (2015) mentioned Gardener’s (1983) theory of multiple intelligences and Jung’s (1971) personality types, as examples of different learning styles and preferable modes of learning. In managing learners’ variation, teachers need to apply different approaches in teaching to ensure success in learning (Davis & Guarino, 2015). In addition, it is important to afford learners the chance to adapt to other types of learning experiences (Davis & Guarino, 2015). These studies suggested that various teaching approaches should be used and learners should be afforded some degree of freedom to choose activities that improve their understanding of the subject. The part played by instruction is well known in teaching of Geometry as indicated by Usiskin (1982). Research on Geometry teaching (Fuys, Geddes, & Tischler, 1988), used the model that was proposed by Pierre and Dina van Hiele in the late 1950s for teaching and learning. The model explained the levels of acquisition of geometric thought as:

a. Visualisation:

Van Hiele (1986) explained this level as the entry reasoning level in which learners identified and explained shapes based on their known appearances.

b. Analysis:

This level was characterised by the learners’ abilities in identifying attributes of shapes, such as sides and corners (van Hiele, 1986).

c. Ordering:

According to van Hiele (1986), learners who operated at this level were able to order properties of shapes logically by providing informal arguments.

d. Deduction:
At this level, learners are able to provide deductive proofs to theorems, construct proof and understand the role played by axioms and definitions (van Hiele, 1986)

e. Rigour

This level is demonstrated through learners’ capabilities to analyse different deductive systems and comparison of such systems (van Hiele, 1986).

The role played by instructional experiences is crucial for progress in the teaching and learning of geometry as indicated by Usiskin (1982), Fuys, Geddes and Tischler (1988), and Messick and Reynolds (1992). It became evident that different teaching approaches for different learning needs help learners improve their knowledge and nurture their problem solving skills. But, more systematic structured instruction would be helpful for middle and high school learners to conquer their errors and improve their understanding of Geometry.

The van Hiele framework above evoked my interest to explore causes of errors and misconceptions in Euclidean Geometry. The purpose of this study is to explore teaching approaches used by teachers to teach Euclidean Geometry, the circle geometry in NC(V) level 4 under the following research question:

What effective instructional frameworks can be employed by teachers to teach NC(v) level 4?

Research sub questions:

1. What are the dominant errors displayed by NC (v) level 4 learners on circle geometry?
2. What could be some of the misconceptions responsible for the dominant errors?
3. What effective instructional approaches can lecturers of NC(v) level 4 use to teach circle geometry effectively?

1.5 AIMS OF THE STUDY

The aim of the research is to examine common errors and misconceptions that learners in NC(V) level 4 have with respect to Euclidean geometry in one TVET college in Gauteng Province.
1.6 OBJECTIVES OF THE STUDY

- To determine NC (V) level 4 learners errors and misconceptions that learners have with respect to Euclidean geometry in a TVET college in Gauteng province
- To determine what could be some of the misconceptions responsible for these errors.
- To discover teaching strategies used by TVET College NC(V) level 4 lecturers to teach Euclidean geometry.
- To recommend pedagogical strategies that can be applied to circle geometry in managing errors and misconceptions to make deductive reasoning more appreciative to NC(V) level 4.

1.7 DEFINITIONS AND TERMS

1.7.1 Conceptual knowledge

A concept is an ‘abstract or generic idea’ generalised from particular instances (Mary-Webster Collegiate Dictionary, 2016). Knowledge of concepts is usually called Conceptual Knowledge (Canobi, 2009). Stohlman, Crammer, Moore, and Maiorca (2014) defined Mathematical conceptual knowledge as knowledge of mathematics concepts, operations and relationships. This knowledge is gained through problem solving practice.

1.7.2 Deductive reasoning

The Mary-Webster Collegiate Dictionary (2016) defined deductive reasoning as a logical process in which a conclusion is based on concordance of multiple premises that are generally assumed to be true. It could also be referred to as top-down logic (Mary-Webster Collegiate Dictionary, 2016). Cueva (2015) argued that logical reasoning skills are an important part of quality Mathematics capabilities. If learners deductive reasoning skills, they are able to find logical connections between mathematical examples and counter examples (cognitive conflict). Lack of deductive reasoning skills can seriously inhibit learners’ critical thinking.

1.7.3 Errors and misconceptions

Luneta (2008, 2015) explained that errors exist because learners experience challenges in understanding instructional strategies sanctioned by the teacher. In addition, transmission of knowledge at various levels of reasoning may also result in misconceptions. For example, when the teacher operates and communicates at a higher level of geometric thinking to that of learners, geometric concepts and
comprehension are not developed in detail (Luneta, 2015). Errors are “simple symptoms of the difficulties a student encounters during the learning experience” (Luneta, 2008:386). Swan (cited in Luneta, 2015), defined an error as a consequence of “carelessness or misinterpretation of symbols or text”. Misconceptions are evident in learners’ written work as errors, which suggest that errors are expressions of underlying misconceptions that learners have (Luneta, 2015). Michael (cited in Luneta, 2015), defined a misconception as a conceptual reasoning difficulty that hampers learners’ abilities to master various disciplines.

1.7.4 Technical Vocational Education & Training College

Technical Vocational Education and Training courses are vocational or occupational by nature, meaning that students receive education and training directed at a specific range of jobs or employment opportunities. Under certain conditions, some students may qualify for admission at a University of Technology to continue their studies at a higher level in the same field of study studied at a TVET College.

1.7.5 National Certificate (Vocational)

The Department of Education introduced the National Certificate Vocational at public Further Education and Training (FET) Colleges in 2007. The NC(V) offered programmes of study in a variety of vocational fields. The programmes were intended to directly respond to the priority skills demands of the South African economy. The National Certificate (Vocational) is offered at levels 2, 3 and 4 of the National Qualifications Framework. The qualification was designed to provide both theory and practical experience in a particular vocational field. The practical component of the study is offered in a simulated workplace environment. Students have the opportunity to experience work situations during their period of study. A learner qualifies for a national Certificate (Vocational) after the completion of all the levels. In order to obtain an NC(V) certificate a student is required to take a total of seven subjects. These include 3 fundamental subjects and 4 vocational/compulsory subjects. The three fundamental subjects are Mathematics for engineering, scientific and agricultural studies or Mathematical Literacy for business studies, English and Life Orientation. The Department of Education has also offered a bursary scheme for all the students who enrol for NC (V), and allocated according to the need analysis.

1.8 BREAKDOWN OF THE RESEARCH REPORT

This research study focused on probing teaching models used by teachers to teach Euclidean geometry (circle geometry), and errors and misconceptions that learners have when solving circle geometry problems. The organisation of the research study was as follows:

- Chapter 1 outlined the background of the study, research problem statement, and rationale, motivation for the study, research question, aims and objectives of the study.
Chapter 2 presented a descriptive conceptual framework. The conceptual framework focused on Mathematics instruction, for instance, definition of geometry, current issues in Mathematics teaching in South Africa, historical account of geometry as a Mathematics topic, van Hiele model for geometric thought, qualities of an effective Mathematics teacher and errors and misconceptions in the learning of Geometry. Data for chapter 2 was collected from scholarly books and journals, Newspapers and Government reports.

Chapter 3 focused on the research methodology and how the research unfolded in the research site.

Chapter 4 focused on the data collection that helped the researcher to understand errors and misconceptions shown by learners. A Geometry performance test, and interviews were used to collect data.

Chapter 5 provided and analysis of errors and misconceptions that learners had with respect to circle Geometry in their performance test, interviews, and classwork and homework books.

Chapter 6 presented the findings of the research study, limitations, recommendations and possible research areas to be pursued.
CHAPTER 2: THEORETICAL FRAMEWORK

2.0 INTRODUCTION

In the previous chapter the researcher presented the background to the research, research problem statement, motivation for the study, the rationale, research question, aims and objectives of the study. In this chapter, the researcher gives an account of theoretical framework to get a detailed overview of the scope of the problem. Henning and Van Der Westhuizen (2007) explain literature review as a process in which various authors present various perspectives about a specific topic. It further presents a detailed account of previous work based on the researcher’s topic (Rockinson-Szapkiw, 2009). Maxwell (2005) further adds that the theoretical framework presents a view or a lens through which to examine the topic. It “provides a clearer sense of the researcher’s theoretical approach to the phenomenon that you propose to study” (Maxwell, 2005: 123). He argues that the theoretical framework serves the following two purposes:

- It outlines how the research relates to what is already known
- It also explains how the research study contributes on the topic

Creswell (1994) also asserts that the theoretical framework informs the research methodology, research question and it justifies the research problem. This is done by “becoming the framework for the entire study, an organising model for research questions or hypotheses for data collection procedure” (Creswell, 1994: 87-88). In this chapter, the theoretical framework will be focusing on the geometry instruction in South Africa, deductive and inductive proof, the role played by proof in geometry, the influence of the van Hiele theory on geometry thinking development, different learner philosophies of mathematics with respect to errors and misconceptions, and the role played by teacher quality, PCK and CK in mathematics instruction.

2.1 GEOMETRY TEACHING AND LEARNING IN SOUTH AFRICA

Bahr et al., Bassarear and DBE (in Luneta, 2015) state that different school curricula for different countries cover the four main common learning outcomes in geometry. The authors argue that by the time learners finish their school geometry curricula, depending on the quality of the instruction, they should be efficiently skilled in:

- Analysing the characteristics, properties and relationships of two-dimensional and three-dimensional shapes (Euclidean geometry)
- Specifying locations and describing spatial relationships using coordinate geometry and other representations
- Applying transformations and use of geometry to analyse mathematical situations, and
- Using visualisation spatial reasoning and geometric modelling to solve problems.

Regardless of these geometric skills that play a crucial role in Mathematics in South Africa, research (Alex & Mammen, 2016) states that there are several obstacles in the teaching and learning of it. There exist discontents with respect to secondary school geometry syllabus and dismal achievement of learners in geometry in South Africa has been a topic of distress over the past forty years (Burger & Shaughnessy, 1986; Mason, 1997; Fortuny, 1991; Mumcu, Cansiz & Aktaz, 2016; Alex & Mammen, 2016; Wessels, 2001; Liu et al., 2015). Additionally, Atebe and Schafer (2009) argue that the teaching and learning of geometry is one of the discouraging experiences in majority of schools across the country mainly because learners experience challenges in apprehending instructional methods used by teachers. For example, in a study (Mji & Makgato, 2006) that was conducted in mathematics learning, one participant revealed “we spend most of the time learning algebra, which is easy, but what about geometry which is difficult? That is why we do little geometry”.

The 2006 TIMSS report further mentions that South African learners achieved dismally in Mathematics and Science out of the 50 countries that took part in the study and that the worst area of achievement was geometry (Reddy, 2006). Local research (Mji & Makgato, 2006) that was conducted outlines reasons that influence dismal geometry achievement as: obsolete instructional practices; inadequate content knowledge by teachers (Luneta, 2015); and traditional teaching and learning methods that deny learners the chance to learn at their own rate and pace; the ignorance with respect to when and why mathematicians do proofs; lack of competency by teachers to teach mathematics, especially the topics that they are not comfortable with; the fact that geometry in the recent past was removed from the curriculum and had to be offered as an optional paper at NCS certificate level (Mji & Ndlovu, 2012).

The existing instructional activities that are characterised by listening, watching and imitating the teacher are not helping in developing adequate learning in geometry (Alex & Mammen, 2016). Furthermore, Bennie (1998) argues that in South African primary schools, geometry instruction is not adequate with respect to affording learners with the appropriate skills to operate at the level of axiomatic thinking in senior secondary schooling. De Villiers (1997) recommends an improved primary school geometry syllabus with the van Hiele levels that would guarantee success in senior secondary schooling. De Villiers (2010) further explains that the systemic geometry syllabus is focused more on secondary school education, and insufficient content is evident in primary school band. As much as tessellations have been recently introduced in primary school geometry education, several teachers and textbook authors are unable to relate them to the van Hiele theory (De Villiers, 2010). These imply that geometry education in South Africa needs drastic attention. In 1997, a group of South African Non-Governmental Organisation by the name of
Mathematics Learning and Teaching Initiative (MALATI) tried to revamp the teaching and learning of geometry (Bennie, 1998). For the reinvention to take place, the group made several suggestions to modify the geometry syllabus. In this way, the group felt that a mechanism is needed to understand and make sense of geometric thought process of learners (Bennie, 1998). As a result, the South African National Curriculum Statement (NCS) for intermediate (grades 4 – 6) phase considered levels 1, 2 and 3 in the van Hiele organisation (De Villiers, 2010). That is, learners in this phase are supposed to outline and represent characteristics and relationships between two-dimensional and three-dimensional objects in various orientations and position (Feza & Webb, 2005). In the revised curriculum which is called CAPS (Curriculum and Assessment Policy Statement) (grades 10 -12) Euclidean geometry was reintroduced (Alex & Mammen, 2016); with the grade 10 learners being required to:

- Examine and develop conclusions with respect to properties of special triangles, quadrilaterals and other polygons.
- Try to prove conjectures by applying any logical method (Euclidean, coordinate or transformation)
- Not approve false conjectures by giving counter examples and 
- Examine other definitions of different polygons (Isosceles, equilateral, right angled triangle, kite, parallelogram, rectangle, rhombus and square) (Department of Basic Education, 2011)

2.2 DEDUCTIVE AND INDUCTIVE PROOF

Mji and Ndlovu (2012) attribute the idea of proof in mathematics to the deductive process. Deductive proof is emphasised in geometry instruction (Harel & Sowder, 1998). Deductive proof is a process that takes place when an individual draw conclusions from logic chain of reasoning from which each statement is a follow-up from the previous one (Simon, 1996). CadwalladerOlsker (2011) further adds that deductive proof is a formal proof that is based on syntactic constructs and manipulations. Its role is to show that certain parts of mathematics do not have any contradictions, through validation of theorems of those parts of mathematics. In simple terms, deductive proof means verifying the truths of discovered theorems. At the same breath, inductive proof means drawing general conclusions from specific occurrences (Mji & Ndlovu, 2012). CadwalladerOlsker (2011) defines inductive proof as proof that is not formal, nor precise. It is what individuals do to make others believe them. This suggests that proof comprises of a subjective part.

2.2.1 THE ROLE PLAYED BY PROOF IN GEOMETRY

Consider the following example:

The teacher has to deliver a lesson on plane geometry to learners. He uses symmetry with respect to a straight line in teaching them relationships between the
equality of segments and angles, perpendicular lines, e.t.c. He tells them that the points on the axis of symmetry are uniform, that the symmetry lines intersect on the axis of symmetry. In checking to see if learners have understood the lesson, he gives them the problem to solve “Let ABC be a triangle for which the extensions of the sides meet at line l. Construct the symmetrical triangle with respect to line l.”

The teacher creates the following mental picture of the solution: Lines AB and AC meet on the axis L in two points that are called P and Q. These points are uniform under symmetry. Then distance AP and AQ are uniform, so that one can create the symmetrical points A’, in the same way one discovers points B’ and C’.

Looking at the mental picture of the solution, the interpretation and reasoning is the result of the teacher’s understanding and knowledge of all relationships. In the same sense, learners become unable to develop the reasoning of interpreting the thought process without help from the teacher. The teacher during teaching, used the concepts of lengths of symmetrical segments that are the same as the basis for his argument, but his argument might not help because learners might have not yet seen the counter example; they might not have seen transformations that change the length of segments. Furthermore, this example requires that learners must reason with the help of a system of relations between ideas; whose meanings they might not even know, which are “points”, “axis of symmetry”, “segments”, “to meet”, “to change length”, “triangle” and “extension”. The teacher might have explained the mentioned concepts, shown them points and segments, showed what is meant by extending a segment and possibly asked them to formulate definitions of vertical angles. It might even be possible that the definition was found not to be correct and as a result, the teacher might have demonstrated that by means of a counter example.

The point the example is trying to demonstrate is that it is the teacher, who gave the counter example; learners would fail to arrive at the correct solution because they are not in a position to give a counter example because they do not have a system of relations at their disposal. It is important to note that the teacher reasons by means of a system of relations that he alone possesses, and learners end up learning this system of relations through rote learning, without understanding and seeing how this system of relations is discovered.

One focal point in the teaching and learning of mathematics for the 21st century is to maintain meaningful mathematics lessons that actively involve learners’ thinking. Al-ebous (2016) believes that modern mathematics is not mere operations or separated skills, but a well organised network with powerful relationships that develop an integrated structure; and the constituents of this structure are mathematics concepts. One of the notable constituents is geometric proof. Wing (in Ramlan, 2016) argues that “geometric reasoning is the process of defining and deducing the properties of a geometric entity using intrinsic properties of that entity, its relationship with other geometric entities, and the rules of inference that bind such properties together in
geometric (Euclidean) space "(Wing, 1985: 6). Drawing from Wing (1985)’s assertion above, this suggests that geometric reasoning is made up of aspects such as:

- Definitions and deductions of properties of geometry
- Relationships of other aspects of geometry, and
- Conclusions with respect to rules that are already present.

This deduction level of geometric thought process focuses on axiomatic formation of mathematics understanding (Bulut, Akcakin & Oflaz, 2016; Mji & Ndlovu, 2012). This suggests that the purpose of the proof process is to create a chain of arguments, for example, from \( X \) (given hypothesis) to \( Y \) (conclusion) with supporting statements. Thus, the proof process may be an easy one, with one step or multistep traversing from given condition to the conclusion. Nurturing geometric thought is directly related with success in writing proof (Moutsios-Rentzos & Spyrou, 2015). Heinze et al.,(2008) explain that the difficulty of a proof process is determined by the number of reasons that a learner has to provide. Various studies reveal that the notion of proof takes a central place in geometry teaching and learning (Hanna, 2000; Martin & Harel, 1989; Usiskin, 1982). The authors suggest that proof is a required public action that follows the attainment of a judgement although it may be undertaken internally against an imaginary possible doubter. Additionally, Mji and Ndlovu (2012) reveal the following four importance of proof in the learning of geometry:

- Geometry affords learners with the chance to learn logic and gain the capability to transfer this logic to other domains. For instance, “law can be reduced to a set of principles, on the order of mathematical axioms and the use of the deductive method. These principles, as in inductive geometric proof can yield all necessary consequences” (Hoeflich, 1986: 96)
- Geometry offers the opportunity for connections to the real world. For instance, learners’ experiences should be related with the demands of their prospective careers (Watts, 1994)
- Proof allows learners to experiences that resemble activities of the mathematicians. For instance, a guided regenerated geometric approach fosters the attitude of experiencing mathematics as a human activity, as opposed to a set of fragmented symbols without any activity for which mathematics is created (Freudenthal, 1971)
- Geometry allows access to opportunities of applying intuition of geometric objects when describing the world. For instance, learners get the chance to learn geometric language that assist them to model the world (Gonzalez & Herbst, 2006)

Harel (2008) explain that proof is a consequence of a mental activity; the particular argument that one gives to establish for oneself or to convince others that the assertion is true. According to Bell (1976), a proof is made up of three senses. They are called:
- verification or justification
  This is the endorsement for why the theorem or hypothesis is true.
- Illumination
  Illumination explains why the theorem or hypothesis is true.
- Systematisation
  Systematisation is the coordination of results into deductive set of axioms, major concepts and theorems.

From the aforementioned senses, it can be concluded that proof assist learners to make sense of the results, highlighting why it has to be true, and demonstrating the logical organisation of ideas and making deductive reasoning obvious. Studies (Bell, 1976; van Dormolen, 1977; Coe & Ruthven, 1994; Harel & Sowder, 1998) classify proof process in terms of various proportions. Of these proportions, Harel and Sowder (1998) explain what is called proof scheme; which is assigned on the basis of learners’ work and historical development. Sowder (in Bulut, Akcakin & Oflaz, 2016) defines proof scheme as a collection of cognitive features that an individual promote. The proof scheme is made up of the following subclasses:

- External conviction
  External conviction proof schemes are held by learners who hold the conviction that a theorem is true due to outside sources. The outside source may be a textbook or a teacher, manipulation symbols; with the manipulation symbols not having logical system of referents (Flores, 2006). Additionally, learners may recall the aspects of an argument that has been proved previously (Harel, 2008). CadwalladerOlsker (2011) argues that there are three categories in external conviction, which are ritual, non-referential symbolic proof scheme and authoritative proof scheme. In ritual proof scheme, learners become convinced by the form or appearance of a proof scheme. For example, learners who have come across two column proofs only may hold a conviction that a proof is valid because it follows a two column format (Harel, 2008). Non referential symbolic proof schemes are believed by learners who are convinced by manipulation of symbols, but these symbols and manipulations hold no system of referents, for example, an incorrect reduction of algebraic reduction $\frac{a+b}{c+b} = (a + b)(c + b)$ transparent evidence of non-referential symbolic proof scheme (CadwalladerOlsker, 2011). Authoritative proof schemes describe proof schemes by learners who are convinced by an external authority such as the teacher or the textbook that a theorem is true.

- Empirical proof scheme
  Empirical proof scheme consists of arguments from which learners make reference to particular examples or perceived patterns for confirmation (Martin, McGone, Bower & Dindyal, 2005). Learners may depend on evidence from examples or direct measurements of quantities, substitution of relevant numbers in algebraic expressions, or may make use of one or more examples
to persuade themselves or others about the truth of a conjecture (Harel, 2008).

- **Deductive proof scheme**
  Harel (2008) argues that the deductive proof scheme is made up of two categories, that is, transformational proof scheme and axiomatic proof scheme. In the transformational proof scheme, if a learner deals with generic features of a situation, and the proof process is aligned towards a conjecture, then the learner is at transformational proof scheme. For instance, learners position their proofs on a developed mental constructs that have the capacity to transform and reason about an object in a way that all necessary relationships are traversed (CadwalladerOlsker, 2011). Transformational proof schemes require deeper comprehension of the concepts under exploration in order to work with mental images that support reasoning in a qualitative relationship (Harel, 2008). In the axiomatic proof scheme, proofs rely on new theorems, axioms and undefined terms (Harel, 2008). Mathematicians assert that deductive proof schemes are the fitting types of justifications in mathematics (Flores, 2006; Martin et al., 2005; Harel, 2008). When a learner operates at a deductive proof scheme, he/ she is able to make operational thought and logical inferences (Martin et al., 2005).

Flores (2006) argues that psychologists discovered the basic way in which humans form concepts is through examples. Learners also value examples as a way of evaluating their understanding of concepts or as a way of understanding a situation (Flores, 2006). Bulut, Akcakin and Oflaz (2016) advice that geometry instruction should not only associate with geometric proofs; but to also take into consideration proof processes with respect to proof schemes. The authors argue that proof schemes play a central role in explaining the meaning of learners’ configuration of geometric knowledge. Comprehending what proving is all about and being able to write proofs successfully is essential for success in mathematics (Aksu & Koruklu, 2015). Geometry instruction should focus on understanding instead of rote learning. Harel (2008) argues that proofs are described as ways in which understanding is related to mental activity of proving. Proof schemes are methods of thinking that represent collective mental characteristics of an individual’s proof. This suggests that means of comprehending that an individual produces influence the quality of ways of thinking that are formed, and these ways of thinking that one has formed influences the quality of the methods of understanding being produced. Since proofs are at the centre of mathematics and the proving is perceived to be complex, teachers are advised to assist learners to develop these processes earlier in their grades (Hanna et al., 2009).
2.3 THE INFLUENCE OF THE VAN HIELE THEORY ON GEOMETRY THINKING DEVELOPMENT

The National diagnostic report (2014: 130) recommends that research discoveries, advanced ideas and strategies be used in teaching and learning of geometry. Understanding theoretical models present the chance to formulate strategies that have significant possibility of victory. Teachers concur with the development of new ideas and approaches; and advocate their implementation in the teaching and learning to overpower the difficulties that are experienced in geometry classrooms. The van Hiele framework for geometric thought that was discovered by Pierre van Hiele and Dina van Hiele-Geldof in 1957 and 1986 focuses on geometry. To date, the framework is a support for curriculum implementation in mathematics learning in many countries such as Netherlands, Germany, Russia and United States (Halat, 2003). Research (Usiskin, 1982; Alex & Mammer, 2015; Kariyana & Sonn, 2016; Dağli & Halat, 2015; Tedla & Sewasew, 2016) confirms the credibility of the framework. From the mid-1980s, there has been an increasing attention in the teaching and learning of geometry (Mayberry, 1981; Burger & Shaughnessy, 1986; Usiskin, 1987; Gutierrez, Jaime, & Fortuny, 1991; Clements & Batista, 1992).

![The van Hiele Theory of Geometric Thought](image)

Figure 2.1: The van Hiele Framework for geometric development, Adapted from Van de Walle (in Luneta, 2015)

Luneta (2015), Alex and Mammen (2016) point out that the van Hiele model of geometric thought is associated with Piaget’s five developmental stages in children; and the part these stages play in learning of geometry. For instance, the neo-constructivism model of conceptual change argue that in order to take in new information that is somehow not compatible with previous information, learners will try to assimilate the new information into their existing schema (Kajander & Lovric, 2010). Learners will undergo cognitive conflict when the new information is absorbed, and learners will have to solve the conflict by interpreting the new
information with respect to the prior one in trying to find balance in Piagetian sense (Kajander & Lovric, 2010). Piaget (1971) argues that a child’s geometric thinking matures with age. For a child to develop ideas with respect to shapes, he/she needs to interact with the objects physically. Clement, Swaminathan, Hannibal and Sarama (1999:193) also emphasise that ‘children’s representation of space is constructed through the progressive organisation of the child’s motor and internalised actions’. However, van Hieles strived to examine different facets involved in learning of geometry and space (Luneta, 2015).

Research (Halat, 2003; Luneta, 2015) states that van Hiele established the existing five levels of geometric thinking. According to the model, learners advance their geometry knowledge according to the following five levels (Figure 2.1): Level 1, Visualisation; Level 2, Analysis; Level 3, Ordering; Level 4, Deduction, and Level 5, Rigour. However, research (Usiskin, 1982; Clement & Batista, 1992; Clement et al., 1999; Mayberry, 1983; Burger & Shaughnnessy, 1986) assert the extant of the first four levels at high school level; and claim that the fifth level is not for high school learners, but more suitable for college or university students. Additionally, Clement and Bastista (1992: 356) discovered the existence of a Level 0, Pre-recognition, which is explained as “children initially perceive geometric shapes, but attend to only a subset of a shape’s visual characteristics” (Davis & Guarino, 2016). At level 1, shapes are determined by the way they look. That is, learners identify shapes based on their mental pictures about that shape as opposed to the mathematical characteristics of the shape. For example, learners can identify a square because of its shape that looks like a box.

![Figure 2.2: Learners identification of shapes at visualisation level. Adapted from Vojkuvkova (2012)](image)

At the Analysis level, learners identify shapes according to the properties the shapes have, but these properties are seen as isolated and not related.

![Figure 2.3: Learners are able to identify shapes based on their properties. Adapted from Vojkuvkova (2012)](image)
The Ordering level comprises of identification of relationships between shapes and comprehension of logical deduction. At this level, learners give informal arguments based on previously uncovered properties and rules. Deduction level is characterised by creating formal deduction and comprehension of logical deduction that include axioms, definitions, theories and postulates. For example, a learner should be able to give reasons for steps in proving. Alex and Mammen (2015) point out that the highest level, Rigour, comprises of the analysis of different axiomatic systems and being able to compare these systems. Van Hieles (1986) conclude that the levels possess five exclusive properties. Other researchers (Usiskin, 1982; Mayberry, 1983; Burger & Shaughnessy, 1986; Senk, 1985) also confirm the properties as:

- **Intrinsic and extrinsic properties:** At level 1, shapes are recognised according to their properties, but an individual’s thinking at this level is unaware of such properties (van Hiele, 1986)

- **Hierarchical organisation:** The manner in which learners think at level 2 is impossible without thinking from level 1; and thinking at level 3 is impossible without thinking at level 2 (van Hiele, 1986). That is, levels are designed in such a way that a learner cannot operate based on an understanding of one level without having gone through the other previous levels. This is seen as one of the reasons for learners finding it difficult to understand the teacher (van Hiele, 1986)

- **Discontinuity:** As van Hiele states “the most distinctive property of the levels of thinking is their discontinuity, the lack of coherence between their networks of relations” (van Hiele, 1986: 49). That is, at a particular point during the teaching and learning process, the learning process stops, and later continues as it was; the learner who has reached a particular level at that time will remain at that level for a while as if he/she reached maturity and at that particular point in time, the teacher is unable to successfully explain the subject.

- **Linguistic property:** Each level has its own set of symbols and language characteristics which is relevant at that particular level and can be seen as irrelevant another level. That is, two people who operate at different levels speak different languages (van Hiele, 1986). This depicts what happens between the teacher and learners. Not one of them can manage the thinking process of the other and their understanding of one another can only start if the teacher tries to understand what learners are thinking. This suggests that teachers who start to teach geometry should position themselves in a language that their learners will comprehend and should not use the language of higher levels, which learners have not yet acquired.
Advancement: Shift from one level to the higher level is not natural but occurs through teaching and learning process. Transition is not possible without the learning of a new language (van Hiele, 1986). This suggests that maturity to the next level takes place in a special way, under adequate and effective teaching and learning processes that are significant at lower levels in order to be able to think and reason at higher levels. For example, the role played by scaffolding in the teaching and learning points to the advancement from one level to the next (Ndlovu & Mji, 2012)

The table 2.1 indicates the van Hiele levels, associated with the ability of learners and applicable school grades as a guide that depends on the quality of instruction.

Table 2.1: The van Hiele framework (adapted from Howse and Howse, 2014)

<table>
<thead>
<tr>
<th>Level</th>
<th>Description</th>
<th>Ability of learners</th>
<th>Possible grades of learners</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Visual</td>
<td>The learner identifies names and compare shapes by their appearance</td>
<td>Grades R - 2</td>
</tr>
<tr>
<td>2</td>
<td>Analysis</td>
<td>Learners analyse shapes and discover properties by observing them</td>
<td>Grades 3 - 6</td>
</tr>
<tr>
<td>3</td>
<td>Abstraction</td>
<td>Learners discover and formulate generalisations with respect to previous studied properties and rules; develop informal arguments to be able to prove generalisations</td>
<td>Grades 7 - 9</td>
</tr>
<tr>
<td>4</td>
<td>Deduction</td>
<td>Learners are able to reason logically and prove theorems deductively</td>
<td>Grades 10- 12</td>
</tr>
<tr>
<td>5</td>
<td>Rigour</td>
<td>Learners establish theorems in various systems of postulates; compare and analyse deductive systems</td>
<td>Advanced school learners and students at universities</td>
</tr>
</tbody>
</table>

Luneta (2015) affirms that the van Hiele levels afford educators the chance to develop and assign learners with geometry activities that are appropriate to their levels of thinking. Furthermore, the van Hiele levels serve as an instrument to gauge present and future geometry performance. Nonetheless, Clement and Bastista (1992) and Clement et al. (1999) disagree with the van Hiele proposition that suggests that children between five and six years old operate within one level at a time. Instead, they claim that learners can operate at more than one level simultaneously. In addition, Dağlı and Halat (2015) reveal that another theory that has helped researchers to explain the how children reason and categorise shapes is the Prototype theory (Rosch, 1973). The main notion of the Prototype theory is that geometric thinking is not primarily defined by formal definitions, rules of features, rather, by the prototypes which represent geometric concepts. Then, categorisation takes place on the basis of similarity to the prototypes. For example, the principal component of triangle concept is the equilateral triangle. And other kinds of triangles
such as the isosceles, scalene or right angled triangle possess the resemblance to the equilateral triangle family. Linchevski (2010) also indicate that young children determine and classify shapes according to the comparison to the visual prototype. He explains that young learners apply a combination of visual prototypes and characteristics that are thrust upon the visual prototype. Findings (Mason, 1989) in a study among fourth and eighth grade learners revealed that there exist three categories of geometric thinking and reasoning in learners’ identification of geometry shapes. The first one is based on the shape’s appearance, for example “this resembles a triangle”. The second one is based on the “imposed attributes” of the figure (Hershkowitz, 1989). The last one is based on the important attributes of the geometric figure such as “a triangle is made up of three sides, three corners, and the sides are closed”. Mason (1989) hypothesises a strong influence of the prototype theory on identification of geometric shapes. Similarly, Hershkowitz (1989) findings in a study with grade five, six, seven and eight learners showed that learners experienced difficulties in determining non-prototyped triangles, which was a twisted equilateral triangle with the base that was not parallel to the bottom of the page. Additionally, studies among three to six years old learners revealed a high rate of non-acceptance of non-prototyped triangles (Hannibal, 1999; Clements & Batista, 1992). The findings imply that young learners determine and categorise geometric shapes by relying on visual prototypes, and this is not dependent on age or school grade.

How geometric concepts are formed is based on learners’ capability to use visual imagery. Learners at young age learn about shapes by determining visual representation of examples of shapes and as they mature, they start to pay attention to similar attributes of figures with informal definitions (Walcott, Morhb & Kastberg, 2009). However, young children identify geometric shapes incorrectly as a result of geometrical misconceptions from which reasons might be lack of adequate exposure to geometric vocabulary and different forms of shapes (Oberdorf & Taylor-Cox, 1999)

2.4 DIFFERENT LEARNER PHILOSOPHIES OF MATHEMATICS WITH RESPECT ERRORS AND MISCONCEPTIONS

Research (Makonye, 2013) reveals that many learners hold different philosophies relative to mathematics although they may not be aware of them. The author advises that it is always important to recognise the philosophical partialities of learners and assist in understanding the errors and misconceptions that learners are susceptible to. Mathematical researchers present diverse and competing perceptions on the nature of mathematics. According to Korner (1986), the philosophy of mathematics provides an interpretation of mathematics. Furthermore, Makonye (2013) explains that the philosophy of mathematics focuses on the epistemological and ontological premises of mathematics which are notable in educational discipline. The Mary
Webster Online dictionary (2016) defines epistemology as a branch of philosophy that investigates the origin, nature, methods, and limits of human knowledge; whereas ontology is the study of metaphysics that deal with the nature of being. Makonye (2013) alludes that epistemology further focuses on methodologies that are employed in accessing the knowledge that is believed to be real. Thus, the philosophy of mathematics bears an influence on the type of mathematics knowledge that is included in the curriculum and how this knowledge is transmitted.

Osei and Eves (1985) again add that the nature of mathematics goes beyond the conceptual and procedural knowledge; it influences how individuals learn mathematics, what they perceive to be important in mathematics and what is not. For instance, if learners hold the belief that mathematics is concerned with application of formulae, then they become eager to memorise formulae; they become attracted to only the deductive application of the formulae in order to arrive to answers to questions. Because learners hold differing views about what is important in mathematics, they tend to copy beliefs of their previous teachers and in turn hold different errors relative to their beliefs (Makonye, 2013). This suggests that the belief systems embraced by learners have an influence to errors and misconceptions that they might have. Makonye (2013) advises that in carrying studies related to errors and misconceptions in mathematics, it is imperative to hold diligent discussions about the nature of mathematics. Additionally, Ernest (1985) emphasises that philosophy is of utmost importance in mathematics instruction because it affect the nature of mathematics, how it is taught and learnt. In short, philosophy accommodates perspectives and belief systems with respect to mathematics. The five philosophies of mathematics that will be discussed are Platonism, Formalism, Logicism, Intuitionism and Fallibilism.

### 2.4.1 Platonism

Platonism is named after the Greek philosopher and mathematician Plato (Makonye, 2013). This has been the dominant philosophy until the 20th century when irrational and complex numbers and counter statements were discovered. The Platonism philosophy perceives mathematics as having a classic existence in the metaphysical world (Makonye, 2013). For instance, numbers are seen to exist firmly in the world. However, Pythagoreans became devastated to uncover that the square root of two is not rational, and could be written as a ratio of two whole numbers (Eves, 1990). This led to the death of the Pythagorean who came with the discovery; because this revelation intimidated their powerful held belief (Eves, 1990). Platonists believe in a firm body of mathematical knowledge when compared to the ever-changing nature of mathematical knowledge as suggested by constructivist philosophers (Makonye, 2013). Furthermore, Platonists believe that knowledge can be narrowed to construct various new forms of other disciplines such as mathematics, geography, natural sciences, e.t.c. (Eves,
Makonye (2013) claims that Platonists still exist, though they encounter problems when they have to explain how mathematics should be taught; because their philosophy suggests that teaching of mathematics has to be through the narration of the mathematical facts that have already proven to exist. He argues that if this belief is of assistance to mathematics educators, then it is important for teachers to identify learning difficulties that learners come across in order to have fixed knowledge of what they are to find out. This suggests that discovering mathematics errors and analysing them is key in the teaching and learning of fixed mathematical knowledge.

### 2.4.2 Logicism

The Logicism philosophy argues that mathematics can be narrowed down to laws of logic (Frege, 1991; Russell, 1919). As a result, mathematics can be gleaned from rules, axioms and postulates; from which results can be accessed through careful and extensive reasoning (Russell, 1919). Axioms and postulates are statements that are seen to be true and need not to be proven (Makonye, 2013). For instance, Russel (1919) makes an example using a claim by Euclid’s principle that the shortest distance between two points is a straight line. But, he uncovered that these laws are sometimes not consistent and appear to be contradictory. For example, Russel (1919) makes use of Euclid’s postulate of the shortest distance between cities; that it does not hold in spherical geometry. That is, the shortest distance between two cities on the globe is not a straight line, but a line on a circle. This discovery led to the revamp of Logicism as the nature of mathematics.

### 2.4.3 Formalism

Formalists claim that mathematics is a game with no meaning (Snapper, 1979). This game is played with marks on “paper, following the rules” (Ernest, 1985: 606). Formalism argues that mathematics is a formal logic system that relies on postulates and axioms in proving theorems (Hilbert, 1862 - 1943). This implies that mathematics is based on sets of symbols and rules that manipulate these symbols. Makonye (2013) believes that this philosophy is an anathema to mathematics instruction. He claims that no concrete learning can be gained from such belief where learning is not dependent on contexts and meaning. He further states that mathematics problem solving is not possible with Formalism belief. Makonye (2013) makes an interesting assertion that mathematics teachers who are poorly trained often endorse this philosophy. Stein et al. (2003) also stress that Formalism is associated with procedural teaching and learning, which emphasise the application of rules without understanding (Skemp, 1978). Learners who support this philosophy of
mathematics hold various misconceptions because they possess slightest comprehension of what mathematics is all about.

2.4.4 Intuitionism

Makonye (2013) argues that Intuitionism counteracts Platonism. Intuitionists argue that people’s beliefs and views occur in their minds (Brouwer, 1881 - 1966). The author views intuition as playing the key role in construction of mathematical knowledge, that is, mathematics is present in the minds of people. Intuitionists believe that the classical, absolutist mathematics is unsafe (Makonye, 2013). Their belief is that mathematical knowledge cannot exist beyond what can be proven. However, Makonye (2013) disagree with their belief, stating that the intuitionists disregard the social nature of mathematical knowledge where mathematical statements are exposed to detailed tests by other mathematicians. He further argues that the highlight of this philosophy is that learners frequently make errors and as a result, misconceptions become evident from their intuitions. Intuition plays a key role because it demonstrates how learners apply their instincts in having alternate conceptions (Makonye, 2013). Brouwer (1966) suggests that instincts that result from generalising correct mathematical solutions create errors, which result in misconceptions. Embracing intuitionism is a good start for teachers because they can hold conclusive mathematical solutions while awaiting learners to re-evaluate their instincts before agreeing to publicly validated mathematical results (Makonye, 2013). Additionally, the author agrees that intuition is a good choice for generating conjectures that can be fiercely examined before they are endorsed.

2.4.5 Fallibilism

This recent philosophy describes the nature of mathematics and is anti-absolutism (Lakatos, 1976; Polya, 1973; Hersh, 1997). Fallibilism is based on the constructivist idea that people create knowledge, including mathematical knowledge (Makonye, 2013). This philosophy suggests that knowledge is the product of human construction and their intelligence; and that this knowledge cannot exist independently like Platonists argue. Fallibilists believe that socio-cultural and historical evolvements are important in describing the nature of mathematics (Eves, 1990). Furthermore, Fallibilists claim that knowledge does not need to present perfect proof; and that mathematics is shaped by recognising that previously discovered results had errors (Eves, 1990). The author makes an example of the Pythagorean discovery that the square root of two is not rational, arguing that mathematical knowledge constructions are
susceptible to errors and misconceptions, even to the most gifted mathematicians such as the Pythagoreans. Eves (1990) again argues that even Godel’s theorems indicated that mathematics is not ontologically confirmed. Research (Makonye, 2013) states that Fallibilism perceive mathematical knowledge to be constructed through ways that are not formal, such as the trial and error method. The Fallibilists perceptions of mathematics include ethnomathematics, equal access to mathematics and different cultural identities (Van der Westhuizen, 2012; Makonye, 2012). This philosophy takes into account that learners make mathematical errors as proof of their fallibility; and this is seen as acceptable and natural.

The above stated philosophies provides a paradigm to study learner errors and misconceptions in mathematics and how these errors and misconceptions are related to philosophies they hold. Makonye (2013) acknowledges that each of the philosophies own a rightful place in mathematics instruction, and each possess its own strengths and weaknesses. He further confirms that philosophies of mathematics serve as an important basis in conducting research on learner errors and misconceptions because they cast light on learners’ thinking about the nature of mathematics. He argues that the inability of learners to understand formal mathematical symbols results in their failure in their inability to comprehend important mathematical concepts; and the lack of understanding these mathematical notation results in syntactical errors and misconceptions (Makonye, 2013). This understanding is important in comprehending how learners conceive of their errors in mathematics. He believes that all the philosophies are relevant to various aspects of mathematics, as well as its instruction.

For educators, it is also important to take into consideration learners current philosophies. This will allow teachers to assist learners more, once they start to comprehend the world-view of mathematics from learners’ perspective. In this way, learners may be supported to adopt other philosophies that assist them in understanding mathematical concepts better. For instance, a learner who holds a Formalist’s view on nature of mathematics might perceive mathematics as a subject that is made up of meaningless symbols which have to be kept in mind and manipulated correctly to get correct answers. Such learner needs assistance to understand that mathematics is a tool that is used to solve every day problems. Finally, in pledging support for the reasoning behind studying learners’ misconceptions, Carpenter (1996) believes that wrong answers or alternative conceptions can be a stepping stone for rigorous discussions about mathematics pedagogy. Fang (2010) also asserts that wrong answers can inculcate the cultural pedagogy of transforming errors in effective instructional resources of logical thought. It is also possible to use misconceptions that learners hold as an important tool in empowering them in proof “checking and construction”, which is a pedagogical method that has been seen to work (Clements & Batista, 1992:55)
2.5 THE ROLE PLAYED BY TEACHER QUALITY, PEDAGOGICAL CONTENT KNOWLEDGE AND CONTENT KNOWLEDGE IN MATHEMATICS INSTRUCTION

Mathematics is a subject that opens doors for human development in professions such as engineering, medicine, statistics, science and accounting. This is confirmed when one considers that of the many scarce skills in South Africa, majority of them needs a decent pass as a prerequisite in grade 12 mathematics (Mhlolo, 2016). What can be deduced from this is that without a pool of mathematicians, South Africa will not be able to take part in the global economic transformations with the rest of the world in terms of economic growth and scientific research. This suggests that as a country, South Africa know its desired goal, but in spite of this vivid results from international mathematics assessments and local national assessments, results show that as a country, we are not yet closer to achieving it (Spaull, 2013; ANA, 2014).

Spaull (2013) affirms this bad reality by presenting an outline of the quality of South African mathematics teachers when compared with their colleagues from other African countries. From his view, the evidence suggests that there is a crisis in the country’s education system, and that the current education system is doing an injustice to the youth. This suggests that as a country, we need to work very smart in achieving our desired goal. The process of venturing in successful mathematics lessons amongst other factors focuses on the quality of the teacher. As the report states “factors that has to do with teachers and teaching are the most important influences on pupil learning” (OECD, 2005:2). This suggests that teacher quality is one of the most significant factors that influence learner achievement. Likewise, evidence suggests that “the main driver of the variation in pupil learning at school is the quality of the teachers” (Barber & Mourshed, 2007). Spaull (2013) emphasises that for one to be classified as a quality teacher, he/she must comply with characteristics such as values, credentials, competency, knowledge and skills amongst others. Wong et al. (2015) state that in creating effective mathematics lessons, teachers need to possess the knowledge for teaching mathematics, for instance how learners typically learn to multiply, and the errors and misconceptions they encounter during that process.

The knowledge should not only focus on mathematics content knowledge, but also consider pedagogical content knowledge (Wong et al., 2015). Wong et al.(2015) define mathematics content knowledge as the content and communication of mathematics that include concepts and procedures and relationships between them; various representations of mathematical concepts and procedures; methods of mathematical reasoning,problem solving strategies; ability to communicate mathematics effectively at different levels; knowledge of the curriculum and knowledge of how learners learn. Similarly, Julie (2015) adds that mathematics content focuses on the discipline, and is made up of domains and knowledge statements. Gardner (1972: 26 & 27) defines disciplines as “the span of alphabets
from aerodynamics to zoology”, and domains are “objects that are studied or explored, such as living things or elements. “Knowledge is what is produced by the discipline and can be a set of assertions or verifiable truth claims”. In simple terms, this is referred to as content knowledge. While pedagogical content knowledge refers to the skills educators use to teach mathematics successfully (Eong, Meng & Rahim, 2015). Anthony and Walshaw (2009) also define pedagogical content knowledge as teachers’ skills, knowledge and beliefs with regard to mathematics and their comprehension about mathematics teaching and learning. Hill (2008) explains pedagogical content knowledge as having three constituencies, that is, knowledge of content, knowledge of the curriculum and knowledge of teaching and learning. Shulman (1986) argues that apart from the subject content knowledge, teachers need to possess specialised skills; that is, unique form of mathematical knowledge that is particular to their work in classrooms. In his view, Shulman (1986) noted that these specialised skills include the understanding of what makes certain topics challenging or easy to learners, conceptions and misconceptions of varying ages of learners and different backgrounds that such learners bring to the classroom; how learners solve mathematical problems; how they develop mathematical knowledge and what contributes to difficulties learners come across in the class.

Learners’ thought processes are made up of many things such as formulae, application, banality, enjoyment and attitudes about mathematics; and problems that may lead to learning difficulties in mathematics are those that may result from previous insufficient learning, informal thinking or poor remembering. Eong et al. (2015) emphasise that teachers need to possess specialised skills in treating misconceptions productively and in a sensible way because such misconceptions can be used in developing conceptual knowledge in learners. Furthermore, AMESA (2015) stresses that if teachers seek ways of understanding why learners have misconceptions, they may come to appreciate learners’ thoughts and find ways of engaging learners’ current knowledge in order to develop new knowledge. This knowledge is what Shulman (1986) calls pedagogical content knowledge. He noted that research on learners’ thinking and mathematical concept formation provides an important foundation for pedagogical content knowledge. For example, he found out that if teachers had PCK as one of their credentials, their classroom practice changed and learning improved when compared to teachers who did not study PCK.

More studies investigated preservice teachers’ knowledge of student learning in Israel. Tirosh (2000) reports that prospective teachers in Israel were aware of common misconceptions and errors (for example, inverting the dividend as opposed to the divisor) made by learners including overgeneralisation of whole number rules to fractions (for example, division makes numbers smaller; multiplication make product bigger). Figure 1 below demonstrate Shulman’s (1986) notion of how the knowledge of how learners learn is related to subject content and pedagogical content knowledge. The left side of the oval that is named Subject Matter knowledge is made up of two strands common content knowledge which is the knowledge used
in the teaching profession in ways that are similar in other professions and occupations that use mathematics; specialised content knowledge allows teachers to participate in teaching such as how to present mathematical ideas, explanations for common rules and procedures and evaluate and understand unusual problem solving methods (Ball et al., 2005); Common content knowledge is what Shulman (1986) calls subject knowledge, this knowledge does not include knowledge of learners and teaching. The right side of the oval presents strands related to PCK and contains KCS, knowledge of content and teaching, and knowledge of the curriculum. KCS is thus the subset of PCK.

![Figure 2.4: Domain chart for mathematical knowledge for teaching and learning. Adapted from Shulman (1986)](image)

These knowledge domains require that effective mathematics teachers draw upon the content knowledge and modify it to meet the teaching and learning demands. Tato et al. (2012) outline two reasons for effective teachers to possess mathematics content knowledge and pedagogical content knowledge; first, teachers’ knowledge has positive influence on learner achievement in mathematics; second, knowledge that future teachers gain in their final year of studies might be an important indicator of the success of their education program. Hill (2008) points out that a teacher may possess great knowledge of the content but possess a weak knowledge of how learners learn or vice versa. This may have implications for developing assessments in determining how learners learn. However, research studies (Hill, 2015) about the relationship between pedagogical content knowledge and learner outcomes is
limited. Even though extensive research has indicated that professional development that is centred around pedagogical content knowledge results in transformed classroom performance and improved learning, these results remain limited (Levi, 2001; Cobb et al., 1991; Saxe, Gearhardt & Nasir, 2001). This suggests that teacher characteristics such as qualifications, experience, and general mathematics knowledge provide limited information for various learner achievements across mathematics classrooms and Hills (2015) advice that more research should focus on developing newer instruments that capture the key teacher qualities. In a unique twist, Luft (2014) discovered a positive correlation between pedagogical content knowledge and learner performance. In his study, he found out that teachers with background in Physical Sciences had influence on learners’ performance in Life Sciences. These studies reveal fascinating hypotheses, which teachers’ knowledge about learners learning is propositional and distinct as opposed to the connectedness relating to mathematics and conceptions about how learners learn.
CHAPTER 3: RESEARCH METHODOLOGY AND DESIGN

3.0 Introduction

In any given study, all research is founded on basic philosophical presumptions with regard to what makes it valid and which research methods are suitable for knowledge development. In carrying out and examining research, it is imperative to have knowledge of what these presumptions are. This chapter presented the methodology that the researcher used in answering the research questions. Research methodology is the systematic, theoretical analysis of methods employed to the field of study (Thomas, 2010). It encompasses a collection of views used in research to collect data. These perspectives are then used as the basis for deduction, conclusions, explanations and predictions (Nicholson, 2011). The chapter followed the following outline:

- research design
- research methodology
- data collection instruments, and
- data analysis method, ethical considerations, validity and reliability of data were taken into account

3.1 Research design

Burns and Groove (in De Langen, 2009) define research design as “a blueprint for conducting a study with maximum control over factors that may interfere with validity of the findings” (Burns & Groove, 2003: 195). Other authors (Bless, Higson-Smith & Kagee, 2006: 71) explain research design as “operations to be performed, in order to test a specific hypothesis under given conditions” or “an overall plan, according to which respondents of the proposed study are selected, as well as the means of data collection or generation” (Welman et al., 2009:46). In this sense, a research as a functional plan from which specific research methods and procedures are linked to gain reliable and valid body of information. This plan is designed in a way that illustrates how other pieces of the research study fit together, that is, samples, measures, variables, e.t.c. to answer the research questions. For instance, a research problem determine methods and procedures, types of measurements, sampling, data collection and analysis that will be used for intended research (Zikmund et al., 2010:66). According to Mouton (1996: 175) the research design “plans, structure and execute” the research to increase the “validity of the findings”. It provides a layout from the underlying philosophical presumptions to research design and collection of data. Yin (2003: 19) also asserts that “colloquially a research design is an action plan for getting from the initial set of questions to be answered to answers".
The document analysis and focus group interview methods were used to collect data. Literature support for each of the data collection methods for the study was also discussed. In order to guarantee trustworthiness of the research study, relevant criteria for qualitative research were explained; several strategies included member checks, peer review and triangulation were proposed and used. The figure below represents the framework design and development of the research methodology and design.

Figure 3.1: Research Design. Adapted from Sekaran & Bougie, 2010

### 3.2 The case study approach

A case study is one of many ways in which researchers can use in conducting research because of its aim of understanding human beings in social situations by interpreting their activities either as a single case, a group, and a community. Gillham (2000a) defines a case study as an examination to answer specified research questions which attempt to find various forms of evidence from case settings. Yin (2003) defines a case study as a factual enquiry that examines extant phenomenon within its real life existence, especially when the borderline between the phenomenon and the context are not defined clearly. Nicholson (2010) argues that the case study is particularly important in cases where contextual conditions of
the occurrence under investigation are analytical and where the researcher does not have any control over the activities as they take place.

Given the interpretive paradigm used in this study, the case study method was selected as the relevant approach to use because it provided the coherent way of collecting data, analysing the data and reporting of the results (Thanh, 2015). Thus, it enabled the researcher to understand particular problem or context in great detail. Interpretive paradigm provided the researcher with the chance to understand participants’ various views on the acquisition of geometric thought, ability to make use of various data collection methods and observe the face to face instruction within the setting. Such detailed descriptions gave the researcher multiple interpretations (Yin, 2003). The researcher chose the case study because it particularistic, descriptive, heuristic and inductive (Merriam, 1991). Particularistic reflected one occurrence which is how NC (v) level 3 learners acquire geometric thinking; descriptive referred to the detailed, rich data gained with respect to the phenomena; heuristic reflected the advancement of understanding the phenomenon, while inductive referred to the form of justification that was used to find generalisations and forms of concepts that appeared from the data (Nicholson, 2010).

3.3 Research methodology

Research methodology refers to the strategy of investigation that traverses from primary presumptions to the research design, and data collection (Myers, 2009). It is also described as the researcher’s perspective in conducting the research project (Leedy & Ormond, 2010; Babbie & Mouton, 2008). Even though there are other distinctive research methods, the most common classifications are quantitative and qualitative (Nicholson, 2010). Quantitative research methodologies were initially used in natural sciences to analyse natural phenomena, whereas qualitative methods were used in social sciences to evaluate social and cultural phenomena (Vosloo, 2014). Neither of these methodologies are better than the other.

Applicability of which methodology to use depends on the context of the research study, nature and the purpose of the research question, and sometimes researchers favour the use of the mixed method to take the advantage of the differences between qualitative and quantitative (Bryman & Burgess, 1999). Qualitative research methodology is realistic as it tries to examine every day activities of different groups and communicates it in their natural settings; it is important in studying educational settings and processes (Nicholson, 2010). According to Fleming (2007:24), “qualitative research aims to explore and to discover issues about the problem at hand, because very little is known about the problem. It uses “soft data to get rich data”. Myers (2009) adds that qualitative research aims to assist researchers to understand people, social and cultural environments from which they live. Within the qualitative approach several data collection strategies and analysis methods are used (Creswell, 2013). These include observations, participants’ observations
(fieldwork), interviews, questionnaires, documents, texts, and researcher’s impressions and reactions (Myers, 2009).

Quantitative approach on the other hand provides statistical results which are represented in numerical and statistical data (Hittleman & Simon, 2002). Furthermore, it uses questionnaires, surveys and experiments to collect data that is revised and presented in a tabular form, which is characterised by the use of statistical analysis (Nicholson, 2010). Quantitative researchers measure variables based on a sample of subjects and demonstrate the relationship between variables using statistical procedures such as correlations, relative frequencies, differences between means (Vosloo, 2010). Furthermore, in quantitative research study, a hypothesis is needed before the research can commence. The table below outlines a summarised version of the main differences between qualitative and quantitative approaches to research.

<table>
<thead>
<tr>
<th>Orientation</th>
<th>Quantitative</th>
<th>Qualitative</th>
</tr>
</thead>
<tbody>
<tr>
<td>Assumption about the world</td>
<td>A singular one, i.e., can be evaluated by an instrument</td>
<td>Multiple realities</td>
</tr>
<tr>
<td>Purpose of the research</td>
<td>Discover relationships between variables that are measured</td>
<td>Comprehension of the social situation from objects point of views</td>
</tr>
<tr>
<td>Methodologies and procedures</td>
<td>- Discovery of relationships between measured variables - Hypothesis needs to be formulated before research is started - Deductive in nature</td>
<td>Flexible - design is established as data is collected - Hypothesis is not required to start the study - Naturally inductive</td>
</tr>
<tr>
<td>Role played by the researcher</td>
<td>Objective observer who neither partake nor have influence on what is being investigated</td>
<td>Becomes participant and get involved in the research environment</td>
</tr>
<tr>
<td>Generalisations</td>
<td>Universally context free generalisations</td>
<td>Descriptive context based generalisations</td>
</tr>
</tbody>
</table>

Table 3.1: Differences between qualitative and quantitative approaches. Adapted from Allwood (2012)

### 3.3.1 Rationale for a qualitative study

The researcher used the qualitative methodology and design for this research. Descriptive statistical approach was used to analyse data (inductive data analysis). Eighty (N=80) NC (v) level 4 learners wrote a geometry performance test (Appendix 5), and 9 of them were selected for focus group interviews (Appendix 7). Focus group interview participants were selected based on their performance in the geometry performance test. Additionally, justification was provided for the use of each of the data collection methods. Scholars (Domegan & Fleming, 2007; Henning, Van Rensburg & Smit, 2004; Denzin & Lincoln, 2011; Richardson, 1995) state that human learning is best evaluated using qualitative data. When a researcher chooses a research methodology, “it is proper to select a paradigm whose assumptions are
This research focused on human learning on acquisition of geometric thought. It should be pointed out that as a qualitative research, the researcher focused on processes rather than only the end results of the study. This approach enabled for thick narrative explanations of the phenomena under investigation and provided the researcher with the chance to consider the perceptions of the participants and the delicacy of complex group interaction and various interpretations in the group’s natural habitat. The researcher discovered the qualitative explanations of their experiences and an inductive analysis of data as relevant for the purpose of this research because all these processes improved the possibility of some objectivity which would have not been evident if quantitative methodology was applied (Delport & De Vos, 2011:65). Additionally, the constructed knowledge is not truth that will remain fixed and generalizable across other possible situations; rather, it existed within this specific context (Kumar, 2005: 12; Jones & Kottler, 2006:83). It was also to be noted that the results of this research study do not favour extensive generalisations; rather, they represented contextual findings that assist in developing knowledge and understanding on acquisition of geometric thought.

Learning is a composite process. For the learning environment to be effective, it needs composite interaction of various variables (Ivankova et al., 2007:57). Assessment of learning was preferably done when learning was in action rather than after it had occurred by observing how the participants are partaking and progressing in their endeavour (Leedy & Ormrod, 2010: 94 - 97). As a consequence, explanation of the processes or occurrences was more of value than the research outcomes (Creswell, 2009:175 -177). Furthermore, it would become challenging to guess with accuracy the behavioural characteristics of complex organisms. Hence, the use of quantitative methodology would make dubious some of those perceptions and views of participants that the researcher needed to comprehend in addressing the complexities of learning processes and contextual factors in the learning environment (Kumar, 2011: 13-20, 104 -105).

3.4 Research paradigm

Within the qualitative research approach there are various paradigms, interpretivism is one of them. Easterby-Smith, Thorpe and Jackson (2012) argue that the research process is made up of three main facets, those are, ontology, epistemology and methodology. According to them, a research paradigm encompasses related practices and thinking that explain the nature of the research along these facets. The term paradigm was derived from the Greek word paradeigma, which denotes a pattern (Kuhn, 1962). In general, a paradigm is explained as “a whole system of thinking”(Neuman, 2011:94).This implies that a paradigm represent a set of values, presumptions, and beliefs that researchers believe in regarding the conduct and nature of research(Mouton, 1996: 203), or a framework of philosophy(Collins & Hussey, 2009:55). Olsen, Lodwick and Dunlop explain a paradigm as “a pattern, structure and framework, or a system of scientific and academic ideas, values and
assumptions” (1992: 16). It is seen as a blueprint for understanding, observing and evaluating problems and finding solutions (Creswell, 2009; Babbie, 2010; Rubin & Babbie, 2010; Babbie, 2011). This suggests that a paradigm is a set of fundamental beliefs that direct the action. Vosloo (2014) emphasises that paradigms play an important role in social science research.

The interpretive paradigm was selected for this study. It is also known as the phenomenological approach (Vosloo, 2014). De Vos et al. (2011) and Neuman (2011) argue that interpretive social science can be tracked back to Max Weber (1864 - 1920) and Wilhelm Dilthey (1833 - 1911). Dilthey stated that there are two main types of science, namely, the natural sciences and the human sciences. The first is based on abstract explanation, and the second is concerned in a understanding based on the lived experiences of people (De Vos et al., 2011b; Neuman, 2011). Weber held the belief that all humans are trying to make sense of their worlds. By so doing, they continue to interpret, create, assign meaning, define, justify and rationalise their daily actions (Babbie & Mouton, 2008: 28).

3.4.1 Rationale for interpretive paradigm

The interpretive paradigm allowed the researcher to understand social interactions between participants’ views on acquisition of geometric thought (Gephart, 1999). The researcher focused on understanding and interpreting everyday occurrences, experiences and social structure (Collis & Hussey, 2009; Rubin & Babbie, 2010), that is, dominant errors displayed by NC (V) level 3 learners and some of the misconceptions responsible for these errors. According to Schwandt (2007), reality must be interpreted through meanings assigned by participants to their world. This meaning can only be found through language and not exclusive quantitative analysis (Schwandt, 2007). Interpretivists further hold the belief that social world cannot be comprehended by using research principles from the natural sciences perspective (Blumberg et al., 2011).

Ontological and epistemological facets are based on an individual’s perspective, which has a key influence on the perceived importance of reality. According to Scotland (2012), two perspectives are objectivistic and constructivist. These differing ways of perceiving the world have consequences in the academic fields; yet, none of them is seen to be superior against each other. Both are seen relevant for some purposes and inadequate or complex for other purposes. Again, an individual may change his/ her view based on a particular situation. This research employed elements from both perspectives and see them as complementary.
3.5 Research area

The research study was undertaken at one satellite campus of the South West Gauteng TVET College in Gauteng Province. South West Gauteng TVET College comprises of one head office and six satellite campuses which are located throughout Soweto to Roodepoort and Randburg and a farm in Sterkfontein. It was initially under the auspice of the Department of Basic Education (2007 - 2015), until the recent function shift of all the public TVET colleges to the Department of Higher Education (2016). Prospective learners who intend to study at this TVET College can either follow Engineering, Business, or Utility studies. Such learners need not to stay at school until they complete their grade 12. To register at a TVET college, one need to have passed grade 10, however, TVET College is more appropriate to learners who have passed grade 12. The college offers vocational training that culminates in an NQF level 4 which opens doors to the world of work. Various vocational subjects are offered at different levels of NQF, which begins at NQF level 2, 3 and 4; with common fundamental compulsory subjects: English, Life Orientation, and Mathematics/ Mathematical Literacy.
3.6 Sampling

A purposeful sample of 80 NC(v) level 4 learners who are enrolled for a course in Mathematics were selected for the study. Purposeful sampling is frequently used when small samples are investigated (Gledhill et al., 2008). It is a non-random sample method which focuses on “information rich” cases for a detailed investigation (Patton, 2002). It is widely applied in qualitative research and focuses on the experience participants possess with regard to the research topic under investigation (Nicholson, 2010; Kyngäs et al., 2011). Participants were selected on the basis that they had completed their grade 12 with mathematics (geometry) as part of their assessment. In this study, focus of interest was based on the type of learners who were in a better position to help in answering the research question (best performing, average performers and poor performers). Age of participants ranged between 19 – 35 years. The table below illustrates gender and ethnical composition of participants.

<table>
<thead>
<tr>
<th>Ethnical classification</th>
<th>Gender</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zulu</td>
<td>Males: 5</td>
</tr>
<tr>
<td></td>
<td>Females: 6</td>
</tr>
<tr>
<td>Tswana</td>
<td>Males: 8</td>
</tr>
<tr>
<td></td>
<td>Females: 5</td>
</tr>
<tr>
<td>South Sotho</td>
<td>Males: 8</td>
</tr>
<tr>
<td></td>
<td>Females: 4</td>
</tr>
<tr>
<td>Xhosa</td>
<td>Males: 7</td>
</tr>
<tr>
<td></td>
<td>Females: 9</td>
</tr>
<tr>
<td>Venda</td>
<td>Males: 6</td>
</tr>
<tr>
<td></td>
<td>Females: 3</td>
</tr>
<tr>
<td>Northern Sotho</td>
<td>Males: 7</td>
</tr>
<tr>
<td></td>
<td>Females: 5</td>
</tr>
<tr>
<td>Coloured</td>
<td>Males: 1</td>
</tr>
<tr>
<td></td>
<td>Females: 2</td>
</tr>
<tr>
<td>Other</td>
<td>Males: 2</td>
</tr>
<tr>
<td></td>
<td>Females: 2</td>
</tr>
<tr>
<td>Total</td>
<td>Males: 44</td>
</tr>
<tr>
<td></td>
<td>Females: 36</td>
</tr>
</tbody>
</table>

Table 3.2: Participants’ gender and ethnical demographics.

There were 55% male and 45% female participants. Ethnic composition of participants was dominantly Black (95%), with Other (5%). They were from surrounding parts of Soweto such as Protea Glen, Chiawelo, Naledi, Dube, Orlando and Zola, within 50 kilometre radius from South West Gauteng TVET College. All participants provided their informed consent together with the consent of their Mathematics teachers and campus manager.
3.7 DATA COLLECTION AND ANALYSIS

3.7.1 Data collection methods

The researcher used the following data collection methods to gather data for the study:

3.7.2 Focus group Interviews

Nicholson (2010) explains interviews as methods of data gathering techniques that uses oral questioning using a set of pre-planned questions. They can be very informative because the interviewer can put forward relevant issues of concern that may lead to constructive suggestions (Shneiderman & Plaisant, 2005). For this study, the researcher used unstructured interviews. This allowed the researcher to pose open ended questions that enabled participants to express their opinions freely (Vosloo, 2010). Unstructured interviews allowed both the researcher and participants to feel comfortable around one another because it felt like a discussion on the given topic. Furthermore, the direction of the interview was determined by the researcher and participants. A total number of nine participants were interviewed (Appendix 7). They were divided in groups of threes based in their performance in the test; that is three high achieves, three moderate achievers and three lowest achievers.

3.7.3 Document analysis

Apart from the geometry performance test (Appendix 5) and focus group interviews (Appendix 7), I studied necessary documents that are relative to the topic under investigation. They included policy documents such as the Mathematics Subject Assessment Guidelines for NC (v) level 4 and Mathematics Subject Guidelines for NC (v) level 4, which are related to specifically circle geometry. I further studied journals, books, and government publications that were meant to provide detailed views of the problem under investigation (Vosloo, 2010).

3.7.4 Data collection process

Before the researcher could start collecting data, she applied for an ethical clearance letter. An ethical clearance involves a variety of best research practices such as protection of human and/or animal subjects in a research study; fairness, intellectual honesty and property, and fairness (DiClemente, 2015). Prospective researchers seek ethical clearance to ensure respect for people (Punch, 2013), beneficence and justice (McGlade, 2014), and justice. I used the ethical clearance together with the request letter to conduct research to apply for an access to undertake the enquiry from the Department of Higher Education, the college principal, the campus manager, through to participants. I visited the research site to introduce myself to the campus manager and participants. I explained to them the importance of this research; why is it important for them to take part and the ethical implications and...
considerations. Once participants understood ethical considerations and agreed to take part in the study, they were given consent forms to sign and return back to the researcher (Appendix 4). Participants were given a period of a week to ponder whether they were interested in taking part in a research study or not. The signing of the consent forms guaranteed that they are taking part in the research, although ethical considerations made a provision that they may withdraw from the study at any time, no penalties would be taken against them. It further built a relationship between the researcher and participants.

3.8 Data analysis process

The researcher used the interpretive qualitative data analysis method which involved inductive data analysis. Qualitative analysis sought to look for patterns, commonalities and / or contrasts (Sgier, 2012) in the contents of data analysed. It continued through a sequence of defined steps, for instance, assignment of codes, building of categories, and themes which are not only relevant and important for those who prefer qualitative analysis, but also for the generalised comprehension of the nature of qualitative analysis. Interpretive component focused at what the data said than what the data do (Sgier, 2012), and how meaning is developed by means of often implicit categorisation, creation of hierarchies and boundaries, and narrative construction of temporal sequences. With reference to inductive approach to data analysis, Thomas (2006) argues that data analysis process develops categories into a framework or a model. Such categories are created from the coding. The researcher adopted the inductive analysis that is suggested by Thomas (Figure 3.2). The Inductive analysis strategy involved five steps: preliminary reading of text data, identifying specific text segments relative to research objectives, labelling of the segments of the text data to create categories, reduction of data redundancy and overlapping among categories and creating a model that incorporated key categories.

<table>
<thead>
<tr>
<th>Preliminary Reading of text data</th>
<th>Identification of specific text segments relative to objectives</th>
<th>Labelling segments of text to create categories</th>
<th>Reduction of redundancy and overlap among categories</th>
<th>Creating a model that incorporate key categories</th>
</tr>
</thead>
<tbody>
<tr>
<td>Many pages of text</td>
<td>Many segments of text</td>
<td>30 to 40 categories</td>
<td>15 to 20 categories</td>
<td>3 to 8 categories</td>
</tr>
</tbody>
</table>

Figure 3.2: Inductive data analysis method of the study. Adapted from Thomas 2006.
3.9 Validity and trustworthiness of data

Validity and trustworthiness are the key elements to the successfulness of any data collection and analysis in any context in research. In ensuring validity and trustworthiness of data, the researcher conducted a pilot study with a group of NC(V) level 4 learners at a different campus to test whether the research instruments (Geometry test and interview questions) would yield intended results. From the pilot study, it was discovered that the research study was valid. McKibben and Silvia (2016) define validity of a research instrument as a degree to which the instrument accurately measures what it intends to measure. Research results were further reported in a transparent manner that represented perceptions of participants and of the documents which were reviewed. Another college was visited to authenticate whether the data received was truthful or not. An assumption was made that same results would be discovered should the research be repeated (Creswell, 2013).

Gastillo (2012) revealed that when a research is conducted over time, participants should also benefit from it. I began to notice this when learners began to reflect about their performance from the test they wrote, based on the errors they made as a result of misconception they held. This made them to gain some knowledge the researcher shared with them from the literature. They acquired some approaches on how to solve circle geometry problems and gain an understanding of geometry thinking. Gastillo (2012) further emphasised that research has the potential to change learners perceptions about the way in which they learn. It is also essential to guarantee a degree of quality in research by considering concepts such as credibility, transferability, authenticity, and dependability (Creswell, 2013).

3.9.1 Credibility

Diane (2014) defines credibility as the establishment that the results of the research are believable. It focuses more on the richness of data collected than on the quantity of data collected. The researcher invested time in revisiting participants to gain detailed knowledge on how they acquire geometric thought before reaching conclusions pertaining to the findings. Another method used was that of data triangulation by multiple analysis and member checking.
3.9.2 Transferability
Moon (2016) describes transferability as the degree to which research can be transferred to other contexts. For this task, the researcher ensured that she/he provided detailed descriptions of the research study, sample, and how data was collected. This allowed those who would have an interest in using the results to transfer them to other contexts.

3.9.3 Dependability
Guanavan (2015) defines dependability as the degree to which research findings are consistent should there a need for them to be repeated. The researcher tried to report research findings in a more detailed manner, making it possible for others to repeat the researcher’s work, but not necessarily to gain the same outcomes.

3.9.4 Conformability
According to Gunavan (2015), conformability questions how research findings are supported by data collected. This is the process of finding out whether the researcher was biased during the study. This is as a result that qualitative research allows the researcher to bring unique perspectives to the study. For this, an external analyst was used to study data collected and how decisions and conclusions were reached during the research study to judge whether the researcher was biased.
CHAPTER 4: DATA COLLECTION

4.0 Introduction

The previous chapter focused on the research design and methodology. It explained the research design, paradigm, methodologies, data collection strategies and tools, analysis methods and data credibility issues used in this study. This chapter focused on data collection.

In general, there are many different ways of gathering data, that is, tests, interviews, diaries, observations, journals, e.t.c. Often enough quantitative designs employ tests and close-ended questionnaires in gathering, analysing and interpreting data. Nonetheless, qualitative methods apply interviews, tests, diaries, journals, observations and open ended questionnaires to obtain, analyse and interpret data. On the other side, mixed methods data collection strategies use close ended questionnaires, interviews, classroom observations, etc. It should be noted that for social research, the choice of data collection technique relies on the needs of the research. In triangulating data, researchers get information using different methods to increase dependability and trustworthiness of that information and its interpretation.

4.1 Data Collection process

Data was gathered from one satellite TVET College and a Technical college to which the researcher assigned herself with. The researcher applied for an ethical clearance letter, with the clearance number from the university. The ethical clearance letter was accompanied with the request letter from the supervisor to request that the researcher be given permission to undertake the research at that satellite TVET College (Appendix 1). The campus manager from the satellite campus granted the researcher permission to undertake the research (Appendix 2). Data collection involves gathering of data to assist in understanding the problem better (Holmes, 2013), and in order to help the researcher to address his/her specific problem (Giollabhui, 2016). The researcher visited the satellite TVET College on numerous occasions before data was collected. The aim was for the researcher to introduce herself to the college community, to explain the purpose of the research study and arrange for appointments to conduct research. Before data collection process could begin, the researcher undertook to explain the ethical procedures that govern academic research to participants (Appendix 3). Because the researcher worked with human subjects, the researcher had to adhere to the University of Johannesburg Research Ethics Committee application, which was approved before the study could start. When this process was completed and participants gained an understanding of what was expected of them, the researcher requested that they complete the consent forms (Appendix 4). Data collection process was divided into three phases. The researcher used the following schedule to collect data:
<table>
<thead>
<tr>
<th>Test Date</th>
<th>Period Number</th>
<th>Time</th>
<th>Name of the teacher</th>
<th>Group Name</th>
<th>Number of learners per group</th>
<th>Location of the Classroom</th>
</tr>
</thead>
<tbody>
<tr>
<td>05.06.2017</td>
<td>Consultation</td>
<td>14:30-15:30</td>
<td>Teacher A</td>
<td>IT 4A</td>
<td>15</td>
<td>J02</td>
</tr>
<tr>
<td>12.06.2017</td>
<td>Consultation</td>
<td>10:30-11:30</td>
<td>Teacher B</td>
<td>IT 4B</td>
<td>20</td>
<td>K02</td>
</tr>
<tr>
<td>17.08.2017</td>
<td>Consultation</td>
<td>12:00-13:00</td>
<td>Teacher C</td>
<td>IT 4C</td>
<td>25</td>
<td>TT06</td>
</tr>
<tr>
<td>17.08.2017</td>
<td>Consultation</td>
<td>09:30-10:30</td>
<td>Teacher D</td>
<td>IT 4D</td>
<td>10</td>
<td>TL01</td>
</tr>
<tr>
<td>17.08.2017</td>
<td>Consultation</td>
<td>14:30-15:30</td>
<td>Teacher E</td>
<td>IT 4D</td>
<td>10</td>
<td>J02</td>
</tr>
</tbody>
</table>

### 4.2 Data Collection instruments

#### 4.2.1 Geometry Test

A performance test (Geometry test appendix 5) was primarily used to collect data. The test was based on circle geometry, which was adapted from 2014 Grade 12 Mathematics Paper 2 National Examination. It consisted of three items which tested the skills with respect to the ability to reason logically and prove theorems deductively (Van Hiele, 1986; Bassarear et al., & DBE in Luneta, 2015). It further sought ways of understanding why NC (v) level 4 learners have misconceptions that led to errors (Shulman, 1986). The purpose of the performance test was to measure the knowledge, skills and changes in knowledge (Jha, 2016) that participants have with respect to circle geometry.

Mornette, Sullivan and DeJong (2013) further add that performance test is used to measure the objectives, that is, to determine errors that NC(v) level 4 learners have with reference to circle geometry and what could be some of the misconceptions responsible for these errors. The following were some of the examples of the data that was collected based on each question in the test:

**Question 1**

This question was made up of two questions, namely, 1.1 and 1.2. The question was set with respect to Cognitive level 1, which is the knowledge level (see Table 5.4). One standard diagram was given:
In the diagram, O is given as the centre of the circle and passes through A, B and C, angle CAB = 48, COB = \( x \). This question examined learners’ knowledge on the theorem which states that the angle at the centre is twice the angle at the circumference. Learners were asked to determine with reasons, value of \( x \). The following are some of the examples of the responses that were provided for this question:

![Figure 4.1: Example of data collected for question1.1](image)

![Figure 4.2: Example of data collected for question1.1](image)

![Figure 4.3: Example of data collected for question1.2](image)
Question 2

In the diagram below, O is the centre of the circle that passes through A, B, C and D. AOD is the straight line and F is the midpoint of chord CD. Angle ODF = 30°

Determine, with reasons the size of:

2.2.1 \( \angle F \)

2.2.2 \( \angle ABC \)

The following are some of the sample responses provided by participants based on the question:

Figure 4.4: Example of data collected for question 1.2

Figure 4.5: Example of data collected for question 2.1
Question 3

Two circles in the diagram below have a common tangent XRY at R. W is a point on the smaller circle. The straight line RWS meet the larger circle at S. The chord STQ is a tangent to the smaller circle, where T is a point of contact. Chord RTP is drawn. Let \( R_1 = x \) and \( R_2 = y \).
3.1 Provide reasons for the following statements:

<table>
<thead>
<tr>
<th>Statement</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{T}_3 = x )</td>
<td>( )</td>
</tr>
<tr>
<td>( \hat{P}_1 = x )</td>
<td>( )</td>
</tr>
<tr>
<td>( WT \parallel SP )</td>
<td>( )</td>
</tr>
<tr>
<td>( \hat{S}_1 = y )</td>
<td>( )</td>
</tr>
<tr>
<td>( \hat{T}_2 = y )</td>
<td>( )</td>
</tr>
</tbody>
</table>

3.2 Prove that \( RT = \frac{WR \cdot RP}{RS} \)

3.3 Determine with reasons, other two angles that are equal to \( y \)

3.4 Prove that angle \( Q_3 = \hat{W}_2 \)

3.5 Prove that \( \frac{WR}{RQ} = \frac{RS^2}{RP^2} \)
Question 3.2 data:
Question 3.3 data:

In $\triangle WTR$ and $\triangle ASPR$

- $W_2 = S_3$ (Corresponding $\triangle$s $WTR$ and $ASPR$)
- $R_3 = R_5$ (Common $\triangle$s $\angle x$)
- $T_5 = P_1$ (Third $\triangle$s $\angle x$)

\[ \frac{RT}{RP} = \frac{WR}{RS} \]

**Figure 4.14:** Example of data collected for question 3.2

**Figure 4.15:** Example of data collected for question 3.2

**Figure 4.16:** Example of data collected for question 3.2
Figure 4.17: Example of data collected for question 3.3

Figure 4.18: Example of data collected for question 3.3

Question 3.4 data:

Figure 4.19: Example of data collected for question 3.4

Figure 4.20: Example of data collected for question 3.4

Figure 4.21: Example of data collected for question 3.4
In minimising potential threats, prior to the test being written, the researcher reassured participants that they are not being judged. It was further emphasised that the results emanating from the performance test will be combined with that of other participants during the research report writing. The test was piloted before it was administered. The purpose was to ensure that wording were clear and not confusing. The sample in this study was selected purposefully, that is, there was reliability and validity of evidence for the type of participants under enquiry (Borghans et al., 2016). This suggested that the test measured the knowledge experiences gained from educational programs (Palinkas et al., 2016).

4.2.2 Focus group Interviews

A focus group interview (Appendix 7) was the second data collection instrument that was used to collect data. It was developed by the researcher. The instrument was made up of questions that were pertinent to the Geometry test. The instrument was divided into two sections. The first section, Section A, collected the biographical data for participants. The second section, Section B, was made up of ten items that were
based on geometry test. Items 1 and 3 tested understanding of basic geometry terminology (Wing, 1985; Luneta, 2015). Items 4, 5, 7 8, and 10 determined strategies used with misconceptions that might have led to errors (Skemp, 1978; Stein et al., 2003; Brouer, 1966, Eves, 1990; Lakatos, 1976; Polya, 1973, Hersh, 1973; Makonye, 2013). Items 6 and 9 tested the analysis of characteristics of properties and their relationships (Bassarear, Department of Education & Bahr et al., in Luneta, 2015). Item 2 evaluated learning experiences with respect to circle geometry (Atebe & Schafer, 2009; Mji & Makgato, 2006; Alex & Mammen, 2016).

A week after the test was written and test scripts were marked and a process of moderation was undertaken, a total of 9 respondents were taken from the sample to be interviewed. The Cambridge Online Dictionary (2017) defines a process of moderation as a review of results in relation to agreed standards so as to ensure consistency of marking. An internal moderator from the research site was chosen to review the marking process (Appendix J). They were grouped as high achievers (3), middle achievers (3) and low achievers (3). Unstructured focus group interviews were undertaken. Interviews were ideal as a second data collection instrument because:

a. Respondents were not likely to forget the answers they gave when they respond in their own words (Di Paola, 2016).

b. It allowed respondents to include adequate information such as feelings, attitudes and their understanding of the subject. Close ended surveys because of their simplicity and limit to the choice of words, would have not offered respondents choices to reflect their real feelings (Rubin & Babbie, 2016).

c. It allowed respondents to reveal candid information and unique insights for researcher because respondents found it less threatening than scaled questions (Basford, 2016)

The following are examples of the transcripts on what participants said based on ten interview questions they were asked:

1. State all the theorems that are associated with circle geometry

Response: A line drawn from the centre of the circle to the midpoint of a chord, then that line is perpendicular to the chord; angles that are in the same segment of a circle are equal; when the diameter of the circle subtends an angle at the circumference, the subtended angle is a right angle; When a tangent to the circle is drawn, it is perpendicular to the radius at the point of contact.. Ngiphelelw, bakhulume ezinye beng’funa kuzibiza (I’ve run out of others because Ntombi and Lihle have already stated other theorems I wanted to state).
2. How would you regard your knowledge of proof in circle geometry?

Response: Well, I can say I enjoyed solving riders in this test. I applied the solid knowledge I gained from previous grades, coupled with the knowledge I gained in grade 12. It was a smooth sailing for me.

3. How would you explain the following terminology?

   Tangent

Response: tangent is a line that touches a curved surface at some point but it does not cross it

   Cyclic quadrilateral

Response: Cyclic quadrilateral is when opposite angles of a cyclic quad are equal to 180˚, supplementary

   Circumference

Response: A circumference is the perimeter.

   Chord

Response: I don’t want to lie. I don’t know the definition

4. Explain the difficulties encountered in question 1.2

Response: Well, I didn’t encounter any problems. For me, after looking at this question, what came to mind were the properties of triangles. I found the value of \( x = 96° \), I then looked at \( \hat{C}_2 \) and added \( \hat{B}_2 \) and equated it to 180°, reasoning behind being the sum of the angles of a triangle equals 180°, and found the value of \( y = \hat{B}_2 = 42° \), with the reason being opposite sides are equal.

5. Point out errors in solving question 3.2. What other alternative approach may be used to solve question 3.2?

Response: I did not attempt to answer the question, but there is no any other approach to the question

6. Explain the theorem associated with the diagram in question 2.

Response: I could not complete this question. I realised that I was not coping. I do not have good background when it comes to proof and theorems.

7. How did you arrive at an answer for question 3.5?
Response: I had already proven that \( \frac{RW}{RS} = \frac{RT}{TP} \), my reason was that WT is parallel to SP, \( \therefore RT = \frac{WR}{RS} \), I then equated \( \hat{Q}_3 \) to \( \hat{P}Sr \), my reason was that these were corresponding angles are equal since WT is parallel to SP, therefore \( \hat{Q}_3 \) is equal to \( \hat{W}_2 \).

8. Describe the strategy (ies) you applied when solving question 3.
Response: To be able to solve question 3, I needed to break down that complex diagram into its smaller parts first, and then identify theorems and properties of each.

9. Deconstruct the diagram in question 3 and explain all the theorems that are associated with it.
Response: I had memorised the theorems the night before the test. When I was supposed to apply them, I had forgotten what I had memorised. I did not know how to approach question 3. I can only prove a theorem on one simple diagram. If a diagram is complicated like this one, I get more confused.

10. Explain the error(s) in 3.3
Response: I initially tried to breakdown the diagram, but ended up getting confused even more. I then left the question unanswered.

However, Pattern (2016) cautioned of possible threats towards focus group interviews:

d. If respondents are not feeling free, they might provide answers that might be minimal and not yield expected outcomes

e. Respondents may find open ended questions time consuming or needing too much effort to answer, as a result, they may give brief and unfulfilling answers.

f. Responses to questions may be difficult to interpret and analyse, limiting the usefulness of the interview session towards the researcher.

g. Focus group interview questions that are too leading and vague may raise red flags for respondents, making them to provide misleading answers. If questions are irritating or offensive to respondents, they may break off and choose not to complete the interview session.

In minimising potential threats towards the interview, the researcher visited the research site on numerous occasions to become part of the campus community. Participants were encouraged to feel free and ask the researcher the questions.
based on the study prior to the interview sessions. The researcher further motivated participants to be as honest as possible when answering the questions by indicating that their honesty would not only guarantee the success of the study, but findings that emerged from this study would enable educators to acknowledge and appreciate learners thoughts and find ways of engaging their current knowledge to develop new knowledge (Hill, 2008). Educators would use misconceptions that learners have to develop a conceptual framework for learning (Shulman, 1986). The researcher further emphasised that participants would benefit from their honest opinions because they may begin to reflect on how they acquire geometry thinking (Siddaiah-Subramanya, Nyandowe & Zubair, 2017).

4.2.3 Document analysis

The final phase of data collection included the analysis of document such as policy documents relative to circle geometry. The researcher chose to analyse these documents as a way to triangulate participants’ perceptions on how they acquire geometric thinking. Patton (in Carter, 2014) explained data triangulation as a use of multiple data sources in a qualitative research to gain detailed understanding of phenomenon under enquiry. It is also viewed as a strategy to improve and test the credibility and validity of the research study (Bekhet, 2012). The researcher compared participants’ geometry performance test and interviews against the policy documents (Subject Assessment Guidelines NC(V) levels 2 – 4 to gain an understanding of how their responses related to their performance. Particular sections of the three policy documents were looked at:

a. National Certificate Vocational Mathematics NQF Level 4
### Topic 3: Space, Shape and Measurement. (Approximately 35 hours)

<table>
<thead>
<tr>
<th>SUBJECT OUTCOME</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1 Use the Cartesian co-ordinate system to derive and apply equations.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>ASSESSMENT STANDARD</th>
<th>LEARNING OUTCOME</th>
</tr>
</thead>
<tbody>
<tr>
<td>• The Cartesian co-ordinate system is used to derive and apply the equation of a circle with any centre.</td>
<td>• Use the Cartesian co-ordinate system to derive and apply the equation of a circle (any centre).</td>
</tr>
<tr>
<td>• The Cartesian co-ordinate system is used to derive and apply the equation of a tangent to a circle given a point on the circle.</td>
<td>• Use the Cartesian co-ordinate system to derive and apply the equation of a tangent to a circle given a point on the circle.</td>
</tr>
</tbody>
</table>

Note:
- Straight lines to be written in the following forms only:
  
  \[ y = mx + c \quad ; \quad y - y_1 = m(x - x_1) \]
  
  and/or \[ ax + by + c = 0 \quad (\text{general form}) \]

- Learners are expected to know and be able to use as an axiom “the tangent to a circle is perpendicular to the radius drawn to the point of contact.”

<table>
<thead>
<tr>
<th>ASSESSMENT TASKS OR ACTIVITIES</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Assignments</td>
</tr>
<tr>
<td>• Test</td>
</tr>
<tr>
<td>• Internal Examination</td>
</tr>
</tbody>
</table>
3.2 Explore, interpret and justify geometric relationships.

**ASSESSMENT STANDARD**

- Geometry involving straight lines and triangles are used to solve problems in geometrical figures.
- The major theorems of circles are stated and applied.

**LEARNING OUTCOME**

- Use geometry of straight lines and triangles to solve problems and to justify relationships in geometric figures.
  - Concepts to include are:
    - angles of a triangle;
    - exterior angles,
    - straight lines,
    - vertically opposite angles;
    - corresponding angles,
    - co-interior angles and
    - alternate angles.
- State and apply the following theorems of circles:
  - If a line is drawn from the centre of a circle to the midpoint of a chord, then that line is perpendicular to the chord.
  - If a line is drawn from the centre of the circle perpendicular to the chord, it bisects the chord.
  - If an arc subtends an angle at the centre of the circle and at any point on the circumference, then the angle at the centre is twice the measure of the angle at the circumference.
  - If the diameter of a circle subtends an angle at the circumference, then the angle subtended is a right angle triangle.
  - If an angle subtended by a chord at a point on the circumference is a right angle, then the chord is a diameter.
  - Angles in the same segment of a circle are equal.
  - The opposite angles of a cyclic quadrilateral are supplementary.
  - An exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.
  - If the exterior angle of a quadrilateral is equal to the interior opposite angle the quadrilateral will be a cyclic quadrilateral.
  - The four vertices of a quadrilateral in which the opposite angles are supplementary will be a cyclic quadrilateral.
  - If a tangent to a circle is drawn, then it is perpendicular to the radius at the point of contact.
  - If a line is drawn perpendicular to a radius at the point where the radius meets the circle, then it is a tangent to the circle.
  - If two tangents are drawn from the same point outside a circle then they are equal in length.
### 3.3 Solve problems by constructing and interpreting trigonometric models.

#### ASSESSMENT STANDARD

- The following compound angle identities are derived and used:
  
  \[
  \sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta \\
  \cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta
  \]

  To derive and apply the following double angle identities:
  
  \[
  \sin 2\alpha = 2\sin \alpha \cos \alpha \\
  \cos 2\alpha = \frac{\cos^2 \alpha - \sin^2 \alpha}{2} - 1 \\
  \frac{1 - 2\sin^2 \alpha}{1}
  \]

- Trigonometric equations are solved using compound and double angle identities without a calculator.

- Compound angle identities are used to simplify trigonometric expressions and to prove trigonometric equations.

- Specific solutions of trigonometric equations are solved.
  
  Range:
  
  - Solutions: \([0;360^\circ]\)
  - Identities limited to:

#### LEARNING OUTCOME

- Use the following compound angle identities:
  
  \[
  \sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta \\
  \cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta
  \]

  To derive and apply the following double angle identities:
  
  \[
  \sin 2\alpha = 2\sin \alpha \cos \alpha \\
  \cos 2\alpha = \frac{\cos^2 \alpha - \sin^2 \alpha}{2} - 1 \\
  \frac{1 - 2\sin^2 \alpha}{1}
  \]

- Determine the specific solutions of trigonometric expressions using compound and double angle identities without a calculator.
  (e.g. \(\sin 120^\circ\), \(\cos 75^\circ\) etc.)

- Use compound angle identities to simplify trigonometric expressions and to prove trigonometric equations.

- Determine the specific solutions of trigonometric equations by using knowledge of compound angles and identities.

**Note:**

- Solutions: \([0;360^\circ]\)
- Identities limited to:
Figure 4.1: Subject Assessment Guideline. Adapted from Department of Higher Education and Training (2013)

b. National Certificate Vocational Mathematics NQF Level 3
## Topic 3: Space, Shape and Measurement.

(Minimum of 26 hours face to face teaching which excludes time for revision, test series and internal and external examination)

<table>
<thead>
<tr>
<th>SUBJECT OUTCOME</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1 Calculate the surface area and volume of two and three dimensional shapes</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>ASSESSMENT STANDARD</th>
<th>LEARNING OUTCOME</th>
</tr>
</thead>
</table>
| • Formulae are used to calculate surface area and volume of geometrical objects. Range: right pyramids (with square, equilateral triangle or regular hexagonal bases), right cones, spheres | • Calculate the surface area and volume of the following geometrical objects:  
  - right pyramids (with square, equilateral triangle or regular hexagonal bases)  
  - right cones  
  - spheres  
  • Calculate the surface area and volume of a combination of the above mentioned geometrical objects. |
| • The surface area and volume of a combination of the above mentioned geometrical objects are calculated. | |

<table>
<thead>
<tr>
<th>ASSESSMENT TASKS OR ACTIVITIES</th>
</tr>
</thead>
</table>
| • Practical tasks in groups with actual objects  
• Assignments  
• Test  
• Examination |

<table>
<thead>
<tr>
<th>SUBJECT OUTCOME</th>
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</thead>
<tbody>
<tr>
<td>3.2 Use the Cartesian co-ordinate system to derive and apply equations.</td>
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</table>

<table>
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<tr>
<th>ASSESSMENT STANDARD</th>
<th>LEARNING OUTCOME</th>
</tr>
</thead>
</table>
| • The Cartesian co-ordinate system is used to derive the equation of a line through two given points.  
• The Cartesian co-ordinate system is used to derive the equation of a line parallel or perpendicular to another line.  
• The Cartesian co-ordinate system is used to find the angle of inclination and apply it to find the equation of a line. | • Use the Cartesian co-ordinate system to derive the equation of a line through two given points.  
• Use the Cartesian co-ordinate system to derive the equation of a line parallel or perpendicular to another line.  
• Use the Cartesian co-ordinate system to derive and use the angle of inclination of a line. |

<table>
<thead>
<tr>
<th>ASSESSMENT TASKS OR ACTIVITIES</th>
</tr>
</thead>
</table>
| • Assignments  
• Test  
• Examination |

<table>
<thead>
<tr>
<th>SUBJECT OUTCOME</th>
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</thead>
<tbody>
<tr>
<td>3.3 Solve problems by constructing and interpreting trigonometric models.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>ASSESSMENT STANDARD</th>
<th>LEARNING OUTCOME</th>
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</table>
The values of the trigonometric functions for special angles are derived

\[ \text{Range: } 30^\circ, 45^\circ \text{ and } 60^\circ \text{ in all four quadrants} \]

- \( \sin(90^\circ \pm \theta); \cos(90^\circ \pm \theta) \)
- \( \sin(180^\circ \pm \theta); \cos(180^\circ \pm \theta); \tan(180^\circ \pm \theta) \)
- \( \sin(360^\circ - \theta); \cos(360^\circ - \theta); \tan(360^\circ - \theta) \)

- Trigonometric identities are used to simplify expressions and prove equations.
  \[ \begin{align*}
  \tan \theta &= \frac{\sin \theta}{\cos \theta} \\
  \sin^2 \theta + \cos^2 \theta &= 1
  \end{align*} \]

- Trigonometric equations are solved for the three trigonometric functions in all four quadrants with a calculator.
  \[ \text{Range: } [\theta; 360^\circ] \]
  Note - positive angles only:
  - \( \sin(90^\circ \pm \theta); \cos(90^\circ \pm \theta) \)
  - \( \sin(180^\circ \pm \theta); \cos(180^\circ \pm \theta); \tan(180^\circ \pm \theta) \)
  - \( \sin(360^\circ - \theta); \cos(360^\circ - \theta); \tan(360^\circ - \theta) \)

- The sine, cosine and area rules are applied.

- Two dimensional problems are solved using the sine, cosine and area rules by interpreting given geometric and trigonometric models.

- Derive and use the values of the trigonometric functions (in surd form where applicable)
  of \( 30^\circ, 45^\circ \) and \( 60^\circ \).

- Use the reduction formulae and special angles to solve trigonometric expressions and prove equations
  in all four quadrants (without the use of a calculator) for the following functions.
  - \( \sin(90^\circ \pm \theta); \cos(90^\circ \pm \theta) \)
  - \( \sin(180^\circ \pm \theta); \cos(180^\circ \pm \theta); \tan(180^\circ \pm \theta) \)
  - \( \sin(360^\circ - \theta); \cos(360^\circ - \theta); \tan(360^\circ - \theta) \)

- Use the following trigonometric identities to simplify expressions and prove equations.
  - \( \tan \theta = \frac{\sin \theta}{\cos \theta} \)
  - \( \sin^2 \theta + \cos^2 \theta = 1 \)

- Solve trigonometric equations (with the use of a calculator) involving reduction formulae using special
  triangles for the three trigonometric functions in all four quadrants.
  - \( \sin(90^\circ \pm \theta); \cos(90^\circ \pm \theta) \)
  - \( \sin(180^\circ \pm \theta); \cos(180^\circ \pm \theta); \tan(180^\circ \pm \theta) \)
  - \( \sin(360^\circ - \theta); \cos(360^\circ - \theta); \tan(360^\circ - \theta) \)

- Apply the sine, cosine and area rules.

- Solve problems in two dimensions by using the sine,
  cosine and area rules by interpreting given geometric
  and trigonometric models.

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**ASSESSMENT TASKS OR ACTIVITIES**

- Practical Exercises
- Assignments
- Test
- Examination

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Figure 4.2: Subject Assessment Guideline. Adapted from the Department of Higher Education and Training (2012)

**c. National Certificate Vocational Mathematics NQF Level 2**

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**SUBJECT OUTCOME**

### 3.1 Measure and calculate physical quantities

<table>
<thead>
<tr>
<th>ASSESSMENT STANDARD</th>
<th>LEARNING OUTCOME</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Scales on measuring instruments are read correctly. Instruments to include are the ruler and protractor.</td>
<td>• Read scales on measuring instruments correctly. Instruments to include are the ruler and protractor.</td>
</tr>
<tr>
<td>• Systeme Internationale (SI) units are understood and used in the appropriate situation.</td>
<td>• Use symbols and Systeme Internationale (SI) units as appropriate to the situation.</td>
</tr>
</tbody>
</table>

*Range: Appropriate situations for SI units include length, height, angles, area, volume, capacity, monetary calculations and time.*

**ASSESSMENT TASKS OR ACTIVITIES**

- Project
- Assignments
- Tests
- Internal examinations

### 3.2 Calculate perimeter, surface area and volume in two and three dimensional geometrical shapes.

<table>
<thead>
<tr>
<th>ASSESSMENT STANDARD</th>
<th>LEARNING OUTCOME</th>
</tr>
</thead>
</table>
| • The perimeter and surface area of the following laminas are calculated:  
  o Square  
  o Rectangle  
  o Circle  
  o Triangle  
  o Parallelogram  
  o Trapezium  
  o Hexagon | • Calculate the perimeter and surface area of the following laminas:  
  o Square  
  o Rectangle  
  o Circle  
  o Triangle  
  o Parallelogram  
  o Trapezium  
  o Hexagon |
| • The volume of the following geometric objects are calculated:  
  o Cubes  
  o Rectangular prisms  
  o Cylinders  
  o Triangular prisms  
  o Hexagonal prisms | • Calculate the volume of the following geometric objects:  
  o Cubes  
  o Rectangular prisms  
  o Cylinders  
  o Triangular prisms  
  o Hexagonal prisms |
| • The effect on area of laminas where one or | • Investigate the effect on area of laminas where |
more dimensions are multiplied by a constant factor $k$ is investigated.
- The effect on the volume and surface area of right prisms and cylinders, where one or more dimensions are multiplied by a constant factor $k$ is investigated.

one or more dimensions are multiplied by a constant factor $k$
- Investigate the effect on the volume and surface area of right prisms and cylinders, where one or more dimensions are multiplied by a constant factor $k$.

**ASSESSMENT TASKS OR ACTIVITIES**
- Draw and cut out exercises/practicals using nets of prisms.
- Tests
- Assignments
- Internal examination

**SUBJECT OUTCOME**

### 3.3 Use the Cartesian co-ordinate system to derive and apply equations.

<table>
<thead>
<tr>
<th>ASSESSMENT STANDARD</th>
<th>LEARNING OUTCOME</th>
</tr>
</thead>
</table>
| The Cartesian co-ordinate system is used to: | - Use the Cartesian co-ordinate system to plot points, lines and polygons.  
- Use the Cartesian co-ordinate system to calculate the distance between two points.  
- Use the Cartesian co-ordinate system to find the gradient of the line joining two points.  
- Use the Cartesian co-ordinate system to find the co-ordinates of the midpoint of a line segment joining two points. |
| - Plot points, lines and polygons.  
- Calculate the distance between two points.  
- Calculate the gradient of a line segment joining two points.  
- Calculate co-ordinates of the midpoint of a line segment joining two points. | |

**ASSESSMENT TASKS OR ACTIVITIES**
- Practical exercises
- Assignments
- Tests
- Internal examination

**SUBJECT OUTCOME**

### 3.4 Use and apply transformations to plot co-ordinates.

<table>
<thead>
<tr>
<th>ASSESSMENT STANDARD</th>
<th>LEARNING OUTCOME</th>
</tr>
</thead>
</table>
| - The coordinates of the point $(x; y)$ is found after translating $p$ units horizontally and $q$ units vertically.  
- The coordinates of the point $(x; y)$ is found after reflecting about the $x$-axis, the $y$-axis, and the line $y = -x$ and the line $y = x$. | - Find the coordinates of the point $(x; y)$ after it is translated $p$ units horizontally and $q$ units vertically.  
- Find the coordinates of the point $(x; y)$ after it is reflected about the $x$-axis, the $y$-axis, and the line $y = -x$ and the line $y = x$. |

**ASSESSMENT TASKS OR ACTIVITIES**
- Practical exercises
- Assignments
- Tests
- Internal examination

**SUBJECT OUTCOME**

### 3.5 Solve problems by constructing and interpreting geometrical models.

<table>
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<th>LEARNING OUTCOME</th>
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4.3 Reliability and validity of data collection instruments

The first phase of data collection was the pilot study. Dikko (2016), Van Wijk and Harrison (2013) define a pilot study as a trial run to evaluate the data collection instrument. Conducting a pilot study was important in this research study as it helped in determining possible flaws with the research instruments (Calitz, 2009), checked if
everyone in the sample did not only understand the questions, but understood them in the same way (Eksteen, Willemse, Mallan & Ellis, 2015), how long it took respondents to complete the measuring instrument, and whether the concepts were adequately operationalised (Newman, Lim & Pineda, 2013).

The pilot study was undertaken by 20 NC (v) level 4 mathematics learners from a neighbouring college to test the reliability and validity of the data collection instrument, that is, the geometry test. With any given research design, instruments that are chosen for data collection must undergo a series of tests (reliability and validity) before they can be considered as acceptable measures (Lim & Khalifa, 2013). Reliability of a research instrument is discovered when the instrument measures concepts it is intended to measure with consistency and without bias (Gbore, 2016). It can further be achieved through test-re-test, where the same measuring instrument is administered to same respondents at different timeframes to attain same results (Creswell, 2013). On the other hand, the purpose of validity of a research instrument is aimed at ensuring that the instrument measures what it is intended to measure (Creswell & Poth, 2017) by including adequate representation of items that operationalise the concept (content validity). It differentiates items on an adequate criterion (criterion-related validity) and guarantees that the measures used fit around the theories for which the test is created (construct validity) (Kane, 2013; Dikko, 2016; Sackett, Shewach, Keiser, 2017).

Before the study was conducted, two research instruments (Geometry test and the Focus group interview questions) were given to a panel of 6 mathematics teachers and two PHD students (Linguistics) who:

1. Highlighted ambiguous and difficult questions to discard or modify them.
2. Determined whether questions elicited adequate and relevant responses.
3. Established whether responses to geometry test and questionnaire could be properly interpreted with respect to information needed (van Teijlingen & Hundley, 2001).
4. Determined whether the researcher included the relevant questions which were necessary to measure concepts intended to be measured (Berg, 2001).

However, it should be noted that geometry test was already validated because it was adapted from 2014 grade 12 Mathematics National Examinations Paper 2. Pre-testing the two data collection instruments was then done on a small number of subjects with similar characteristics as those in the actual study. The goal was to ascertain how well the instruments worked in an actual study. Sekaran (in Dikko, 2016) argues that respondents can bias data that is collected if they do not understand questions put across them; as a result, pre-testing of questions assisted in identifying unclear and ambiguous statements in research. Van Wijk and Harrison (2013) also believe that pre-testing research instruments adds value and credibility to the entire research process.
CHAPTER 5: INTERPRETATION OF DATA

5.0 Introduction

The previous chapter focused primarily on ways in which data was collected from the research site. The researcher used the geometry performance test and interviews to gather evidence on the misconceptions that NC(v) level 4 Mathematics with respect to circle geometry and the errors displayed as a result of such misconceptions. Document analysis was used to triangulate results obtained using data collection methods adopted from literature.

In this chapter, the researcher interpreted and analysed the data. Data analysis takes into account the process of breaking down the data into practical themes, patterns, inclinations and relationships (Mouton, 2001: 108). Luneta (2013) further defined data analysis as a process of noting and drawing out patterns in the data, making enquiries about the discovered patterns. This includes creating a theory or opinions based on the data with respect to the research topic. This process is termed inductive analysis (Luneta, 2013). The researcher chose the inductive approach to data analysis. In doing so, the researcher understood data interpretation and analysis.

The first step in data analysis is the data interpretation (Creswell, 2014). This preliminary exploratory analysis in qualitative research is made up of looking into the data to get the general sense of the data, memoing of concepts and thinking about the organisation of data to be analysed; and whether more data is needed or not (Creswell, 2014). For instance, as Agar (1980) suggested “…… read the transcripts in their entirety several times. Immerse yourself in the details, trying to get a sense of the interview as a whole before breaking it into parts” (p103). Lawrence and Tar (2013) explained the process of data analysis as involving working with data by sorting it, break it down, arranging it, searching for patterns, uncovering what is important and to be learned and deciding what should be generalised. Creating memos in the margins of transcripts, field notes and photographs assist in this first step of data interpretation.

Data was interpreted and analysed with respect to the research question, namely what are the dominant errors displayed by NC(v) level 4 learners in circle geometry? , and what could be some of the misconceptions responsible for these errors?. Interpretation and analysis also focused on the misconceptions that NC(v) level 4 learners have with respect to circle geometry, which were evident as errors in the geometry performance test and the interviews. Samples were taken from the geometry test as proof of what had transpired in learners work. Sample analysis was made according to individual questions, for instance, determining with reasons values of x and y, sizes of angles and proof.
5.1 Inductive data analysis

The researcher focused on the generation of the new theory resulting from the data (Creswell, 2012). Inductive data analysis associated with qualitative research (Lawrence & Tar, 2013). One relative approach that is associated with inductive data analysis is the Grounded Theory, pioneered by Glaser and Strauss (Cho & Lee, 2014). Grounded Theory approach required the researcher to be completely open-minded without any preconceived ideas of what would be discovered (Creswell, 2012). Once analysis was complete, the researcher had to examine the existing theories in order to place his/her new theory within the discipline. This process further required the researcher to turn the constructed contexts of participants into the series of presentations, field notes, conversations and memos to provide meaning of the phenomenon under inquiry (Creswell, 2007: 36). It further enabled the researcher to create a credible and rich detailed narrative using field notes and interview transcripts and researcher interpretations (Lawrence & Tar, 2013).

Within the inductive analysis method, there exists the qualitative content analysis method (Bengtsson, 2016). Krippendorff (2004) defined content analysis as “a research technique for making replicable and valid inferences from texts (or other meaningful matter) to the contexts of their use” (p: 18). Downe-Wamboldt (in Bengtsson, 2016) argued that content analysis is more than a counting process, but the goal is to link the results to their context or to the environment in which they were produced “content analysis is a research method that provides a systematic and objective means to make valid inferences from verbal, visual or written data in order to describe and quantify specific phenomena” (p: 314). Krippendorff (2012) further alluded that it is a research technique for making replicable and valid inferences from texts or other meaningful (matter to the context of their use); as a research technique, it provides new insights, increases the researcher’s understanding of particular phenomena, or informs practical actions. According to Saldana (2009), content analysis provides the following two kinds of definitions:

1. Definitions that takes content to be contained in a text. In this study, the content was the learners’ work from the geometry performance test and focus group interviews that was contained from their texts.

2. Definitions that were as they were defined in circle geometry section of the textbooks but learners were unable to relate to them.

The researcher has to choose whether the analysis is a manifest analysis or a latent analysis. With the manifest analysis, the researcher describes what the participants actually say, stays very close to the text, uses the words themselves, and describes the visible and obvious in the text (Krippendorff, 2012). Latent analysis is extended to an interpretive level whereby the researcher seeks to find the underlying meaning of the text, that is, what the text is talking about (Berg, 2001; Catanzaro, 1988; Downe-Wambolt, 1992). The researcher chose latent analysis for this study.
The choice of data analysis for this research was the content analysis. Participants’
geometry test and focus group interviews responses were analysed. The researcher
sought to understand the misconceptions that NC(v) level 4 learners displayed
through the errors they made as well as the responses to the interview questions.
Makonye and Mashaka (2016) highlighted the importance of identifying learning
difficulties that learners experience by identifying specific errors that learners
frequently make to try and understand why learners make such errors. Error analysis
was done with the purpose of finding type of errors NC(v) level 4 learners have with
regard to circle geometry. The content data analysis process allowed the researcher
to give participants remedial support in correcting the misconceptions they hold
(Sonn, 2013).

5.2 Coding the geometry performance test and focus group interviews.

Corbin and Strauss (1990) defined coding as a way in which concepts or ideas are
labelled and tagged. The authors regard a concept or an idea as a “basic unit of
analysis” (p.7). Charmaz (2006) explained coding as “categorising segments of data
with a short label or name that simultaneously summarises and accounts for each
piece of data” (p.43) and as “the pivotal link between collecting data and developing
an emergent theory to explain these data” (p.46). Saldana (2010) defined a code in
qualitative research as a word or a phrase that assigns a summative, major attribute
for a piece of language-based or visual data.

For this research study, coding was done by assigning labels or tags for participants,
geometry performance test scripts, interview transcripts as suggested by Charmaz
(1983), Saldana (2010), and Corbin and Strauss (1990). Coding assisted the
researcher in identifying participants without using their main names, campuses
where they came from. For instance, I coded participants’ from P_1 to P_80, with
respect to their performance in the Geometry test. The P indicated the participant
and 1 indicated the number of participants who took part in the study. The aim was
to protect the anonymity of participants as guaranteed in the consent forms (See
attached Appendix 8).

The next step was to categorise participants’ responses from CA to CE. When one
solves geometry riders, one has to start with the calculations and a reason. This
category was concerned with the calculations by participants and the corresponding
reasons (see attached Appendix 9).

<table>
<thead>
<tr>
<th>Category Number</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>C_A</td>
<td>Correct answer, correct reason</td>
</tr>
<tr>
<td>C_B</td>
<td>Correct answer, Incorrect reason</td>
</tr>
<tr>
<td>C_C</td>
<td>Incorrect answer, Incorrect reason</td>
</tr>
<tr>
<td>C_D</td>
<td>Incorrect answer, Correct reason</td>
</tr>
<tr>
<td>C_E</td>
<td>NO answer</td>
</tr>
</tbody>
</table>
Participants’ responses were further categorised with respect to Van Hiele model of geometric thinking. Van Hiele (1986) classified levels of geometric thinking in terms of the following:

A. Level 1: Visualisation: Learners are able to identify and name common geometric shapes
B. Level 2: Analysis: Learners identify a shape according to its properties, but are unable to identify relationships between classes of figures.
C. Level 3: Abstraction: learners recognise class inclusion of shapes, that is, they can give definitions and recognise how definitions identify a concept.
D. Level 4: Deduction: Learners know the importance of proofs and possess the abilities to construct geometric proof.
E. Level 5: Rigour: Learners understand relationships between various axiomatic systems.

Table 5.2: Category of participants' answers with respect to Van Hiele levels (Van Hiele, 1986)

<table>
<thead>
<tr>
<th>Category No.</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Van_Hiele_A</td>
<td>Van Hiele level 1</td>
</tr>
<tr>
<td>Van_Hiele_B</td>
<td>Van Hiele level 2</td>
</tr>
<tr>
<td>Van_Hiele_C</td>
<td>Van Hiele level 3</td>
</tr>
<tr>
<td>Van_Hiele_D</td>
<td>Van Hiele level 4</td>
</tr>
<tr>
<td>Van_Hiele_E</td>
<td>Van Hiele level 5</td>
</tr>
</tbody>
</table>

A summary of participants’ errors was summarised per question (see Appendix 9), per participant in a table format with reference to the above categories respectively. For instance, if P_19 is coded C_E, ERR_D, and Van_Hiele_A for his/her test, it would mean that Participant 19 has provided no answer, and shows lack of knowledge of the concepts or procedures and is giving an indication of operating at Van Hiele level 1.

Participants’ answers relative to the type of errors made, for example, simple mistakes, conceptual errors, procedural errors and language errors were categorised as follows:

Table 5.3: Type of errors

<table>
<thead>
<tr>
<th>Category</th>
<th>Type of error</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>ERR_A</td>
<td>No error</td>
<td>No error made</td>
</tr>
<tr>
<td>ERR_B</td>
<td>Simple Mistake</td>
<td>Simple error made due to a limited concentration; incorrect answers which learners can self-correct; simple error because learners were in a hurry</td>
</tr>
<tr>
<td>ERR_C</td>
<td>Conceptual error</td>
<td>Errors caused by lack of basic knowledge such as basic concepts, terminology, mastery skills.</td>
</tr>
<tr>
<td>ERR_D</td>
<td>Procedural error</td>
<td>Learners have conceptual knowledge but apply it incorrectly; concepts are memorised without knowing when and how to apply them or why are they applied.</td>
</tr>
<tr>
<td>ERR_D</td>
<td>Language error</td>
<td>Lack of geometry technical terms</td>
</tr>
</tbody>
</table>
Geometry performance test was adapted from the grade 12 November 2014 Mathematics Paper 2 examinations. As a result, it conformed to the assessment guidelines for Mathematics NC (v) level 4 as stated in the policy documents (DHET, 2013). According to the Department of Higher Education and Training (2013), an assessment should consist of questions that cover all cognitive levels. Such questions should be knowledge based, routine based procedures, complex procedures and problem solving based questions. The table below provides a summary of the cognitive levels:

Table 5.4: Cognitive skills (DBE, 2011a)

<table>
<thead>
<tr>
<th>Cognitive levels</th>
<th>Explanation of the skills displayed</th>
</tr>
</thead>
<tbody>
<tr>
<td>Knowledge</td>
<td>Straight recalling</td>
</tr>
<tr>
<td></td>
<td>Correct use of mathematical vocabulary</td>
</tr>
<tr>
<td></td>
<td>Proof of prescribed theorems</td>
</tr>
<tr>
<td>Routine procedure</td>
<td>Use known procedures</td>
</tr>
<tr>
<td></td>
<td>Simple applications and calculations which might involve many steps</td>
</tr>
<tr>
<td>Complex procedure</td>
<td>Problems involving complex calculations and/or higher thinking order reasoning.</td>
</tr>
<tr>
<td></td>
<td>There is often not an obvious route to the solution</td>
</tr>
<tr>
<td></td>
<td>Require conceptual understanding</td>
</tr>
<tr>
<td>Problem solving</td>
<td>Higher order thinking, understanding and processes are involved</td>
</tr>
<tr>
<td></td>
<td>May require the ability and skills to break the problem down into its components</td>
</tr>
</tbody>
</table>

5.3 Geometry Performance Test Discussion

The test was based on the cognitive levels 1 to 4 in terms of the Mathematics NC (v) Level 4 Assessment Guidelines and Subject Guidelines (DHET, 2013).

Question 1

This question was made up of two questions, namely, 1.1 and 1.2. The question was set with respect to Cognitive level 1, which is the knowledge level (see Table 5.4). One standard diagram was given:

In the diagram, O is given as the centre of the circle and passes through A, B and C, angle CAB = 48, COB = x. This question examined learners' knowledge on the
theorem which states that the angle at the centre of the circle is twice the angle at the circumference. Learners were asked to determine with reasons, value of $x$. The correct answer for this question is given as:

1.1 $x = 2 \times 48^\circ = 96^\circ$ [angle at the centre of the circle is twice the angle at the circumference]

Of the eighty participants who took part in the study, fifty-five of them gave the correct answer, which is $96^\circ$ and a correct reason. Participants seemed to understand and applied the theorem that state that the angle at the centre of the circle is twice the angle at the circumference. They might have read and understood the textbook or the teacher. Flores (2006) stated that external conviction proof schemes are held by learners who hold the conviction that a theorem is true due to outside sources; the outside source maybe a teacher or a textbook. Wing (in Ramlan, 2016) argued that one of the notable component of the successful proof process is the ability to define and deduce properties of a geometric entity by applying intrinsic properties, relationships with other entities and rules of inference that bind the properties together. Figure 5.1 below is a typical sample of one of the correct responses given provided by 55 learners who got the answer correct:

![Figure 5.1: Sample of correct response](image)

Out of eighty participants, thirteen participants provided a correct answer and the reason ‘angle at the centre’ instead of ‘angle at the centre is twice the angle at the circumference’. ‘Angle at the centre’ statement is incomplete; as a result, it is rendered incorrect. The correct statement was supposed to be ‘the angle at the centre is twice the angle at the circumference’. It appeared that learners knew how to apply the theorem ‘angle at the centre is twice the angle at the circumference’ because they gave the correct value of $x = 96^\circ$, but failed to provide the correct reason. When participants were interviewed, they provided the complete correct reason. The reason for an incomplete statement may have been as a result of carelessness or being in a hurry. Luneta (2013) classified this type of an error as a simple error that occurs as a result of limited concentration, incorrect answers which learners can self-correct, or a simple error because learners were in a hurry. Below is a sample of a partially correct response given by 13 participants:
Twelve participants wrote the correct answer without providing a reason. These learners may have failed to understand the relationship between the stated theorem and how they arrived at an answer. This may be attributed to an insufficient learning and understanding of the theorem from previous grades. Bennie (1998) cautioned that geometry instruction at lower level grades is insufficient in terms of affording learners the skills and knowledge to operate at the level of axiomatic thinking in senior secondary schooling. Furthermore, Henning (2014), made an intriguing discovery that there is an underlying conclusion that this is a sick ‘Mathless’ and ‘Scienceless’ nation and its pathology can be seen in how learners perform in examinations and tests as a result of their competence in various knowledge domains. This suggests that learners fail right at the beginning in previous grades where they were learning to learn. They failed because most teachers have not been trained to teach the most challenging fields of knowledge such as Mathematics (Henning, 2014). The correct solution for this question is given as:

1.1 \( x = 96^\circ \) \( (\text{angle at the centre is twice the angle at the circumference}) \)

Question 1.2 was set according to cognitive level 2, which is the routine procedure (Table 5.4). The question examined routine procedures and simple applications and calculations which may involve many steps. In this question, participants were required to find the value of \( y \) with reasons. Out of eighty participants who wrote the test, 25% provided the correct value of \( y \), with the correct reason. These learners may have understood the theorem that states ‘angles of a triangle are equal to 180°’, and applied correct manipulation of symbols to arrive to \( y = 42^\circ \). These learners may have remembered the aspects of this argument maybe because they might have proved a similar question previously. Moutsios-Rentzos and Spyrou (2015) advised that it is important to nurture geometric thinking because it is directly related with success in writing proof. Below is a sample of correct solution given by 20 participants:

Figure 5.3: Sample of correct response

Thirty-eight percent provided the value of \( y \) as 78° and a reason as ‘sum of a triangle’ to give solution to this question. This is wrong. Participants could not realise that since the value of \( x = 96^\circ \), from 1.1 above, they needed to remember the theorem that states ‘the sum of angles of a triangle is 180°’. From the diagram, \( \hat{B}_2 \) and \( \hat{C}_2 \) are equal reason being they are angles opposite sides of a triangle. From there, they needed to indicate that \( y = \frac{180^\circ - 96^\circ}{2} = 42^\circ \)
therefore \( y = 42^\circ \). It was not the case with 31 participants. These learners might have lacked knowledge of concepts and lack of procedures as explained by the researcher above. Alex and Mammen (2016), warned about learners’ lack of conceptual and procedural knowledge due to the current instructional activities that are characterised by listening, watching and imitating what the teacher does. This seemed to indicate that participants experienced challenges with regard to proof assessments. Figure 5.4 below is a sample response provided by P_56:

![Sample response image](image)

**Figure 5.4: Sample of incorrect response**

Nineteen participants did not give the answer of \( y \) and the reason. These learners might have lacked knowledge of concepts, which led to the procedure being wrong. They might have not studied because they viewed proof as a game with no meaning. Snapper (in Makonye, 2013), and this game is played with marks on ‘paper following the rules’ as a result, they possessed slightest understanding of coming up with the correct value of \( y \) and the reason. This demonstrates lack of logical thinking and incompetence. The correct solution for this question is given as:

\[
\hat{C}_2 + \hat{B}_2 = 180^\circ - 96^\circ = 84^\circ \ (\text{Sum of angles in a triangle})
\]

\[
y = \frac{180^\circ - 96^\circ}{2} = 42^\circ
\]

\( y = \hat{B}_2 = 42^\circ \ [\text{angles opposite sides are equal}]\)

Question 2.2.1 was set according to cognitive level 2, the routine procedure (Table 5.4). This question examined routine procedures and simple calculations which might have involved many steps. The diagram below represented the question:
In the diagram, $O$ is the centre of the circle that passes through $A$, $B$, $C$ and $D$. $AOD$ is the straight line and $F$ is the midpoint of chord $CD$. Angle $ODF = 30^\circ$. Participants were required to determine with reasons, $F_1$ and $\overset{\frown}{ABCD}$. Seventeen out of eighty participants gave the correct value of $F_1$ and a correct reason. These learners might have had good memory recall of the theorem that states ‘a line from the centre to the midpoint bisects the chord’. Figure 5.5 below depicts a sample of one of the correct responses given by participants:

Figure 5.5: Sample of correct response.

Seventeen participants out of eighty gave the value of $F_1$ as $60^\circ$ and the reason as ‘angles of triangle’. These seventeen participants could not even recognise that $F_1$ ‘looks’ $90$ degrees. Their reasons might be attributed to poor visualisation. Twenty-four participants gave the reason that $CF = FD$, and 19 gave the reason as ‘an angle subtended by the diameter’. These responses were incorrect. They seemed to have made unjustified statements, suggesting that they were confused between the theorem and the converse and guessed the reason as ‘line from the centre to chord. Learners could not deduce that line $OF$ cuts $CD$ in the middle, and forms a right angled triangle $OFD$. As a result, $F_1 = 90^\circ$. As van Hiele level 1 highly referes to visualisation, it was common and unfortunate to see most participants were still operating below this level. This is despite the fact that this content is covered in secondary schools; participants at TVET College could not recall this basic theorem that appeal to visualisation. When interviewed these participants could not explain basic circle geometry concepts like ‘midpoint, bisect, chord’. Lack of definition to these circle geometry concepts might have resulted from lack of competency by teachers to teach mathematical topics that they are not comfortable with (Luneta, 2013; Mji & Ndlovu, 2012; Henning, 2014; Dhlamini, 2012). For instance, Dhlamini (2012) revealed that secondary school teachers regarded geometry as a ‘difficult
section” to teach. From this view, evidence suggests that there is a crisis in mathematics because quality of a mathematics teacher should be a significant factor that influences learner achievement. Figure 5.6 below is a sample of an incorrect response given by P_78:

![Figure 5.6: Sample of an incorrect response](image)

The correct solution for this question is:

$$F_1 = 90^\circ \text{ (Line from the centre to Midpoint chord)}$$

For question 2.2.2 only 5 participants out of 80 provided the answer as $$A\hat{B}C = 150^\circ$$ and the reason as ‘opposite angles of a cyclic quadrilateral are supplementary’. These learners may have demonstrated a good understanding and the application of the theorem ‘opposite angles of a cyclic quadrilateral are supplementary’, they might have had good background of proof from previous grades. Borrowing from Wing (1895) assertion, these learners were able to define and deduce the properties of a cyclic quadrilateral, relate other aspects such as supplementary angles, and draw conclusions with respect to rules that were already presented by the question. The figure below is a sample of correct response as given by P_04:

![Figure 5.7: Sample of a correct response](image)

Thirty-one participants out of 80 provided the value of $$\hat{B}$$ as $$30^\circ$$ and an incorrect reason as ‘opposite angles of a cyclic quad are equal’. $$\hat{B} = 30^\circ$$ is incorrect.
Participants might have an understanding of the theorem that states ‘opposite angles of the cyclic quadrilateral are supplementary’ but from the diagram, they could not recognise that since $\hat{B}$ is given as $30^\circ$, then $\hat{A}\hat{B}\hat{C}$ is $150^\circ$. Poor visualisation might have been the cause that led participants arriving at the value of $\hat{A}\hat{B}\hat{C}$ $30^\circ$. Figure 5.8 below provides a sample of the partially correct responses given by 31 participants:

Figure 5.8: Sample of a partially correct response

Nineteen participants gave the value of $\hat{B}$ as $90^\circ$ and the incorrect reason as ‘angles in the semi-circle’. This is wrong. It might be because these participants did not know what a cyclic quadrilateral is as a result they could not recognise it from the diagram. This might be as a result of poor visualisation. Thirteen participants did not answer nor provide reason. This might be because they could not link properties to diagrams. Oberdorf and Taylor - Cox (1999) cautioned that learners identify geometric shapes incorrectly because of the geometrical misconceptions from which reasons might be lack of adequate exposure to geometric vocabulary and different forms of shapes.

Leong et al., (2015), argued that learners thought process is made up of many things such as formulae, application, banality, enjoyment and attitudes about mathematics. Problems that may lead to learning difficulties are those that result from informal thinking and lack of conceptual knowledge (Leong et al., 2015). Figure 5.9 below is a sample of the incorrect responses given by P_49.

Figure 5.9: Sample of an incorrect response

The correct solution for this question is:

$\hat{A}\hat{B}\hat{C} = 150^\circ$ (opposite angles of a cyclic quadrilateral are supplementary)

Question 3 was made up of five sub questions as indicated below. Two circles in the diagram have a common tangent $XYR$ at $R$. $W$ is any point on the small circle. The straight line $RWS$ meets the large circle at $S$. The chord $STQ$ is a tangent to the small circle where $T$ is a point of contact. Chord $RTP$ is drawn. Let $R_4 = x$ and $R_2 = y$. 
This question was developed with respect to cognitive levels 3 (complex procedures) and 4 (Problem solving) (Table 5.4). This question required participants to perform complex procedures that involved complex calculations and apply higher order thinking skills. Sub questions included questions required the skills to break the question into its parts (Dhet, 2013). Cognitive levels 3 and 4 focused on the understanding category that did not only involve recall or definition of series of steps but also an understanding of underlying concepts of logarithms. Learners were expected to develop their own techniques for solving such problems. Sub questions in this question were divided into the following sub-sections:

3.1 Provide reasons for the following statements:

<table>
<thead>
<tr>
<th>Statement</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{T}_3 = x )</td>
<td>( _ _ _ _ _ _ _ _ _ _ _ _ _ (1) )</td>
</tr>
<tr>
<td>( \hat{P}_1 = x )</td>
<td>( _ _ _ _ _ _ _ _ _ _ _ _ _ (1) )</td>
</tr>
<tr>
<td>( WT \parallel SP )</td>
<td>( _ _ _ _ _ _ _ _ _ _ _ _ _ (1) )</td>
</tr>
<tr>
<td>( \hat{S}_1 = y )</td>
<td>( _ _ _ _ _ _ _ _ _ _ _ _ _ (1) )</td>
</tr>
<tr>
<td>( \hat{T}_2 = y )</td>
<td>( _ _ _ _ _ _ _ _ _ _ _ _ _ (1) )</td>
</tr>
</tbody>
</table>

3.2 Prove that \( RT = \frac{WR \cdot RP}{RS} \) \( (2) \)

3.3 Determine with reasons, other two angles that are equal to \( y \) \( (4) \)

3.4 Prove that angle \( \hat{Q}_3 = \hat{W}_2 \) \( (3) \)

3.5 Prove that \( \frac{WR}{RQ} = \frac{RS^2}{RP^2} \) \( (3) \)
For sub question $T_3 = x$, and $P_1 = x$, five out of 80 participants gave the reason as ‘tan-chord theorem’. They might have had a good comprehension of the theorem and applied it correctly. Figure 5.10 below is a sample of the 5 participants who provided the correct reason.

![Figure 5.10: Sample of a correct response](image)

**Figure 5.10: Sample of a correct response**

However, 55 participants provided irrelevant reasons as answers such as ‘corresponding angles with $T_3$’. This is incorrect. Participants’ inability to recognise that $T_3$ and $P_1$ are equal corresponding angles might have been as a result of poor visualisation because these angles are in a non-stereotypical position. Such learners appear not to be operating at van Hiele level 1. Twenty participants did not provide answers. This might have been as a result of lack of the ability to apply higher thinking order and poor visualisation. Figure 5.11 below is a sample of the incorrect reason given by $P_{55}$.

![Figure 5.11: Sample of an incorrect response](image)

**Figure 5.11: Sample of an incorrect response**

Sixteen participants provided the reason as ‘corresponding angles’ instead of ‘corresponding angles are equal’. Participants could not realise that in arriving at the conclusion, they needed to apply the proportionality theorem but, they wasted time in trying to prove that a pair of triangles is similar. Sixty-eight participants did not provide an answer. Inability to provide answers might be attributed to the opinion that they failed to devise strategies to decompose the complex diagram into manageable parts before attempting to answer. Lack of knowledge of proportionality theorem and cyclic quadrilateral definitions might have played a central role in participants’ inability to provide correct responses. During the interviews, their definitions of a cyclic quadrilateral were either incomplete or completely wrong altogether.

Sub question $S_1 = y$, only 6 out of 80 participants provided the correct reason as ‘corresponding angles are equal’. Thirty-seven out of 80 participants gave the reason as ‘corresponding angles’ instead of ‘corresponding angles are equal’. This reason is incomplete, and as a result, it is incorrect. 37 participants did not provide the reason. This might be because they could not realise that $S_1$ and $y$ were subtended by the same chord PQ. Figure 5.12 below is a sample of the correct response that was given by one of the respondents:
Figure 5.12: Sample of a correct response

Figure 5.13 below is a sample of lack of response of 37 participants who did not provide the answer for the sub-question.

Figure 5.13 Sample of a blank

Sub question $T_2 = y$, 12 participants provided the correct reason as ‘alternate angles WT II SP. 16 participants provided the incorrect reason ‘corresponding angles’ instead of ‘corresponding angles are equal’. These participants could not realise that in arriving at the correct conclusion, they needed to apply proportionality theorem, instead, they wasted time in trying to prove that a pair of triangles is similar. 68 of participants did not provide an answer. Inability to provide correct reasons might be attributed to the opinion that participants failed to devise strategies to decompose the complex diagram into the manageable parts before attempting to give answers for each sub-question. Lack of procedural knowledge might also have played a central role in participants’ inability to give correct responses, and as a result, majority of them left blank spaces. De Villiers (1998), cautioned about the traditional methods of teachers whereby learners are given ready – made definitions of proof in geometry, as one of the reasons that inhibit learners’ competency and mastery skills in proof.

Solutions to sub-question 3.1 are given as:

**Tangent chord theorem**

**Tangent chord theorem**

**Corresponding angles are equal**

**Angles subtended by chord PQ or angles in the same segment**

**Alternate angles; WT II SP**

In sub question 3.2, five participants provided the correct solution and the correct reasoning. They might have had a good comprehension of the proportionality theorem from previously proved examples (Flores, 2006). Figure 5.14 below is a sample of the correct answer and the correct reason as provided by 5 participants who got the responses correct.
Figure 5.14: Sample of a correct response

Half of the participants provided the correct solution but wasted time trying to prove a pair of triangles is similar.

Twenty participants did not provide an answer. Participants might have not known the steps to use to arrive at the solution. Thirty-four of them worked backwards and could not reach the solution as shown below. These participants might have had the slightest knowledge of the proportionality theorem and as a result may have lacked the skills to solve the problem. Dhlamini (2012) highlighted the instructional activities that focus on the lowest levels of geometric knowledge and routine problems. This is likely to have an undesirable influence on learner performance and understanding.

Figure 5.15 provides a sample of P_33 response that could not arrive at the correct solution.

Figure 5.15: Sample of an incorrect response

. The correct solution to this question is:

\[
\frac{RW}{RS} = \frac{RT}{RP} \quad \text{(Line parallel to one side of the triangle; Proportionality theorem; WT II SP)} \quad \text{OR}
\]

\[
\frac{RW}{RS} = \frac{RT}{RP} \quad \text{(Triangle RTW III Triangle RPS)}
\]
Therefore \( RT = \frac{RW \cdot RP}{RS} \)

Sub question 3.3, fifteen participants gave the correct answer and the correct reason. Figure 5.16 is a sample of one of the correct responses provided by P_03. These participants indicated to have possessed prior knowledge of tan-chord theorem and angles in the same segment are equal.

![Figure 5.16: Sample of a correct response](image)

Thirty-seven participants gave the incorrect solution. They might have read the question but could not understand what was required from them. Figure 5.17 below is a sample of one of the incorrect responses given P_69:

![Figure 5.17: Sample of an incorrect response](image)

Twenty-eight participants did not attempt the question. Participants could not realise that \( T_2 = R_3 \); that is, angle \( R_3 \) lies on one point of contact of tangent \( XRY \) and is equal to angle \( T_2 \) reason being \( T_2 \) lies in alternating segment. Therefore, \( R_3 = Q_1 \) because they are angles in the same segment. Inability to provide answers might have been due to participants’ inadequate prior knowledge of tangent-chord properties and poor visualisation. As Wing (in Ramlan, 2016), mentioned, inability to define and deduce properties of a geometric entity by applying intrinsic properties and relationships with other entities and rules of inference that bind properties together, lead to poor achievement and understanding of geometry.

The correct solution to this question is given as:
\[ Y = \hat{T}_2 = \hat{R}_3 \text{ (Tan-Chord theorem)} \]

\[ Y = \hat{R}_3 = \hat{Q}_1 \text{ (Angles in the same segment)} \]

From sub question 3.4 forty-five participants could not establish relationships between angles. Figure 5.19 below is a sample of the incorrect response provided by P_{77}:

![Figure 5.18: Sample of an incorrect response](image)

Forty-five out eighty could not establish relationships between angles, with some merely stating that \( \hat{Q}_3 = \hat{W}_2 \) because the angle of a cyclic quadrilateral is equal to the interior opposite angles. Fifteen participants did not answer the question. When interviewed, participants could not come up with the alternative solution to the question. The correct solution is given as:

\[ \hat{Q}_3 = P\hat{S}R \text{ (ext angle of a cyclic quad)} \]

\[ P\hat{S}R = \hat{W}_2 \text{ (Corresponding angles; WTII SP)} \]

\[ \text{OR} \]

\[ \hat{Q}_2 = x \text{ (angles in the same segment)} \]

\[ \hat{Q}_3 = 180^\circ - (x + y) \text{ (angles on a straight line)} \]

\[ \hat{W}_2 = 180^\circ - (x + y) \text{ (angles of a Triangle WRT)} \]

**Therefore** \( \hat{Q}_3 = \hat{W}_2 \)

From question 3.1, 3.3 and 3.4 above, participants who got incorrect answers and those who left the questions unanswered might have been unable to link the questions together and therefore found them challenging to answer. Some merely stated that \( \hat{Q}_3 = \hat{W}_2 \), giving the reason as ‘the exterior angle of a cyclic quadrilateral is equal to the interior opposite angles.’ This is incorrect because candidates could not realise that \( RWTQ \) is not a cyclic quadrilateral. Others assumed that \( SQ \) and \( PR \) are diameters of the circle. Some candidates made reference of \( P\hat{S}R \) as \( \hat{S}_{1+2} \). Participants might have lacked necessary strategies and skills to approach the questions. As Bennie (1998) previously indicated that in South African primary
schools, geometry instructional activities are insufficient with regard to equipping learners with relevant skills to operate at the levels of axiomatic thinking when they reach senior secondary schools. Henning (2014) again made a significant discovery that as much as South African primary schools have sound foundation phase curriculum, it tends to go way too fast for majority of learners. It requires rapid progress through important concept-learning that should, for majority of learners, go much slower. As a result, learners are denied the opportunity to learn at their own rate and pace (Luneta, 2015), because teachers are required to religiously stay on curriculum schedule (Henning, 2014). De Villiers (2010) attested that the systemic geometry syllabus is focused more on the secondary school education, and insufficient content is evident in primary school band. The author further made claims that as much as tessellations have been recently introduced in primary school geometry education, majority of teachers and text book authors are unable to relate them to the van Hiele theory. This is an undesirable situation because geometry taught in the first years of schooling should lay a solid and sufficient foundation on which more advanced learning of concepts can be based.

In sub question, 3.5, only 5 participants provided the correct solution accompanied by the correct reasons. Figure 5.20 is a sample of P_01’s correct response provided:

![Image of a correct response]

**Figure 5.19: Sample of a correct response**

Fifty-two participants did not attempt to answer the question. Thirteen participants provided reason as ‘proven above’ and two and ‘two triangles are congruent’. Ten candidates provided the reason ‘proven above’, as a reason for a pair of angles claimed to be equal but these angles were not proven to be equal anywhere. None
of these were correct. Participants made claims that $\hat{R}$ is common without being aware that the two angles in question actually occur in two different triangles and therefore are not common. This might be as a result of poor visualisation by learners. For instance, the neo-constructivism model of conceptual change argued that in order to take in new information that is relatively not compatible with the previous information; learners tend to assimilate the new information into their existing schema (Kajander & Lovric, 2010), learners will undergo cognitive conflict, when the new information is absorbed, and they have to solve the conflict by interpreting the new information in relation to the prior information in trying to find balance. As such, problems that may lead to learning difficulties in mathematics are those that may result from informal thinking and poor remembering (Eong et al., 2015). The correct solution is given as:

**In $\Delta RTS$ and $\Delta RQP$:**

$\hat{R}_3 = \hat{R}_2 = Y$ (Proven)

$\hat{S}_2 = \hat{P}_2$ (angles in the same segment)

$\hat{R}TS = RQP$ ($3^{rd}$ angle of $\Delta$)

**Therefore $\Delta RTS \cong \Delta RQP$**

The Table below shows the frequency of errors made with respect to slips, conceptual and procedural errors.

**Table 5.5 Frequency of errors made by participants in relation to slips, conceptual and procedural errors**

<table>
<thead>
<tr>
<th>Category</th>
<th>Question Numbers</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>ERR 1</td>
<td>1.1 1.2 2.1 2.2 3.1 3.2 3.3 3.4 3.5</td>
<td>23</td>
</tr>
<tr>
<td>ERR 2</td>
<td>6 6 5 9 39 48 50 67 67</td>
<td>297</td>
</tr>
<tr>
<td>ERR 3</td>
<td>6 12 38 26 45 52 40 64 63</td>
<td>346</td>
</tr>
</tbody>
</table>
5.4 Focus group interview discussion

The researcher conducted the focus group interviews with nine participants (See Appendix 7). Participants were chosen based on their geometry test performance; 3 outstanding performances (80% – 100%), Meritorious performance (70%-79%), Substantial performance (60% -69%), Adequate performance (50%-59%), Moderate performances (40% - 49%) and Not achieved (0% – 29 %). Figure 5.21 below is an example of the scale of achievement that was used to grade and select participants according to their achievement.

![Scale of achievement and rating](http://www.swgc.co.za)

The researcher conducted face to face interviews, in this case, the focus group interviews with participants. The purpose of the focus group interviews was clarified in chapter 3. In creating the interview questions, the researcher designed probing questions and made sure that same questions were presented to all participants to compare results. In each interview discussion, the researcher made sure that notes were taken. This is termed memoing (Miles & Huberman, 1984). These reflective notes assisted the researcher to make sense of participants’ feelings, beliefs and convictions about the topic under enquiry (Corbin & Strauss, 2008). Errors were categorised into expositions such as telling approaches, and explanations. Direct quotations were used to capture participants’ responses.

The researcher prepared the data by creating a database with the first column that contained all the data transcripts of the interviews (Luneta, 2014). The next step was to segment transcripts according to participants’ characteristics, that is, HIGH_ACHIEVERS, AVERAGE_ACHIEVERS and LOW_ACHIEVERS. The second column contained the preliminary codes (Layder, 1998). These were the bolded jotted down preliminary phrases or words for codes from the transcripts that were worthy of the attention of the researcher (Boyatzis, 1998). The third column contained the final codes, which were derived from preliminary codes. These were in a form of a word or short phrases, which were the basic topic of a transcript (Saldana, 2009). According to Tech (1990), it is important that these codes are identifications of the topic, which is what is talked about or written about. The fourth column was made up of the generic categories. These categories were derived from
the groupings imposed on the final codes, reducing the number of different pieces of data in the analysis (Vaismoradi, 2013). The last and fifth column was made up of themes (Saldana, 2009). Themes were derived from generic categories, which were used to identify major elements of the entire content analysis (Vaismoradi, 2013).

Interview questions were divided into three main categories. Questions 1, 2, 3 and 4 were based on the definitions of terms and concepts. Question 5 evaluated participants’ levels knowledge of geometry. Questions 6, 7, and 8 focused on learner errors. Questions 9 and 10 examined the strategies used by participants in arriving at their conclusions with respect to responses in the geometry performance test.

Geometry syllabus and Curriculum (DBE, 2011) required that learners should be fluent with the terminology of each chapter they encounter during learning. According to the National Diagnostic Report (2014), one of the leading causes of learner confusion is lack of understanding of basic geometry terminology and poor visualisation. Assuming that terms and theorems were learned, answers would enable the researcher to determine if participants gained knowledge of such concepts or not. Questions 1, 2, 3, and 4 focused on definitions and explanations. Below are the extracts in direct quotations, of learner responses. Extracts are presented in terms of the order of the questions.

11. “State all the theorems that are associated with circle geometry”

HIGH_ACHIEVERS

P_01: “A line drawn from the centre of the circle to the midpoint of a chord, then that line is perpendicular to the chord; angles that are in the same segment of a circle are equal; when the diameter of the circle subtends an angle at the circumference, the subtended angle is a right angle; When a tangent to the circle is drawn, it is perpendicular to the radius at the point of contact. Ngiphelelelw, bakhulume ezinye beng’funa kuzibiza (I’ve run out of others because Ntombi and Lihle have already stated other theorems I wanted to state)”

P_04: “An arc subtended an angle at the centre of the circle and at any point on the circumference, and then the angle at the centre is twice the angle at the circumference; Opposite angles of a cyclic quadrilateral are supplementary; The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle; A line that is drawn perpendicular to the radius at the point the radius intersect the circle, that line is a tangent to the circle”

P_05: “The line that is drawn from the centre of the circle and is perpendicular to the chord, it bisects the chord; An angle subtended by a chord at a point on the circumference is a right angle, that chord is a diameter; Four
vertices of a quadrilateral from which the opposite angles are supplementary, it will be a cyclic quadrilateral; Two tangents that are drawn from the same point outside the circle are equal; An angle between a tangent and a chord that is drawn from the point of contact is equal to an angle in the alternate segment.

**AVERAGE_ACHIEVERS**

**P_11:** “Tan-chord theorem; the exterior angle of a cyclic quadrilateral equals the interior opposite angle. Ya! I know of other theorems, but I will have to go through them before I can state them”

**P_13:** “An angle at the centre of a circle is twice the angle at the circumference; the line drawn from the centre of the circle and is perpendicular to the chord bisect the chord”

**P_18:** “Angles in the same segment of a circle are equal; opposite angles of a cyclic quadrilateral are supplementary. That is all I can think of now”

**LOW_ACHIEVERS**

**P_26:** “The angle which an arc of a circle subtends at the centre is twice the angle it subtends at the circumference”

**P_59:** “The angle subtended at the centre of a circle by a diameter is a right-angled triangle”

**P_74:** “A tangent to the circle is perpendicular to the radius at the point of contact”

**12. How would you explain the following terminology? Tangent Cyclic quadrilateral, circumference, chord?”**

**HIGH_ACHIEVERS**

**P_01:** “A tangent is a line that passes through the circumference, but touches only one point at that circumference. A cyclic quadrilateral refers to the lines/diagrams which touch the circumference at four points or a diagram with four corners that touches the circumference of a circle at four points. Circumference refers to the points of a full circle or points around a circle. A chord is a line which starts from the circumference and end at the other point of the circumference but doesn’t pass through the centre”

**P_04:** “A tangent is a line that touches a curved surface at some point but it does not cross it. A chord in geometry is a straight line from which the two endpoints lie on the circle. A circumference is the borderline of the
circle. A cyclic quadrilateral is a four sided figure, like a square or a rectangular shape, whose vertices lie on a circle.

P_05: “May I answer the questions without following any specific order. A circumference is the perimeter of an object or outer dimensions of an object. Hmmmm, a tangent is a line that touches a curve at a given point; however, it does not cross it, like, the tangent graph in terms of sine and cosine waves. A cyclic quadrilateral is a four sided figure, where its lines intersect, lying on a circle. Yeah Eish!”

AVERAGE_ACHIEVERS

P_11: “A tangent is a line that touches the circumference at one point. Circumference is the distance around the outer rim of a circle. Errrr…I don’t know how to define a chord” (hides face)

P_13: “For me, a tangent is a line that touched the circle only on one point. A cyclic quadrilateral is something as given by triangle around it. A chord is a line with end-points on the circumference”

P_18: “(Hides the face with both hands) I forgot the definitions, I don’t wanna lie. I am able to point out these terms when given a diagram to work with”

LOW_ACHIEVERS

P_26: “Tangent is the line drawn from the radii of the circle to another side of the circle. Cyclic quadrilateral is when all angles that are in 8 circle touch or its attached to the circle. Circumference is when the angle forms the degree of 90˚ or more. Chord is the centre of the circle.”

P_59: “Cyclic quadrilateral is when opposite angles of a cyclic quad are equal to 180˚, supplementary. Chord is a line cutting across a circle or is at a position other than the diameter. ”

P_74: “A tangent is a line that passes through…. A cyclic quad has two sides that add up to 180; I do not know how to explain the circumference and a chord”

13.”Explain the theorem associated with the diagram in question 2.”

HIGH_ACHIEVERS

P_01: “I used the theorem associated with isosceles triangles. This theorem uses the properties of an isosceles triangle such as: if two sides are equal, also their angles are equal, and where the two sides meet, they form 90˚ (perpendicular)”

P_04: “Line from the centre to midpoint is perpendicular to the chord”
P_05: “Line from the centre to midpoint is perpendicular to the chord”

AVERAGE_ACHIEVERS

P_11: “Perpendicular from the centre to chord, that is, the line from the centre to the midpoint or chord”

P_13: “B is equal to 150˚, opposite angles of a cyclic quadrilateral ABCD, since D is given as 30˚”

P_18: “F₁ is 90˚, because OF bisect CD, or OF is perpendicular to CD, Yes”

LOW_ACHIEVERS

P_26:” F₁ is 90˚, because CF is equal to FD, this is given”

P_59:” F₁ is 60˚ because it is the angle that is subtended by the diameter”

P_74:” F₁ is 90˚, from the diagram, it is given that CF = FD”

14. “Deconstruct the diagram in question 3 and explain all the theorems that are associated with it.”

HIGH_ACHIEVERS

P_01: “Tangent-chord theorem. Errrr.. from the given information, we are told that two circles in the diagram share a common tangent XYZ at R. W is any point on the small circle. Therefore, T₃ = x = P₁. Then again, there is y = R₃ = Q₁ because they are angles in the same segment.”

P_04: “Opposite angles of a cyclic quadrilateral equal to 180˚, R + P = 180˚, and S + Q = 180˚, meaning that SPQR is a cyclic quadrilateral. But in being more specific, Q₃ = P₃ the reason is that Q₃ is an exterior angle of a cyclic quadrilateral that is equal to the opposite interior angle. Therefore, P₅R is equal to W₂ they are corresponding angles: WT // SP”

P_05: “Proportionality theorem, from question 3.3 prove that RT = \( \frac{WR \cdot RP}{RS} \), \( \frac{RW}{RS} = \frac{RT}{RP} \) (Line parallel to one side of the triangle; Proportionality theorem; WT II SP) or one can explain it this way:

\[ RW \frac{RW}{RS} = RT \frac{RT}{RP} \] (Triangle RTW III Triangle RPS)

Therefore \( RT = \frac{RW \cdot RP}{RS} \) ”

AVERAGE_ACHIEVERS
P_11: “Tangent theorem (Tangent XRY), Tan-chord theorem”

P_13: ”Cyclic quad theorem, Corresponding angles $S_1 = y$”

P_18: “Alternating angles TQR”

LOW_ACHIEVERS

P_26: “Mam, I did not know the theorems, I did not study for the test”

P_59: “Only if I knew how to explain the theorems, I would have answered the question correctly; I had never enjoyed any geometry lesson because I am always confused when the topic is brought up in class, as a result, I decide to skip geometry lessons”

P_74: “I felt frustrated and confused; as a result I answered this question incorrectly. I could not understand what I had to do”

Analysis of learner responses

HIGH_ACHIEVERS

Three participants in this category represented 6% of participants who achieved between 80% - 100% in the geometry performance test. These participants seemed to have an outstanding comprehension of basic knowledge of circle geometry theorems, terms and definitions and the ability to link diagrams to properties. This suggested that participants were able to operate at a deductive proof scheme, where they were able to make operational thoughts and logical inferences (Martin et al., 2005).

AVERAGE_ACHIEVERS

Participants in this category represented 10% of participants who achieved 49% - 59%. This group seemed to have acquired a limited knowledge of basic circle geometry theorems, terms and definitions and the limited ability to link properties of diagrams to properties to diagrams. This was evident in the responses these participants gave, for instance, “Ya! I know of other theorems, but I will have to go through them before I can state them”, Participants seemed to have lacked appropriate vocabulary to explain the terms, for instance “I am able to point out these terms when given a diagram”. This lack of appropriate vocabulary was again evident in the response given by one of the participants when he/she explained the cyclic quadrilateral as “something given by a triangle around it”. This definition is incorrect because, a cyclic quadrilateral cannot be a “thing” and one cannot make sense of out of the definition. This might have been due to lack of reinforcement by the teachers in the previous grades. There might have been an assumption that learners are fluent with these terms, definitions, and theorems from previous grades, as a result, this important section of learning might have been rushed through. As Henning (2014) had suggested, that teachers often rush through important sections
of the syllabus because they have to keep up with the schedules provided by the district; and learners miss out on important sections of the knowledge that was supposed to have been properly learned and acquired.

LOW_ACHIEVERS

Participants in this category represented majority of participants, that is 84% who achieved between 0% - 25% in the geometry performance test. Participants in this category seemed to have insufficient knowledge of theorems, terms and definitions. This was indicated in the responses participants provided, for instance, “A chord is the centre of the circle”, “A cyclic quadrilateral is when all angles that are in 8 circle touch, or its attached to the circle”, and “Circumference is when the angle forms the degree of 90˚ or more”. Participants seemed to have been clueless when coming to stating, defining the terminology associated with circle geometry. These responses are an example of what Atebe and Schafer (2009) referred to when they argued that the teaching and learning of geometry is one of the discouraging experiences in majority of South African schools mainly because learners experience challenges in understanding instructional methods used by teachers. Participants' performance might be attributed to numerous reasons such as inadequate circle geometry knowledge, the fact that geometry in the past was removed from the curriculum and had to be offered as an optional paper, and the instructional activities that are characterised by listening, watching and imitating what the teacher does (Mji & Ndlovu, 2012).

What should be done to support learners to understand circle geometry concepts?

Basic work such as explanations of concepts and theorems should be thoroughly covered, for instance, each definition of a concept or a theorem should be accompanied by the explanation of the relationship using the diagram (DBE, 2014). Furthermore, Burger and Shaughnessy (1986) suggested that the sequencing of learning activities had a positive influence on learner achievement and understanding. For instance, if preliminary learning activities are frustrating, the learning process is affected in a negative way. This suggests that learning should begin with learning activities where learners experience success at mastering the concepts while gradually progressing to more challenging activities. Kariyana and Sonn (2016) highlighted the importance of incorporating various learning approaches such as discovery learning, learner-centered approaches and cooperative learning to accommodate various interests, learning needs, talents and backgrounds in managing learner variations for learning to be effective. According to Aird and Van Duyn (2012), textbook authors provide proof exercises and examples as finished products in textbooks. A figure below is an example of a finished product of a proof example, with the solution provided on the next page of the same material:
The traditional approach where learners are provided with questions and answers inhibits their thinking as well as inability to solve problems as shown above, and as a result they consistently operate below the van Hiele level 1. De-Villiers (1997) advise that instead of religiously following the current finished product approach to geometry, teachers should aspire to develop learners’ abilities to define. Research (Freudenthal, 1973) advocates that learners should be active participants in coming up with definitions and explanations of geometric concepts and theorems. Educators
should design assessments that promote higher thinking skills that will result in
learners acquiring logical and deductive reasoning skills.

As mentioned in the previous chapter 2, teaching and learning of geometry is one of
the discouraging experiences in the majority of schools across South Africa (Atebe &
Schafer, 2009). Assuming that participants were aware of their learning, the following
question by the researcher sought to understand participants’ views on their
knowledge of geometry.

15. “How would you regard your knowledge of proof in circle geometry?”

HIGH_ACHIEVERS

P_01: “I regard my circle geometry knowledge very good because hhmmm,
during the lesson I listen very carefully to what the educator says. I always
consult different textbooks. Infact, I enjoyed proof questions”

P_04: “Well, I can say I enjoyed solving riders in this test. I applied the solid
knowledge I gained from previous grades, coupled with the knowledge I
gained in grade 12. It was a smooth sailing for me”

P_05: “I cannot say I am good at proof problems, again, looking at my
performance in the test, I did not do bad also. I needed to be tactical
when answering proof question, for example, read the question carefully
to gain an understanding of what is needed, I looked for the additional
given information, and applied direct reasoning, from there I solved the
problems and from there I looked back at what I did and did not”

AVERAGE_ACHIEVERS

P_11: “My knowledge of proof is moderate, at times, I do well in proof, and some
other times, I perform poorly. I think I can do well if I can put more effort
and practice regularly”

P_13: “I notice some improvement on this test. I generally perform poorly, but
because I had studied, my performance changed. I think that the group
discussions on proof I held with my classmate had played a role. I
learned a lot, although time was limited. I believe that I can still improve”

P_18: “I do not enjoy proof at all. I get confused most of the time, so I put more
effort on algebra, and finances. Even when I write exams and tests,
geometry section becomes the last that I answer because most of the
time, I get incorrect answer, so I channel my time to those questions that
I understand and that I know I will get correct. But, I worked on my
attitude towards geometry last year; here I am now, noticing major
improvement on Euclidean geometry”
LOW_ACHIEVERS

P_26: “My knowledge of proof is limited, I have never got it right, even from lower grades, I used to struggle, and as a result, I have never liked proof.”

P_59: “I find it hard to prove, we only focus on easier and straightforward proof and examples that are done in class, so it becomes difficult for me to solve complex problems”

P_74: “I am not so confident when I solve geometry riders. I know less, some of them I cannot prove. I understand when the teacher does an example of a proof question, but when I’m alone, I get totally confused.”

Analysis of learner responses

HIGH_ACHIEVERS

Participants in this group seemed to have solid knowledge of proof, which they might have acquired through paying careful attention to details, consulting different study materials and through basic knowledge gained from previous grades. One participant highlighted the importance of being ‘tactical’ when answering proof questions. This might mean that they were able to develop geometric concepts by using their visual imagery, as these concepts developed, they might have paid attention to similar attributes of figures with informal definitions (Walcott, Morhb & Kastberg, 2009). This suggested that the way in which geometry concepts largely depends on the visual imagery.

AVERAGE_ACHIEVERS

Responses from the participants might have suggested that they were unsure of their levels of knowledge of geometry. This was revealed in the responses participants provided, for instance, “……sometimes I do well, sometimes I perform poorly”. One participant made a statement “I understand when the teacher does an example of a proof question, but when I’m alone, I get totally confused”. This might be because of the ‘one size fits all’ approach to learning, where assessments are not developed in line with the different learning needs in a classroom. This approach to learning is characterised by rote learning (Reddy, 2006). Stein et al., (in Makonye, 2013) also argued that learners who conform with this approach to learning hold various misconceptions because they do not have adequate knowledge of the nature of mathematics.

LOW_ACHIEVERS

Responses from the group suggested that participants might have inadequate knowledge of concepts, and the procedures. This was evident in the type of responses which were provided. For instance, one participant mentioned “I understand when the teacher does an example of a proof question, but when I am
alone, I totally get confused”. Other participant made the statement “I find it hard to prove, we only focus on easier and straightforward proof and examples that are done in class, so it becomes difficult for me to solve complex problems”. These responses are in line with the research that was conducted by Mji and Makgato (2006) where participants who took part in the study gave responses such as “we spend most of time learning algebra, which is easy, but what about geometry which is difficult?. That is why we do little geometry”. As a result, learners are unable to operate at axiomatic thinking (Bennie, 1998). Inability to acquire knowledge of procedures and concepts might be related to lack of competency by teachers to teach mathematical concepts that they are not comfortable with (Mji & Ndlovu, 2012). According to Makonye (2013), mathematical teachers who are poorly trained also train their learners to apply rules without conceptual understanding. This results in learners holding various misconceptions about mathematics.

**What should be done to support learners to understand circle geometry concepts?**

Teachers should find balance between learning activities, learner abilities and tasks at hand. For instance, the van Hiele framework for geometric thinking was developed by Pierre and Dina van Hiele after coming to a realisation that their students struggled with geometry (van Hiele, 1986). According to the van Hieles, each level has its own set of symbols and language characteristics which are relevant at that particular level and can be seen as irrelevant at another level. This demonstrates two people who operate at different levels speak different languages (Van Hiele, 1986); this is an example of what happens in the classroom between the teacher and learners; none of them can manage the thinking process of the other and their understanding can only begin if the teacher tries to understand what the learners are thinking. According to Luneta (2015), the framework provide the platform for teachers to develop and assign learners with learning activities that are relevant to their levels of thinking, and serves as an instrument to measure present and future learning. As Jones (2003:128) mentioned ‘the van Hiele model has become a proved descriptor of progress of students’ reasoning in geometry and is a valid framework for the design of teaching sequences in school geometry’. This suggests that the framework can inform mathematics teachers and schools about the learning support in circle geometry that students need when they engage in Euclidean geometry.

Evaluating misconceptions that learners have that result in errors is important in building improved geometric understanding in learning. But, there are limited studies that focused on areas of learners looking at their own misconceptions as a way of advancing their learning (Kembitzku, 2009; Drake & Amspaugh, 1994; Eggleton & Molayar, 2001). Assuming that participants were aware of their own learning, answers to this question would enable the researcher to determine if participants were aware of their errors.
16. “Explain the difficulties encountered in question 1.2”

HIGH_ACHIEVERS

P_01: “Well, I didn’t encounter any problems. For me, after looking at this question, what came to mind were the properties of triangles. I found the value of \( x = 96^\circ \), I then looked at \( \hat{C} \) and added \( \hat{B} \) and equated it to \( 180^\circ \), reasoning behind being the sum of the angles of a triangle equals \( 180^\circ \), and found the value of \( \hat{y} = \hat{B} = 42^\circ \), with the reason being opposite sides are equal”

P_04: “I think this question was straight forward. For me to get the correct answer, I had to get question 1.1, correct first. In question 1.1, I found the value of \( x \) to be \( 96^\circ \) by just looking at the diagram. From there, I remembered that angles of a triangle equal to \( 180^\circ \). I subtracted \( 96^\circ \) from \( 180^\circ \) and equated it to \( \hat{C} + \hat{B} \), that is, \( \hat{y} = \hat{B} = 42^\circ \). That was it”

P_05: “I did not experience any problems. Like I said earlier, it was about a strategy. I read the question, and applied the properties of triangles.”

AVERAGE_ACHIVERS

P_11: “I did not have any problem with this question, \( y = \hat{B} = 42^\circ \)”

P_13: “\( 180^\circ - 96^\circ = \hat{C} + \hat{B}, \) therefore \( \hat{y} = \hat{B} = 42^\circ \)”

P_18: “\( y = \hat{B} = 42^\circ \)”

LOW_ACHIEVERS

P_26: “I did not have any problem with this question”

P_59: “I guessed the answers; hence I did not get any correct answer for the question.”

P_74: “I did not know what I was doing. I got the answers incorrect. I think it is because I panicked when I came across question 3. To be honest, I have never enjoyed geometry because in class, we only focus on simple diagrams and problems”

17. “Point out errors in solving question 3.2. and 3.4 What other alternative approach may be used to solve question 3.2 and 3.4?”

HIGH_ACHIEVERS
P_01:” Well, I did not encounter any problems with the questions. Err…, I can’t think of the alternative approach now (giggles). But I approached the question from the proportionality point. That is, I looked at $\frac{RW}{RS} = \frac{RT}{TP}$, my reason was that WT is parallel to SP, $\therefore RT = \frac{WR\cdot RP}{RS}$”

P_04: “I did not experience any challenges. I also approached the question like (Name of the previous learner) did, but I can also solve it this way: From the diagram, I can identify triangle RTW and triangle RPS which are equal in sides, from there $\frac{RW}{RS} = \frac{RT}{TP}$, $\therefore RT = \frac{WR\cdot RP}{RS}$”

P_05: “No challenges experienced, I looked at $\frac{RW}{RS} = \frac{RT}{TP}$, my reason was that WT is parallel to SP, $\therefore RT = \frac{WR\cdot RP}{RS}$”

AVERAGE_ACHIVERS

P_11:”I did not experience any challenges, but I don’t know of another approach to solve the problem”

P_13: “My mistake in the solution was to equate $R_3$ to $R_3$, and stated that they are similar”

P_18: “I did not attempt to answer the question, but there is no any other approach to the question”

LOW_ACHIEVERS

P_26: “I did not answer the question because I did not know what I was doing.”

P_59: “I cannot think of any other approach. I answered the question incorrect”

P_74: “I did not answer the question. I did not know what to do. My mind went blank when I came across question 3.2”

18.”Explain errors in 3.3”

HIGH_ACHIEVERS

P_01: “I did not encounter any”

P_04: “I did not come across any….”

P_05: “None so far”

AVERAGE_ACHIVERS
P_11: “I couldn’t understand the meaning of the word tangent; I think that is why I got the answer wrong”

P_13: “I am unable to explain the errors because I honestly did not know what I was doing. It was for the first time I came across this type of question. We normally do easier exercises in class, not this”

P_18: ”I did not answer this question. If we can do revision on this work, I will do better”

LOW_ACHIEVERS

P_26: “I know very little when it comes to theorems; hence I am unable to prove them”

P_59: “I really did not know how to answer this question. Maybe having extra classes on geometry will help because I feel that our teacher rushed through this section. I was always left behind”

P_74: “I could not complete this question. I realised that I was not coping. I do not have good background when it comes to proof and theorems”

Analysis of learner responses

HIGH_ACHIEVERS

There were no evidence of errors based on the participants’ responses on questions 1.2, 3.2 and 3.3. Participants demonstrated understanding of properties of triangles when answering 1.2. For 3.2, participants seemed to have a good comprehension of the proportionality theorem, and a good understanding of tan-chord theorem. Comprehending what proving is about and the ability to write proof successful is important for success in Mathematics (Aksu & Koruklu, 2015).

AVERAGE_ACHIEVERS

Participants seemed to have not experienced any difficulties in answering 1.2. There were no evident errors in participants’ responses for question 3.2, but one participant indicated that there is no any other approach to answer the question. One participant was able to recognise his/her error as the inability to correctly define the term tangent, as the reason he/she could not answer question 3.3 correctly. This realisation, according to Elisero (2012) might be beneficial to motivating learners to draw conclusions concerning future learning behaviours.

LOW_ACHIEVERS

Participants seemed to have performed well in question 1.2 because there was no evidence of errors in their responses. One participant stated that he/she guessed the answer for question 3.2. For question 3.3, participants mentioned lack of concepts
and procedures as reasons they did not answer the question. One participant stated that he/she guessed the answer. This might have been an indication that the participant was not aware of errors.

**What should be done to support learners to understand reflecting on their errors?**

The notion of having learners reflecting on their individual misconceptions and errors might assist with improving their awareness and self-regulation (Kembitzku, 2009). According to Eliserio (2012), encouraging learners to evaluate their learning results and draw conclusions from such, might have a positive influence on their achievement. Learners who are self-regulated believe in opportunities to take on challenging learning tasks, practice their learning to develop deeper understanding of the subject matter and exert more effort give them rise to academic success (Leidinger, 2012). This can be achieved through allowing them to think out loud, that is, allowing them to explain their thought process when solving a geometry problem. Schunk and Zimmerman (1998) outlined specific ways in which learners can become reflective of their learning such as planning of their learning, for instance, analyse the learning task, set the learning goals and planning the strategies (what is the goal of this task?, what strategies are most effective with this type of task?). The next phase is the monitoring phase (Is the strategy working?, Am I staying focused, do I need to adjust the strategy?), and finally the evaluation phase (What did I feel about this strategy? Did I use it properly? How well did it work, and was the strategy a good match with the learning task?). Schunk and Zimmerman (1998) emphasised that these strategies might serve as a motivation especially in cases of poor performance which is attributed to insufficient effort and poor task strategies.

For an assessment to be valid, fair and reliable, it should adhere to all the assessment guidelines (DBE, 2011; DHET, 2013). This meant that an assessment should cover all the cognitive levels, for instance, knowledge, routine, complex procedure and problem solving. Assuming that participants were exposed to assessments that incorporated all the cognitive levels, responses to questions 9 and 10 would enable the researcher to determine if participants acquired all these skills or not.

19. “How did you arrive at an answer for question 3.5?”

**HIGH_ACHIEVERS**

P_01: “I had already proven that \( \frac{RW}{RS} = \frac{RT}{TP} \), my reason was that WT is parallel to SP, \( \therefore RT = \frac{WR\cdotRP}{RS} \), I then equated \( \hat{Q}_3 \) to PŠR, my reason was that these were corresponding angles are equal since WT is parallel to SP, therefore \( \hat{Q}_3 \) is equal to \( \hat{W}_2 \)”
P_04: “I have proven from 3.4, since RT is equal WR.RP; I stated that, that statement is correct”

P_05: I had already proven that \( \frac{RW}{RS} \cdot \frac{RT}{TP} \), my reason was that WT is parallel to SP, \( \therefore RT = \frac{WR.RP}{RS} \), I then equated \( \hat{Q} \) to \( \hat{P} \), my reason was that these were corresponding angles are equal since WT is parallel to SP, therefore \( \hat{Q} \) is equal to \( \hat{W}_2 \)

AVERAGE_ACHIVERS

P_11: “Eish.....This question was difficult for me”

P_13: I left the question incomplete, as I did not know what to write””

P_18: “I got stuck at question 3.1 and could not reach to question 3.5”

LOW_ACHIEVERS

P_26: “I did not answer the question”

P_59: “I did not know how to answer the question, I got it wrong”

P_74: “Time was up by the time I was trying to figure out the answers”

20. “Describe the strategy (ies) you applied when solving question 3.”

HIGH_ACHIEVERS

P_01: “To be able to solve question 3, I needed to break down that complex diagram into its smaller parts first, and then identify theorems and properties of each”

P_04: “To be able to solve question 3, I needed to break down that complex diagram into its smaller parts first, and then identify theorems and properties of each”

P_05: “As (Name withheld) stated the trick in answering this question lied in breaking the complex diagram into smaller parts first, then determining the properties of each part, as shown in the solution”

AVERAGE_ACHIVERS

P_11: “I got this question wrong, but now that I’m discussing it, I come to realise my mistake, \( P_1 = x_1 \) reason being the tan-chord theorem, I initially provided the incorrect reason as exterior angle of a cyclic quad”
P_13: “I did not have any plan or approach. Proof is difficult to master for me”

P_18: “I initially tried to breakdown the diagram, but ended up getting confused even more. I then left the question unanswered”

LOW_ACHIEVERS

P_26: “By the time it was announced time up, I was still struggling to make sense of question 3. That is why I did not answer it”

P_59: “I had memorised the theorems the night before the test. When I was supposed to apply them, I had forgotten what I had memorised. I did not know how to approach question 3. I can only prove a theorem on one simple diagram, If a diagram is complicated like this one, I get more confused”

P_74: “Question 3 was the most challenging for me. I did not know how to approach it. I was threatened by the diagram; as a result, I found answering questions to be difficult. I did not answer it”

Analysis of learner responses

HIGH_ACHIEVERS

Participants mentioned the skill to deconstruct the complex diagram in question 3, and proceeded to identify diagram. This might have suggested that they spent more time understanding lower cognitive level questions and applied this information to solve complex problems (Hanna et al., 2009).

AVERAGE_ACHIVERS

Participants seemed to have not had any skill or approach to answering question 3. This was evident in the responses that they provided, for instance “I did not have any plan or approach. Proof is difficult to master for me”. One participant realised his/her mistake during the interview session, “I got this question wrong, but now that I’m discussing it, I come to realise my mistake, \( P_1 = x_1 \) reason being the tan-chord theorem, I initially provided the incorrect reason as exterior angle of a cyclic quad”. Realisation of this alternative conception can be used as an important tool to empower learners in “proof checking and construction” (Clements & Battista, 1992:55).

LOW_ACHIEVERS

Participants in this group might have had no clue in approaching the question. This is evident in one of the responses provided “By the time it was announced time up, I was still struggling to make sense of question 3. That is why I did not answer it” This suggested that the participant spent time trying to figure out how to approach the question. Another participant made a mention of memorisation as a reason that
he/she was unable to solve the problem. Harel (2008) suggested that geometry learning should focus on understanding rather than rote learning.

What should be done to support learners to understand problem-solving strategies?

The way in which learners operate from one level is impossible without thinking from previous levels, for instance, thinking at level 3 is impossible without thinking at level 2 and 1 (van Hiele, 1986). As a result, shift from one cognitive level to the next is not natural, but should take place under effective teaching and learning processes (Mayberry, 1983; Burger & Shaughnessy, 1986). This implies that maturity to the next cognitive levels should happen under adequate learning processes that are necessary at lower levels in order to be able to think and reason at higher levels. Pegg (in Luneta, 2015), explained that the hierarchical sequence of the levels indicated that each level must be fully developed before proceeding to the next. This could be achieved through rigorous learning activities where learners are encouraged to come up with their workable definitions of their own activities stimulated by appropriate questions that are interesting and educational (De Villiers, 1997:46). Levels are designed in such a way that a learner cannot operate based on one level without having gone through other previous levels. The role played by scaffolding in the learning of geometry points to the advancement from one level to the next (Ndlovu & Mji, 2012).
CHAPTER 6: FINDINGS, RECOMMENDATIONS AND CONCLUSIONS

6.0 Introduction

The preceding chapter analysed participants’ responses with focus on the misconceptions they hold that resulted in the type of errors shown. Question by question analysis was done on 80 answer sheets, and was followed by focus group interviews based on the learners’ performance in the test. In this chapter, summary of the findings is given which is followed by conclusions. The researcher suggests some recommendations.

The theoretical point of view that guided this study is different learner philosophies with respect to errors and misconceptions. Makonye (2013) argued that majority of learners hold differing views relative to mathematics although they may not be aware. According to him, it is important to acknowledge these philosophical partialities that learners have and assist in understanding the misconceptions they are susceptible to, that result in errors. Korner (1986) mentioned that the philosophy of mathematics provides an interpretive view of mathematics; whereas Makonye (2013) argued that the philosophy of mathematics is central to the epistemological and ontological premises of mathematics which are widely known in mathematics discipline. Osei and Eves (1985) also affirmed that the nature of mathematics goes beyond conceptual and procedural knowledge; it influences how individuals learn mathematics, what they see to be important and what is not. These views suggest that mathematical researchers present different competing perceptions on the nature of mathematics.

Mamba (2011) and Makonye (2013) agreed that misconceptions and errors play a crucial role in the learning because they form part of learners’ conceptual structures that will engage on the new concepts and influence learning. From a constructivist point of view, misconceptions and errors are seen as the natural results of learners’ efforts in constructing their knowledge; these misconceptions are intelligent constructs based on correct but insufficient, not wrong previous knowledge (Makhubele, 2014). This view is in line with Makonye (2013) who argued that Fallibilists view mathematics knowledge to be constructed through informal ways such as the trial and error methods; as a result learners commit errors as proof of their fallibility, which is seen as natural and acceptable. These perceptions imply that learners’ errors should be seen as the normal part of the learning process and should be used as important sources for the learning process. Teachers should acknowledge and accept learners’ errors as cues for discovering what learners know and how that knowledge was constructed (Borasi, 1994; Mamba, 2011; Makonye, 2013; Luneta, 2015).
Another theoretical point of view that guided the study is the role played by teacher quality, content knowledge and pedagogical content knowledge. From chapter two, the researcher outlined from a theoretical perspective, what makes a quality teacher. In conducting mathematics lessons, teachers need knowledge for teaching mathematics, how learners learn, misconceptions they might hold that lead to errors during learning. This knowledge is what Shulman (1986) termed pedagogical content knowledge. Eong et al., (2015) emphasised that teachers needed PCK to help in remedying misconceptions in an effective and sensible way. According to Shulman (1986), if teachers had PCK as one of their credentials, classroom practice changed and learning improved when compared to teachers who did not have PCK. Tirosh (2000) also noticed the improvement in teachers who had PCK in Israel who were able to notice the misconceptions (inverting the dividend as opposed to the divisor, overgeneralisation of whole number rules to fractions) held by learners when compared to the teachers who did not have PCK. These studies showed interesting hypotheses about teachers’ knowledge on learners’ learning.

The study was further guided by the geometry teaching and learning in South Africa. Research by Alex and Mammen (2016) discovered serious concerns in the teaching and learning of geometry in South African secondary schools. Learners experienced difficulties in understanding instructional approaches used by their teachers (Atebe & Schafer, 2009). This was further confirmed in the TIMMS (2006) report which highlighted the dismal achievement of secondary schools learners in Science and Mathematics, with the worst area of performance being geometry (Reddy, 2006). The researcher mentioned numerous reasons in chapter 2 as some of the root causes of poor performance in geometry. De Villiers (1997) recommended an improved primary school geometry syllabus that would guarantee success in secondary school geometry. As a result, the NCS for Intermediate phase took into consideration the levels 1, 2, and 3 of the van Hiele framework (Howse & Howse, 2014).

The study was again guided by the van Hiele framework for geometric thinking. While analysing learners’ errors, the framework aided the researcher in understanding the levels in which learners operated. The framework was built on a model that was made of five levels of geometry development. Van Hieles were concerned about the challenges their learners came across in secondary school geometry. The model derived three main constituents; insight, phases of learning and thought levels (Usiskin, 1982). According to Usiskin (1982), insight takes place when an individual acts in a new situation adequately and with intention; phases of learning explain the phases through which learners progress in attaining higher levels of thinking and the thought process emerged when the van Hieles noticed problems learners went through repeatedly with sections of the subject matter even after various explanations were provided. Their interest lied in wanting to improve the learning outcome, which is when a theoretical framework that included five levels was developed. Based on the framework, according to the van Hieles, based on the
quality of the learning activities, learners should go through the five phases, where movement to the next level is impossible without acquiring full understanding previous levels. It is evident that throughout these levels, teachers play significant role, i.e. planning and development of the learning tasks, directing attention to the characteristics of the geometric figures, introducing terminology and facilitating discussions using these terms and encouraging explanations (Luneta, 2015).

Discussions and findings of this research study were guided by the coding system that was used and explained in chapter 4. This coding system was inspired by various scholars like Dhlamini (2012), Luneta (2015), Makhubele (2014), and Alex and Mammen (2016), and many others who undertook studies on errors and misconceptions in Mathematics.

6.1 Summary of the findings

Findings of this research provided answers to the research questions. Chapter 2 provided different philosophies learners hold with regard to errors and misconceptions. The errors differed with contexts and cognitive levels of questions. Luneta (2015) argued that errors hide behind misconceptions. The following section explains errors and misconceptions identified, and an attempt is made to provide explanations why the learners made the identified errors.

The following major errors were identified:

1. **Lack of logical procedures when solving circle geometry problems.**

   One component of errors that were uncovered in this research is the lack of correct use of procedural knowledge when providing answers that involved mathematical procedures steps. For instance, in question 2 and 3 where participants had to use answers for question 2.2.1 to answer question 2.2.2 and question 3.1.1 to answer questions 3.3 and 3.4. Out of the 80 sampled participants, only eight got question 2.2.1 and 2.2.2 correct and five participants got question 3.1, 3.3 and 3.4 correct. 67 participants performed poorly in question 3 because they failed to score minimum marks in this question. It appeared that there was a decreasing tendency in marks scored by participants from question 2.2.1 - 2.2.2 and 3.1, 3.3 and 3.4 because participants had to get the preceding answer correct before arriving at the next answer. The answers lacked logic in simplification that lead to the answer. Figure below is an example of a solution by P_56 who wrote the error this way:
It was clear from the responses provided that many learners not only lacked knowledge of concepts, but it appeared that they also seemed to lack logical problem solving abilities in mathematical proof. This could be the misconception that cased the errors learners made. This finding is consistent with the studies (de Villiers, Cassim, 2006; Wu; 2006) which discovered that learners experienced challenges with deductive reasoning in a logical and coherent way. Wu (2006:13) advised that learners need to be taught to reason logically because “logical reasoning is the backbone of problem solving”. One of the reasons why learners at secondary school level make such errors in proof is because of the teacher instruction at primary school level (Cassim, 2006).

As a remedy, mathematics curriculum and classroom activities should incorporate learning activities that are central to the “why” in geometry (Luneta, 2015), that would illustrate an idea of how knowledge is constructed. This way, teachers will get a true picture of the learning of concepts and procedures. When learning is centralised on procedures with lack of conceptual understanding, it poses danger because incorrect procedures will be practised. When learners learn without understanding, they learn through rote learning; such learning is not linked to any previous knowledge (Bennie,1998). As a result, new concepts are not well understood and become dissociated knowledge which becomes difficult to remember. Learning by rote becomes a root cause of misconceptions in mathematics because learners try to remember distorted rules (Oliver, 1989). Another problem to learning without understanding is the dissociation of what is learnt in the class and everyday life activities. According to Hoeflich (1994), proof
enables learners to experience activities of mathematicians. The likely misconception identified is insufficient knowledge of procedures. For instance, learners had to use the answer from the question 2.2.1 to arrive at the answer for question 2.2.2, but this was not the case with participants. Furthermore, learners were supposed to use the answer obtained from question 3.1.1 to arrive at the correct answers for questions 3.3 and 3.4. Inability to use the preceding answers to arrive at other conclusions resulted in errors due lack of logical reasoning.

2. Application of circle geometry concepts in incorrect contexts to solve circle geometry problems.

The researcher found that although participants knew concepts, they applied them incorrectly to solve problems. For instance, P_15 provided the following solution to the question:

![Image](image.png)

Figure 6.3: Error type wrong statement given

This reflects lack of conceptual knowledge (Kilpatrick, 2014). This is called the misapplication of concepts (Makhubele, 2014). The misapplication of concepts in circle geometry suggested that participants were unable to apply properties of shapes to solve problems. The kind of reasoning in the figure above implied geometry reasoning of a learner operating at van Hiele level 1. This was evident in 75% of responses to the test items. During the interview, when participants were asked to define the terms (See Appendix), P_17 gave an answer “cyclic quadrilateral is when opposite angles of a cyclic quadrilateral are equal to 180°”. Another participant, P_67 provided the definition of a chord as “chord is a line that cut across a circle or is at a position other than the diameter”. From these definitions, the researcher could deduce that participants knew concepts, but used them incorrectly to answer questions. 65% of participants indicated the blind use of concepts in solving problems. These errors might have occurred as a result of inadequate knowledge of concepts and procedures. Several studies (Cunningham & Roberts, 2010; Karadag & Aktumen, 2011; Siyepu, 2005; Renne, 2004) affirmed that learners make errors due to the lack of appropriate vocabulary to express properties of figures. In the United States of America, Batista (1992) also found out that learners failed to learn basic geometry terms, as a result, are not prepared for advanced proof in geometry. In correcting the problem, in order for learners to be able to apply concepts correctly, they need knowledge, understanding and mathematical wisdom. Özerem (2012) identified five factors that are central in an understanding of geometric
concepts as: imagery, wording, anecdotes, cases in point and formal principles as shown by the figure below:

![Figure 6.4: Factors involved in understanding geometric concepts. Adapted from Özerem (2012)]

A study that was conducted by Aydoğan (2007) also affirmed that understanding concepts in geometry require the use of figures. Various names need to be assigned to concepts and their prototypes (Makhubele, 2014). This suggested that assigning names to concepts and prototypes have a positive influence on geometry thinking. De Villiers (1997) identified the advantage of hierarchical definition of a concept as that all the theorems proved for that concept applies to its special cases.

3. **Participants experience problems with proof questions.**

These test items were based on deductive proof (see Appendix E). 75 out of 80% experienced challenges in applying deductive reasoning to solve problems. For instance, P_54’s response to question 3.4 revealed that participants were not fully nurtured in deductive reasoning. The figure below represents the solution provided by P_54:

![Figure 6.5: Error type wrong proof reasoning](image)

These answers are consistent with research (Healy & Hoyles, 2000; Lin, 2005; Webber, 2004) which indicated that learners experience difficulties when constructing proof. This might have been due to insufficient concept images for doing proof, inability to use own examples, inability to get overall structure of proof, and the inability to use mathematical notation (Moore, 1994). The proof process teaches individuals to reason logically (Hoeflich, 1994), as
result, coherent formulation of arguments allows individuals to see how mathematical results are related to broader mathematical ideas. De Villiers (2004) argued that proof provide insights into why an argument is true, invention of new results with the intellectual challenge and the systematisation of axioms, concepts and theorems. One of the reasons why proof is challenging to learners is that theorems are provided as finished products to learners (Aird & Van Duyn, 2012). This does not give learners enough challenge to think and reason deductively. De Villiers (1997:45) referred to this method as the “end –result of some mathematical activity that preceded it”. Learners are not given the chance to nurture the deductive and logical reasoning skills if teachers religiously following learning approaches that use proof as a finished product.

In correcting the problem, more effort should be put on the curriculum and the syllabus in such a way that activities focus on real mathematical activities rather than mere assimilation of finished products. Research in mathematics education (Blandford, 1908; Freudenthal, 1973) advocated the active participation of learners in coming up with the definitions of geometry concepts. Blandford (in De Villiers, 1998) regarded the product-driven method as “radically vicious method” (1997: 46), and in so doing, learners are denied of the intellectually enriching learning activities. This suggests that teachers should design learning activities that promote higher order thinking.

4. **Participants experience problems with class inclusion, identification and linking of different properties of shapes.**

Analysis for this study revealed that participants had problems in identifying class inclusion of shapes. Identification of a shape means to discover its type and the associated properties (Makhubele, 2014), while class inclusion means the ability to sort and classify different shapes based on their appearances and properties. Within a diagram, participants should be able identify shapes and their respective properties. According to Makhubele (2014), learners who are unable to identify and classify shapes experience difficulties when constructing proof. That is, lack of skills to identify and classify shapes result in errors. For instance, 67 participants failed to link questions 3.1, 3.3 and 3.4. Below is an example of response that was provided by P_40:

![Figure 6.6: Error type Wrong supporting statement](image-url)
These responses are almost similar to other 66 solutions provided by participants. These responses are in line with the findings by Feza and Webb (2005) who argued that learners have problems to recognise class inclusions of shapes. For instance, they might argue that a square is not a rectangle. This was also affirmed by de Villiers (1997), Siyepu (2005) and Roux (2003) who argued that the South African high school learners perform poorly when it comes to questions that include features and properties of class inclusion. Siyepu (2005) further added that secondary school learners in South Africa are unable to identify and name shapes such as kite, rhombus, trapezium, parallelogram and triangle. Learners are unable to identify correct features of these shapes. Van de Walle (2001) indicated that there is a need to develop skills in identifying class inclusion of shapes if learners are to progress through van Hiele levels. According to the van Hiele (1999) capabilities to identify and name shapes is seen as fundamental for geometric conceptualisation. But, learners are unable to correctly identify, name and classify basic geometric shapes (Marchis, 2008; de Villiers, 1997; Siyepu, 2005; Roux, 2003). Makhubele (2014) argued that one of the reasons why learners fail to identify class inclusion of shapes results from learners having developed a concept image without the concept definition; they frequently fail to identify examples that are not the same to their developed concept image.

5. **Negative attitudes towards circle geometry**

The analysis of the focus group interviews revealed that participants held negative attitudes towards geometry and were often anxious. Participants expressed their feelings of nervousness and uneasiness towards the topic. For instance, when asked to deconstruct the diagram in question 3 and
explain all the theorems that are associated with it, P_74 gave the response “I felt frustrated and confused; as a result I answered this question incorrectly. I could not understand what I had to do” Another participant, P18 gave the response “I do not enjoy proof at all. I get confused most of the time, so I put more effort on algebra and finance…….” P_74 gave an answer “I did know what I was doing. I got the answers wrong. I think it is because I not panicked when I came across question 3. To be honest, I never enjoyed geometry……” When asked to point out errors in solving question 3.2, P_74 gave a response “I did not answer the question. I did not know what to do. My mind went blank when I came across question 3.2”. These revelations are in line with studies (Hlalele, 2012; Tella, 2007; Dörfler, 2007; Rossman, 2006; Tsanwani, 2009; Khatoon & Mahmood, 2010) which discovered that learners have negative attitudes towards mathematics. Oludipe (2009) explained that the feelings of nervousness, uneasiness and tension resulted in misconceptions that limit learners’ academic progress in mathematics. Haris and Coy (2003) revealed that affective feelings and worry are some of the factors that led to the decline in learner performance in mathematics. Huhtala (2000) confirmed these when he said “when a learner experiences negative emotions, the learning process can be disabled”. Tella (2007) held the same conviction when he said “students who are anxious, angry or depressed do not learn; people who are caught in these states do not take in information efficiently or deal with it well”.

6. Inability to use $XY$ as a given tangent to prove two angles are equal to $x$ using the tangent-chord theorem

Participants seemed to have held a misconception when they answered question 3.1.1 and 3.1.2 which tested their knowledge and skills on how to prove a tangent. Questions required them to provide reasons for statements:

$\hat{T}_3 = x$

$\hat{P}_1 = x$

55 out of 80 participants gave the reason as ‘corresponding angles with $\hat{T}_3$’. This is surprising because $\hat{P}_1$ and $x$ are not occupying the same relative position at each intersection where the line crosses two others. Below is an example of P_36 response:

Figure 6.8: Error type wrong reasoning.

Twenty participants did not provide reasons for the sub questions. This may have been due to lack of knowledge of concepts. For instance, during the interviews, when asked to define a tangent, P_08 gave the answer as “A tangent is a line that passes through …., I forgot the definition, I am able to point out to it when given a diagram to work with”. This indicated that the
question posed a challenge to participants to answer judging by the responses and marks scored for this question. It would have appeared that this section was rushed through with the belief that learners had a prerequisite knowledge of terms from previous levels (Henning, 2014). Participants’ responses to this question are in line with the studies (Hoffer, 1981; Senk, 1989; Dhlamini, 2012; Siyepu, 2005) which discovered that learning to do proof in geometry is “one of the challenging tasks” for many learners. From the South African point of view, Siyepu (2005) and Dhlamini (2012) indicated that many of the grade 11 and 12 learners experienced difficulties with regard to circle geometry. This might be the reason learners struggle to move to the van Hiele level 4 and 5. Level 4 for Van Hiele is deduction. This level requires learners to be fully conversant with proofs. Level 5 of the van Hiele is rigour; in this level learners should be able to understand relationships between geometry concepts and be able to identify them in an abstract system.

Participants’ achievement such as the lack of ability to define circle geometry terminology in this question further affirmed Atebe’s (2005) discovery that the first three levels are within the thinking abilities of elementary school learners while the last two levels involve mathematical thinking needed at high school and institutions of higher learning, but learners find it difficult to acquire. This suggested that learners lack abilities to answer the questions based on cognitive levels 3 and 4. Cognitive requires learners to answer questions that involve higher order thinking and reasoning. There is no obvious route towards the solution and the solutions require basic understanding and knowledge of concepts (DBE, 2011).

As a remedy, learners need to be exposed to all the characteristics of a tangent. They need to be made aware that for a line to be a tangent, it needs to conform to certain properties, whether there is a visible circle or not. For instance, basic definitions of geometry concepts should be accompanied by clear labelled diagrams illustrating the concepts under definition.

![Figure 6.9: Examples of a tangent](image)
7. Poor visualisation

Learners could not visually identify \(ABCD\) as a cyclic quadrilateral and could not recognise that \(RWTQ\) is not a cyclic quadrilateral

A fair number of participants expressed answers to question 2.2.2 that revealed a misconception. The question tested whether they could visually identify \(ABCD\) as a cyclic quadrilateral since it lies on a visible circle and then use the properties to prove or determine the required angle. Participants were required to determine with reasons the size of \(\widehat{ABC}\). The question proved to be challenging and confusing because they could not identify the cyclic quadrilateral and as a result provided the size of \(\widehat{ABC}\) as 30° and the reason as ‘angles in the semi-circle’. This misconception might have been caused by learners’ lack of problem solving skills and inadequate knowledge of concepts. The focus group interview that was undertaken further revealed that participants could not provide a definition of a cyclic quadrilateral. For example, when asked to define a cyclic quadrilateral, P_08 gave the definition as “Cyclic quadrilateral is when opposite angles of a cyclic quadrilateral are equal to 180°” Another participant, P_12 defined cyclic quadrilateral as “When all angles that are in eight circle touch or its attached to the circle.

Participants showed another misconception, which is inability to visually recognise that \(PQRS\) is a cyclic quadrilateral when answering question 3.4. The question tested participants’ knowledge of proving properties of a cyclic quadrilateral cyclic quadrilateral by proving that \(\hat{Q}_3 = \hat{W}_2\). Seventy five percent provided incorrect answers and could not realise that \(\hat{Q}_3 = \hat{W}_2\); reason being \(\hat{Q}_3\) is an external angle of a cyclic quadrilateral. Participants could not identify that \(\hat{PSR} = \hat{W}_2\) because they are corresponding angles (\(WT // SP\)). This is not surprising because during the focus group interviews when asked to provide any other solution to the problem, P_18 provided the reason as “I did not attempt to answer the question, but there is no any other approach to the question”. This type of reasoning signalled poor visualisation. Fifteen percent of participants did not attempt to answer the question. This example represented 45 participants who could not give the correct solution. Participants could not identify the relationships between the angles, with 19 participants merely stating that \(\hat{Q}_3 = \hat{W}_2\) and the reason being ‘an exterior angle of a cyclic quadrilateral is equal to the interior opposite angle’. This is incorrect because participants could not realise that RWTQ is not a cyclic quadrilateral; which also an indication of poor visualisation by participants. Fifteen percent participants made an assumption that SQ and PR are the diameters of the circle. The focus group interview that was undertaken further revealed that participants could not provide a definition of a cyclic quadrilateral. For example, when asked to define a cyclic quadrilateral, P_08 gave the definition as “Cyclic quadrilateral is when opposite angles of a cyclic...
quadrilateral are equal to 180°”. Another participant, P_12 defined cyclic quadrilateral as “When all angles that are in eight circle touch or its attached to the circle”. These misconceptions might have been caused by inability of participants to do proof. Their responses confirmed the discoveries made by Dhlamini (2012), Siyepu (2005), Mji and Ndlovu (2012) whose studies revealed that learning to write proof is a challenging task for most learners. Senk (1989) in the United States indicated that 70% of secondary school learners lack understanding of concepts of proof. Another study in Nigeria (WAEC, 2003: 175) indicated that proof was removed from the syllabus because it was challenging for both teachers and learners. In South Africa, Siyepu (2005) found out that grade 11 learners experienced difficulties with circle geometry proof. This might be the reason why learners failed to move to levels 5 of the van Hiele model of geometry thinking. Pierre and Dina van Hiele (1986) posited that the first three levels were within the thought level of the elementary school learners, while the last two levels required thinking at high school and higher education. This implies that the primary school geometry in South Africa does not prepare learners adequately for them to operate at advanced levels in secondary school geometry (De Villiers, 2010). As a remedy, learners should be taught all the qualities that characterises a shape to be called a cyclic quadrilateral. For instance, they should be taught that for a quadrilateral to be cyclic, its opposite angles should be equal to 180°; the exterior angle should be equal to the opposite interior angle and angles on the same segment should be equal. All these qualities should hold even if there is no visible circle.

6.2 RECOMMENDATIONS
The following recommendations were suggested based on the findings from the study.

Improve the time allocated for geometry
Data analysis for the study indicated that learners at TVET colleges reach NC (v) level 4 with insufficient knowledge of geometric concepts. These inadequate geometric skills become a stumbling block that hinders the problem solving ability of learners and geometry achievement in NC (v) level 4. This might be as a result that the NC (v) level 4 learners lack the necessary pre-requisites of geometric skills from NC (v) levels 2 and 3. TVET colleges Mathematics policy documents (Mathematics Assessment & Subject Guidelines NC (v) Level 2, 2011 & Mathematics Assessment & Subject Guidelines NC (v) Level 3, 2012) lack fundamental Euclidean geometry concepts that will enable learners to be equipped with higher mental order thinking skills to be successful with problem solving at NC (v) level 4. This suggested that the policy documents are not clear when it comes to what learners need to have done in circle geometry as prior knowledge from level 2
that would enable them to progress to level 3 and through to level 4. Furthermore, at NC (v) level 4, the geometry section (Mathematics Assessment & Subject Guidelines, 2013) is allocated approximately 35 hours. These 35 hours are divided between the sub-sections:

I. Analytical geometry – the use of the Cartesian co-ordinate system to derive and apply equations.

II. Euclidean geometry – Exploring, interpreting and justifying geometric relationships, and

III. Trigonometric identities – solving problems by constructing and interpreting trigonometric models.

Teachers decided to focus on the analytical geometry section of the syllabus. It is to be noted that participants used in this study had completed grade 12 with Euclidean geometry as part of their assessment. Adequate time should be spent on instruction at NC(v) level 2 to reinforce the already learned geometry concepts from grade 12 mathematics. This might remedy some of the misconceptions that learners enter TVET colleges with, that resulted in errors. Early intervention at NC (v) levels 2 and 3 is important for learners experiencing challenges so that they become successful at NC(v) level 4. TVET colleges’ mathematics policy documents should be clear on what is that learners need to have done in circle geometry as prior knowledge. For instance, the Curriculum and Assessment Policy Statement (DBE, 2011) can be used as a reference to guide the time allocation and what Mathematical knowledge and skills should be acquired from one NC(v) level before progressing to the next.

**Improve learners’ problem solving skills**

This recommendation is made because participants performed dismally on question 3 which were based on cognitive levels 3 and 4 (Appendix E). Out of 80 participants, only 5 provided the correct answers. Majority of participants did not attempt the question; those who attempted it provided incorrect answers. This suggested that participants experienced problems with higher order thinking questions. The challenges participants faced in this question suggested that teachers are still using the traditional methods of rote learning. It is therefore recommended that teachers design assessments that focuses on cognitive levels 3 and 4 (DBE, 2011a; DHET, 2013). These are characterised by performing complex procedures and problem solving. They include the type of questions that consist of complex calculations and higher order reasoning, there is no straight route to the answer, and include making relationships between different representations.

**Use of the correct geometry vocabulary**

The recommendation is made because findings from the data analysis showed that participants experienced challenges with regard to geometric concepts and properties. For instance, participants used concepts in incorrect
contexts during problem solving. Interview transcripts further revealed that participants were unable to define the terms like cyclic quadrilateral, chord, circumference, and tangent. Recent research (Makhubele, 2014) argued that for many South African learners, language barrier is one of the common factors that affect Mathematics learning. This is because South Africa like many other African countries, base their Mathematics curricula and content from the western countries which use English and Afrikaans as their medium of instruction (Makhubele, 2014). Geometry teaching and learning has its own set of mathematical notations and vocabulary; as a result, second language learners in public institutions experience problems in transition from the van Hiele level 1 to level 2. Teachers should assist learners by using the appropriate use of terms and concepts during learning when they explain theorems and their relationships. This can be achieved through thorough demonstrations of the roles of definitions, conjectures, theorems, proofs and counter examples using illustrations.

Learning should be aligned with the van Hiele Framework
The researcher made this recommendation because many participants were capable of answering questions that needed one answer step. More than one step answers seemed to be a challenge to participants. This was evident in the number of unanswered test items from this study. This implied that participants were operating at levels 1 and 2 of the van Hiele model. According to the model (van Hiele, 1986), one main reason why learners perform poorly in tests and examinations in geometry in secondary schools is that the syllabus is presented either at the lower levels than that of the learners or at levels that are higher than that learners operate within. This suggests that the teacher and learners in the same class operate at different levels, which makes it difficult for one to understand the thinking of the other. Geometry teachers are encouraged to determine the thinking levels of their learners and try to position their thinking to the same levels of their learners. Learning activities should also be designed according to the different learning needs of learners in the class. These will improve the conceptual understanding of geometry concepts. For example, when learning is centralised on concepts, the teacher should make sure that all learners understand and know properties of shapes; once they master them, then it will become easier for them to identify class inclusion, which according to this study, is inadequate based on participants’ responses. Makhubele (2014) argued that learners are only able to recognise, describe, and differentiate shapes from one another by understanding their properties.

Learners with Mathematics anxiety should be provided with emotional support
This recommendation stemmed from the interview transcripts analysis which showed that participants experienced anxiety feelings. For instance, P_74
gave the following response when asked to explain difficulties he/she faced in question 1.2 “I did not know what I was doing. I think I panicked when I came across the questions. To be honest, I have never enjoyed geometry because in class we focus on simpler diagrams and problems”. Sarason (1988) defined anxiety as feelings of fear, nervousness and uncertainty that individuals experience when they are faced with what they see as a threat to their self-esteem. When learners develop feelings of worry on performing poorly in the test, they experience anxiety. Sarason (1988) further argued that test anxiety is a contributing factor to various negative outcomes such as psychological distress and academic failures. Learners are capable of performing well in assessments but might not do well because of the levels of test anxiety they experience. Learners who experience mathematical learning difficulties develop feelings of disinterest towards mathematics, which lead to worry that result in errors and misconceptions. They in turn develop negative attitudes which make them to shy away from being active participants during learning. Adequate and continuous care and support should be shown towards such learners through constructive individual feedback sessions. Before the start of the geometry learning, the teacher can develop and use the attitudes towards mathematics (geometry section) to measure the attitudes of the learners towards geometry and come up with methods to change negative attitudes. Methods that can be used to change negative attitudes might include beginning the lesson with humorous activities, providing learners with the brief history and the importance of mathematics; how mathematics influence their everyday living and how they can contribute meaningfully towards the lesson to make them appreciate the subject. While it is important to show care and provide support for emotional learners, it should also be important to make them aware to nurture their patience, persistence and tolerance towards Mathematics.

Use of Manipulatives during geometry learning

The van Hiele Framework is a strong advocate of the use of manipulatives because they assist learners to transition from one level to the next. Manipulatives are physical objects that are used as teaching aids to engage learners through hands on learning (Makhubele, 2014). When learners use these models, they become more aware of positions and locations in space.

6.3 CONCLUSIONS

The aim of the research was to identify effective instructional strategies that lecturers at TVET colleges can use to teach NC(v) level 4 learners Euclidean geometry in Gauteng Province. The study aimed to explore the misconceptions and errors learners held with regard to circle geometry. The study was central to providing narrative description of errors and misconceptions. This research can bring awareness to TVET college sector
teachers, and policy makers to assist learners to overcome the type of errors and misconceptions identified.

In pursuing these aims, responses were aligned to the research question and its sub-questions which are:

Research question:

1. What are the dominant errors displayed by NC(v) level 4 learners in circle geometry?
2. What could be some of the misconceptions responsible for the dominant errors?
3. What effective instructional approaches can lecturers of NC(V) level 4 use to teach circle geometry effectively?

Chapter 1 highlighted the main aim of the study was:

1. To examine common errors and the likely misconceptions that learners in NC(v) level 4 have with respect to circle geometry in one TVET college in Gauteng Province.

The importance of this study was highlighted in various ways. It was emphasised that it is necessary for teachers at TVET colleges to identify, diagnose (so that they understand), and remedy NC(v) level 4 learners misconceptions that result in errors. Teaching becomes a futile exercise if it does not remedy different misconceptions learners hold. These misconceptions become parasitic and can be fatal if they are not attended to on time. This made the study of errors and misconceptions highly important in mathematics, with special interest in Euclidean geometry, which is considered to be an undesirable section of the syllabus. Analysis of errors provides teachers with the direction with regard to learners’ procedural and conceptual misunderstandings. These errors can also be a source of knowledge because they provide teachers with insights into learners’ difficulties about certain mathematics skills.

Chapter 2 provided the theoretical background that guided the study. Different views from different researchers based on geometry teaching and learning in South Africa, the role played by proof in geometry, the influence of the van Hiele model on geometry thinking and development, different learner philosophies of mathematics with respect to errors and misconceptions and the role of teacher quality, content knowledge and pedagogical content knowledge on mathematics instruction were presented.
Chapter 3 outlined the research methodology that was applied in the study. Phenomenology was selected as the research design. The researcher chose it because phenomenological methods are relevant at providing detailed explanations on participants’ experiences in their natural settings.

Chapter 4 presented the analysis of participants’ responses to the geometry performance test and the focus group interview transcripts. Numerous tables were designed that enabled the researcher to code and code errors and misconceptions.

Chapter 5 presented the findings and discussions that emanated from the data analysis. Findings were consistent with research (Shannon, 2002) that misconceptions have a negative influence on meaningful learning process. Errors and misconceptions result in learners having negative perceptions towards geometry and these negative perceptions generate negative attitudes towards Mathematics as a subject. Similar to the research (Burger & Shaughnessy, 1985), focus group interviews revealed that errors and misconceptions are related to negative feelings. That is, learners with higher levels of misconceptions expressed their dis-interests and frustrations towards geometry. These findings were further confirmed by Mestre (1989) who argued that there is an attachment in the form of emotions and intellect between learners and their misconceptions because they actively constructed them. These misconceptions end up in emotional dispositions like fear, anxiety and frustrations; which become a threat to a meaningful participation in Mathematics (Buxton, 1981).

In light of the findings that resulted from the study, errors and misconceptions cannot be ignored. Instead, they should be viewed as the stepping stone that transform the learning process. That is, if these misconceptions and errors are identified and diagnosed, teachers can use them as tools that help them to understand the thinking process and mathematical concept formation of learners. Borasi (1994: 69) argued that errors serve as springboards for further investigation. Makonye (2013) also affirmed that errors and misconceptions are a natural part of the learning process because they reveal how learners construct mathematical knowledge as part of their fallibility. Teachers should take it upon themselves to acknowledge and understand learner errors, identify their causes and correct them. It is therefore important for continuous professional development for mathematics teachers, more especially on how to identify and handle errors in geometry.

In concluding, the study advocates the claims put forward by previous researches that the van Hiele framework is one of the best frameworks that can be used in exploring learners’ geometric thinking and development. It seems that Mathematics curriculum developers for the TVET college sector do not consider the levels at which geometric concepts should be introduced.
They should ensure that knowledge about the van Hiele framework is adequately covered in the syllabus and curriculum. Training workshops can be scheduled with the specialists on the van Hiele framework to address Mathematics teachers from NC(v) levels 2, 3 and 4. Teacher need to have sufficient information of the stages and levels at which concepts should be introduced in relation to the framework. They should be further given training in terms of identifying and diagnosing and dealing with them. This will bring about the desired change in terms of learning and achievement in geometry.
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APPENDICES

Appendix 1: Letter to Campus Management requesting permission to conduct the study

1588 Mtipa Street
Dube Location
Soweto
1801

The Deputy Campus Manager
South West Gauteng College
George Tabor Campus
1440 Mncube Drive
Dube Location
Soweto
1868
16 April 2017

REQUEST FOR PERMISSION TO CONDUCT RESEARCH IN SWGC TVET COLLEGE (GEORGE TABOR CAMPUS) WITH THE NC(V) LEVEL 4 MATHEMATICS LEARNERS

Dear Mr Diphare A.M

My name is Puleng Motseki. I am a second year student at the University of Johannesburg registered for Master's degree (Mathematics Education).

I am conducting research on errors and misconceptions that NC(v) level 4 mathematics learners display with respect to Euclidean geometry. Euclidean geometry is an official topic for NC(v) Level 4 TVET Mathematics. The aim of this study is to identify the types of errors learners make and the likely misconceptions as a result of these errors.

I have taught Mathematics for the past nine years and realised that learners do not have a good grasp of Euclidean geometry as a topic. During this time, I came to a realisation that learners displayed misconceptions during problem-solving.
The research will involve a purposive sample of 80 NC(v) level 4 mathematics learners. Participants will write a geometry performance test, which will be followed by focus group interviews based on their performance from the test. In writing this test, learners will use their alias names as a way of protecting their anonymity and identification on their answer sheets. No actual names will be used.

This group is the Engineering (IT) learners whose official timetables reveal that they attend Mathematics every day of the week. The test will be written at 14:30 so as not to interrupt the daily running of the campus.

After the test, 9 learners will be chosen based on their performance for the focus group interviews.

I look forward to your response as soon as it is convenient for you.

Regards

P. Motseki

Research supervisor: Professor Kakoma Luneta
Department of Childhood Development
011 559 2563/5246
082 635 6259
kluneta@uj.ac.za
Appendix 2: Letter from the campus management

Date: 12 May 2017

PERMISSION LETTER BY THE DEPUTY CAMPUS MANAGER

PERMISSION FROM THE SWGC TVET COLLEGE TO CARRY OUT THE RESEARCH

Title of the research: Acquisition of the geometric thought: A case study of the TVET College learners.

I, _______ Assef Aphiwe, the deputy campus manager of the campus, have read and understand the contents of the letter seeking permission to conduct research on the NC(v) level 04 Mathematics learners in Euclidean Geometry.

I therefore permit Puleng Dorah Motseki to carry the above mentioned research study at George Tabor Campus.

Signature: _______
Appendix 3: Signed information sheets for learners

Dear Student

12 May 2017

My name is Motseki P.D. I am a second year student registered for a Master’s degree (Mathematics Education) at the University Of Johannesburg Faculty Of Education. I am conducting a research under the research topic: Acquisition of geometric thought: A case study of TVET college learners.

The research aims to identify the types of errors that NV(v) level 4 Mathematics learners display with regard to circle geometry. The research is to be conducted on purposeful sample of 80 NC(v) level 4 mathematics learners. I have been a Mathematics teacher for the past 8 years and during this time, realised that learners do not have a good understanding on Euclidean geometry. I strongly believe in diagnosing these errors, I can recommend learning strategies to understand circle geometry better.

80 participants will write a geometry performance test. Nine participants will undergo focus group interviews based on their performance in the test (3 best, 3 average and 3 poor). Feedback will be given to all participants.

The test will not replace your term test, nor will it be for marks. It will be a voluntary test, meaning that you do not have to take part if you do not want to. Should you decide half-way through that you no longer want to participate, it will be completely accepted and you will not be affected in any way. Your original names will not be used, but an alias name will be used as a form of identity to protect you. All information about you will be kept confidential and be safely disposed between 3 – 5 years after the completion of the study.

I look forward to working with you. Feel free to contact me should you have further questions.

Regards

Motseki P.D. (076 611 8892)

Signature: [Signature]  Date: 12/05/2017
Appendix 4: Signed learner consent forms

CONSENT FORM

I have read the information presented to me in the letter about the acquisition of geometric thought: a case study of TVET College learners. I was afforded the opportunity to ask any questions related to this study, and to receive satisfactory answers. I am aware that I have the option of some of my responses to be audio recorded to ensure accurate recording and to avoid misunderstandings in my responses. I was informed that I may withdraw my consent at any time without any penalties by the researcher. With full knowledge, I agree, out of my own free will, to participate in this study.

Participant’s Surname and name (IN PRINT):

Tshu ELIZABETH

Participant’s Signature:

Date:

12 Mar 17

Researcher’s Surname and Name (IN PRINT):

Moiseki PLENA

Researcher’s Signature:

Date:

12 Mar 2017

UNIVERSITY OF JOHANNESBURG
CONSENT FORM

I have read the information presented to me in the letter about the acquisition of geometric thought: a case study of TVET College learners. I was afforded the opportunity to ask any questions related to this study, and to receive satisfactory answers. I am aware that I have the option of some of my responses to be audio recorded to ensure accurate recording and to avoid misunderstandings in my responses. I was informed that I may withdraw my consent at any time without any penalties by the researcher. With full knowledge, I agree, out of my own free will, to participate in this study.

Participant's Surname and name (IN PRINT):

Mulaudzi Bensen

Participant's Signature:

Date:

12/05/17

Researcher's Surname and Name (IN PRINT):

Pulelo Motseki

Researcher's Signature:

Date:

12/05/17
CONSENT FORM

I have read the information presented to me in the letter about the acquisition of geometric thought: a case study of TVET College learners. I was afforded the opportunity to ask any questions related to this study, and to receive satisfactory answers. I am aware that I have the option of some of my responses to be audio recorded to ensure accurate recording and to avoid misunderstandings in my responses. I was informed that I may withdraw my consent at any time without any penalties by the researcher. With full knowledge, I agree, out of my own free will, to participate in this study.

Participant’s Surname and name (IN PRINT):
MOLOTSI CALON

Participant’s Signature:

Date:
12/05/2017

Researcher’s Surname and Name (IN PRINT):
MOISEI ALENGA

Researcher’s Signature:

Date:
13/05/2017

UNIVERSITY OF JOHANNESBURG
Appendix 5: Geometry performance test

Test questions on Circle Geometry (Adapted From 2014 National Examination Grade 12 Paper 2)

Date: 05/06/2017

Marks: /25

Duration: 1 Hour

1. In the diagram below, O is the centre of the circle and passes through A, B and C. CÂB = 48°, CÔB = x and C₂ = y

Determine with reasons, values of:

1.1. x  
1.2. y  

2. In the diagram, O is the centre of the circle that passes through A, B, C and D. AOD is the straight line and F is the midpoint of chord CD. Angle ODF = 30°
Determine, with reasons the size of:

2.2.1 \( \angle F \)  

2.2.2 \( \angle ABC \)

3. Two circles in the diagram below have a common tangent XRY at R. W is a point on the smaller circle. The straight line RWS meet the larger circle at S. The chord STQ is a tangent to the smaller circle, where T is a point of contact. Chord RTP is drawn.

Let \( R_1 = x \) and \( R_2 = y \)
3.1 Provide reasons for the following statements:

<table>
<thead>
<tr>
<th>Statement</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{T}_3 = x )</td>
<td>( \text{________} )</td>
</tr>
<tr>
<td>( \hat{P}_1 = x )</td>
<td>( \text{________} )</td>
</tr>
<tr>
<td>( \text{WT} \parallel \text{SP} )</td>
<td>( \text{________} )</td>
</tr>
<tr>
<td>( \hat{S}_1 = y )</td>
<td>( \text{________} )</td>
</tr>
<tr>
<td>( \hat{T}_2 = y )</td>
<td>( \text{________} )</td>
</tr>
</tbody>
</table>

3.2 Prove that \( \text{RT} = \frac{\text{WR} \cdot \text{RP}}{\text{RS}} \) (2)

3.3 Determine with reasons, other two angles that are equal to \( y \) (4)

3.4 Prove that angle \( Q_3 = W_2 \) (3)

3.5 Prove that \( \frac{\text{WR}}{\text{RQ}} = \frac{\text{RS}^2}{\text{RP}^2} \) (3)
Appendix 6: Marking guideline

Geometry Performance Test Solution

1.2 \( x = 96^\circ \) (angle at the centre is twice the angle at the circumference) \( \checkmark \)

1.2 \( \hat{C}_2 + \hat{B}_2 = 180^\circ - 96^\circ = 84^\circ \) (Sum of angles in a triangle) \( \checkmark \)
\[ Y = \hat{B}_2 = 42^\circ \]

2.2.1 \( F_1 = 90^\circ \) (Line from the centre to Midpoint chord) \( \checkmark \)

2.2.2 \( \triangle ABC = 150^\circ \) (opposite angles of a cyclic quad) \( \checkmark \)

3.1 Tangent chord theorem \( \checkmark \)
Tangent chord theorem \( \checkmark \)
Corresponding angles are equal \( \checkmark \)
Angles subtended by chord PQ or angles in the same segment \( \checkmark \)
Alternate angles; WT II SP \( \checkmark \)

3.2 \( \frac{RW}{RS} = \frac{RT}{RP} \) (Line parallel to one side of the triangle; Proportionality theorem; WT II SP) OR
\( \triangle RW \triangle RS \triangle RP \) (angle; angle; angle)
\[ \frac{RW}{RS} = \frac{RT}{RP} \] (Triangle RTW III Triangle RPS)
Therefore \( RT = \frac{RW \cdot RP}{RS} \)

3.3 \( Y = \hat{T}_2 = \hat{R}_3 \) (Tan-Chord theorem)
\( Y = \hat{R}_3 = \hat{Q}_1 \) (Angles in the same segment)
3.4 $\hat{Q}_3 = \text{P}\hat{S}\text{R} \text{ (ext angle of a cyclic quad)}$

$\text{P}\hat{S}\text{R} = \hat{W}_2 \text{ (Corresponding angles; WTI SP)}$

OR

$\hat{Q}_2 = x \text{ (angles in the same segment)}$

$\hat{Q}_3 = 180^\circ - (x + y) \text{ (angles on a straight line)}$

$\hat{W}_2 = 180^\circ - (x + y) \text{ (angles of a Triangle WRT)}$

Therefore $\hat{Q}_3 = \hat{W}_2$

3.5 In $\Delta \text{RTS and } \Delta \text{RQP}$:

$\hat{R}_3 = \hat{R}_2 = Y \text{ (Proven)}$

$\hat{S}_2 = \hat{P}_2 \text{ (angles in the same segment)}$

$\text{RTS} = \text{RQP} \text{ (3rd angle of } \Delta)$

Therefore $\Delta \text{RTS III } \Delta \text{RQP}$
Appendix 7: Focus group interview questions

Focus Group Interview Questions

1. State all the theorems that are associated with circle geometry

_______________________________________________________________
_______________________________________________________________
_______________________________________________________________
_______________________________________________________________
_______________________________________________________________
_______________________________________________________________
_______________________________________________________________
_______________________________________________________________
_______________________________________________________________
_______________________________________________________________

2. How would you regard your knowledge of proof in circle geometry?

_______________________________________________________________
_______________________________________________________________
_______________________________________________________________
_______________________________________________________________

3. How would you explain the following terminology?
   - Tangent
     __________________________________________________________
     __________________________________________________________
     __________________________________________________________
   - Cyclic quadrilateral
     __________________________________________________________
     __________________________________________________________
4. Explain the difficulties encountered in question 1.2

5. Point out errors in solving question 3.2. What other alternative approach may be used to solve question 3.2?

6. Explain the theorem associated with the diagram in question 2.
7. How did you arrive at an answer for question 3.5?

8. Describe the strategy (ies) you applied when solving question 3.

9. Deconstruct the diagram in question 3 and explain all the theorems that are associated with it.
10. Explain the error(s) in 3.3
Table 5.7 Data analysis transcripts

<table>
<thead>
<tr>
<th>RAW DATA</th>
<th>PRELIMINARY CODES</th>
<th>FINAL CODES</th>
<th>GENERIC CATEGORY</th>
<th>THEMES (Burnard, 1991)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
A line drawn from the centre of the circle to the midpoint of a chord, then that line is perpendicular to the chord; angles that are in the same segment of a circle are equal; when the diameter of the circle subtends an angle at the circumference, the subtended angle is a right angle; When a tangent to the circle is drawn, it is perpendicular to the radius at the point of contact. Ngiphelelwe, bakhulume ezinye beng’funga kuzibiza (I’ve run out of others because Ntombi and Lihle have already stated other theorems I wanted to state).

An arc subtended an angle at the centre of the circle and at any point on the circumference, and then the angle at the centre is twice the angle at the circumference; Opposite angles of a cyclic quadrilateral are supplementary; The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle; A line that is drawn perpendicular to the radius at the point the radius intersect the circle, that line is a tangent to the circle.

The line that is drawn from the centre of the circle and is perpendicular to the chord, it bisects the chord; An angle subtended by a chord at a point on the circumference is a right angle, that chord is a diameter; Four vertices of a quadrilateral from which the opposite angles are supplementary, it will be a cyclic quadrilateral; Two tangents that are drawn from the same point outside the circle are equal; An angle between a tangent and a chord that is drawn from the point of contact is equal to an angle in the alternate segment.

My knowledge of proof is moderate, at times, I do well in proof, and some other times, I perform poorly. I think I can do well if I can put more effort and practice regularly.
An angle at the centre of a circle is twice the angle at the circumference; the line drawn from the centre of the circle and is perpendicular to the chord bisect the chord”

| ∠ at centre = 2 ∠ at circumference. | ∠ at centre = 2 ∠ at circumference. | Linking properties of theorems to diagrams | Ability to link theorems to properties of diagrams is linked to success in writing proof. |
| Mid-point chord. | Mid-point chord. | Limited knowledge of circle geometry theorems. | Insufficient knowledge of proof is related to poor performance in geometry. |

Angles in the same segment of a circle are equal; opposite angles of a cyclic quadrilateral are supplementary. That is all I can think of now”

| ∠ in same seg, are equal. | ∠ in same seg, are equal. | Limited knowledge of theorems. | Limited knowledge of circle geometry theorems. |
| Supplementary ∠s | Supplementary ∠s | Limited knowledge of theorems. | Limited knowledge of circle geometry theorems. |
| Limited knowledge of theorems. | Limited knowledge of theorems. | Limited knowledge of circle geometry theorems. | Limited knowledge of circle geometry theorems. |

I did not have any problem with this question.

I did not know what I was doing. I got the answers incorrect. I think it is because I panicked when I came across question 3.

To be honest, I have never enjoyed geometry because in class, we only focus on simple diagrams and problems.
A tangent is a line that passes through the circumference, but touches only one point at that circumference. A cyclic quadrilateral refers to the lines/diagrams which touch the circumference at four points or a diagram with four corners that touches the circumference of a circle at four points. Circumference refers to the points of a full circle or points around a circle. A chord is a line which starts from the circumference and end at the other point of the circumference but doesn't pass through the centre.

A tangent is a line that touches a curved surface at some point but it does not cross it. A chord in geometry is a straight line from which the two endpoints lie on the circle. A circumference is the borderline of the circle. A cyclic quadrilateral is a four sided figure neh, like a square or a rectangular shape, whose vertices lie on a circle.

May I answer the questions without following any specific order? A circumference is the perimeter of an object or outer dimensions of an object. Hhmmm, a tangent is a line that touches a curve at a given point; however, it does not cross it, like, the tangent graph in terms of sine and cosine waves. A cyclic quadrilateral is a four sided figure, where its lines intersect, lying on a circle. Yeah Eish! "

<table>
<thead>
<tr>
<th>Definition of the terms:</th>
<th>Ability to correctly explain:</th>
<th>Ability to recall and explain theorems and terminology and definitions linked with circle geometry is related to correct association in diagrams.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tangent, cyclic quad, circumference, and chord.</td>
<td>Tangent, cyclic quad, circumference, and chord.</td>
<td></td>
</tr>
</tbody>
</table>

May I answer the questions without following any specific order? A circumference is the perimeter of an object or outer dimensions of an object. Hhmmm, a tangent is a line that touches a curve at a given point; however, it does not cross it, like, the tangent graph in terms of sine and cosine waves. A cyclic quadrilateral is a four sided figure, where its lines intersect, lying on a circle. Yeah Eish! "

<table>
<thead>
<tr>
<th>Definition of the term:</th>
<th>Insufficient knowledge of concepts.</th>
<th>Limited recall of circle geometry terms/ definitions.</th>
<th>Limited recall of circle geometry terms leads to feelings of regret.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tangent.</td>
<td>Feeling ashamed</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Feeling ashamed</td>
<td>Unable to define chord</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Errrr…I don’t know how to define a chord. (hides face)
For me, a tangent is a line that touched the circle only on one point. A cyclic quadrilateral is something as given by triangle around it. A chord is a line with end-points on the circumference.

(Hides the face with both hands) I forgot the definitions, I don’t wanna lie. I am able to point out these terms when given a diagram to work with.

Tangent is the line drawn from the radii of the circle to another side of the circle. Cyclic quadrilateral is when all angles that are in 8 circle touch or its attached to the circle. Circumference is when the angle forms the degree of 90˚ or more. Chord is the centre of the circle.

Cyclic quadrilateral is when opposite angles of a cyclic quad are equal to 180°, supplementary. Chord is a line cutting across a circle or is at a position other than the diameter.

A tangent is a line that passes through…. A cyclic quad has two sides that add up to 180; I do not know how to explain the circumference and a chord.

I used the theorem associated with isosceles triangles. This theorem uses the properties of an isosceles triangle such as:

<table>
<thead>
<tr>
<th>Definition of the terms: Tangent, incorrect definition of the term cyclic quad</th>
<th>Incorrect definition of terms, lack of correct vocabulary.</th>
<th>Incorrect definition of terms, lack of correct vocabulary.</th>
<th>Inadequate knowledge of terms/definitions and lack of relevant vocabulary lead to poor achievement and feelings of regret.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Feeling embarrassed, cannot remember concepts.</td>
<td>Insufficient knowledge of circle geometry concepts.</td>
<td>Lack of conceptual knowledge of theorems in circle geometry.</td>
<td>Inadequate knowledge of concepts and definitions lead to poor achievement in circle geometry.</td>
</tr>
</tbody>
</table>

| Incorrect definitions of tangent, cyclic quad and circumference. | Incorrect definitions of cyclic quad, and chord. | Incomplete definition of tangent, incorrect definition of cyclic quad, unable to define the chord. | Properties of isosceles triangle | Correct linking of theorems to diagrams. | Relationship between properties of theorems to theorems linking | Ability to link properties of theorems to theorems linking |

I 155
If two sides are equal, also their angles are equal, and where the two sides meet, they form 90° (perpendicular).

Line from the centre to midpoint is perpendicular to the chord.

<table>
<thead>
<tr>
<th>Midpoint-chord</th>
<th>Midpoint-chord</th>
</tr>
</thead>
<tbody>
<tr>
<td>Perpendicular from the centre to chord, that is, the line from the centre to the midpoint or chord.</td>
<td></td>
</tr>
<tr>
<td>( \hat{B} ) is equal to 150°, opposite angles of a cyclic quadrilateral ABCD, since ( \hat{D} ) is given as 30°.</td>
<td></td>
</tr>
<tr>
<td>Correct linking of theorems to diagrams.</td>
<td></td>
</tr>
<tr>
<td>Relationship between properties of theorems to diagrams.</td>
<td></td>
</tr>
<tr>
<td>Ability to link properties of theorems to diagrams leads to feelings of success in proof.</td>
<td></td>
</tr>
<tr>
<td>( F_1 ) is 90°, because OF bisects CD, or OF is perpendicular to CD. Yes</td>
<td></td>
</tr>
<tr>
<td>Correct value of ( F_1 ), accompanied by incorrect reason.</td>
<td></td>
</tr>
<tr>
<td>Inability to use relevant reasons to conclusions.</td>
<td></td>
</tr>
<tr>
<td>Inability to link answers to correct conclusions.</td>
<td></td>
</tr>
<tr>
<td>Inability to provide relevant supporting reasons to procedures used.</td>
<td></td>
</tr>
<tr>
<td>( F_1 ) is 90°, because ( CF = FD ), this is given</td>
<td></td>
</tr>
<tr>
<td>Incorrect value of ( F_1 ), incorrect reason.</td>
<td></td>
</tr>
<tr>
<td>( F_1 ) is 60° because it is the angle that is subtended by the diameter</td>
<td></td>
</tr>
<tr>
<td>( F_1 ) is 90°, from the diagram, it is given that ( CF = FD ).</td>
<td></td>
</tr>
<tr>
<td>Tangent XRY is associated with the Tangent theorem, diagram SPQR associated with cyclic quad theorem, SPR and XYR associated with tan-chord theorem, WTR and XYR associated with tan-chord theorem, SPR, SP, and WT associated with corresponding angles and TQR associated with alternating angles.</td>
<td></td>
</tr>
<tr>
<td>Correct use of the tan-chord theorem, and properties of alternating Angles.</td>
<td></td>
</tr>
<tr>
<td>Correct linking of theorems to diagrams.</td>
<td></td>
</tr>
<tr>
<td>Relationship between properties of theorems to diagrams.</td>
<td></td>
</tr>
<tr>
<td>Ability to link properties of theorems to diagrams leads to feelings of success in proof.</td>
<td></td>
</tr>
<tr>
<td>Tan-chord theorem, tan xy and chord WR, Corresponding angles are equal in $S_1 = y$; Proportionality theorem WT is parallel to SP; angles in the same segment and exterior angle of a cyclic quadrilateral equal to opposite interior angle.</td>
<td>Correct use of the tan-chord theorem, proportionality theorem and properties of cyclic quad.</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
</tr>
</tbody>
</table>
| Tangent XRY is associated with the Tangent theorem, diagram SPQR associated with cyclic quad theorem, SPR and XYR associated with tan-chord theorem, WTR and XYR associated with tan-chord theorem, SPR, SP, and WT associated with corresponding angles and TQR associated with alternating angles. | Correct use of the tan-chord theorem, proportionality theorem and properties of cyclic quad. | Correct linking of theorems to diagrams. | Relationship between properties of theorems to diagrams. | Ability to link properties of theorems to diagrams leads to feelings of success in proof. |}

**Mam, I did not know the theorems, I did not study for the test.**

Only if I knew how to explain the theorems, I would have answered the question correctly; I had never enjoyed any geometry lesson because I am always confused when the topic is brought up in class, as a result, I decide to skip geometry lessons.

I felt frustrated and confused; as a result I answered this question incorrectly. I could not understand what I had to do.

**I regard my circle geometry knowledge very good because**

<table>
<thead>
<tr>
<th>Feelings of confidence</th>
<th>Use of key learning</th>
<th>Application of</th>
</tr>
</thead>
</table>
hhmmm, during the lesson I listen very carefully to what the educator says. I always consult different textbooks. Infact, I enjoyed proof questions.

Well, I can say I enjoyed solving riders in this test. I applied the solid knowledge I gained from previous grades, coupled with the knowledge I gained in grade 12. It was a smooth sailing for me.

I cannot say I am good at proof problems, again, looking at my performance in the test, I did not do bad also. I needed to be tactical when answering proof question, for example, read the question carefully to gain an understanding of what is needed, I looked for the additional given information, and applied direct reasoning, from there I solved the problems and from there I looked back at what I did and did not.

My knowledge of proof is moderate, at times, I do well in proof, and some other times, I perform poorly. I think I can do well if I can put more effort and practice regularly.

I notice some improvement on this test. I generally perform poorly, but because I had studied, my performance changed. I think that the group discussions on proof I held with my classmate had played a role. I learned a lot, although time was limited. I believe that I can still improve.

I do not enjoy proof at all. I get confused most of the time, so I put more effort on algebra, and finances. Even when I write exams and tests, geometry section becomes the last that I answer because most of the time, I get incorrect answer, so I channel my time to those questions that I understand and that

| Limited knowledge of proof | Limited knowledge of proof | Insufficient knowledge of concepts. |
| Proof is the last choice in any assessment. Channel time and effort to algebra and finance. Improved proof results because of changed strategies. | Proof is the last choice in any assessment. Channel time and effort to algebra. | Proof is regarded difficult, so it is made last choice. Sense of Improvement. |

A good foundation of previous knowledge, Understanding the question before answering it, consulting different study materials, using the given information given as it may contain certain clues, and reflection leads to feelings of confidence.
I know I will get correct. But, I worked on my attitude towards geometry last year; here I am now, noticing major improvement on Euclidean geometry.

My knowledge of proof is limited, I have never got it right, even from lower grades, I used to struggle, and as a result, I have never liked proof.

I find it hard to prove, we only focus on easier and straightforward proof and examples that are done in class, so it becomes difficult for me to solve complex problems.

I am not so confident when I solve geometry riders. I know less, some of them I cannot prove. I understand when the teacher does an example of a proof question, but when I'm alone, I get totally confused.

Well, I didn't encounter any problems. For me, after looking at this question, what came to mind were the properties of triangles. I found the value of x = 96˚, I then looked at $\hat{C_2}$ and added $\hat{B_2}$ and equated it to 180˚, reasoning behind being the sum of the angles of a triangle equals 180˚, and found the value of $y = \hat{B_2} = 42˚$, with the reason being opposite sides are equal.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Improved proof results because of changed attitude.</td>
<td>Difficult to do proof. Proof is central to easier and straightforward assessments.</td>
<td>Difficult to prove. Proof is central to easier and straightforward assessments.</td>
<td>Simple proof questions. Learning difficulty.</td>
</tr>
<tr>
<td>Remembered and applied properties of triangles.</td>
<td>Properties of triangles.</td>
<td>Ability to recall properties of theorems. Ability to link questions together.</td>
<td>Ability to link properties of triangles lead to success in proof questions.</td>
</tr>
<tr>
<td>Lack of self-confidence, imitation and negative attitude and focusing on simple proof exercises in proof lead to insufficient understanding.</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
I think this question was straight forward. For me to get the correct answer, I had to get question 1.1, correct first. In question 1.1, I found the value of $x$ to be 96° by just looking at the diagram. From there, I remembered that angles of a triangle equal to 180°. I subtracted 96° from 180° and equated it to $\hat{C}_2$ added to $\hat{B}_2$; that is, $y = \hat{B}_2 = 42°$. That was it.

I did not experience any problems. Like I said earlier, it was about a strategy. I read the question, and applied the properties of triangles.

<table>
<thead>
<tr>
<th>I did not have any problem with this question, $y = \hat{B}_2 = 42°$</th>
<th>Ability to get first question correct, and use the answer to get the value of $\hat{B}_2$. Remembered and applied properties of triangles.</th>
<th>Properties of triangles. Ability to read and understand the question.</th>
<th>Properties of triangles. Ability to recall properties of theorems. Ability to link questions together.</th>
<th>Correct use of properties of triangles is related to achievement.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$180° - 96° = \hat{C}_2 + \hat{B}_2$, therefore $y = \hat{B}_2 = 42°$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$y = \hat{B}_2 = 42°$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

I did not have any problem with this question.

I guessed the answers; hence I did not get any correct answer for the question.

I did not know what I was doing. I got the answers incorrect. I think it is because I panicked when I came across question 3. To be honest, I have never enjoyed geometry because in class, we only focus on simple diagrams and problems.
Well, I did not encounter any problems with the questions. Err..., I can’t think of the alternative approach now (giggles). But I approached the question from the proportionality point. That is, I looked at \( \frac{RW}{RS} = \frac{RT}{TP} \), my reason was that WT is parallel to SP, \( \therefore RT = \frac{WR\cdot RP}{RS} \).

I did not experience any challenges. I also approached the question like (Name of the previous learner) did, but I can also solve it this way: From the diagram, I can identify triangle RTW and triangle RPS which are equal in sides, from there \( \frac{RW}{RS} = \frac{RT}{TP} \), \( \therefore RT = \frac{WR\cdot RP}{RS} \).

No challenges experienced, I looked at \( \frac{RW}{RS} = \frac{RT}{TP} \), my reason was that WT is parallel to SP, \( \therefore RT = \frac{WR\cdot RP}{RS} \).

| Ability to solve the question using proportionality theorem. Lack of alternative approach to solve the problem. | Proportionality theorem. Limited problem solving approach. | Ability to recall properties of theorems. Ability to link questions together. Limited problem solving approach. | Correct application of the properties of theorems lead to improved performance in proof. |
| Used two approaches to arrive at the same answer. | Knowledge of problem solving approaches. | Knowledge of problem solving approaches. | Limited of problem solving approaches lead to limited knowledge. |

| Ability to apply the properties of parallel lines to arrive to the answer. | Lack of alternative approach to solve the problem. | Incorrect use of similarity as a property of triangles. Inability to answer the question. Conviction that there is no any other approach to the question. | Ability to recognise errors lead to improved performance |


| Correct application of the properties of theorems lead to improved performance in proof. | Lack of procedural knowledge. Insufficient conceptual knowledge. | Limited knowledge of problem solving approaches lead to poor performance. | Insufficient procedural knowledge. |

| I did not answer the question because I did not know what I was doing. | Lack of knowledge on how to solve the problem. | Lack of procedural knowledge | Incorrect use of similarity as a property of triangles. Inability to answer the question. Conviction that there is no any other approach to the question. |

| I did not experience any challenges, but I don’t know of another approach to solve the problem. | I did not attempt to answer the question, but there is no any other approach to the question. | I did not experience any challenges, but I don’t know of another approach to solve the problem. | Incorrect use of similarity as a property of triangles. Inability to answer the question. Conviction that there is no any other approach to the question. |

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>I did not answer the question. I did not know what to do. My mind went blank when I came across question 3.2</td>
<td>Question unanswered. Mental block.</td>
<td>No errors made</td>
<td>Lack of errors</td>
<td>Lack of errors lead to improved achievement.</td>
</tr>
<tr>
<td>I did not encounter any. I did not come across any. None so far.</td>
<td>The question was easy to solve.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>I couldn’t understand the meaning of the word tangent; I think that is why I got the answer wrong.</td>
<td>Lack of understanding of geometry concepts.</td>
<td>Limited conceptual understanding.</td>
<td>Lack of conceptual understanding.</td>
<td>Lack of basic terminology associated with circle geometry. Inability to identify and correct own errors and lack of higher thinking levels lead to poor achievement.</td>
</tr>
<tr>
<td>I am unable to explain the errors because I honestly did not know what I was doing. It was for the first time I came across this type of question. We normally do easier exercises in class, not this.</td>
<td>Inability to identify the error(s). First encounter with high order question.</td>
<td>Inability to operate at higher thinking level.</td>
<td>Lack of higher thinking levels</td>
<td></td>
</tr>
<tr>
<td>I did not answer this question. If we can do revision on this work, I will do better.</td>
<td>Unanswered question. Propose revision on the work.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>I know very little when it comes to theorems; hence I am unable to prove them.</td>
<td>Limited knowledge of circle geometry theorems.</td>
<td>Limited knowledge of circle geometry theorems.</td>
<td>Lack of circle geometry concepts.</td>
<td>Lack of procedural knowledge. Geometry section rushed through. Slow capturing and registering of concepts. Limited background knowledge of theorems lead to poor achievement.</td>
</tr>
<tr>
<td>I really did not know how to answer this question. Maybe having extra classes on geometry will help because I feel that our teacher rushed through this section. I was always left behind.</td>
<td>Lack of procedural knowledge. Geometry section went too fast. Slow to grasping concepts.</td>
<td>Lack of procedural knowledge. Geometry section went too fast. Slow to grasping concepts.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>I could not complete this question. I realised that I was not coping. I do not have good background when it comes to proof and theorems.</td>
<td>Question left unanswered. Poor background of</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>I had already proven that $\frac{RW}{RS} = \frac{RT}{TP}$, my reason was that WT is parallel to SP, therefore $\hat{Q}_3$ is equal to $\hat{W}_2$.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>---</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Theorems.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Able to recognise parallel lines (WT // SP).</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ability to remember the already proved theorems.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ability to recall properties of theorems.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ability to link questions together.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Linking of questions that were previously solved lead to successful proof process.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>I have proven from 3.4, since RT is equal WR.RP; I stated that, that statement is correct.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Experienced learning difficulty.</td>
<td>Lack of procedural knowledge</td>
<td>Lack of procedural knowledge</td>
<td>Lack of procedures leading to poor performance.</td>
<td></td>
</tr>
<tr>
<td>Eish.....This question was difficult for me.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Unanswered question.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>To be able to solve question 3, I needed to break down that complex diagram into its smaller parts first, and then identify theorem properties of each.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Deconstruction of the complex diagram to its smaller constituents made it easy to identify theorems.</td>
<td>Deconstruct complex diagrams to smaller ones and assign properties of theorems.</td>
<td>Deconstruction of the complex diagram to its smaller constituents made it easy to identify theorems.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
theorems and properties of each.

As (Name withheld) stated the trick in answering this question lied in breaking the complex diagram into smaller parts first, then determining the properties of each part, as shown in the solution.

I got this question wrong, but now that I’m discussing it, I come to realise my mistake, \( P_1 = x_1 \) reason being the tan-chord theorem, I initially provided the incorrect reason as exterior angle of a cyclic quad.

I did not have any plan or approach. Proof is difficult to master for me.
I initially tried to breakdown the diagram, but ended up getting confused even more. I then left the question unanswered.

By the time it was announced time up, I was still struggling to make sense of question 3. That is why I did not answer it.

I had memorised the theorems the night before the test. When I was supposed to apply them, I had forgotten what I had memorised. I did not know how to approach question 3. I can only prove a theorem on one simple diagram. If a diagram is complicated like this one, I get more confused.

Question 3 was the most challenging for me. I did not know how to approach it. I was threatened by the diagram; as a result, I found answering questions to be difficult. I did not answer it.
| experienced. | Question not answered. |  |
Table 5: Evaluating participants' test responses per question with respect to identified codes, categories and Van Hiele levels

<table>
<thead>
<tr>
<th>Participant</th>
<th>Question 1</th>
<th>Question 2</th>
<th>Question 3</th>
<th>Van Hiele</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1.1</td>
<td>1.2</td>
<td>2.2.1</td>
<td>2.2.2</td>
</tr>
<tr>
<td>P_01</td>
<td>C_A, ERR_A</td>
<td>C_A, ERR_A</td>
<td>C_A, ERR_A</td>
<td>C_A, ERR_A</td>
</tr>
<tr>
<td>P_02</td>
<td>C_A, ERR_A</td>
<td>C_A, ERR_A</td>
<td>C_A, ERR_A</td>
<td>C_A, ERR_A</td>
</tr>
<tr>
<td>P_03</td>
<td>C_A, ERR_A</td>
<td>C_A, ERR_A</td>
<td>C_A, ERR_A</td>
<td>C_A, ERR_A</td>
</tr>
<tr>
<td>P_04</td>
<td>C_A, ERR_A</td>
<td>C_A, ERR_A</td>
<td>C_A, ERR_A</td>
<td>C_A, ERR_A</td>
</tr>
<tr>
<td>P_05</td>
<td>C_A, ERR_A</td>
<td>C_A, ERR_A</td>
<td>C_A, ERR_A</td>
<td>C_A, ERR_A</td>
</tr>
<tr>
<td>P_06</td>
<td>C_A, ERR_A</td>
<td>C_A, ERR_A</td>
<td>C_C, ERR_D</td>
<td>C_B, ERR_B</td>
</tr>
<tr>
<td>P_07</td>
<td>C_A, ERR_A</td>
<td>C_A, ERR_A</td>
<td>C_C, ERR_D</td>
<td>C_B, ERR_B</td>
</tr>
<tr>
<td>P_08</td>
<td>C_A, ERR_A</td>
<td>C_A, ERR_A</td>
<td>C_C, ERR_D</td>
<td>C_B, ERR_B</td>
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Appendix 8: Ethical Clearance

NHREC Registration Number REC-110613-036

ETHICS CLEARANCE

Dear Puleng Motseki

Ethical Clearance Number: 2018-070

ACQUISITION OF GEOMETRIC THOUGHT: A CASE STUDY OF TVET COLLEGE LEARNERS

Ethical clearance for this study is granted subject to the following conditions:

- If there are major revisions to the research proposal based on recommendations from the Faculty Higher Degrees Committee, a new application for ethical clearance must be submitted.
- If the research question changes significantly so as to alter the nature of the study, it remains the duty of the student to submit a new application.
- It remains the student’s responsibility to ensure that all ethical forms and documents related to the research are kept in a safe and secure facility and are available on demand.
- Please quote the reference number above in all future communications and documents.

The Faculty of Education Research Ethics Committee has decided to

☑ Grant ethical clearance for the proposed research.
☐ Provisionally grant ethical clearance for the proposed research
☐ Recommend revision and resubmission of the ethical clearance documents

Sincerely,

Dr David Robinson
Chair: FACULTY OF EDUCATION RESEARCH ETHICS COMMITTEE
11 September 2018